PRACTICAL SHEET - 7

RANDOM VARIABLES AND MATHEMATICAL EXPECTATION

1. For the given p.d.f, find the value of k.

(i) $f(x) = kx^2$, $0 \leq x \leq 10$

(ii) $f(x) = \begin{cases} k, & x = 0 \\ 2k, & x = 1 \\ 3k, & x = 2 \end{cases}$

Solution:

(i) we know that $\int_{-\infty}^{\infty} f(x) = 1$.

$$\therefore \int_{0}^{10} kx^2 = 1$$

$$\left[\frac{x^3}{3}\right]_0^{10} = \frac{1}{k}$$

$$k = 0.003$$

(ii) For discrete: $\Sigma f(x) = 1$

$$k + 2k + 3k = 1$$

$$k = 1/6$$

2. Use mathematical expectation, to find mean and variance for the pdf.

$$f(x) = \begin{cases} pq^x & ; x = 0,1,2,\dots \\ 0 & ; \text{otherwise} \end{cases}$$

Solution:

$$\text{Mean} = E(x) = \sum_{x=0}^{\infty} x f(x)$$

$$= \sum_{x=0}^{\infty} x p q^x$$

$$= pq[1 + 2q + 3q^2 + \dots]$$

$$= \frac{q}{p}$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 f(x)$$

$$= p[0 + 1 \cdot q + 4 \cdot q^2 + 9 \cdot q^3 + \dots]$$

$$= pq[1 + 4q + 9q^2 + 16q^3 + \dots]$$

$$= pq[(1 + 3q + 6q^2 + 10q^3 + \dots) + (q + 3q^2 + 6q^3 + \dots)]$$

$$= pq[(1-q)^{-3} + q(1-q)^{-3}]$$

$$= \frac{\cancel{p}q(1+q)}{p^{3\,2}}$$

$$= \frac{q}{p^2} + \frac{q^2}{p^2}$$

$$\therefore \text{variance} = \frac{q}{p^2} + \frac{q^2}{p^2} - \left(\frac{q}{p}\right)^2$$

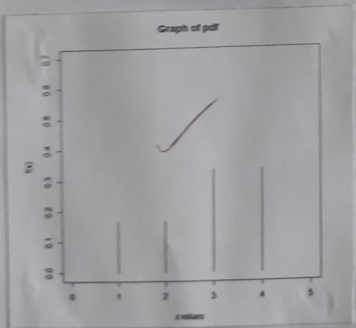
$$= \frac{q}{p^2}$$

Program

```
x <- 1:4 ; x
f <- c(1/6, 1/6, 2/6, 2/6); f
cf <- cumsum(f); cf
df <- data.frame(x,f,cf); df
plot(x,f, type="h", xlab="x values", ylab="f(x)", main="Graph of pdf", xlim=range(0,5),
    ylim=range(0,0.7))
plot(x, cf, type="s", xlab="x values", ylab="F(x)", main="Graph of CDF",
    xlim=range(0,5), ylim=range(0,1.5))
```

Output

x	f	cf
1	0.1666667	0.1666667
2	0.1666667	0.3333333
3	0.3333333	0.6666667
4	0.3333333	1.0000000



3. From the pdf of a discrete random variable, calculate the cdf and then plot the pdf and cdf.

x	1	2	3	4
f(x)	1/6	1/6	2/6	2/6

Solution:

We know that CDF, $F(x) = P(X \le x) = \sum_{x=-\infty}^{x} f(x)$

$F(1) = 1/6$

$F(2) = 2/6$

$F(3) = f(1) + f(2) + f(3)$

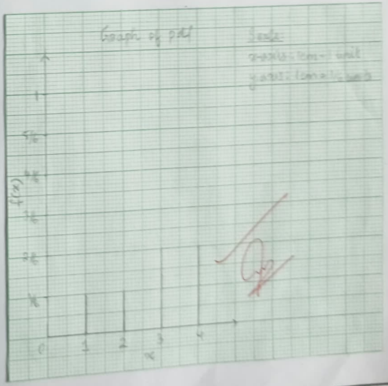
$\quad = 4/6$

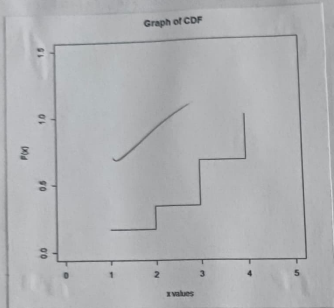
$F(4) = f(1) + f(2) + f(3) + f(4)$

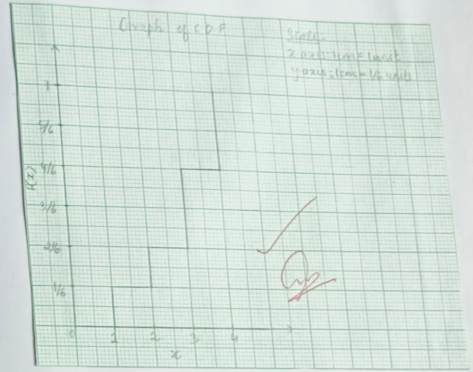
$\quad = 1$

$\therefore$ C.D.F is as follows:

x	1	2	3	4
F(x)	1/6	2/6	4/6	1




Graph of CDF
F(x)
x values
0.0
0.5
1.0
1.5
0 1 2 3 4 5


Graph of CDF
Scale:
x axis: 1cm = 1 unit
y axis: 1cm = 1/6 units
F(x)
x
1/6
2/6
3/6
4/6
5/6
1
0 1 2 3 4

4. For the given p.d.f, calculate its C.D.F and hence calculate the first and second raw moments.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Solution:

$$F(x) = \int_0^x f(x)\,dx$$

$$= \int_0^x \lambda e^{-\lambda x}\,dx$$

$$= \cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_0^x$$

$$= \underline{\underline{1 - e^{-\lambda x}}}$$

First raw moment, $\mu_1' = \int_0^\infty x f(x)\,dx$

$$= \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

$$= \lambda\left\{\left[x\frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty + \int_0^\infty \frac{e^{-\lambda x}}{\lambda}\,dx\right\}$$

$$= \cancel{\lambda} \times \frac{1}{\cancel{\lambda}}\left\{0 + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty\right\}$$

$$= \underline{\underline{\frac{1}{\lambda}}}$$

Second raw moment, $\mu_2' = \int_0^\infty x^2 f(x)\,dx$

$$= \lambda\left\{\left[\frac{x^2 e^{-\lambda x}}{-\lambda}\right]_0^\infty + \frac{2}{\lambda}\int_0^\infty x e^{-\lambda x}\,dx\right\}$$

$$= \cancel{\lambda} \times \frac{1}{\cancel{\lambda}}\left\{0 + \frac{2}{\lambda^2}\right\}$$

$$= \underline{\underline{\frac{2}{\lambda^2}}}$$

5. For a given p.d.f calculate the m.g.f and hence calculate its first raw moment and second raw moment.

$$f(x) = \begin{cases} {}^nC_x p^x q^{n-x} & ; x = 0, 1, 2, \dots, n \\ 0 & ; \text{otherwise} \end{cases}$$

Solution:

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx}\, {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= 1 \cdot q^n + {}^nC_1 \cdot pe^t \cdot q^{n-1} + \dots + 1 \cdot (pe^t)^n$$

$$= \underline{\underline{(q+pe^t)^n}}$$

$$\mu_1' = \frac{d}{dt}[M_x(t)]\Big|_{t=0}$$

$$= \left(n(q+pe^t)^{n-1} \times pe^t\right)_{t=0}$$

$$= \underline{\underline{np}}$$

$$\mu_2' = \frac{d^2}{dt^2}[M_x(t)]\Big|_{t=0}$$

$$= np\left[(q+pe^t)^{n-1}e^t + e^t(n-1)(q+pe^t)^{n-2}pe^t\right]\Big|_{t=0}$$

$$= np[1 + (n-1)p]$$

$$= \underline{\underline{np + n(n-1)p^2}}$$