The final answer is highlighted with color yellow answer
The work is presented with text color dark blue work

Question 3:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.1.3, sections b, c

(b) $f(x) = 1 / (x^2 - 4)$

If $x = \pm 2$, it maps to 1 / (4 - 4), which is not a real number. Therefore, this is not a function.

A: Not a function. If $x = \pm 2$, then f(x) is not a real number.

(c) $f(x) = \sqrt{x^2}$

If x is a real number, then x2 ≥ 0. So $\sqrt{x^2} = |x|$,which is a real number, which is . Therefore, for x $\in$ R, every x maps to a real number. It is a function. The range of the function is { x $\in$ R : $|x|$ }

A: It's a function. The range is { x $\in$ R : $|x|$ }.

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b) Exercise 4.1.5, sections b, d, h, i, l

(b) Let A = {2, 3, 4, 5}. f: A $\rightarrow$ Z such that $f(x) = x^2$.

x = 2, f(x) = 4; x = 3, f(x) = 9; x = 4, f(x) = 16; x = 5, f(x) = 25

A: {4, 9, 16, 25}

(d) f: $\{0,1\}^5 \rightarrow$ Z. For x $\in \{0,1\}^5$, f(x) is the number of 1's that occur in x. For example f(01101) = 3, because there are three 1's in the string "01101".

x $\in \{0,1\}^5$, so x is a string with length 5. The number of 1's that occur in x is from 0 to 5.
Ex: f(00000) = 0, f(00001) = 1, f(00011) = 2, f(00111) = 3, f(01111) = 4, f(11111) = 5

A: {0, 1, 2, 3, 4, 5}

(h) Let A = {1, 2, 3}. f: $A \times A \rightarrow Z \times Z$, where f(x,y) = (y, x).

x = 1, y = 1, f(1, 1) = (1, 1); x = 1, y = 2, f(1, 2) = (2, 1); x = 1, y = 3, f(1, 3) = (3, 1);
x = 2, y = 1, f(2, 1) = (1, 2); x = 2, y = 2, f(2, 2) = (2, 2); x = 2, y = 3, f(2, 3) = (3, 2);
x = 3, y = 1, f(3, 1) = (1, 3); x = 3, y = 2, f(3, 2) = (2, 3); x = 3, y = 3, f(3, 3) = (3, 3);

A: {(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)}

(i) Let A = {1, 2, 3}. f: $A \times A \rightarrow Z \times Z$, where f(x,y) = (x,y+1).

x = 1, y = 1, f(1, 1) = (1, 2); x = 1, y = 2, f(1, 2) = (1, 3); x = 1, y = 3, f(1, 3) = (1, 4);
x = 2, y = 1, f(2, 1) = (2, 2); x = 2, y = 2, f(2, 2) = (2, 3); x = 2, y = 3, f(2, 3) = (2, 4);
x = 3, y = 1, f(3, 1) = (3, 2); x = 3, y = 2, f(3, 2) = (3, 3); x = 3, y = 3, f(3, 3) = (3, 4);

A: {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

(i) Let A = {1, 2, 3}. f: $P(A) \rightarrow P(A)$. For $X \subseteq A$, f(X) = X - {1}.

P(A) = { ϕ , {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}
X = ϕ , f(x) = ϕ - {1} = ϕ ; X = {1}, f(x) = {1} - {1} = ϕ ; X = {2}, f(x) = {2} - {1} = {2};
X = {3}, f(x) = {3} - {1} = {3}; X = {1, 2}, f(x) = {1, 2} - {1} = {2}; X = {1, 3}, f(x) = {1, 3} - {1} = {3};
X = {2, 3}, f(x) = {2, 3} - {1} = {2, 3}; X = {1, 2, 3}, f(x) = {1, 2, 3} - {1} = {2, 3};

A: { ϕ , {2}, {3}, {2, 3}}

Question 4:

I. Solve the following questions from the Discrete Math zyBook:

a. Exercise 4.2.2, sections c, g, k

(c) h: Z → Z. $h(x) = x^3$

When x = 0, h(0) = 0;
When x > 0, x = 1 and h(1) = 1, x = 2 and h(2) = 8, ...and so on.
When x < 0, x = -1 and h(-1) = -1, x = -2 and h(-2) = -8, …..and so on.
Every x maps to a unique h(x), and if $x \neq y$, then $h(x) \neq h(y)$, so it's a one-to-one function.
However, it's not an onto function, ex: there is no x for h(x) = 3.

A: one-to-one but not onto

(g) f: Z × Z → Z × Z, f(x, y) = (x+1, 2y)

When x = 0, f(0, y) = (1, 2y);
When x > 0, x = 1 and h(1, y) = (2, 2y), x = 2 and h(2, y) = (3, 2y), ...and so on.
When x < 0, x = -1 and h(-1, y) = (0, 2y), x = -2 and h(-2, y) = (-1, 2y), …..and so on.
The above formula can also apply on the variable y.
Every (x, y) map to a unique h(x, y), and if $(x_1, y_1) \neq (x_2, y_2)$, then $f(x_1, y_1) \neq f(x_2, y_2)$, so it's a one-to-one function. However, it's not an onto function, ex: there is no y for h(x, y) = (x, 3).

A: one-to-one but not onto

(k) f: Z+ × Z+ → Z+, f(x, y) = 2x + y.

When x = 1 and y = 1, f(1, 1) = 3.
When x = 1 and y = 2, f(1, 2) = 4; When x = 1 and y = 3, f(1, 3) = 5;... and so on.
When x = 2 and y = 1, f(2, 1) = 5; When x = 2 and y = 2, f(2, 2) = 6;... and so on.
We can observe that $(1, 3) \neq (2, 1)$, but f(1, 3) = f(2, 1) = 5. So it's not a one-to-one function.
Moreover, it's not onto. Ex: there is no (x, y) such that f(x, y) = 1

A:Neither one-to-one nor onto

b. Exercise 4.2.4, sections b, c, d, g

(b) f: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.

We replace every input string's first bit by 1, so f(001) = 101 and f(101) = 101. 001 $\neq$ 101, but f(001) = f(101) = 101. Therefore, it's not a one-to-one function. Moreover, Because the range of the function is the bit string starting with 1, which means there is no bit string that starts with 0. Therefore, the range is smaller than the target and it's not an onto function.

A:Neither one-to-one nor onto

(c) f: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example f(011) = 110.

We can list all elements to check the properties. f(000) = 000. f(001) = 100. f(010) = 010. f(100) = 001. f(011) = 110. f(110) = 011. f(101) = 101. f(111) = 111.
Every input bit string maps to an unique bit string, so it's an one-to-one function.
Moreover, the range of the function equals the target, so it's an onto function.

A:one-to-one and onto

(d) f: $\{0, 1\}^3 \rightarrow \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, f(100) = 1001.

We can list all elements to check the properties. f(000) = 0000. f(001) = 0010. f(010) = 0100. f(100) = 1001. f(011) = 0110. f(110) = 1101. f(101) = 1011. f(111) = 1111.
Every input bit string maps to an unique bit string, so it's an one-to-one function.
However, the cardinality of the range is obviously smaller than the cardinality of the target, so it's not an onto function.

A:one-to-one but not onto

(g) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8} and let B = {1}. f: P(A) → P(A). For X ⊆ A, f(X) = X - B. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

When X = ϕ, f(ϕ) = ϕ. When X = {1}, f({1}) = ϕ. Therefore, it's obvious that the function is not one-to-one. Since the function is X - B, and B = {1}, the range of the function must not include the subset {1} and the all subset containing the element 1. Therefore, it's not onto.

A:Neither one-to-one nor onto

II. Give an example of a function from the set of integers to the set of positive integers that is :

f: $Z \to Z^+$

a. one-to-one, but not onto.

A: $f(x) = 2x + 3$ if $x \geq 0$

$-2x$ if $x < 0$ (not onto since there is no x for $f(x) = 1$)

b. onto, but not one-to-one.

A: $f(x) = |x| + 1$ (not one-to-one since $f(1) = f(-1) = 2$)

c. one-to-one and onto.

A: $f(x) = 2x + 1$ if $x \geq 0$

$-2x$ if $x < 0$

d. neither one-to-one nor onto

A: $f(x) = 1$

Question 5:

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.3.2, sections c, d, g, i

(c) f: R → R. f(x) = 2x + 3

$f(x_1) = 2x_1 + 3$. $f(x_2) = 2x_2 + 3$. If $f(x_1) = f(x_2)$ and $x_1\ x_2 \in R$, then $2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$. It's one-to-one. Moreover, y is an element in the target and $y \in R$. $f(x) = 2x + 3 = y \Rightarrow x = (y - 3) / 2$, which means for every y in the target, there is a $x \in R$ such that f(x) = y. Therefore, it's onto. Now we know that it's bijection, and the function has a well-defined inverse. $f^{-1}(x) = (x - 3) / 2$.

A: The function has a well-defined inverse. $f^{-1}(x) = (x - 3) / 2$

(d) Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8}. f: P(A) → {0, 1, 2, 3, 4, 5, 6, 7, 8}. For $X \subseteq A$, f(X) = |X|. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

It's obvious that the function is not one-to-one since f({1}) = f({2}) = |{1}| = |{2}| = 1. Therefore, f^{-1} is not well-defined.

A: The function is not one-to-one, so f^{-1} is not well-defined.

(g) f: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, f(011) = 110.

We can list all elements to check the properties. f(000) = 000. f(001) = 100. f(010) = 010. f(100) = 001. f(011) = 110. f(110) = 011. f(101) = 101. f(111) = 111.
Every input bit string maps to an unique bit string, so it's an one-to-one function.
Moreover, the range of the function equals the target, so it's an onto function.
The function is a bijection, so it has a well-defined inverse. We can reverse the output string to obtain the input string. Therefore, the output of f^{-1} is obtained by taking the input string and reversing the bits.

A: The function has a well-defined inverse. the output of f^{-1} is obtained by taking the input string and reversing the bits.

(g) f: Z × Z → Z × Z, f(x, y) = (x+5, y-2)

$f(x_1, y_1) = (x_1 + 5, y_1 - 2)$. $f(x_2, y_2) = (x_2 + 5, y_2 - 2)$. If $f(x_1) = f(x_2)$, then $(x_1 + 5, y_1 - 2) = (x_2 + 5, y_2 - 2) \Rightarrow x_1 = x_2$ and $y_1 = y_2$. It's one-to-one.
Moreover, (a, b) is an element in the target and (a, b) ∈ Z × Z. f(x, y) = (x+5, y-2) = (a, b) ⇒ x = a - 5 and y = b + 2, which means for every (a, b) in the target, there is a pair of (x , y) ∈ Z × Z, such that f(x, y) = (a, b). Therefore, it's onto. Now we know that it's bijection, and the function has a well-defined inverse. $f^{-1}(x, y) = (x - 5, y + 2)$.

A: The function has a well-defined inverse. $f^{-1}(x, y) = (x - 5, y + 2)$.

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b) Exercise 4.4.8, sections c, d

f(x) = 2x + 3
g(x) = 5x + 7
h(x) = x^2 + 1

(c) f o h

$f(h(x)) = f(x^2 + 1) = 2(x^2 + 1) + 3 = 2x^2 + 2 + 3 = 2x^2 + 5$

A: f o h = $2x^2 + 5$

(d) h o f

$h(f(x)) = h(2x + 3) = (2x + 3)^2 + 1 = 4x^2 + 12x + 9 + 1 = 4x^2 + 12x + 10$

A: f o h = $4x^2 + 12x + 10$

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c) Exercise 4.4.2, sections b-d

f(x) = x^2
g(x) = 2^x
h(x) = $\lceil x / 5 \rceil$

(b) Evaluate f o h(52)

f o h(52)= $f(\lceil 52 / 5 \rceil) = f(11) = 11^2 = 121$

A: f o h(52) = 121

(c) Evaluate g o h o f(4)

g o h o f(4) = g(h(f(4))) = g(h(42) = g(h(16)) = g(⌈16 / 5⌉) = g(4) = 2^4 = 16

A:g o h o f(4) = 16

(d) Give a mathematical expression for h o f.

h o f(x) = h(f(x)) = h(x^2) = ⌈x^2 / 5⌉.

A:h o f(x) = ⌈x^2 / 5⌉

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d) Exercise 4.4.6, sections c-e

f: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.

g: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, g(011) = 110.

h: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, h(011) = 010.

(c) What is h o f(010)?

h o f(010) = h(f(010)) = h(110) = 111

A:h o f(010) = 111

(d) What is the range of h o f?

We can list the range of f(x) first. f(000) = f(100) = 100; f(001) = f(101) = 101; f(010) = f(110) = 110; f(111) = f(011) = 111. The range of f(x) = {100, 101, 110, 111}. Now we use the range of f(x) as the domain of h(x) to find the range of h o f. h(100) = 101; h(101) = 101; h(110) = 111; h(111) = 111. Therefore, the range of h o f is {101, 111}.

A: {101, 111}

(e) What is the range of g o f?

We can list the range of f(x) first.f(000) = f(100) = 100; f(001) = f(101) = 101; f(010) = f(110) = 110; f(111) = f(011) = 111. The range of f(x) = {100, 101, 110, 111}. Now we use the range of f(x) as the domain of h(x) to find the range of g o f. g(100) = 001; h(101) = 101; h(110) = 011; h(111) = 111. Therefore, the range of h o f is {001, 101, 011, 111}.

A: {001, 101, 011, 111}

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e) Extra Credit: Exercise 4.4.4, sections c, d

Let f: X → Y and g: Y → Z be two functions.

(c) Is it possible that f is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

A: No. Since f is not one-to-one, we assume x and y ∈ X, and x ≠ y but f(x) = f(y). f(x) = f(y) = z ∈ Y, and g(z) = w, w ∈ Z. Therefore, g o f (x) = g(z) = w = g(z) = f(y). Then we know x ≠ y but g o f (x) = g o f (y). Therefore, it’s impossible that f is not one-to-one and g o f is one-to-one.

(d) Is it possible that g is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

A: Yes, g o f can be one-to-one even g is not one-to-one. The diagram below illustrates an example:

$$\begin{tikzcd} x \arrow[r, "f"] & 1 \arrow[r, "g"] & a \\ y \arrow[r] & 2 \arrow[r] & b \\ & 3 \arrow[ur] & \end{tikzcd}$$

g is not one-to-one but g o f is one-to-one.