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Soft Computing ENGG6570 **Homework-1** 2021/05/28

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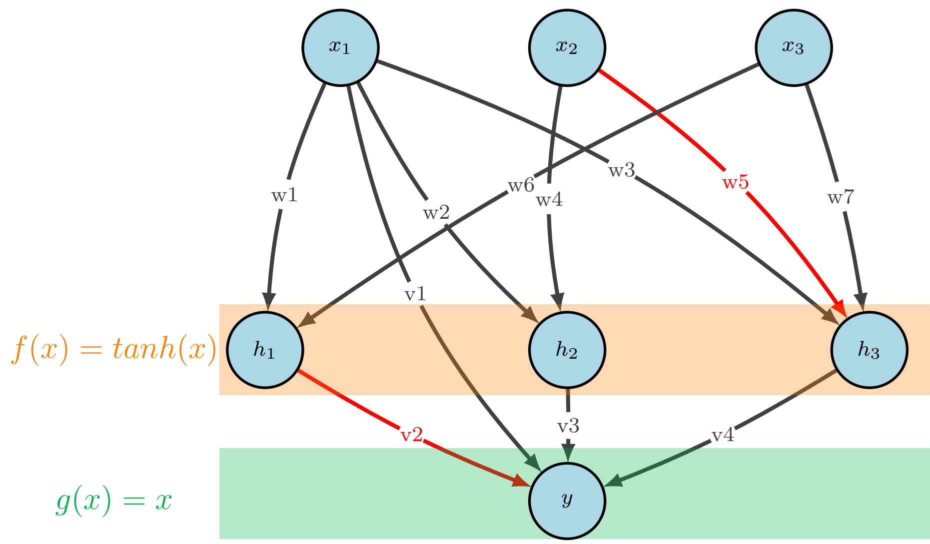
Question 1-1.**Write the feedforward equations of h1, h2, h3 and y, as a function of the input variables x1, x2 and x3. Note that the thresholds must be included:**



Solution 1-1:
Directly read from the diagram above, for hidden layer, we can write:

$$\boxed{h_1 = f\left(w_1 x_1 + w_6 x_3 + \theta_{h1}\right)} \tag{1}$$

$$\boxed{h_2 = f\left(w_2 x_1 + w_4 x_2 + \theta_{h2}\right)} \tag{2}$$

$$\boxed{h_3 = f\left(w_3 x_1 + w_5 x_2 + w_7 x_3 + \theta_{h3}\right)} \tag{3}$$

For the output layer, we can write:

$$y = g\left(v_1 x_1 + v_2 h_1 + v_3 h_2 + v_4 h_3\right) \tag{4}$$

Question 1-2. **Derive a learning algorithm for v2, w5, using LMS (Least Mean Square) method, i.e., by minimizing the output error e = t - y, where t is the target output.**

Solution 1-2:
Given that $f(x) = tanh(x)$ and $g(x) = x$, we have f' and g':

$$f' = 1 - f^2$$

$$g' = 1$$

By observation from the diagram, v_2 is a directed Edge from Vertex h_1 to Vertex y; while, w_5 is a directed Edge from Vertex x_2 to h_3.

From the lecture, we can proofed that the LMS Error strategy leads to a uniform learning equation for both the hidden layer and the output layer:

$$\Delta Edge = \eta \times \underbrace{(ActivationFunction)'}_{\text{Activationfunction@Dist}} \times \underbrace{(error)}_{\text{error@Dist}} \times \underbrace{(input)}_{\text{value@Source}}$$

$$\boxed{\Delta v_2 = \eta g' e_y h_1 = \eta\,(t-y)\,h_1} \tag{5}$$

$$\boxed{\Delta w_5 = \eta f' e_{h3} x_2 = \eta\left(1-f^2\right) e_{h3} x_2} \tag{6}$$

Where:

$$e_{h3} = g' \times e \times v_4 = g'\,(t-y)\,v_4 = (t-y)\,v_4$$

The following details are given if one is interested in seeing the mathematical details for getting equation (5), and equation (6) directly from the defination of E:

The LMS Error E is defined as:

$$E = \frac{1}{2}(t-y)^2$$

It is trivial to proof equation (5), $\Delta v_2 = \eta\,(t-y)\,h_1$, from the definition of E:

$$\Delta v_2 = -\eta \frac{\partial E}{\partial v_2} = -\eta\,(t-y)\left(\frac{\partial\,(t-y)}{\partial v_2}\right) = \eta\,(t-y)\left(\frac{\partial y}{\partial v_2}\right) = \eta\,(t-y)\,(h_1 g') = \eta\,(t-y)\,h_1 \quad \blacksquare$$

It is tricky to proof equation (6) from the definition of E.
One can write:

$$\Delta w_5 = -\eta \frac{\partial E}{\partial w_5} = -\eta\,(t-y)\left(\frac{\partial\,(t-y)}{\partial w_5}\right) = \eta\,(t-y)\left(\frac{\partial y}{\partial w_5}\right) \tag{7}$$

Using equation (4) to re-write y in equation (7) so that y explicitly contains w_5:

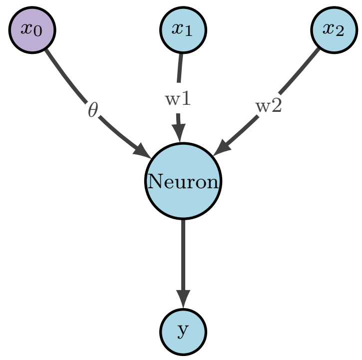
$$y = g\,(v_1x_1 + v_2h_1 + v_3h_2 + v_4h_3) = g\left[(v_1x_1 + v_2h_1 + v_3h_2) + v_4 f\,(w_3x_1 + w_5x_2 + w_7x_3 + \theta_{h3})\right]$$

$$\frac{\partial y}{\partial w_5} = g' \frac{\partial\left[(v_1x_1 + v_2h_1 + v_3h_2) + v_4 f\,(w_3x_1 + w_5x_2 + w_7x_3 + \theta_{h3})\right]}{\partial w_5} = g' v_4 f' x_2 = v_4 f' x_2 \tag{8}$$

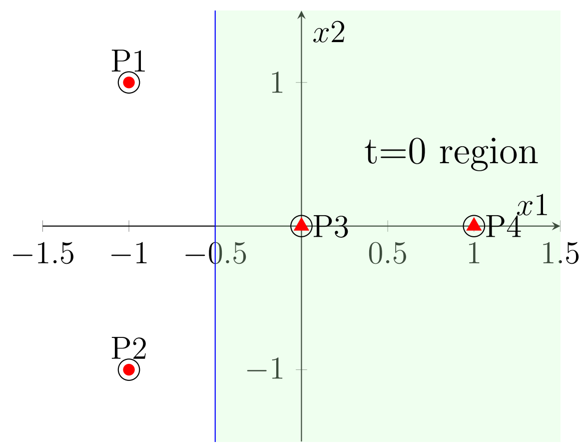
Now, put equation (8) back to equation (7), we can get:

$$\Delta w_5 = \eta\,(t-y)\left(\frac{\partial y}{\partial w_5}\right) = \eta\,\underbrace{(t-y)\,v_4}_{e_{h3}} f' x_2 = \eta e_{h3} f' x_2 = \eta\left(1-f^2\right) e_{h3} x_2 \quad \blacksquare$$

Question 2-1. Design a single-neuron perceptron to solve this problem.



	X1	X2	t
P1	-1	1	1
P2	-1	-1	1
P3	0	0	0
P4	1	0	0



Solution2-1:

For hardlimitor activation function $y = h.l.(\mathbb{W}\mathbb{X} + \theta)$, the decision boundary is determined by setting the independent variable at phase transition point (0):

$$\mathbb{W}\mathbb{X} + \theta = 0 \tag{9}$$

By observation, we are going to choose the line $x_1 = -0.5$ as our boundary. The weight vector that is orthogonal to our decision boundary is:

$$\mathbb{W} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

We know the line, $x_1 = -0.5$, is passing the point $(-0.5, 1)^T$, so we can find θ by

$$\begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} + \theta = 0.5 + \theta = 0 \Rightarrow \theta = -0.5$$

Finally, we can write our model as:

$$\boxed{y = h.l.\left(\begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 0.5 \right)} \tag{10}$$

Question-2-2: Test your solution with all four input vectors.

Solution-2-2:

Use all 4 input points to test our model:

1. Test for P1:

$$y_1 = h.l.\left(\begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0.5\right) = h.l.\,(1 - 0.5) = 1 \qquad ☑$$

2. Test for P2:

$$y_2 = h.l.\left(\begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} - 0.5\right) = h.l.\,(1 - 0.5) = 1 \qquad ☑$$

3. Test for P3:

$$y_3 = h.l.\left(\begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0.5\right) = h.l.\,(0 - 0.5) = 0 \qquad ☑$$

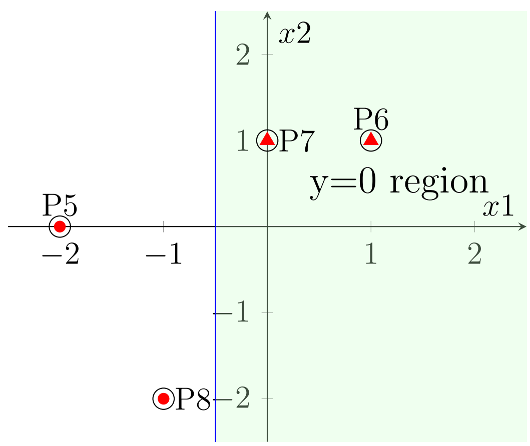
4. Test for P4:

$$y_4 = h.l.\left(\begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0.5\right) = h.l.\,(-1 - 0.5) = 0 \qquad ☑$$

Question-2-3: Classify the following input vectors with your solution. You can either perform the calculations manually or with Matlab.

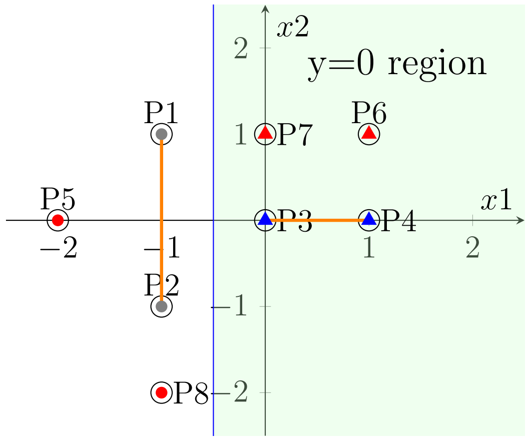
Solution-2-3:

	X1	**X2**	$Y_{Calc-from-MatLab}$
P5	-2	0	1
P6	1	1	0
P7	0	1	0
P8	-1	-2	1



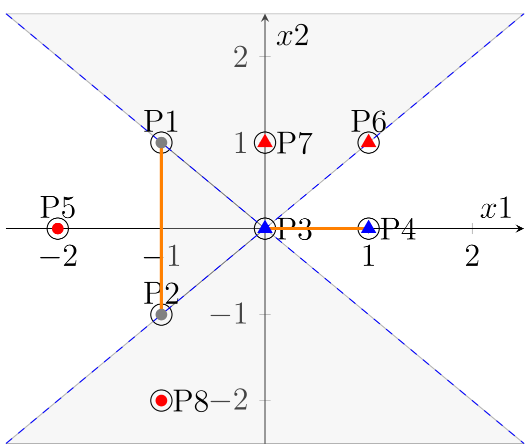

```
% Learning Model
w=[-1,0]
theta=-0.5

% Training DataSet P1-P4 for question2-1 to question2-2
pointset1=[-1,-1,0,1;1,-1,0,0]
result1 = hardlim(w*pointset1+theta)

% DataSet P5--P8 for question2-3
pointset2=[-2,1,0,-1;0,1,1,-2]
result2 = hardlim(w*pointset2+theta)
% result2 =
%       1      0      0      1
```

Question-2-4: Which of the input vectors in Part (3) is always classified the same way, regardless of the solution values of the weigh $\mathbb{W}$ and the threshold θ? Which may vary depending on the solution? Why?



Solution: We can see from the plot below, no decision boundary shall across the two thick orange lines that respectively connect Class-1 (Point-1 and Point-2) and Class-2 (Point-3 and Point-4).

- Any data points inside of the gray zone (open boundary) are **not** always classified the same way.
- Any data points inside of the white zone (close boundary) are always classified the same way



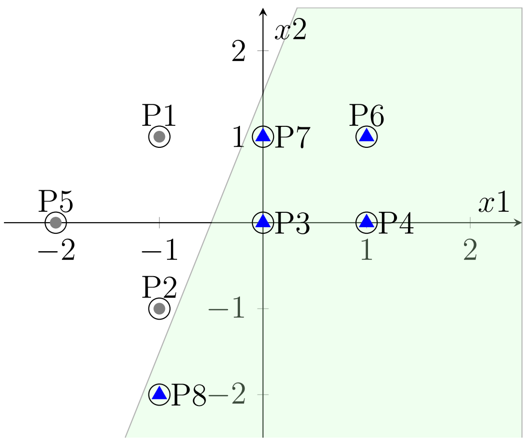
The proofs of these conclusion above are too complicated that must beyond the scope of this course. However, just for answering this particular question, I have listed the result as blow, and I can give a proof of these.

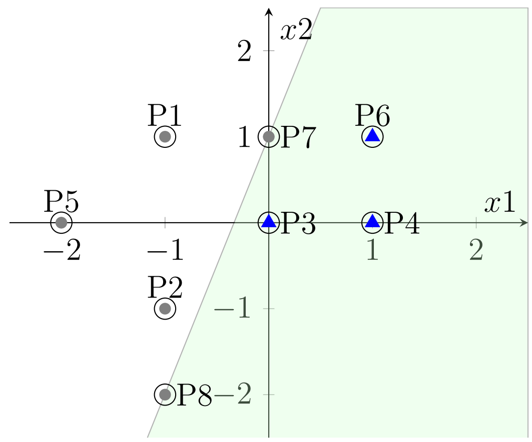
$$\boxed{\text{Point 7 and Point 8 may vary depending on the decision boundary}} \tag{11}$$

$$\boxed{\text{Point 5 and Point 6 can always be classified the same way (5 in class 1 and 6 in class 2).}} \tag{12}$$

Theorem 1. *For a single-neuron preceptron system, with Point 1, Point 2, Point 3 and Point 4 as the training set, the classification of Point 7 and Point 8 may vary depending on choice of the decision boundary.*

Proof. I claim that activation function $y = h.l.\left(\begin{bmatrix} -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1.5\right)$ will classify Point 7 and Point 8 as Class-2. While, activation function $y = h.l.\left(\begin{bmatrix} -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1\right)$ will classify Point 7 and Point 8 as Class-1. As we can see from the diagram and MatLab code below, this is true. Therefore, the classification of Point 7 and Point 8 may vary depending on decision of boundary. ■






```
% Learning Model
w=[-3,1]
theta=-1.5
% Training DataSet P1-P4 for question2-1 to question2-2
% target values are 1,1,0,0
pointset1=[-1, -1, 0, 1;
            1, -1, 0, 0]
result1 = hardlim(w*pointset1+theta)
% DataSet P5--P8 for question2-3
pointset2=[-2, 1, 0, -1;
            0, 1, 1, -2]
result2 = hardlim(w*pointset2+theta)
% result2 =
%       1     0     0     0
```

```
% Learning Model
w=[-3,1]
theta=-1
% Training DataSet P1-P4 for question2-1 to question2-2
% target values are 1,1,0,0
pointset1=[-1, -1, 0, 1;
            1, -1, 0, 0]
result1 = hardlim(w*pointset1+theta)
```

```
% DataSet P5--P8 for question2-3
pointset2=[-2, 1, 0, -1;
            0, 1, 1, -2]
result2 = hardlim(w*pointset2+theta)
% result2 =
%      1     0     1     1
```

Lemma 2 (Intersection Polygon Lemma)**.** On a 2-D plane, if two polygons(1-D for points, 2-D for lines, 3-D for triangles, ...etc) intersect with each other, then there shall not exist any line can completely separate these two polygons.

Proof. Polygon A and Polygon B intersect with each other, means there exist at least one element in A that is also in B; While, there exist at least one line that can separate these two polygon. means there is no elements in A and is also in B. There two sentence are conflicting with each other naturally. ∎

Theorem 3. *For a single-neuron preceptron system, with Point 1, Point 2, Point 3 and Point 4 as the training set (as stated in the first part of this question), the classification of Point 5 and Point 6 have to be always classified the same way. (Point 5 is in the same class with Point 1 and Point 2; while, Point 6 is always in the same class with Point 3, and Point 4)*

Proof. Suppose there exist a decision boundary (a line) that can classify Point 5 in Class-2, or can classify Point 6 in Class-1, then, as shown in the diagram below, the polygon (orange lines) that in-close Class-1 and the polygon (green lines) that in-close Class-2 will always intersect with each other.

By the Intersection Polygon Lemma, there does not exist any line (decision boundary) that can separate two intersecting polygons (classes).

As our assumption is conflicting with the Intersection Polygon Lemma, our assumption is wrong. There is no decision boundary that can put Point 5 in class 2 or Point 6 in class 1. Attempting on doing this will lead to a linear non-separable problem. ∎