The final answer is highlighted with color yellow answer
The work is presented with text color dark blue work

Question 7:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 6.1.5, sections b-d

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(b) What is the probability that the hand is a three of a kind? A three of a kind has 3 cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank. For example, {4♠, 4♦, 4♣, J♠, 8♥} is a three of a kind.

> We choose ranks first. There are C(13,1) ways to select a rank for the three. The rest two cards are both a single card and they have different ranks, so there are C(12,2) ways to choose two ranks for the rest two cards.
> Then We choose suits. There are C(4,3) ways to choose suits for the three. There are separately C(4, 1) ways to choose suits for the both.
> The probability is C(13,1) × C(12,2) × C(4,3) × C(4,1) × C(4,1) / C(52,5) = 264 / 12495.
>
> Ans:C(13,1) × C(12,2) × C(4,3) × C(4,1) × C(4,1) / C(52,5) = 264 / 12495.

(c) What is the probability that all 5 cards have the same suit?

> We choose a suit first, then there are C(4,1) ways to choose the suit.
> Then we choose 5 ranks, and there are C(13,5) ways to choose 5 ranks.
> Therefore, The probability is C(4,1) × C(13,5) / C(52,5) = 99 / 49980
>
> Ans:C(4, 1) × C(13,5) / C(52,5) = 99 / 49980

(d) What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair. For example, {4♠, 4♦, J♠, K♣, 8♥} is a two of a kind.

> We choose ranks first. There are C(13,1) ways to choose a rank for the pair. Then there are C(12,3) ways to choose the ranks for the remaining three cards.
> Then we choose suits. There are C(4,2) ways to choose the suits for the pair. There are separately C(4,1) ways to choose the suit for the remaining three cards.
> The probability is C(13,1) × C(12,3) × C(4,2) × C(4,1) × C(4,1) × C(4,1) / C(52,5) = 352 / 833.
>
> Ans:C(13,1) × C(12,3) × C(4,2) × C(4,1) × C(4,1) × C(4,1) / C(52,5) = 352 / 833

b) Exercise 6.2.4, sections a-d

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(a) The hand has at least one club.

Let C be the event that the hand has at least one club and $\overline{C}$ be the event that the hand has no club. P(C) = 1 - P($\overline{C}$). P($\overline{C}$) = C(39,5) / C(52,5) = 2109 / 13720. P(C) = 1 - 2109 / 13720 = 11611 / 13720

Ans:P(C) = 1 - P($\overline{C}$) = 1 - (C(39,5) / C(52,5)) = 11611 / 13720

(b) The hand has at least two cards with the same rank.

Let C be the event that the hand has at least two cards with the same rank and $\overline{C}$ be the event that the hand has 5 cards with different ranks. P(C) = 1 - P($\overline{C}$). P($\overline{C}$) = C(13,5) $\times 4^5$ / C(52,5) = 2112 / 4165.
P(C) = 1 - 2112 / 4165 = 2053 / 4165

Ans:P(C) = 1 - P($\overline{C}$) = 1 - (C(13,5) $\times 4^5$ / C(52,5)) = 2053 / 4165

(c) The hand has exactly one club or exactly one spade.

Let C be the event that the hand has exactly one club and S be the event that the hand has exactly one spade. C and E are not mutually exclusive. P(C∩E) means the hand has exactly one club and one spade. P(C∪E) = P(C) + P(E) - P(C∩E).
P(C) = C(13,1)×C(39,4) / C(52,5) . P(E) = P(C).
P(C∩E) = C(13,1)×C(13,1)×C(26,3) / C(52,5)
P(C∪E) = P(C) + P(E) - P(C∩E) = (2×C(13,1)×C(39,4) - C(13,1)×C(13,1)×C(26,3)) / C(52,5) = 65351 / 99960

Ans:P(C∪E) = P(C) + P(E) - P(C∩E) = (2×C(13,1)×C(39,4) - C(13,1)×C(13,1)×C(26,3)) / C(52,5) = 65351 / 99960

(d) The hand has at least one club or at least one spade.

Let C be the event that the hand has at least one club or at least one spade and $\overline{C}$ be the event that the hand has no club and no spade. P(C) = 1 - P($\overline{C}$). P($\overline{C}$) = C(26,5) / C(52,5) = 253 / 996. P(C) = 1 - P($\overline{C}$) = 1 - 253 / 996 = 743 / 996

Ans:P(C) = 1 - P($\overline{C}$) = 1 - (C(26,5) / C(52,5)) = 743 / 996

Question 8:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 6.3.2, sections a-e

The letters {a, b, c, d, e, f, g} are put in a random order. Each permutation is equally likely. Define the following events:

A: The letter b falls in the middle (with three before it and three after it)
B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "agbdcef" would be an outcome in this event.
C: The letters "def" occur together in that order (e.g. "gdefbca")

(a) Calculate the probability of each individual event. That is, calculate p(A), p(B), and p(C).

Sample space is 7! = 5040

P(A): The letter b at the middle is fixed, and there are 6! ways to permute the other letters. Therefore, P(A) = 6! / 7! = 1 / 7

P(B): When we arrange the letter b to be at the left of the letter c, there is a corresponding arrangement whose letter c is at the left of the letter b. The sample space can be divided into these two arrangements. Therefore, P(B) = 1 / 2.

P(C): We set "def" as a letter z, so we can imagine that we permute {a, b, c, z, g}. There are 5! Ways to arrange the letters. Therefore, P(C) = 5! / 7! = 1 / 42.

Ans:
P(A) = 1 / 7
P(B) = 1 / 2
P(C) = 1 / 42

(b) What is p(A|C)?

P(A|C) = P(A∩C) / P(C). P(A∩C) =The probability that b is at the middle and "def" appears together. If "def" is at the left of b, there are 3! arrangements. If "def" is at the right of b, there are also 3! arrangements. Therefore, P(A∩C) = (3! + 3!) / 7! = 1/ 420
P(A|C) = P(A∩C) / P(C) = (1 / 420) / (1 / 42) = 1 / 10.

Ans:P(A|C) = P(A∩C) / P(C) = 1 / 10.

(c) What is p(B|C)?

$P(B|C) = P(B\cap C) / P(C)$. $P(B\cap C)$ = The probability that b is at the left of c and "def" appears together. We set "def" as a letter z, so we can imagine that we permute {a, b, c, z, g}. There are 5! Ways to arrange the letters. Moreover, we can divide the sample space into half for the event that b is left of c. Therefore, $P(B\cap C) = (5! / 2) / 7! = 1/ 84$

$P(B|C) = P(B\cap C) / P(C) = (1 / 84) / (1 / 42) = 1 / 2$.

Ans: $P(B|C) = P(B\cap C) / P(C) = 1 / 2$.

(d) What is p(A|B)?

$P(A|B) = P(A\cap B) / P(B)$. $P(A\cap B)$ = The probability that b is at the middle and b is at the left of c. When b is at the middle, c only has C(3,1) options. And the other letters have 5! arrangements. Therefore, $P(A\cap B) = (C(3,1)\times 5!) / 7! = 1/ 14$

$P(A|B) = P(A\cap B) / P(B) = (1 / 14) / (1 / 2) = 1 / 7$.

Ans: $P(A|B) = P(A\cap B) / P(B) = 1 / 7$.

(e) Which pairs of events among A, B, and C are independent?

$P(A\cap B) = 1 / 14 = P(A)\times P(B)$

$P(B\cap C) = 1 / 84 = P(B)\times P(C)$

$P(A\cap C) = 1 / 420 \neq P(A)\times P(C) = 1 / 294$

Therefore, A and B are independent, B and C are independent.

Ans: A and B are independent, B and C are independent.

---

b) Exercise 6.3.6, sections b, c

A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is 1/3 and the probability of tails is 2/3. The outcomes of the coin flips are mutually independent. What is the probability of each event?

(b) The first 5 flips come up heads. The last 5 flips come up tails.

The first 5 flips come up heads and the last 5 flips come up are required for the total 10 times, the product rule applies. The probability is $(1 / 3)^5\times(2 / 3)^5$

Ans: $(1 / 3)^5\times(2 / 3)^5$

(c) The first flip comes up heads. The rest of the flips come up tails.

The probability of the first flip come up heads is 1 / 3. The probability of the rest of the flips come up tails is $(2 / 3)^9$. Both conditions are required, so the probability is $(1 / 3)\times(2 / 3)^9$

Ans: $(1 / 3)\times(2 / 3)^9$

---

c) Exercise 6.4.2, section a

(a) Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1, 2, 3, 4 and 5. You choose a die at random, and roll it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

> Let D be the event that chooses the fair dice, $\overline{D}$ be the event that chooses the biased dice. P(D) = P($\overline{D}$) = 1 / 2. Let S be the event that the outcome is 4, 3, 6, 6, 5, 5. P(S|D) = $(1 / 6)^6$, and P(S|$\overline{D}$) = $(1 / 4)^2(3 / 20)^4$. P(D|S) = P(S|D) $\times$ P(D) / (P(S|D) $\times$ P(D) + P(S|$\overline{D}$) $\times$ P($\overline{D}$))= $(1 / 6)^6$ / $((1 / 6)^6 + (1 / 4)^2(3 / 20)^4)$ $\approx$ 0.404
>
> Ans: P(D|S) = P(S|D) $\times$ P(D) / (P(S|D) $\times$ P(D) + P(S|$\overline{D}$) $\times$ P($\overline{D}$)) $\approx$ 0.404

Question 9:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 6.5.2, sections a, b

A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

(a) What is the range of A?

> There are five possible outcomes of A: 0 ace, 1 ace, 2 aces, 3 aces, 4 aces. Therefore, the range of A = {0, 1, 2, 3, 4}
>
> Ans: {0,1,2,3,4}

(b) Give the distribution over the random variable A.

> The distribution of a random variable is the set of all pairs (r, P(A = r)).
> P(A = 0) = C(48,5) / C(52,5)
> P(A = 1) = C(4,1) × C(48,4) / C(52,5)
> P(A = 2) = C(4,2) × C(48,3) / C(52,5)
> P(A = 3) = C(4,3) × C(48,2) / C(52,5)
> P(A = 4) = C(4,4) × C(48,1) / C(52,5)
>
> Ans: ((0, C(48,5) / C(52,5)), (1,C(4,1) × C(48,4) / C(52,5)), (2, C(4,2) × C(48,3) / C(52,5)), (3, C(4,3) × C(48,2) / C(52,5)), (4, C(4,4) × C(48,1) / C(52,5)))

---

b) Exercise 6.6.1, section a

(a) Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is E[G]?

> E[G] = 2 × P(G = 2) + 1 × P(G = 1) + 0 × P(G = 0) = 2 × (C(7, 2) / C(10, 2)) + 1 × (C(7,1) × C(3,1) / C(10,2) = (42 / 45) + (21 / 45) = 63 / 45 = 1.4
>
> Ans: E[G] = 1.4

c) Exercise 6.6.4, sections a, b

(a) A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then X = 25. What is E[X]?

The range of X= {1, 4, 9, 16, 25, 36}. E[X} = 1 × (1 / 6) + 4 × (1 / 6) + 9 × (1 / 6) + 16 × (1 / 6) + 25 × (1 / 6) + 36 × (1 / 6) = 91 / 6 = 15.17

Ans: E[X] = 15.17

(b) A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and Y = 4. What is E[Y]?

The range of Y= {0, 1, 4, 9}. E[X} = 0 × (1 / 8) + 1 × (3 / 8) + 4 × (3 / 8) + 9 × (1 / 8) = 24 / 8 = 3

Ans: E[Y] = 3

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d) Exercise 6.7.4, section a

(a) A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?

Let P be the number of the students who get his or her own coat. Let Pi be the random variable that is equal to 1 if the ith student gets his or her own coat. E[P] = E[P1] + E[P2] +...+E[P10].
E[P1] = 1 × (1 / 10) + 0 × (9 / 10) = 1 / 10, and the other E[Pi] are the same.
E[P] = 1 / 10 + 1 / 10 + … + 1 / 10 = 1.

Ans: 1

Question 10:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 6.8.1, sections a-d

The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

(a) What is the probability that out of 100 circuit boards made exactly 2 have defects?

We use Bernoulli trial probabilities to solve this question. The probability that out of 100 circuit boards made exactly 2 have defects is : $C(100, 2) \times (1 / 100)^2 \times (99 / 100)^{98} \approx 0.185$

Ans: $C(100, 2) \times (1 / 100)^2 \times (99 / 100)^{98} \approx 0.185$

(b) What is the probability that out of 100 circuit boards made at least 2 have defects?

“At least 2 have defects” means we can minus the event that has no defect and the event that has exactly one defect.

P(no defect) = $(99 / 100)^{100}$

P(exactly 1 defect) = $C(100,1) \times (1 / 100)^1 \times (99 / 100)^{99}$

Therefore, the probability that out of 100 circuit boards made at least 2 have defects is $1 - (99 / 100)^{100} - C(100,1) \times (1 / 100)^1 \times (99 / 100)^{99} \approx 0.264$

Ans: $1 - (99 / 100)^{100} - (C(100,1) \times (1 / 100)^1 \times (99 / 100)^{99}) \approx 0.264$

(c) What is the expected number of circuit boards with defects out of the 100 made?

E[circuit boards with defects out of the 100 made] = $n \times p$, which n = 100 and p = 1 / 100.

Therefore, E[circuit boards with defects out of the 100 made] = $100 \times (1 / 100) = 1$

Ans: 1

(d) Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compared to the situation in which each circuit board is made separately?

Since either both circuit boards in a batch have a defect or they are both free of defects, P(out of 100 circuit boards at least 2 have defects) = P(out of 50 circuit boards at least 1 has defect). P(out of 50 circuit boards at least 1 has defect) = 1 - P(out of 50 circuit boards no defect).

P(out of 50 circuit boards no defect) = $(99 / 100)^{50}$

Therefore, P(out of 50 circuit boards at least 1 has defect) = 1 - $(99 / 100)^{50} \approx 0.395$.

E[circuit boards with defects out of the 100 made] = E[(baches with defects out of the 50 made) $\times$ 2] = 2 $\times$ E[baches with defects out of the 50 made].
E[baches with defects out of the 50 made] = E[bache 1] + E[bache 2] +.... + E[bache 50] = (1 / 100) $\times$ 50 = 1 / 2
Therefore, E[circuit boards with defects out of the 100 made] = 2 $\times$ (1 / 2) = 1

If the circuit boards are made separately, P(out of 100 circuit boards at least 2 have defects) = 1 - $(99 / 100)^{100}$ - (C(100,1) $\times (1 / 100)^1 \times (99 / 100)^{99}$) $\approx 0.264 < 0.395$
Moreover, E[circuit boards with defects out of the 100 made] = 1

Therefore, when the circuit boards are made separately, the expected numbers are the same. However, the probability that out of 100 circuit boards at least 2 have defects is higher when the circuit boards are made in batches of two.

Ans:
1. P(the probability that out of 100 circuit boards (50 batches) at least 2 have defects) = 1 - $(99 / 100)^{50} \approx 0.395$
2. E[circuit boards with defects out of the 100 made] = 1
3. Compared to the situation in which each circuit board is made separately, the expected number are the same, but the probability that out of 100 circuit boards at least 2 have defects is higher when the circuit boards are made in batches of two.

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b) Exercise 6.8.3, section b

A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

(b) What is the probability that you reach an incorrect conclusion if the coin is biased?

The incorrect conclusion is reached if there are at least four heads. The probability that there are 4, 5, 6, 7, 8, 9 or 10 heads is
$C(10,4) \times (0.3)^4 \times (0.7)^6 + C(10,5) \times (0.3)^5 \times (0.7)^5 + C(10,6) \times (0.3)^6 \times (0.7)^4 + C(10,7) \times (0.3)^7 \times (0.7)^3 + C(10,8) \times (0.3)^8 \times (0.7)^2 + C(10,9) \times (0.3)^9 \times (0.7)^1 + C(10,10) \times (0.3)^{10} \times (0.7)^0 \approx 0.35$

Or we can use 1 - P(there are 0, 1, 2 or 3 heads):
$1 - (C(10,0) \times (0.3)^0 \times (0.7)^{10} + C(10,1) \times (0.3)^1 \times (0.7)^9 + C(10,2) \times (0.3)^2 \times (0.7)^8 + C(10,3) \times (0.3)^3 \times (0.7)^7) \approx 0.35$

Ans: $C(10,4)(0.3)^4(0.7)^6 + C(10,5)(0.3)^5(0.7)^5 + C(10,6)(0.3)^6(0.7)^4 + C(10,7)(0.3)^7(0.7)^3 + C(10,8)(0.3)^8(0.7)^2 + C(10,9)(0.3)^9(0.7)^1 + C(10,10)(0.3)^{10}(0.7)^0 \approx 0.35$