University of Waterloo
CS240 Fall 2017
Assignment 3

Due Date: Wednesday, November 1, at 5pm

Please read `http://www.student.cs.uwaterloo.ca/~cs240/f17/guidelines.pdf` for guidelines on submission. This assignment contains written questions and a programming question. Submit your written solutions electronically as a PDF with file name a03wp.pdf using MarkUs. We will also accept individual question files named a03q1w.pdf, a03q2w.pdf, a03q3w.pdf, a03q4w.pdf and a03q5w if you wish to submit questions as you complete them.

Problem 1

See additional pages.

Problem 2

a) See additional pages.

b) $m_i = 1 + m_{i-1} + m_{i-3}$ for $i \geq 3$ where $m_0 = 1$, $m_1 = 2$ and $m_2 = 3$.

c) $1 = 2^0$,$2 \geq 2^{1/3} \approx 1.26$ and $e \geq 2^{2/3} \approx 1.59$ so the property holds at $i = 0, 1, 2$. For any $i \geq 3$, suppose our claim holds for all $k = 0, \ldots, i-1$. Then $m_i = 1 + m_{i-1} + m_{i-3}$ satisfies $m_i \geq 1 + 2^{(i-1)/3} + 2^{(i-3)/3}$ by the induction hypothesis. The right-side is $2^{i/3}(2^{-i/3}+2^{-1/3}+2^{-3/3})$, and the term in parentheses is greater than 1. So by induction, we obtain that $m_i \geq 2^{i/3}$ for all $i \geq 0$.

d) Consider an AVL-2 tree of height i with n nodes. By definition of m_i, we have $m_i \leq n$ and $m_i \geq 2^{i/3}$. Taking logarithms in base 2, $i/3 \leq \log n$, so height i is $O(\log n)$. Also, in any binary tree of height i, there are at most $2^i - 1$ nodes, so height i is $\Omega(\log n)$.

Problem 3

a) 3,5,2,7,10,1,4,6,8,9
9,4,6,1,3,5,2,7,10,8
9,1,8,4,6,3,5,2,7,10

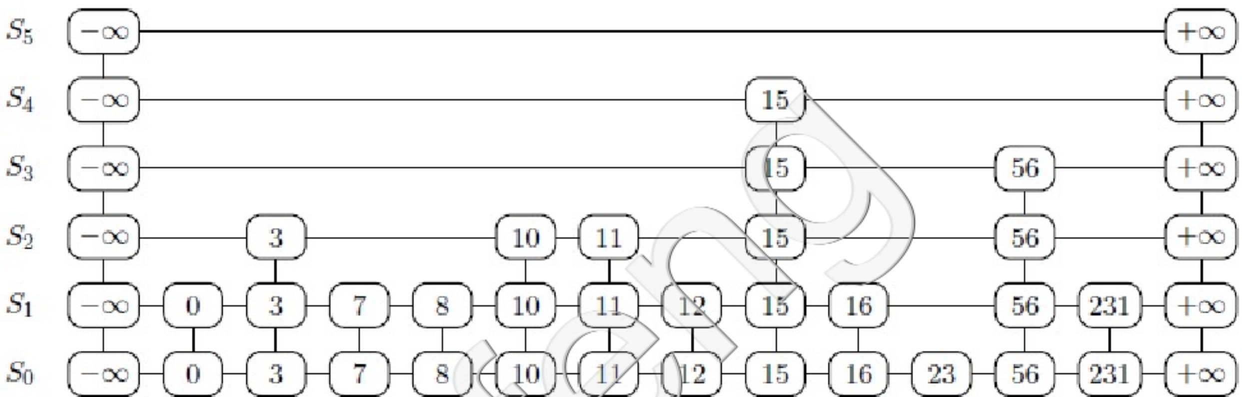
10	7	2	5	3	3	1	6	4	9	1	8	1	9	Total
10	8	4	7	6	1	6	8	8	10	4	10	2	3	87

b) 2,3,1,5,4,7,6,8,10,9
3,1,2,4,5,6,7,8,9,10
1,3,2,4,5,6,8,9,7,10

10	7	2	5	3	3	1	6	4	9	1	8	1	9	Total
10	7	2	5	3	2	3	7	5	10	2	8	1	9	74

Problem 4

a) The resulting skiplist:



Key	23	12	56	231	3	16	8	10	15	7	0	11
Comparisons	8	9	9	8	7	7	8	9	10	7	6	8

b) **i)** In order to have a tower of at least level i, we must have i flips of H in a row. The probability of each flip is p and they are independent so p^i.

ii) The expected height of a node is $1/p$. So the expected number of nodes on level i is np^i. Sum over all levels (i goes to infinity), to obtain $n/(1-p)$ which is $\Theta(n)$.

iii) $C(k) = 1 + (p)C(k-1) + (1-p)C(k)$
$pC(k) = (p)C(k-1) + 1$
$C(k) = C(k-1) + (1/p)$

iv) $C(k) = C(k-1) + (1/p) = C(k-2) + (1/p) + (1/p) = C(0) + k(1/p) = k/p$

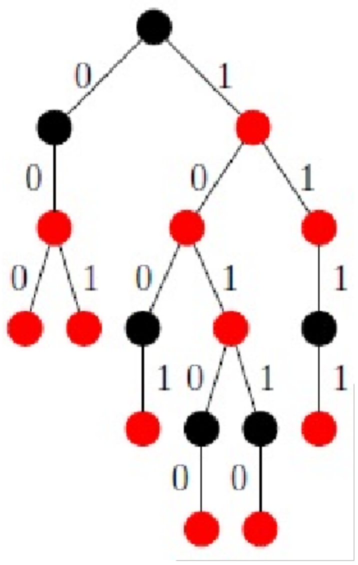
Problem 5

a) Worst case: search for last element: $\Theta(k+m) = \Theta(k+n/k)$;i.e. visit k nodes in top level and m nodes in bottom level.

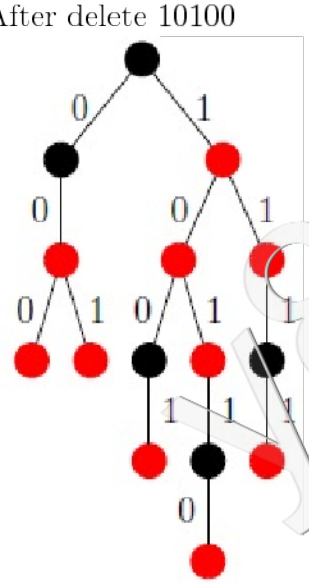
b) The minimum is obtained for $k \approx n/k$, so $k \approx \sqrt{n}$ (we assume that k divides n, so it is either $\lceil\sqrt{n}\rceil$ or $\lfloor\sqrt{n}\rfloor$). This gives $\Theta(\sqrt{n})$.

Problem 6

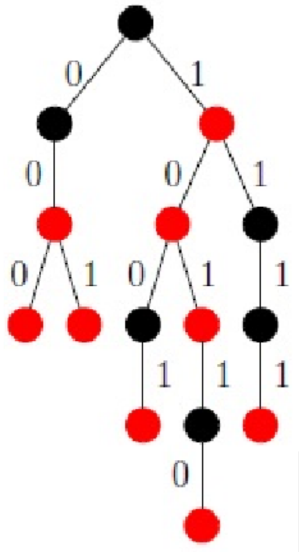
a) The resulting trie:



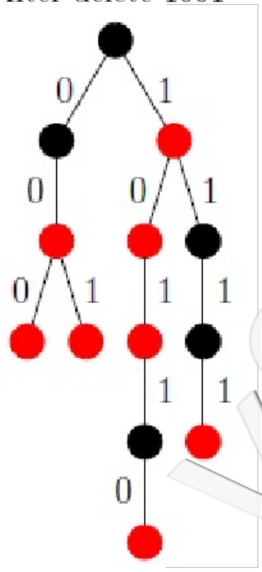
b) After delete 10100



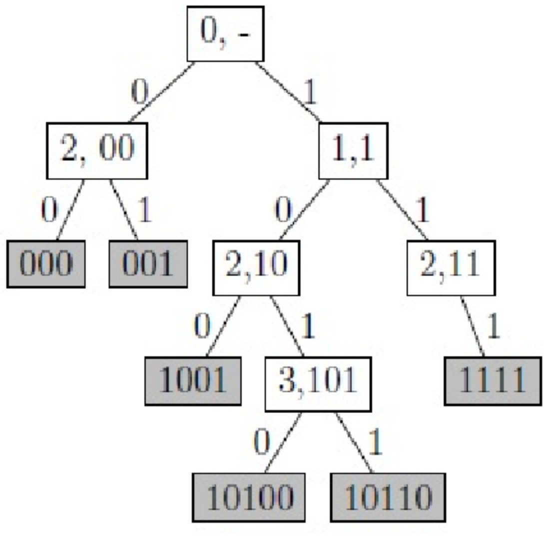
After delete 11



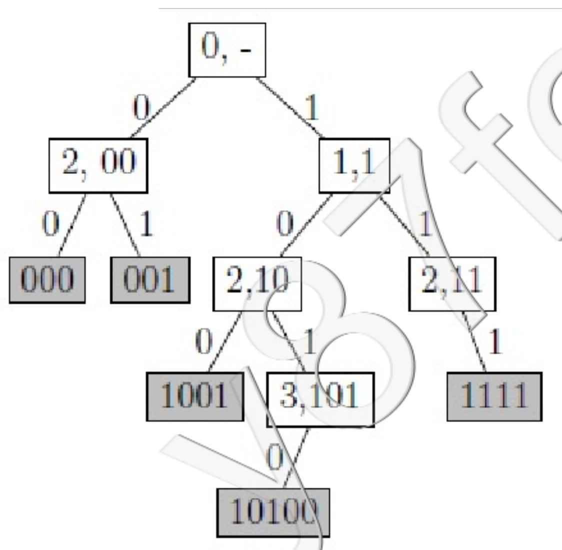
After delete 1001



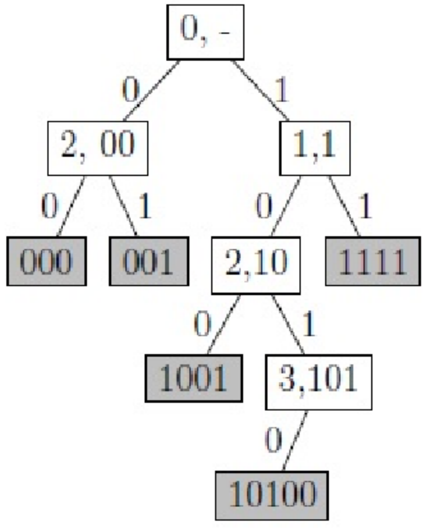
c) The resulting compressed trie:



d) After delete 10100



After delete 11



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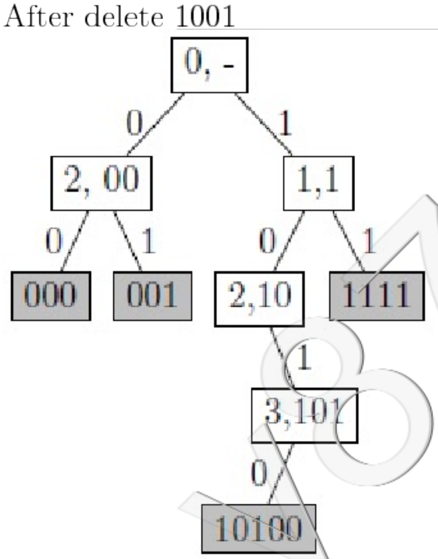


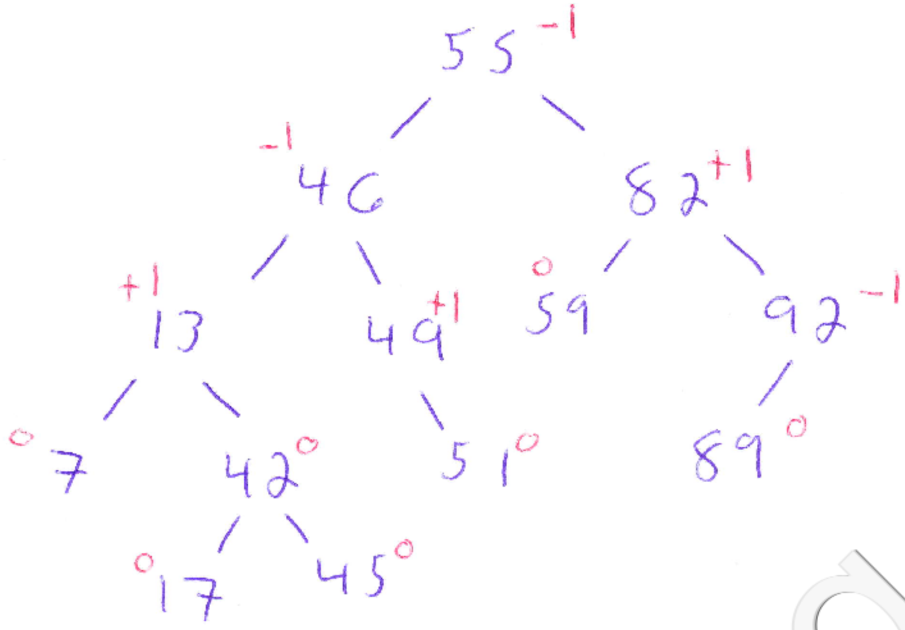
e,f) Note that the order that items are inserted is not important so we will not prove the following.
The height of a trie to encode the numbers 0 through $2^k - 1$ in binary is k.

The height of a trie is k (except when $k = 0$ the height is 1). Prove by induction on the number of bits (k).
Base case: When $k = 1$, the trie is a node with two children (0 and 1) and has height 1. When $k = 0$ the trie is a node with child 0 and has height 1.

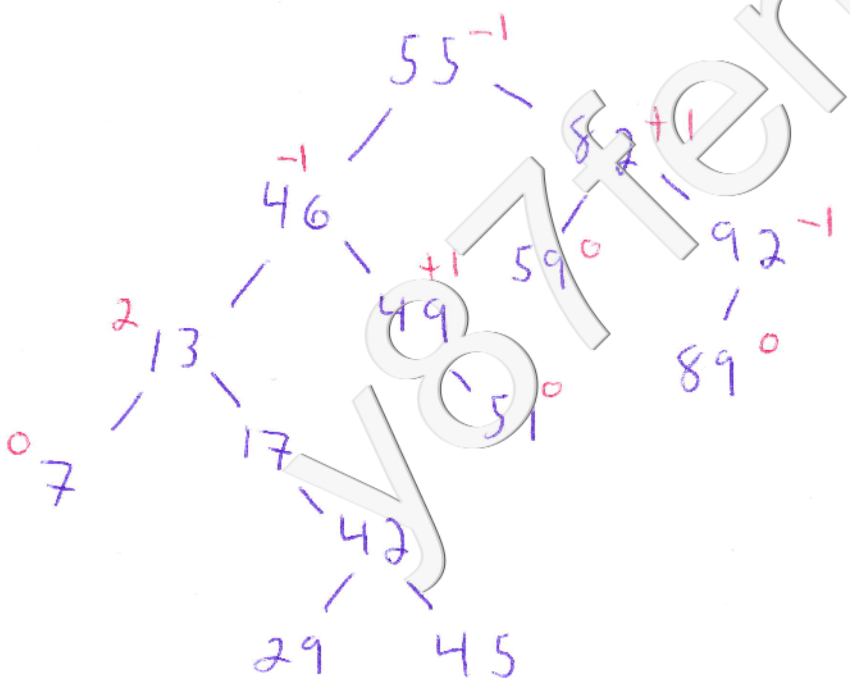
Induction hypothesis: Assume that for $k - 1$ the height is $k - 1$ and the trie includes a left subtree containing only 0 and a complete right subtree containing all other numbers with every node marked.

Induction Step: Notice that when inserting the numbers $2^{k-1}, \ldots, 2^k - 1$, every number starts with a 1 and may have any other combination of binary digits. Thus is formed from a $(k-1)$-bit prefix (beginning with a 1) and a 1-bit suffix (either 1 or 0); i.e., the number is a $(k-1)$-bit number with one more bit added. We represent this by adding anew children with paths 0 and 1 to each leaf in the right subtree of the original trie give by the induction hypothesis, while leaving the old leaves marked since they still represent $(k-1)$-bit numbers. As a result, the new leaves do not allow any compression and the height of the right subtree goes up by 1. Since by assumption the right subtree had all but one of the nodes, this raises the height of the entire tree by 1, so the new height is $(k-1) + 1 = k$.

1a)

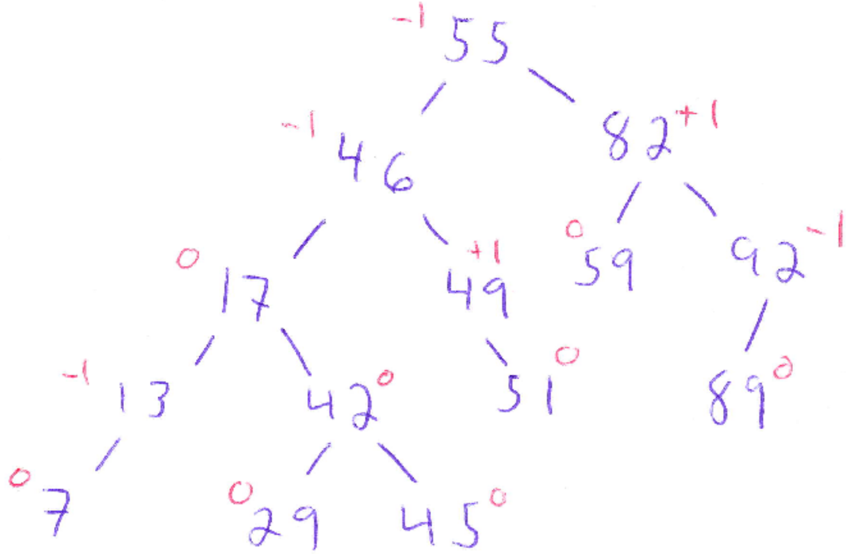


b)



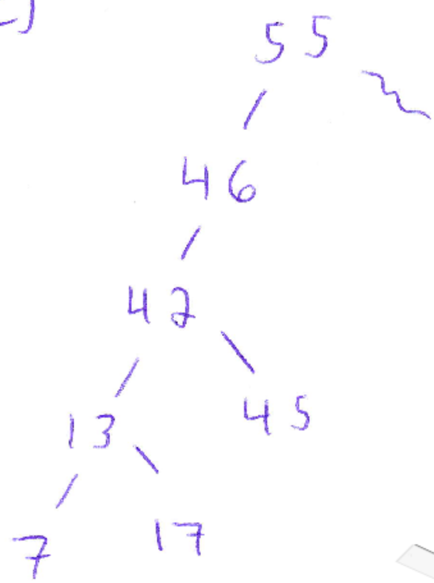
After Right Rotation on 42

b)

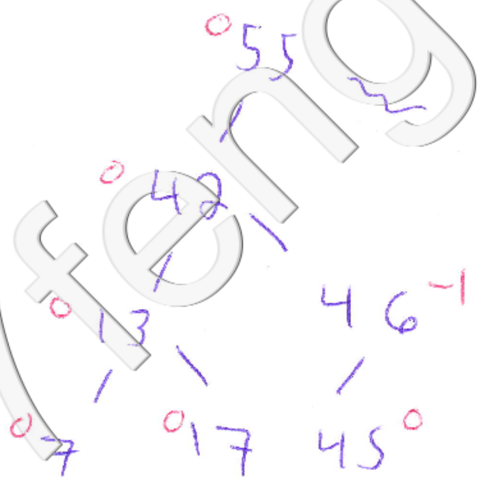


After left Rotation on 13

c)

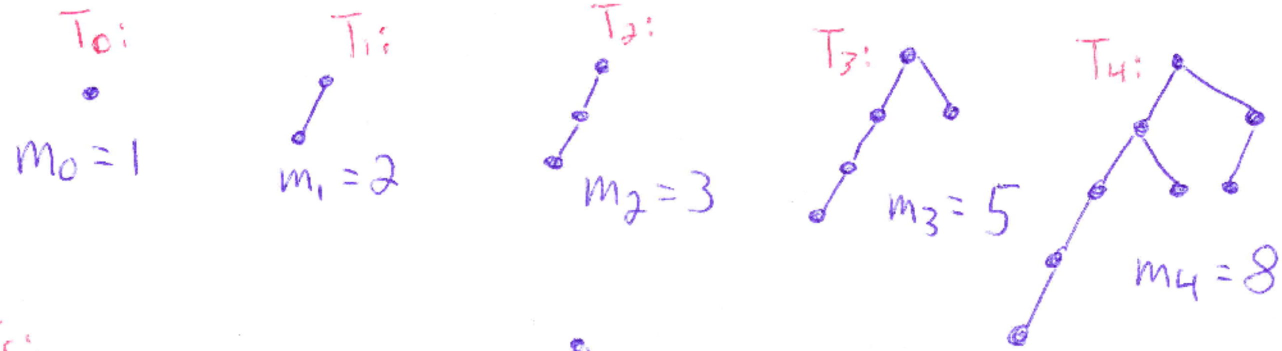


After left Rotation on 13

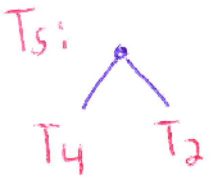


After Right Rotation on 46

2a)



$m_0 = 1$ $m_1 = 2$ $m_2 = 3$ $m_3 = 5$ $m_4 = 8$



$m_5 = 1 + 3 + 8 = 12$

$m_6 = 1 + 5 + 12 = 18$

4a)

S_5: $-\infty$ — $+\infty$

S_4: $-\infty$ — 15 — $+\infty$

S_3: $-\infty$ — 15 — 56 — $+\infty$

S_2: $-\infty$ — 3 — 10 — 11 — 15 — 56 — $+\infty$

S_1: $-\infty$ — 0 — 3 — 7 — 8 — 10 — 11 — 12 — 15 — 16 — 56 — 231 — $+\infty$

S_0: $-\infty$ — 0 — 3 — 7 — 8 — 10 — 11 — 12 — 15 — 16 — 23 — 56 — 231 — $+\infty$