Boundedness and Stability

第二次　**SDEM 5.3**

杨徵羽

2022 年 11 月 24 日

markovianSwitching.m

```matlab
function [rGrid] = markovianSwitching(Gamma, T, stepSize)
N = length(Gamma); % the quantity of states
iJump = 1; tJump(1) = 0; rJump(1) = randi(N); % initial value setting
while tJump(iJump) < T
    tJump(iJump+1) = tJump(iJump) + exprnd(-1/Gamma(rJump(iJump),rJump(iJump)));
    distribution = Gamma(rJump(iJump),:);
    distribution(rJump(iJump)) = 0;
    distribution = distribution/sum(distribution);
    rJump(iJump+1) = discretize(rand, cumsum([0 distribution]));
    iJump = iJump + 1;
end
%% convert to time grid in order to be compatible with numerical solutions
nJump = length(tJump);
tGrid = 0:stepSize:T; nGrid = length(tGrid); rGrid = zeros(1, nGrid);
tJumpGrid = ceil(tJump/stepSize);
for iJump = 1:nJump-1
  for iGrid = tJumpGrid(iJump)+1:tJumpGrid(iJump+1)
    rGrid(iGrid) = rJump(iJump);
  end
end
rGrid = rGrid(1:nGrid); % delete calculation beyond T
```

SDEMexample5_4.m

```matlab
clear;clc;close all;
nSample = 25; T = 20; stepSize = 0.01;
alpha = @(r)(1*(r==1)+(-1/2)*(r==2)); % a simple way of representing piecewise function
sigma = 5; gamma = 1.5; Gamma = [[-4 4]; [gamma -gamma]];
r = markovianSwitching(Gamma, T, stepSize); % one r(t) for all sample paths
tGrid = 0:stepSize:T;
nGrid = length(tGrid);
f = @(x,t,r) alpha(r) * x;
g = @(x,t,r) sigma;
x = zeros(nSample, nGrid); % not `zeros(nGrid)`!!!
for iSample = 1:nSample
    x(1) = 1;
    for iGrid = 1:nGrid - 1
        x(iSample, iGrid + 1) = x(iSample, iGrid) + f(x(iSample, iGrid), iGrid, r(iGrid)) * stepSize + g(x(iSample, iGrid), iGrid, r(iGrid)) * normrnd(0, stepSize);
    end
end
for iSample = 1:nSample
    plot(tGrid,x(iSample, :)); hold on
end
p = 2; pthmoment = mean(x.^p, 1);
plot(tGrid,pthmoment)
```

全章结构

稳定性种类	节	定义	V 函数判别法	系数判别法	例子
p 阶矩渐近有界	5.2	1	2	3	4,5
p 阶矩指数稳定	5.3	7	8	10,12,16	25,26,27
p 阶矩渐近稳定	5.4	28	29,30,31		32,33
a.s. 指数稳定	5.3	7	9	10,12,14,16	
a.s. 渐近稳定	5.4	28	29		
依概率稳定	5.5	34	35		
依概率渐近稳定	5.5	34	36		38
依概率渐近大范围稳定	5.5	34	37		
依分布渐近稳定	5.6	40	43	44	45,46

5.3 Exponential Stability

- Definition 5.7 p 阶矩/a.s. 指数稳定性定义
- Lemma 5.1: 初值不为 0 $\leftrightarrow$ 解不为 0（反证法，LV 计算，辅助函数 $e^{\lambda t}V$，含停时截断的 Itô 公式）
- Theorem 5.8: p 阶矩指数稳定性 V 函数判别法（辅助函数 $e^{\lambda t}V$，含停时截断的 Itô 公式）
- Theorem 5.9: p 阶矩指数稳定 $\rightarrow$ a.s. 指数稳定（均值不等式，Hölder 不等式，BDG 不等式，Chebyshev-Borel-Cantelli 三板斧）
- Corollary 5.10: 系数判别法（LV 计算，最大特征值）

上次的 LV 计算 (SDEM p.159)

By (5.7) and (5.9) we compute the operator LV from $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ to $\mathbb{R}$ as follows:

$$\begin{aligned}
LV(x,t,i) &= pq_i|x|^{p-2}x^T f(x,t,i) + \tfrac{1}{2}pq_i|x|^{p-2}|g(x,t,i)|^2 \\
&\quad + \tfrac{1}{2}p(p-2)q_i|x|^{p-4}|x^T g(x,t,i)|^2 + \sum_{j=1}^{N}\gamma_{ij}q_j|x|^p \\
&\le pq_i|x|^{p-2}\left[x^T f(x,t,i) + \frac{p-1}{2}|g(x,t,i)|^2\right] + \sum_{j=1}^{N}\gamma_{ij}q_j|x|^p \\
&\le pq_i|x|^{p-2}(\beta_i|x|^2+\alpha) + \sum_{j=1}^{N}\gamma_{ij}q_j|x|^p \\
&\le \left(p\beta_i q_i + \sum_{i=1}^{N}\gamma_{ij}q_j\right)|x|^p + \alpha pq_i|x|^{p-2} \\
&= -\lambda_i|x|^p + \alpha pq_i|x|^{p-2}. \qquad (5.10)
\end{aligned}$$

BDG 不等式 (SDE p.127)

Theorem 7.3 *Let* $g \in \mathcal{L}^2(R_+; R^{d\times m})$. *Define, for* $t \geq 0$,

$$x(t) = \int_0^t g(s)dB(s) \qquad \text{and} \qquad A(t) = \int_0^t |g(s)|^2 ds.$$

Then for every $p > 0$, *there exist universal positive constants* c_p, C_p *(depending only on* p*), such that*

$$c_p E|A(t)|^{\frac{p}{2}} \leq E\Big(\sup_{0\leq s\leq t} |x(s)|^p\Big) \leq C_p E|A(t)|^{\frac{p}{2}} \tag{7.5}$$

for all $t \geq 0$. *In particular, one may take*

$$\begin{array}{lll} c_p = (p/2)^p, & C_p = (32/p)^{p/2} & \text{if } 0 < p < 2; \\ c_p = 1, & C_p = 4 & \text{if } p = 2; \\ c_p = (2p)^{-p/2}, & C_p = \left[p^{p+1}/2(p-1)^{p-1}\right]^{p/2} & \text{if } p > 2. \end{array}$$

Chebyshev 不等式 (SDEM p.7)

(iii) **Chebyshev's inequality**

$$P\{\omega : |X(\omega)| \geq c\} \leq c^{-p} E|X|^p$$

if $c > 0,\ p > 0,\ X \in L^p$.

A simple application of Hölder's inequality implies

$$(E|X|^r)^{1/r} \leq (E|X|^p)^{1/p}$$

if $0 < r < p < \infty,\ X \in L^p$.

Borel-Cantelli 引理 (SDEM p.10)

Lemma 1.2 *(Borel–Cantelli's lemma)*

(1) If $\{A_k\} \subset \mathcal{F}$ *and* $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, *then*

$$\mathbb{P}\Big(\limsup_{k\to\infty} A_k\Big) = 0.$$

That is, there exists a set $\Omega_1 \in \mathcal{F}$ *with* $\mathbb{P}(\Omega_1) = 1$ *and an integer-valued random variable* k_1 *such that for every* $\omega \in \Omega_1$ *we have* $\omega \notin A_k$ *whenever* $k \geq k_1(\omega)$.

(2) If the sequence $\{A_k\} \subset \mathcal{F}$ *is independent and* $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$, *then*

$$\mathbb{P}\Big(\limsup_{k\to\infty} A_k\Big) = 1.$$

That is, there exists a set $\Omega_2 \in \mathcal{F}$ *with* $\mathbb{P}(\Omega_2) = 1$ *such that for every* $\omega \in \Omega_2$, *there exists a sub-sequence* $\{A_{k_i}\}$ *such that the* ω *belongs to every* A_{k_i}.

最大/小特征值 (SDEM p.59)

A square matrix A is said to be symmetric if $A = A^T$. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, its eigenvalues are all real numbers, namely $\lambda(A) \subset \mathbb{R}$. Denoted its largest and smallest eigenvalue by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. In this symmetric case, the spectral radius of A becomes $\rho(A) = |\lambda_{\max}(A)| \vee |\lambda_{\min}(A)|$. Moreover,

$$\lambda_{\max}(A) = \sup_{x\in\mathbb{R}^n, |x|=1} x^T A x \quad \text{and} \quad \lambda_{\min}(A) = \inf_{x\in\mathbb{R}^n, |x|=1} x^T A x. \tag{2.29}$$

The following inequality is also useful

$$\lambda_{\min}(A)|x|^2 \le x^T A x \le \lambda_{\max}(A)|x|^2, \quad \forall x \in \mathbb{R}^n. \tag{2.30}$$