The final answer is highlighted with color yellow answer
The work is presented with text color dark blue work

Question 5:

a) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.12.2, sections b, e

1.12.2 - b

Ans.

1.	$p \rightarrow (q \land r)$	Hypothesis
2.	$\neg p \lor (q \land r)$	Conditional identity 1
3.	$(\neg p \lor q) \land (\neg p \lor r)$	Distributive laws 2
4.	$(\neg p \lor q)$	Simplification 3
5.	$(q \lor \neg p)$	Commutative laws 4
6.	$\neg q$	Hypothesis
7.	$\neg p$	Disjunctive syllogism 5, 6

1.12.2 - e

Ans.

1.	$p \lor q$	Hypothesis
2.	$q \lor p$	Commutative laws 2
3.	$\neg q$	Hypothesis
4.	p	Disjunctive syllogism 2, 3
5.	$\neg p \lor r$	Hypothesis
6.	$\neg\neg p$	Double negation 4
7.	r	Disjunctive syllogism 5, 6

2. Exercise 1.12.3, section c

1.12.3 - c

Ans.

1.	$p \vee q$	Hypothesis
2.	$\neg\neg p \vee q$	Double negation 1
3.	$\neg p \rightarrow q$	Conditional identity 2
4.	$\neg p$	Hypothesis
5.	q	Modus ponens 3, 4

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3. Exercise 1.12.5, sections c, d

1.12.5 - c

Ans.

- j: I will get a job
- c: I will buy a new car
- h: I will buy a new house

The argument can be presented as "((I will buy a new car) and (I will buy a new house)) only if (I get a job)" which is "$(c \wedge h) \rightarrow j$".

The form of the argument is:

$(c \wedge h) \rightarrow j$

$\neg j$

____

$\neg c$

We use the truth table to check whether the argument is valid

c	h	j	$c \wedge h$	$(c \wedge h) \rightarrow j$ (Hypothesis)	$\neg j$ (Hypothesis)	$\neg c$ (Conclusion)
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
F	T	T	F	T	F	T
T	**F**	**F**	**F**	**T**	**T**	**F**

F	**T**	**F**	**F**	**T**	**T**	**T**
F	F	T	F	T	F	T
F	**F**	**F**	**F**	**T**	**T**	**T**

Based on the truth table above, we found that the argument is not valid when c = T, h = j = F.

The argument is not valid. When c = T, h = j = F, the hypotheses are both true but the conclusion ¬c is false.

1.12.5 - d

Ans.

- j: I will get a job
- c: I will buy a new car
- h: I will buy a new house

The argument can be presented as “((I will buy a new car) and (I will buy a new house)) only if (I get a job)” which is “$(c \land h) \to j$”.

The form of the argument is:

$(c \land h) \to j$

$\neg j$

h

____

$\neg c$

We use the truth table to check whether the argument is valid

c	h	j	$c \land h$	$(c \land h) \to j$ (Hypothesis)	$\neg j$ (Hypothesis)	h (Hypothesis)	$\neg c$ (Conclusion)
T	T	T	T	T	F	T	F
T	T	F	T	F	T	T	F
T	F	T	F	T	F	F	F
F	T	T	F	T	F	T	T
T	F	F	F	T	T	F	F
F	**T**	**F**	**F**	**T**	**T**	**T**	**T**
F	F	T	F	T	F	F	T

F	F	F	F	T	T	F	T

Based on the truth table above, we found that the argument valid. When c = j = F, and h = T, all three hypotheses are true, and the conclusion is true as well.

Now, we use the rules of inference to prove that it's valid:

1.	$(c \land h) \to j$	Hypothesis
2.	$\neg j$	Hypothesis
3.	$\neg(c \land h)$	Modus tollens 1, 2
4.	$\neg c \lor \neg h$	De Morgan's laws 3
5.	$\neg h \lor \neg c$	Commutative laws 4
6.	h	Hypothesis
7.	$\neg\neg h$	Double negation 6
8.	$\neg c$	Disjunctive syllogism

Based on the rules of inference above, we can conclude that the argument is valid.

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b) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3, section b

1.13.3 - b

When the argument is invalid, the hypotheses are true and the conclusion is false. Therefore, the conclusion P(x) is false for inputs a and b. Moreover, the hypothesis $\exists x \neg Q(x)$ should be true, so there is at least one Q(x) should be false. We assume that Q(b) is false. Finally, the hypothesis $\exists x(P(x) \lor Q(x))$ should be true, so Q(a) must be true. Based on the assumption above, we have a truth table below

	P	Q
a	F	T
b	F	F

The above table can prove that the argument is false.

Ans.

$\exists x(P(x) \vee Q(x))$ is true when x = a, Q(a) is true. $\exists x \neg Q(x)$ is true when x = b, $\neg Q(b)$ is true. However, since P(a) = P(b) = F, $\exists x P(x)$ is false. Therefore, the argument is not valid.

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2. Exercise 1.13.5, sections d, e

1.13.5 - d

Ans.

- M(x): x missed class
- A(x): x received an A.
- D(x): x got a detention.

"Every student who missed class got detention" can be presented as "For every student, if he missed a class, then he got detention", which is "$\forall x(M(x) \rightarrow D(x))$". "Penelope is a student in the class" is an element definition. "Penelope did not miss class." can be presented as "$\neg M(\text{Penelope})$". "Penelope did not get detention." can be presented as "$\neg D(\text{Penelope})$".

The form of the argument is:

$\forall x(M(x) \rightarrow D(x))$
Penelope is an arbitrary student
$\neg M(\text{Penelope})$
____
$\neg D(\text{Penelope})$

Because the hypothesis $\neg M(\text{Penelope})$ should be true, M(Penelope) must be false. However, we have no idea about the true value of D(Penelope). Moreover, we consider that Penelope is the only student in the class. We use the truth table to check whether the argument is valid.

M(Penelope) (Hypothesis)	D(Penelope)	$M(x) \rightarrow D(x)$ (Hypothesis)	$\neg D(\text{Penelope})$ (Conclusion)
F	T	T	**F**
F	F	T	T

Based on the table above, when M(Penelope) = F, and D(Penelope) = T, the conclusion is false.

The argument is not valid. When M(Penelope) = F and D(Penelope) = T, the hypothesis $\forall x(M(x) \rightarrow D(x))$ is true and the hypothesis $\neg M(\text{Penelope})$ is also true. However, the conclusion $\neg D(\text{Penelope})$ is false.

1.13.5 - e

Ans.

- M(x): x missed class
- A(x): x received an A.
- D(x): x got a detention.

“Every student who missed class got detention” can be presented as “For every student, if he missed a class or got detention, then he didn’t get an A”, which is “ $\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$”. “Penelope is a student in the class” is an element definition. “Penelope got an A.” can be presented as “A(Penelope)”. “Penelope did not get detention.” can be presented as “$\neg$D(Penelope)”.

The form of the argument is:

$\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$
Penelope is an arbitrary student
A(Penelope)
____
$\neg$D(Penelope)

The argument is valid, and we use the rules of inference to prove that it’s valid

1.	$\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$	Hypothesis
2.	Penelope is an arbitrary student	Hypothesis
3.	$M(Penelope) \vee D(Penelope) \rightarrow \neg A(Penelope)$	Universal instantiation 1, 2
4.	A(Penelope)	Hypothesis
5.	$\neg\neg A(Penelope)$	Double negation 4
6.	$\neg(M(Penelope) \vee D(Penelope))$	Modus tollens 3, 5
7.	$\neg M(Penelope) \wedge \neg D(Penelope)$	De Morgan’s laws 6
8.	$\neg D(Penelope)$	Simplification 7

Based on the rule of inference table above, we can conclude that the argument is valid

Question 6:

Solve Exercise 2.2.1, sections d, c, from the Discrete Math zyBook:

2.2.1 - d

The product of two odd integers is an odd integer.

Ans.

Direct proof. We assume that a and b are odd integers. We will show that $a \times b$ is also an odd integer.
Since a is an odd number, $a = 2k + 1$ for some integer k. Moreover, b is also an odd number, so $b = 2n + 1$ for some integer n.
$a \times b = (2k + 1)(2n + 1) = 4kn + 2k + 2n + 1 = 2(2kn + k + n) + 1$
Since k and n are integers, $2kn + k + n$ is also an integer. Since $a \times b = 2c + 1$, where $c = (2kn + k + n)$ is an integer, then $a \times b$ is an odd integer.

2.2.1 - c

If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Ans.

Direct proof. We assume that $x \leq 3$. We will show that $12 - 7x + x2 \geq 0$
Since x is a real number and x-3 is also a real number. Therefore, $(x-3)^2 \geq 0$ because the square of a real number is larger or equal to 0. And we have $(x-3)^2 = x^2-6x+9 \geq 0$.
Then we subtract x to both side, and we have $x^2-7x+9 \geq -x$
Then we add 3 to both side, and we have $x^2-7x+12 \geq -x + 3$
Since x is a real number and $x \leq 3$, subtract x to both side and we have $0 \leq -x+3$
Therefore, we have $x^2-7x+12 \geq -x + 3 \geq 0$, then $x^2-7x+12 \geq 0$.

Question 7:

Solve Exercise 2.3.1, sections d, f, g, l, from the Discrete Math zyBook:

2.3.1 - d

For every integer n, if n^2-2n+7 is even, then n is odd.

Ans.

Proof by contrapositive. We assume that n is an even integer. We will show that n^2-2n+7 is odd.

Since n is an even integer, n = 2k for some integer k.

n^2-2n+7 = $4k^2$-4k+7 = $4k^2$-4k+6+1 = 2($2k^2$-2k+3)+1.

Since n^2-2n+7 = 2c + 1, where c = ($2k^2$-2k+3) is an integer, then n^2-2n+7 is an odd integer.

2.3.1 - f

For every non-zero real number x, if x is irrational, then 1/x is also irrational.

Ans.

Proof by contrapositive. We assume that 1/x is a rational number. We will show that x is also rational.

Since 1/x is a rational number, 1/x = a/b for some two integers, a and b, where b ≠ 0. Since x is a non-zero real number, a is also ≠ 0.

Then, we multiply x to both side, and we have 1 = (a × x)/b

Multiply b to both side, and we have b = a × x

Divide a to both side, and we have b/a = x

Since a and b are both integers and ≠ 0, x is a rational number.

2.3.1 - g

For every real number x and y, if $x^3+xy^2 \leq x^2y+y^3$, then $x \leq y$.

Ans.

Proof by contrapositive. We assume that x > y. We will show that $x^3+xy^2 > x^2y+y^3$.

Then, multiply x^2 to both sides, and we have $x^3 > x^2y$

Add xy^2 to both sides, and we have $x^3+xy^2 > x^2y+ xy^2$

Because x > y, $xy^2 = y^2 \times x > y^2 \times y = y^3$

Therefore, $x^3+xy^2 > x^2y+ xy^2 > x^2y+ y^3$, so we have $x^3+xy^2 > x^2y+y^3$.

Because the square of real number is larger than or equal to 0, $x^2 \geq 0$, $y^2 \geq 0$. Because $x > y$, at least one of them is not 0. So $x^2+y^2 > 0$. Since $x > y$, $x^3+xy^2 = (x^2+y^2)x > (x^2+y^2)y = x^2y+y^3$. Therefore, we have $x^3+xy^2 > x^2y+y^3$.

2.3.1 - I

For every pair of real number x and y, if $x+y>20$, then $x>10$ or $y>10$

Ans.

Proof by contrapositive. We assume that $x \leq 10$ and $y \leq 10$. We will show that $x+y \leq 20$.
For $x \leq 10$, we subtract 10 to both sides and we have $x-10 \leq 0$.
For $y \leq 10$, we subtract 10 to both sides and we have $y-10 \leq 0$.
Since $(x-10)$ and $(y-10)$ are both ≤ 0, $(x-10)+(y-10) \leq 0$ as well.
Now we have $x+y-20 \leq 0$.
Add 20 to both sides, and we have $x+y \leq 20$.

Question 8:

Solve Exercise 2.4.1, sections c, e, from the Discrete Math zyBook:

2.4.1 - c

The average of three real numbers is greater than or equal to at least one of the numbers.

Ans.

Proof by contradiction. We assume that the average of three real numbers is not greater than or equal to at least one of the numbers, which means the average of three real numbers is less than the all three numbers.
We assume x, y and z are three real numbers, and $(x+y+z)/3<x$, $(x+y+z)/3<y$ and $(x+y+z)/3<z$.
For $(x+y+z)/3<x$, we multiply 3 to both sides and we have $x+y+z<3x$
For $(x+y+z)/3<y$, we multiply 3 to both sides and we have $x+y+z<3y$
For $(x+y+z)/3<z$, we multiply 3 to both sides and we have $x+y+z<3z$
Now we add the three above quotations together, and we have $3x+3y+3z<3x+3y+3z$, which is impossible. This contradicts the fact that $3x+3y+3z$ is impossible to be less than $3x+3y+3z$.

2.4.1 - e

There is no smallest integer.

Ans.

Proof by contradiction. We assume that there is a smallest integer.
We assume that this smallest integer number is x.
Since x is an integer, x-1 is also an integer. We assume y is an integer and $y=x-1$.
Now we calculate $y-x=(x-1)-x=x-1-x=-1$.
Since the result of y-x is -1, which is a negative integer, we know that y is less than x.
However, it contradicts the fact that x is the smallest integer.

Question 9:

Solve Exercise 2.5.1, section c, from the Discrete Math zyBook:

2.4.1 - c

If integers x and y have the same parity, then x + y is even.The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Ans.

Proof by cases. We consider two cases: 1.Both integers are even integers 2. Both integers are odd integers

Case 1:
x and y are both even integers. $x=2k$ for some integer k, $y=2n$ for some integer n. $x+y=2k+2n=2(k+n)$. Therefore, $x+y=2c$, where $c=k+n$ is an integer. We can conclude that x+y is an even integer.

Case 2:
x and y are both odd integers. $x=2k+1$ for some integer k, $y=2n+1$ for some integer n. $x+y=(2k+1)+(2n+1)=2k+2n+2=2(k+n+1)$. Therefore, $x+y=2c$, where $c=k+n+1$ is an integer. We can conclude that x+y is an even integer.

Based on case 1 and case 2, we know that if integers x and y have the same parity, then x + y is even.