PRACTICAL SHEET - 8

BIVARIATE DISTRIBUTION

1.

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

The above table is the joint distribution of (x, y). Find:

(i) $f_1(x)$ and $f_2(y)$

(ii) $P(Y \leq 3)$

(iii) $P(Y \leq 3, X \leq 1)$

(iv) $P(Y \leq 3 \mid X \leq 1)$

(v) $P(X \mid Y = 3)$

(vi) $P(X + Y \leq 4)$

Solution:

(i) $f_1(0) = \frac{1}{4}$, $f_1(1) = \frac{5}{8}$, $f_1(2) = \frac{1}{8}$

$f_2(1) = \frac{3}{32}$, $f_2(2) = \frac{3}{32}$, $f_2(3) = \frac{11}{64}$, $f_2(4) = \frac{13}{64}$, $f_2(5) = \frac{3}{16}$, $f_2(6) = \frac{1}{4}$

(ii) $P(Y \leq 3) = \frac{23}{64}$

(iii) $P(Y \leq 3, X \leq 1) = \frac{9}{32}$

(iv) $P(Y \leq 3, X \leq 1) = \frac{f(y \leq 3, x \leq 1)}{f_1(x \leq 1)}$

$= \frac{9}{28}$

(v) $P(X \mid Y = 3) = P(X = 0 \mid Y = 3)$

$= f(x = 0, y = 3) / f_2(3)$

$P(X=0|Y=3) = \frac{2}{11}$

$P(X=1|Y=3) = \frac{f(x=1, y=3)}{f_2(3)}$

$= \frac{8}{11}$

$P(X=2|Y=3) = \frac{f(x=2, y=3)}{f_2(3)}$

$= \frac{1}{11}$

(vi) $P(X+Y \leq 4) = P(X=0, Y=4) + P(X=1, Y=3) + P(X=2, Y=2) + P(X=0, Y=1) + P(X=0, Y=2)$

$+ P(X=0, Y=3) + P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=1)$

$= \frac{13}{32}$

$$f(x,y) = \begin{cases} \frac{x+y}{21} & , x = 1,2,3 \\ & y = 1,2 \end{cases}$$

Find $f_1(x)$ and $f_2(y)$ and check for independence.

Solution:

$$f_1(x) = \sum_{y=1}^{2} f(x,y)$$

$$= \frac{x+1+x+2}{21}$$

$$= \frac{2x+3}{21} \quad ; \quad x = 1,2,3$$

$$f_2(y) = \sum_{x=1}^{3} f(x,y)$$

$$= \frac{1+y+2+y+3+y}{21}$$

$$= \frac{6+3y}{21} \quad ; \quad y = 1,2$$

$$f_1(x) \cdot f_2(y) = \frac{(2x+3)(6+3y)}{21 \times 21} \neq \frac{x+y}{21}$$

$\therefore$ X and Y are not independent.

3. $f(x,y) = cxy$, $0 < x < 4$

$1 < y < 5$

is a joint probability distribution function. Determine c, also check for independence.

Solution:

$$\iint_{x\,y} f(x,y)\,dy\,dx = 1$$

$$\int_0^4\int_1^5 cxy\,dy\,dx = 1$$

$$c\int_0^4 x\left[\frac{y^2}{2}\right]_1^5 dx = 1$$

$$\int_0^4 x\,dx = \frac{1}{12c}$$

$$\left[\frac{x^2}{2}\right]_0^4 = \frac{1}{12c}$$

$$c = \frac{1}{96}$$

$$\therefore f(x,y) = \frac{xy}{96} \quad , \quad 0 < x < 4, \; 1 < y < 5$$

$$f_1(x) = \int_y f(x,y)\,dy$$

$$= \int_1^5 \frac{xy}{96}\,dy$$

$$= \frac{x}{8}$$

$$f_2(y) = \int_x f(x,y)\,dx$$

$$= \int_0^4 \frac{xy}{96}\,dx$$

$$= \frac{y}{12}$$

$$f_1(x) \cdot f_2(y) = \frac{xy}{96} = f(x,y)$$

X and Y are independent.

4. If $f(x,y) = cx(1-y)$; $0<x<y<1$. Find c; also examine whether the variables are independent.

Solution:

$$\int_0^1\int_x^1 cx(1-y)\,dy\,dx = 1$$

$$\int_0^1 x\left[y-\frac{y^2}{2}\right]_x^1 dx = \frac{1}{c}$$

$$\int_0^1 x\left[\frac{1}{2}-x+\frac{x^2}{2}\right]dx = \frac{1}{c}$$

$$\left[\frac{x^2}{4}-\frac{x^3}{3}+\frac{x^4}{8}\right]_0^1 = \frac{1}{c}$$

$$\underline{c = 24}$$

$$f_1(x) = \int_y f(x,y)\,dy$$

$$= 24x\int_x^1 (1-y)\,dy$$

$$= 24x\left[y-\frac{y^2}{2}\right]_x^1$$

$$= 24\left(\frac{x}{2}-x^2+\frac{x^3}{2}\right)$$

$$= 12x(1-2x+x^2)$$

$$= \underline{12x(1-x)^2}$$

$$f_2(y) = \int_x f(x,y)\,dx$$

$$= \int_0^y 24x(1-y)\,dx$$

$$= 24(1-y)\left[\frac{x^2}{2}\right]_0^y$$

$$= \underline{12(1-y)\cdot y^2}$$

$$f_1(x)\cdot f_2(y) \neq f(x,y)$$

$\therefore$ X and Y are not independent.

5. Let $f(x_1, x_2) = \begin{cases} 21x_1^2 x_2^3 & ; \ 0 < x_1 < x_2 < 1 \\ 0 & , \text{ otherwise} \end{cases}$, be the joint p.d.f of x_1 and x_2.

Find the conditional mean and variance of x_1 given $x_2 = \frac{1}{2}$, $0 < x_2 < 1$.

Solution:

$f(x_1, x_2) = 21x_1^2 x_2^3 \ ; \ 0 < x_1 < x_2 < 1$

$E(X_1 | X_2) = \int_{x_1} x_1 f(x_1 | x_2)\, dx_1$

But $f(x_1 | x_2) = \dfrac{f(x_1, x_2)}{f_2(x_2)}$

Now, $f_2(x_2) = \int_{x_1} f(x_1, x_2)\, dx_1$

$= \int_0^{x_2} 21 x_1^2 x_2^3\, dx_1$

$= 7x_2^3 \cdot x_2^3$

$= 7x_2^6$

$\therefore f(x_1 | x_2) = \dfrac{21 x_1^2 \cancel{x_2^3}}{7 x_2^{\cancel{6}\,3}}$

$= \dfrac{3x_1^2}{x_2^3}$

$E(X_1 | X_2) = \int_0^{x_2} x_1 \cdot \dfrac{3x_1^2}{x_2^3}\, dx_1$

$= \dfrac{3}{x_2^3} \left[\dfrac{x_1^4}{4} \right]_0^{x_2}$

$= \dfrac{3}{4} x_2 \ , \ 0 < x_2 < 1,$

$E(X_1^2 | X_2) = \int_0^{x_2} x_1^2 \cdot \dfrac{3x_1^2}{x_2^3}\, dx_1$

$= \dfrac{3}{x_2^3} \left[\dfrac{x_1^5}{5} \right]_0^{x_2}$

$= \dfrac{3}{5} x_2^2 \ , \ 0 < x_2 < 1$

$$V(X_1|X_2) = E(X_1^2|X_2) - [E(X_1|X_2)]^2$$

$$= \frac{3}{5}x_2^2 - \left(\frac{3}{4}x_2\right)^2$$

$$= \frac{3}{5}x_2^2 - \frac{9}{16}x_2^2$$

$$= \frac{3}{80}x_2^2 \; ; \; 0 < x_2 < 1$$

When $x_2 = \frac{1}{2}$,

$$E(X_1|X_2 = \tfrac{1}{2}) = \frac{3}{4} \times \frac{1}{2}$$

$$= 3/8$$

$$V(X_1|X_2 = \tfrac{1}{2}) = \frac{3}{80} \times \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{320}$$