Boundedness and Stability

第一次　**SDEM 5.1–5.2**

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5.1 Introduction

- 稳定性引入（圆形壁上的小球）
- 稳定性定义（类比极限，先按 ω 分类，再按 t 分类）
- Lyapunov 方法（V 相当于势能，LV 相当于求导）
- 全章结构（两步建立稳定性判据）
- 对系统的假设：$f(0,t)=0, g(0,t)=0$（不为 0 就作差）
- 对系统的假设：初值不随机（利用全概率公式）

全章结构

稳定性种类	节	定义	V 函数判别法	系数判别法	例子
p 阶矩渐近有界	5.2	1	2	3	4,5
p 阶矩指数稳定	5.3	7	8	10,12,16	25,26,27
p 阶矩渐近稳定	5.4	28	29,30,31		32,33
a.s. 指数稳定	5.3	7	9	10,12,14,16	
a.s. 渐近稳定	5.4	28	29		
依概率稳定	5.5	34	35		
依概率渐近稳定	5.5	34	36		38
依概率渐近大范围稳定	5.5	34	37		
依分布渐近稳定	5.6	40	43	44	45,46

初值不随机 (SDE p.110)

Definition 2.1 *(i) The trivial solution of equation (1.2) is said to be stochastically stable or stable in probability if for every pair of* $\varepsilon \in (0,1)$ *and* $r > 0$, *there exists a* $\delta = \delta(\varepsilon, r, t_0) > 0$ *such that*

$$P\{|x(t; t_0, x_0)| < r \text{ for all } t \geq t_0\} \geq 1 - \varepsilon$$

Let us now explain why we need only to discuss the case of constant initial values. Suppose one would like to let the initial value x_0 be a random variable. He then should replace e.g. "$|x_0| < \delta$" by "$|x_0| < \delta$ a.s." in the definition accordingly. This seems more general but is in fact equivalent to the above definition. For example, suppose we have (i), then for any random variable x_0 with $|x_0| < \delta$ a.s., we have

$$\begin{aligned}
& P\{|x(t; t_0, x_0)| < r \text{ for all } t \geq t_0\} \\
= & \int_{S_\delta} P\{|x(t; t_0, y)| < r \text{ for all } t \geq t_0\} P\{x_0 \in dy\} \\
\geq & \int_{S_\delta} (1 - \varepsilon) P\{x_0 \in dy\} = 1 - \varepsilon.
\end{aligned}$$

初值不随机 (SDE p.119)

Definition 3.1 *The trivial solution of equation (1.2) is said to be almost surely exponentially stable if*

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t;t_0,x_0)| < 0 \qquad a.s. \tag{3.1}$$

to the equilibrium position $x = 0$ exponentially fast. Moreover, let us explain once again why we only need to discuss the case of constant initial values. For a general initial value x_0 (i.e. x_0 is $\mathcal{F}_{t_0}$-measurable and belongs to $L^2(\Omega; R^d)$), it follows from (3.1) that

$$\begin{aligned}
&P\Big\{\limsup_{t\to\infty} \frac{1}{t} \log |x(t;t_0,x_0)| < 0\Big\} \\
&= \int_{R^d} P\Big\{\limsup_{t\to\infty} \frac{1}{t} \log |x(t;t_0,y)| < 0\Big\} P\{x_0 \in dy\} \\
&= \int_{R^d} P\{x_0 \in dy\} = 1,
\end{aligned}$$

初值不随机 (SDE p.127)

Definition 4.1 *The trivial solution of equation (1.2) is said to be pth moment exponentially stable if there is a pair of positive constants* λ *and* C *such that*

$$E|x(t;t_0,x_0)|^p \le C|x_0|^p e^{-\lambda(t-t_0)} \qquad \text{on } t \ge t_0 \tag{4.1}$$

for all $x_0 \in R^d$. *When* $p = 2$, *it is usually said to be exponentially stable in mean square.*

exponent is negative. Moreover, if one wishes to consider the initial value of an $\mathcal{F}_{t_0}$-measurable random variable $x_0 \in L^p(\Omega; R^d)$, then, by (4.1),

$$E|x(t;t_0,x_0)|^p = \int_{R^d} E|x(t;t_0,y)|^p P\{x_0 \in dy\}$$

$$\le \int_{R^d} C|y|^p e^{-\lambda(t-t_0)} P\{x_0 \in dy\} = CE|x_0|^p e^{-\lambda(t-t_0)}.$$

Besides, noting $(E|x(t)|^{\hat{p}})^{1/\hat{p}} \le (E|x(t)|^p)^{1/p}$ for $0 < \hat{p} < p$ we see that the pth moment exponential stability implies the $\hat{p}$th moment exponential stability.

5.2 Asymptotic Boundedness

- Definition 5.1: p 阶矩渐近有界性定义
- Theorem 5.2: V 函数判别法（辅助函数 $e^{\lambda t}V$，停时截断，$\mathscr{K}$ 类函数，Jensen 不等式）
- Theorem 5.3: 系数判别法（LV 计算，Young 不等式，M-矩阵）
- Examples 5.4, 5.5: 数值算例

Example 5.4（α 为常数）

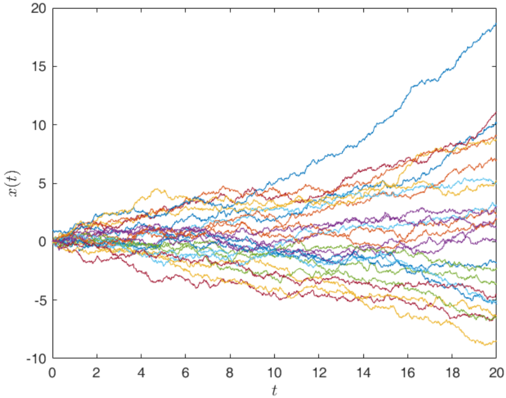

图 1: 样本轨迹，$\alpha = 0.1, \sigma = 0.5$

Example 5.4（α 为常数）

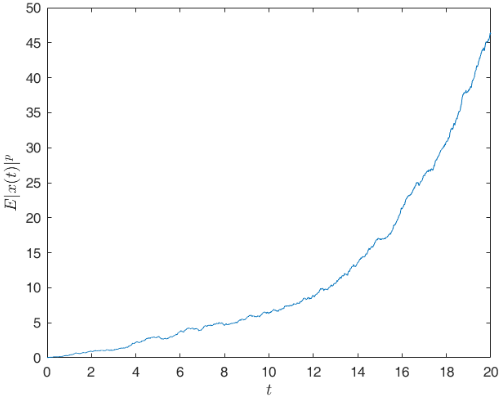

图 2: 2 阶矩，$\alpha = 0.1, \sigma = 0.5$

Example 5.4（α 为常数）

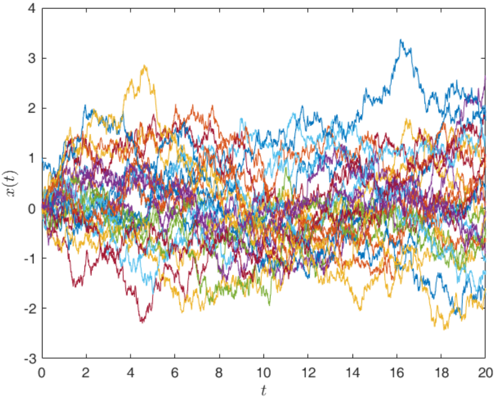

图 3: 样本轨迹，$\alpha = -0.1, \sigma = 0.5$

Example 5.4（α 为常数）

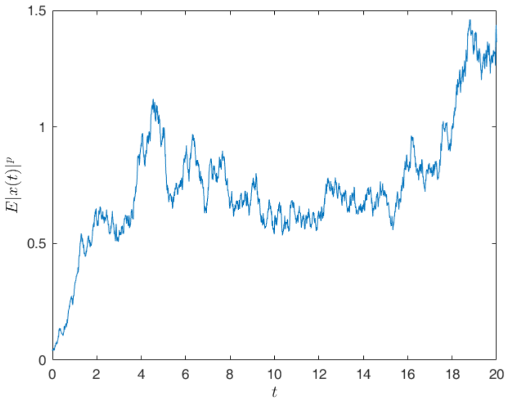

图 4: 2 阶矩，$\alpha = -0.1, \sigma = 0.5$

Example 5.4（α 与 $r(t)$ 有关）

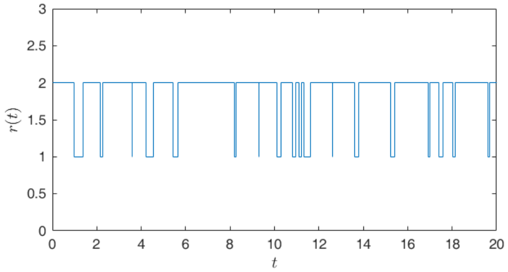

图 5: 马尔可夫切换，$\gamma = 0.5, \sigma = 0.5$

Example 5.4（α 与 $r(t)$ 有关）

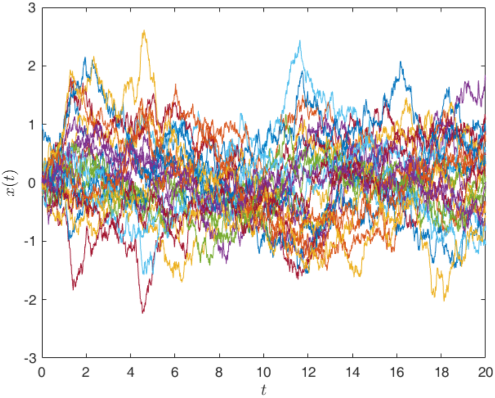

图 6: 样本轨迹，$\gamma = 0.5, \sigma = 0.5$

Example 5.4（α 与 $r(t)$ 有关）

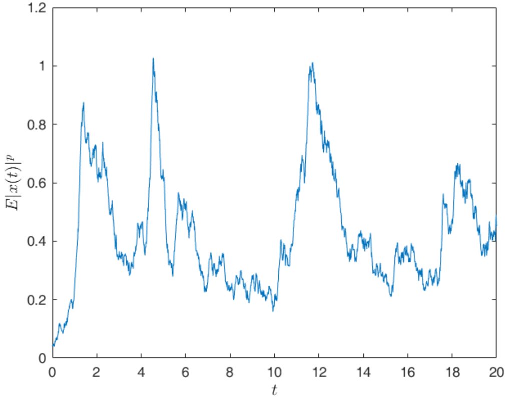

图 7: 2 阶矩，$\gamma = 0.5, \sigma = 0.5$

Example 5.4（α 与 $r(t)$ 有关）

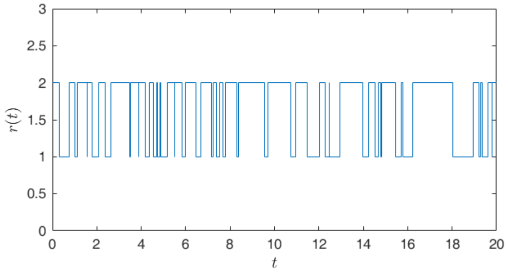

图 8: 马尔可夫切换，$\gamma = 1.5, \sigma = 0.5$

Example 5.4（α 与 $r(t)$ 有关）

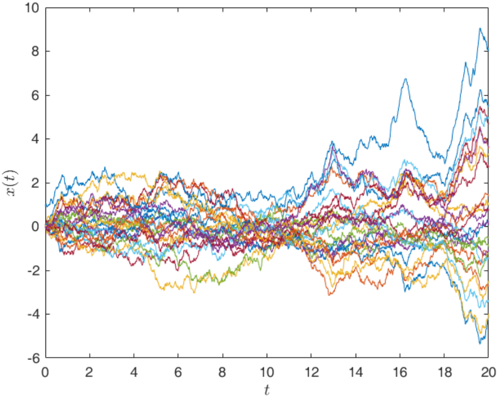

图 9: 样本轨迹，$\gamma = 1.5, \sigma = 0.5$

Example 5.4（α 与 $r(t)$ 有关）

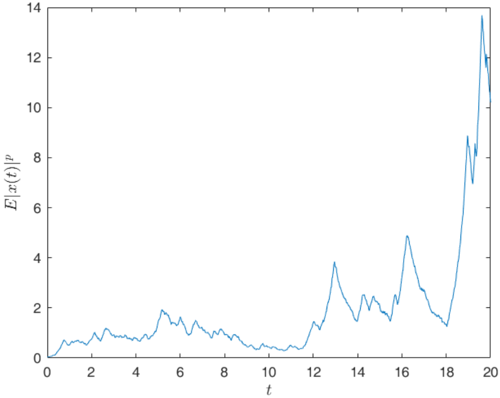

图 10: 2 阶矩，$\gamma = 1.5, \sigma = 0.5$

markovianSwitching.m

```matlab
function [rGrid] = markovianSwitching(Gamma, T, stepSize)
N = length(Gamma); % the quantity of states
iJump = 1; tJump(1) = 0; rJump(1) = randi(N); % initial value setting
while tJump(iJump) < T
     tJump(iJump+1) = tJump(iJump) + exprnd(-1/Gamma(rJump(iJump),rJump(iJump))
          );
     distribution = Gamma(rJump(iJump),:);
     distribution(rJump(iJump)) = 0;
     distribution = distribution/sum(distribution);
     rJump(iJump+1) = discretize(rand, cumsum([0 distribution]));
     iJump = iJump + 1;
end
%% convert to time grid in order to be compatible with numerical solutions
nJump = length(tJump);
tGrid = 0:stepSize:T; nGrid = length(tGrid); rGrid = zeros(1, nGrid);
tJumpGrid = ceil(tJump/stepSize);
for iJump = 1:nJump-1
   for iGrid = tJumpGrid(iJump)+1:tJumpGrid(iJump+1)
      rGrid(iGrid) = rJump(iJump);
   end
end
rGrid = rGrid(1:nGrid); % delete calculation beyond T
```

SDEMexample5_4.m

```matlab
clear;clc;close all;
nSample = 25; T = 20; stepSize = 0.01;
alpha = @(r)(1*(r==1)+(-1/2)*(r==2)); % a simple way of representing piecewise function
sigma = 5; gamma = 1.5; Gamma = [[-4 4]; [gamma -gamma]];
r = markovianSwitching(Gamma, T, stepSize); % one r(t) for all sample paths
tGrid = 0:stepSize:T;
nGrid = length(tGrid);
f = @(x,t,r) alpha(r) * x;
g = @(x,t,r) sigma;
x = zeros(nSample, nGrid); % not `zeros(nGrid)`!!!
for iSample = 1:nSample
    x(1) = 1;
    for iGrid = 1:nGrid - 1
        x(iSample, iGrid + 1) = x(iSample, iGrid) + f(x(iSample, iGrid), iGrid, r(iGrid)) * stepSize + g(x(iSample, iGrid), iGrid, r(iGrid)) * normrnd(0, stepSize);
    end
end
for iSample = 1:nSample
    plot(tGrid,x(iSample, :)); hold on
end
p = 2; pthmoment = mean(x.^p, 1);
plot(tGrid,pthmoment)
```