The final answer is highlighted with color yellow answer
The work is presented with text color dark blue work

Question 5:

a. Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$ (leaving no remainder).

Answer:

Theorem: For any positive integer n, 3 divide $n^3 + 2n$.

Proof: By induction on n

Base case: n = 1, $n^3 + 2n = 3$. Since 3 divides 3, the theorem holds for the case n = 1.

Inductive Step: Assume that for positive integer k, 3 can divide $k^3 + 2k$, which means $(k^3 + 2k) = 3m$ for some integer m. Then we need to prove that 3 divides (k+1)3 + 2(k+1).

For any integer $k \geq 1$:

$$
\begin{aligned}
(k+1)^3 + 2(k+1) &= (k^3 + 3k^2 + 3k + 1) + (2k + 2) \\
&= (k^3 + 2k) + (3k^2 + 3k + 3) \\
&= 3m + (3k^2 + 3k + 3) \quad \text{// By the inductive hypothesis} \\
&= 3m + 3(k^2 + k + 1) \\
&= 3(m + k^2 + k + 1)
\end{aligned}
$$

Since m and k are both integers, $(m + k^2 + k + 1)$ is an integer as well.
Therefore, $(k+1)^3 + 2(k+1)$ can be divided by 3.

b. Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes.

Answer:

Theorem: any positive integer n ($n \geq 2$) can be written as a product of primes.

Proof: By strong induction on n

Base case: n = 2, since 2 is a prime number, it's the product of primes.

Inductive Step: Assume that for k ≥ 2, any integer n from 2 through k can be written as the product of primes. We need to prove that k+1 can be written as the product of primes as well.

If k + 1 is a prime number, then it can be written as a product of a prime number, which is itself k + 1.

If k + 1 is not a prime number, which is a composite number, then it can be expressed as the product of 2 integers a and b, a ≥ 2 and b ≥ 2.
If k + 1 = ab, then a = (k + 1) / b. Because b ≥ 2, a = (k + 1) / b < k + 1. Therefore, a≤k.

If $k + 1 = ab$, then $b = (k + 1) / a$. Because $a \geq 2$, $b = (k + 1) / a < k + 1$. Therefore, $b \leq k$.
Now we know that both a and b are less than or equal to k, by the inductive hypothesis, a and b can be expressed as a product of primes.

$a = p1 \times p2 \times ... \times pn$

$b = q1 \times q2 \times ... \times qm$

Since $k + 1 = ab$, $k + 1$ can be written as a product of primes as well.

$k + 1 = ab = (p1 \times p2 \times ... \times pn)(q1 \times q2 \times ... \times qm)$.

Question 6:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 7.4.1, sections a-g

Define P(n) to be the assertion that: $\sum_{j=1}^{n} j^2 = n(n + 1)(2n + 1) / 6$

(a) Verify that P(3) is true.

Answer:

When n = 3, the left side is $\sum_{j=1}^{3} j^2 = 1 + 4 + 9 = 14$. The right side is $3(3 + 1)(2\times3 + 1) / 6 = 14$.

The left side = 14 = the right side. Therefore, $\sum_{j=1}^{3} j^2 = 3(3 + 1)(2\times3 + 1) / 6$

(b) Express P(k).

Answer:

For the positive integer k, $P(k) = \sum_{j=1}^{k} j^2 = k(k + 1)(2k + 1) / 6$

(c) Express P(k + 1).

Answer:

For the positive integer k + 1, $P(k + 1) = \sum_{j=1}^{k+1} j^2 = (k + 1)(k + 2)(2(k+1) + 1) / 6$

(d) What must be proven in the base case?

Answer:

Base case is when n = 1, and we need to prove that P(1) is true in the base case.

(e) What must be proven in the inductive step?

Answer:

Assume that P(k) is true for all positive integer k, then we need to prove that P(k+1) is true in the inductive step. $P(k) = \sum_{j=1}^{k} j^2 = k(k + 1)(2k + 1) / 6$ is true, then we need to show that $P(k+1)= \sum_{j=1}^{k+1} j^2 = (k + 1)(k + 2)(2(k + 1) + 1) / 6$ is true.

To prove that P(k+1) is true, we'll use the inductive hypothesis.

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

Answer:

The inductive hypothesis is that P(k) is true.

(g) Prove by induction that for any positive integer n, Answer:

Base case:

When n = 1, the left side is $\sum_{j=1}^{1} j^2 = 1$. The right side is $1(1 + 1)(2\times1 + 1) / 6 = 1$.

The left side = 1 = the right side. Therefore, $\sum_{j=1}^{1} j^2 = 1(1 + 1)(2\times1 + 1) / 6$

Inductive Step:

Assume that P(k) is true for all positive integer k, then we need to prove that P(k+1) is true.

P(k) = $\sum_{j=1}^{k} j^2 = k(k + 1)(2k + 1) / 6$ is true, then we need to show that P(k+1)= $\sum_{j=1}^{k+1} j^2 = (k + 1)(k + 2)(2(k + 1) + 1) / 6$ is true.

$$P(k+1)= \sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k + 1)^2$$

$$= k(k + 1)(2k + 1) / 6 + (k + 1)^2$$ // By the inductive hypothesis, $\sum_{j=1}^{k} j^2 = k(k + 1)(2k + 1) / 6$

$$= (k(k + 1)(2k + 1) + 6(k + 1)^2) / 6$$
$$= ((k +1)(k(2k + 1) +6(k + 1)) / 6$$
$$= (k + 1)(2k^2 + k + 6k + 6) / 6$$
$$= (k + 1)(2k^2 + 7k + 6) / 6$$
$$= (k + 1)((k + 2)(2k + 3)) / 6$$
$$= (k + 1)(k + 2)(2(k + 1) + 1) / 6$$

Therefore, P(k+1)= $\sum_{j=1}^{k+1} j^2 = (k + 1)(k + 2)(2(k + 1) + 1) / 6$.

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b) Exercise 7.4.3, section c

(c) Prove that for $n \geq 1$, $\sum_{j=1}^{n} 1 / j^2 \leq 2 - 1/n$

Answer:

Theorem: For any positive integer n, $\sum_{j=1}^{n} 1 / j^2 \leq 2 - 1/n$

Proof: By induction on n

Base case: n = 1, the left side is $\sum_{j=1}^{1} 1 / 1^2 = 1$. The right side is $2 - 1/1 = 1$

The left side = 1 = right side. Therefore, the theorem holds for the base case.

Inductive Step: Assume that for positive integer k, $\sum_{j=1}^{k} 1 / j^2 \le 2 - 1/k$ is true, then we need to prove that $\sum_{j=1}^{k+1} 1 / j^2 \le 2 - 1/(k+1)$ is true.

For any integer k≥1:

$$\sum_{j=1}^{k+1} 1 / j^2 = \sum_{j=1}^{k} 1 / j^2 + 1 / (k + 1)^2$$

$\le 2 - 1/k + 1 / (k + 1)^2$ // By the inductive hypothesis, $\sum_{j=1}^{k} 1 / j^2 \le 2 - 1/k$

$\le 2 - 1/k + 1 / k(k + 1)$ // Because k≥1, $1 / (k + 1)^2 \le 1 / k(k + 1)$

$= 2 - (1/k - 1 / k(k + 1))$

$= 2 - (((k + 1) - 1) / k(k + 1))$

$= 2 - (k / k(k + 1))$

$= 2 - 1 / (k + 1)$

Therefore, $\sum_{j=1}^{k+1} 1 / j^2 \le 2 - 1/(k+1)$ is true.

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c) Exercise 7.5.1, section a

Prove each of the following statements using mathematical induction.

(a) Prove that for any positive integer n, 4 evenly divides $3^{2n}-1$.

Answer:

Theorem: For any positive integer n, 4 evenly divides $3^{2n}-1$.

Proof: By induction on n

Base case: n = 1, $3^{2n}-1 = 3^2 - 1 = 8$. Since 4 evenly divides 8, the theorem holds for the case n = 1.

Inductive Step: Assume that for positive integer k, 4 can divide $3^{2k} - 1$. Then we need to prove that 4 divides $3^{2(k+1)} - 1$.

For any integer k≥1:

$3^{2(k+1)} - 1 = 3^{2k+2} - 1$

$= 3^{2k} \times 3^2 - 1$

$= 9(3^{2k}) - 1$

$= 9(4m + 1) - 1$ // By the inductive hypothesis, $3^{2k} - 1 = 4m$ for some integer m

So $3^{2k} = 4m + 1$

$= 36m + 8$

$=4(9m + 2)$

Since m is an integer, (9m + 2) is an integer as well.
Therefore, $3^{2(k+1)} - 1$ can be divided by 4.