An axiom is a formula
that is assumed to hold.

An axiom schema is a
pattern that describes
an entire set of axioms.

For example in first-order logic we can discribe the axioms of equality.

$$\forall x\ (x \approx x)$$

$$\forall x\, \forall y\ (x \approx y \rightarrow A(x) \approx A(y))$$

where $A(u) \in Form(L)$, does not contain x or y and $A(x)$, respectively, $A(y)$, is formed from $A(u)$ by replacying u by x, respectively, y.

We can extend these ideas with respect to particular domains.

For instance, consider the domain $\mathbb{N}$ of the natural numbers.

Each element of $\mathbb{N}$ is given by a term:

$$0,\ S(0),\ SS(0),\ SSS(0), \ldots$$

Elements of $\mathbb{N}$ described by the successor function satisfy certain axioms.

PA 1: $\forall x \; \neg (s(x) = 0)$

Zero is not the successor of any natural number.

PA 2: $\forall x \forall y \, (s(x) \doteq s(y) \rightarrow x \doteq y)$

If the successor of x equals the successor of y then x equals y.

P3: $\forall x \, (x + 0 \approx x)$

If b is in $\mathbb{N}$ then b plus 0 equals b.

P4: $\forall x \, \forall y \, (x + s(y) = s(x+y))$

If b and c are in $\mathbb{N}$ then b plus the successor of c equals the successor of b plus c.

P3 and P4 are the axioms of addition.

P5: $\forall x\,(x \cdot 0 = 0)$

Multiplication by 0 results in 0.

P6: $\forall x\, \forall y\,(x \cdot s(y) = x \cdot y + x)$

For b and c in $\mathbb{N}$, b multiplied by the successor of c is b multiplied by c, plus b.

These are the axioms of multiplication.

P7: Let $A(u)$ be a formula of arithmetic with free variable u.

$$A(0) \to (\forall x\,(A(x) \to A(s(x))) \to \forall x\, A(x))$$

This is the induction axiom for Peano arithmetic.

Notice that the base case is given as $A(0)$.

The induction step is

$$\forall x\,(A(x) \rightarrow A(s(x)))$$

Familiar facts about the natural numbers can be proven with the above axioms.

E.g. $\forall x \forall y \, (x+y = y+x)$.

Let $A(u_1, \ldots, u_n)$ be a formula whose free variables are $u_1, \ldots, u_n$.

Recall that an interpretation consists of a domain and a function mapping individual symbols, n-ary relation symbols, and m-ary function symbols, respectively, to individuals in the domain, n-ary relations in the domain, and m-ary total functions in the domain.

Given an interpretation $\mathcal{I}$, a formula $A(u_1, \ldots, u_k)$ defines the k-ary relation of k-tuples over domain D that make $A(u_1, \ldots, u_k)$ evaluate to 1.

$$R_{A(u_1, \ldots, u_k)} = \left\{ \langle d_1, \ldots, d_k \rangle \in D^k \;\middle|\; A(u_1, \ldots, u_k)^{(\mathcal{I}, (u_1/d_1, \ldots, u_k/d_k))} = 1 \right\}$$

A relation R is definable in $\mathfrak{I}$ if $R = R_A$ for some A.

Example: In Peano Arithmetic the relation $\leq$ is defined by: $\exists x\,(u_1 + x = u_2)$

We can also define the $\leq$ relation.

$$x < y \text{ iff } x \leq y \wedge x \neq y .$$