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Soft Computing ENGG6570

Homework-3

2021/06/09

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Question-1: For a neural network for robot path planning, the neural dynamics of each neuron is described by an additive equation,

$$\frac{dx_i}{dt} = -Ax_i + \sum_{j=1, j\neq i}^{n} w_{ij} f\left(x_j\right) + I_i \tag{1}$$

where $w_{ij} = w_{ji}$; $f(a) = \max\{a, 0\}$ is a linear-above threshold function; and I_i is the external input from the environment. Prove this neural network system is stable using the Lyapunov stability theory.

Solution-1: Recall the Grossberg's general model's :

$$\frac{dx_i}{dt} = a_i\left(x_i\right)\left[b_i\left(x_i\right) - \sum_{j=1, j\neq i}^{n} C_{ij} d_j\left(x_j\right)\right] \tag{2}$$

Where: a_i is the amplification function for neuron i; b_i is the self-signal function for neuron i; d_j is the other-signal function (activation function of other neuron j), c_{ij} is the connection weight from neuron j to neuron i.

Let's try to re-write the original additive equation in Grossberg's general model's format. Note that for all the equations below, we will assume the principal diagonal elements in matrix w and C are all zero so that we can ignore the term $i \neq j$ from the summation.

$$\frac{dx_i}{dt} = \left[\left(-Ax_i + I_i\right) - \sum_{j=1}^{n} \left(-w_{ij}\right) f\left(x_j\right)\right] \tag{3}$$

By comparing equation 3 with equation 2, one can proof the additive equation's stability by showing all the 3 stability criteria are satisfied accroding to Grossberg version's Lyapunov stability theorem:

$$C_{ij} = -w_{ij} \equiv -w_{ji} = C_{ji} \qquad \text{(Proof for symmetry)}$$

$$a_i\left(x_i\right) = 1 > 0 \qquad \text{(Proof for positivity)}$$

$$d_j(x_j) = f(x_j) = \max\{a, 0\} \geq 0 \qquad \text{(Proof for monotonicity)} \quad \blacksquare$$

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Question 2 A neural network is characterized by an input-output equation,

$$y_i\left(t+1\right)=\phi\left(\sum_{j=1}^{n}w_{ij}y_j\left(t\right)+\theta_i\right) \tag{4}$$

Where $w_{ij}=w_{ji}$, and $\phi(a)=(1+e^{-a})^{-1}$ is the sigmoid function, θ_i is the threshold. (1) Use the function-summation exchange to transform this equation into an additive equation; (2) Prove the stability of this system.

Solution 2—-Step-1: transform this equation into an additive equation:

$$\dot{y}_i=\frac{\Delta y_i}{\Delta t}=\frac{y_i\left(t+1\right)-y_i\left(t\right)}{1}=-y_i\left(t\right)+\phi\left(x_i\right) \tag{5}$$

Where a_i is defined as:

$$x_i\equiv\sum_{j=1}^{n}w_{ij}y_j\left(t\right)+\theta_i \tag{6}$$

Multiply $\sum\limits_{i=1}^{n}w_{ji}$ on each side of equation 5:

$$\sum_{i=1}^{n}w_{ji}\dot{y}_i=-\sum_{i=1}^{n}w_{ji}y_i\left(t\right)+\sum_{i=1}^{n}w_{ji}\phi\left(x_i\right) \tag{7}$$

From equation 6, the definition of a_i, one can write:

$$\dot{x}_i=\frac{d\left(\sum\limits_{j=1}^{n}w_{ij}y_j\left(t\right)\right)}{dt}=\sum_{j=1}^{n}w_{ij}\dot{y}_j\left(t\right) \tag{8}$$

Put equation 8 and equation 6 into equation 7:

$$\dot{x}_j=x_j+\theta_j+\sum_{i=1}^{n}w_{ji}\phi\left(x_i\right) \tag{9}$$

Finally, switch i and j, we can write equation in the additive equation format:

$$\dot{x}_i=x_i+\sum_{j=1}^{n}w_{ij}\phi\left(x_j\right)+\theta_i \tag{10}$$

Solution 2—-Step-2: Note that equation 10 is identical with equation 1 in Question 1, except for:

$$A = -1$$

$$I_i = \theta_i$$

$$f(x_j) \text{ is replaced by } \phi(x_j)$$

Equation 10 can be re-write as:

$$\frac{dx_i}{dt} = \left[(x_i + \theta) - \sum_{j=1}^{n} (-w_{ij})\, \phi(x_j) \right] \tag{11}$$

Comparing equation 11 with equation 2, one can proof the additive equation's stability by showing all the 3 stability criteria are satisfied according to Grossberg version's Lyapunov stability theorem:

$$C_{ij} = -w_{ij} \equiv -w_{ji} = C_{ji} \qquad \text{(Proof for symmetry)}$$

$$a_i(x_i) = 1 > 0 \qquad \text{(Proof for positivity)}$$

$$d_j(x_j) = \phi(x_j) = {}^{1}\!/_{1+e^{-x_j}} \geq 0 \qquad \text{(Proof for monotonicity)} \quad \blacksquare$$