Propositional Logic

Notes for Zhongwan

Ch. 1 & 2

Propositional logic is a symbolic logic.

That is, we use strings of symbols and connectives to represent arguments.

Arguments will be given using the tools of deductive reasoning.

Given a set of premises that are assumed to hold show that a given conclusion necessarilly holds.

For our logic we should provide a description of the formulae that are in the language of the logic. That is the syntax of propositional logic.

In addition, for each formula of the logic we should provide a semantic interpretation saying what the formula means.

Finally, we should provide a proof procedure that shows us how to deduce that a formula is 'true' or that allows us to deduce that an argument is true.

That is, if a given set of formulae representing the assumptions of the argument are all assumed to be true then the formula representing the arguments conclusion necessarily holds true.

Propositions, in general, compound propositions are formed from simple propositions and propositional connectives.

Examples of simple propositions: A

B

C

Examples of compound propositions with simple propositions and connectives:

A or B

not A

Example inference

If

A or B

and

not A

then

B

Propositions, both simple and compound, may be either 'true' or 'false.'

We typically represent that the simple proposition A is 'true' by giving A the value $\underline{1}$.

When A is 'False' it has the value 0. These two possible values can be conveniently represented by the 'truth table'

A	
1	
0	

Similarly the possible values of the compound proposition not A are represented by

A	not A
1	0
0	1

The compounb proposition A and B is described as follows:

A	B	A and B
1	1	1
1	0	0
0	1	0
0	0	0

The compound proposition A or B is true if one or both of A and B are true.

A	B	A or B
1	1	1
1	0	1
0	1	1
0	0	0

The compound proposition
A implies B
(alternatively: if A then B)
means: if A is true then
B is true

A	B	A implies B
1	1	1
1	0	0
0	1	1
0	0	1

Examples: $x > 3$ implies $x^2 > 9$

Denote the empty set by $\emptyset$.

Then for all sets S,
if $x \in \emptyset$ then $x \in S$.

Thus $\emptyset \subseteq S$, that is
the empty set is a subset
of each set S.

Proposition A iff B stands for: if A then B, and if B then A.

A	B	A iff B
1	1	1
1	0	0
0	1	0
0	0	1

In general an n-ary function with the domain all n-tuples in $\{0, 1\}^n$ and range $\{0, 1\}$ is called an n-ary truth function.

L^P is the set of all expressions of propositional logic.

L^P is built up from three types of symbols.

Propositional symbols

p

q

r

p_1

etc.

There is an unbounded number of these symbols.

Connective symbols

$\neg$	negation
$\wedge$	and
$\vee$	or
$\to$	implication
$\leftrightarrow$	iff equivalence

Punctuation symbols

$($ $)$

the left and right parentheses.

Then the expressions of L^P are the finite strings of symbols where each symbol is either a proposition, a connective or a punctuation symbol.

So, for instance

$$p \in L^{\varphi}$$

$$(r) \in L^{\varphi}$$

$$\neg(p \vee q) \in L^{\varphi}$$

$$p \wedge \rightarrow q \in L^{\varphi}$$

The length of an expression is the number of symbols in the expression.

So p has length $\underline{1}$

(r) has length 3

$p \wedge \rightarrow q$ has length 4.

The empty expression the expression with no symbols, has length 0.

By convention, the empty string is denoted by $\emptyset$.

Suppose U is an expression of length n. Then we can describe $U = u_1 u_2 \ldots u_n$ where for all $i \in [1..n]$: u_i is a symbol.

Two expressions, U and V, are equal, written $U = V$, if the length of U equals the length of V and if $U = u_1 \dots u_n$ and $V = v_1 \dots v_n$ then for all $i \in [1..n]$: $u_i = v_i$.

Given expressions U and V they can be concatenated to form the expression UV.

The length of UV is the length of U plus the length of V.

Notice that for the empty string $\emptyset$ we have that $U\emptyset = U$ and that

$$\emptyset U = U.$$

If $U = W_1 V W_2$ where

W_1, W_2 and V are expressions

then V is a segment of U.

If in addition $V \neq U$ then

V is a proper segment of U.

If V and W are expressions and $U = VW$ then V is an initial segment of U.

W is a terminal segment of U.

If $W \neq \emptyset$ then V is a proper initial segment.

while if $V \neq \emptyset$ then W is a proper terminal segment.

L^P consists of the expressions of propositional logic.

Some expressions, for instance, $p \wedge \rightarrow q$, seem problematic.

Therefore we define the well-formed formulae of L^p, namely $\mathrm{Form}(L^p)$.

The smallest formulae in $\mathrm{Form}(L^p)$ are the atomic formulae or atoms.

Def. Atom(L^P) consists of the expressions in L^P whose length is one and whose single element is a propositional symbol.

Def. Form(L^p) consists of the expressions in L^p that can be formed by (repeated) application of the following rules:

[1] $\text{Atom}(L^p) \subseteq \text{Form}(L^p)$.

[2] If $A \in \text{Form}(L^p)$ then $(\neg A) \in \text{Form}(L^p)$.

[3] If $A, B \in \text{Form}(L^p)$ then $(A * B) \in \text{Form}(L^p)$.

In the previous slide $*$ is used as a shorthand to refer to $\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$.

So for instance rule [3] includes $(A \rightarrow B) \in Form(\mathcal{L}^P)$ when $A, B \in Form(\mathcal{L}^P)$.

Rules [1], [2] and [3] from the previous definition are called the formation rules of Form(L^P).

An alternative definition for Form(L^P) is the following.

Form(L^P) is the smallest class of expressions of L^P closed under the formation rules for formulae of L^P.

We can now show that the expression

$$((p \vee q) \rightarrow ((\neg p) \leftrightarrow (q \wedge r)))$$

is in fact a formula in $\mathrm{Form}(L^{p})$.

Notice that p, q and r are in $Form(L^p)$ because they are atomic and in $Atom(L^p)$, rule [1].

Therefore by [2], $(\neg p) \in Form(L^p)$.
By [3] we get that $(p \vee q)$, $(p \wedge q) \in Form(L^p)$.

Then again by [3] we obtain that $((\neg p) \longleftrightarrow (q \wedge r)) \in Form(L^p)$ and with one further application of [3]

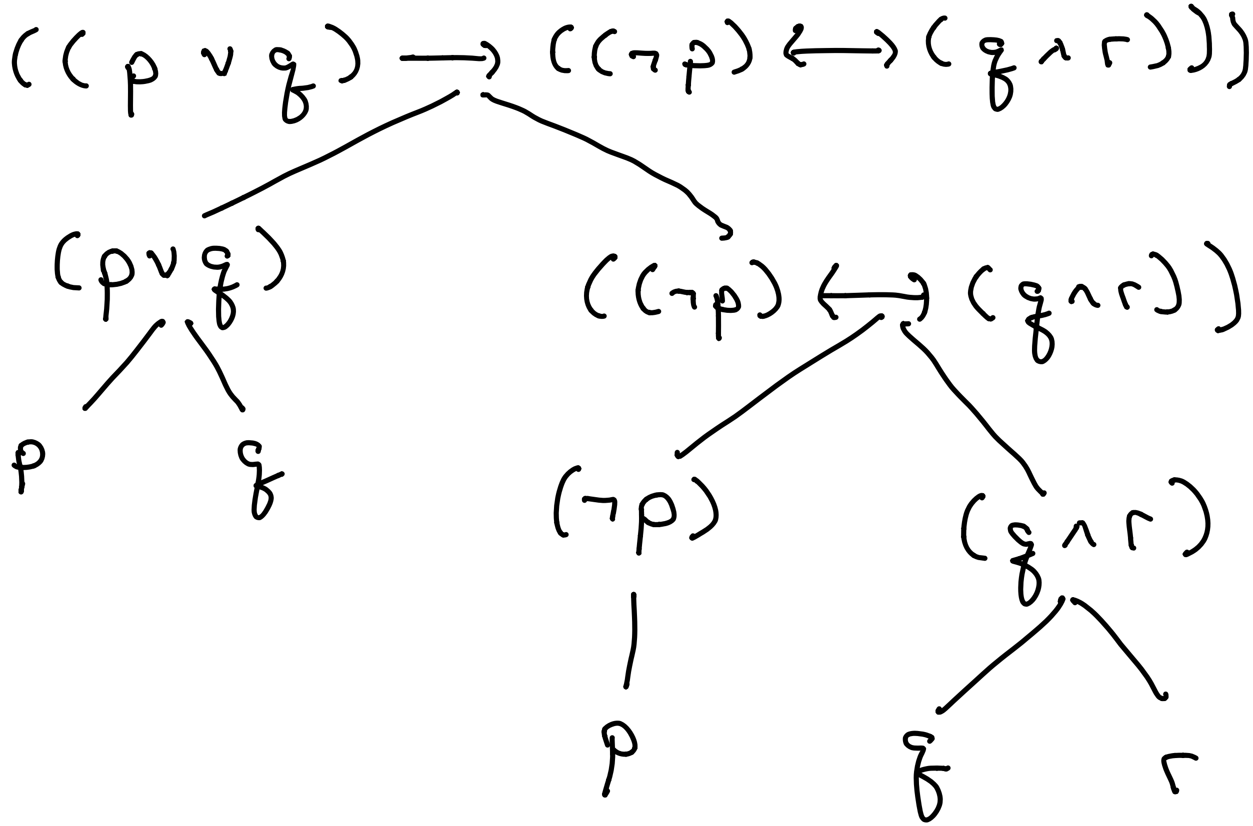
$$((p \vee q) \longrightarrow ((\neg p) \longleftrightarrow (q \wedge r)))$$

is in $Form(L^p)$.

We can organize this justification that

$$((p \vee q) \to ((\neg p) \leftrightarrow (q \wedge r)))$$

is in Form($\mathcal{L}^P$) as follows.


((p ∨ q) → ((¬p) ↔ (q ∧ r)))
(p ∨ q)
((¬p) ↔ (q ∧ r))
p
q
(¬p)
(q ∧ r)
p
q
r

Recall that a proposition is used to represent a declarative sentence, something that in any given environment is either true or false.

A few example propositions.

- The sum of 3 and 5 is 8.
- The sum of 3 and 5 is 7.
- Program P terminates.

- Jane reacted violently to Jacks accusations.

In each case, we can represent the sentence by a single propositional atom.

A more interesting example is:

She is clever and hard working.

Can be represented by

$(C \wedge H)$

He is clever but not
hard working.

He didn't write the letter
or the letter was lost.

If they do not practice hard
then they will lose.

She will win unless she
is injured.

Red sun at night only if
Sailors delight.

Fact: Every formula in $\mathrm{Form}(L^P)$ is of one of six forms: an atom, $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, or $(A \leftrightarrow B)$.

We can prove this fact by induction on the structure of the formulae in $\mathrm{Form}(\mathcal{L}^P)$.

Such a proof would have the following form.

Suppose R is a property of formulae.

If [1] for all $p \in Atom(L^P)$ it is the case that $R(p)$, and

[2] for all $A \in Form(L^P)$ if $R(A)$ then $R((\neg A))$, and

[3] for all $A, B \in Form(L^P)$ if $R(A)$ and $R(B)$ then $R((A * B))$.

Then $R(A)$ for all $A \in Form(L^P)$.

Recall that the formulae in $\mathrm{Form}(\mathcal{L}^p)$ were given by rule [1], [2] and [3].

Formulae generated by rule [1] are of the form $p \in \mathrm{Atom}(\mathcal{L}^p)$.

Formulae generated by rule [2] are of the form $(\neg A)$ where $A \in \mathrm{Form}(\mathcal{L}^p)$.

Formulae generated by rule [3] are of the form $(A * B)$ where $*$ is one of $\wedge, \vee, \rightarrow$ or $\leftrightarrow$ and $A, B \in Form(L^P)$.

Hence the generated formula is of the form $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ or $(A \leftrightarrow B)$.

Therefore all formulae in $Form(L^P)$ are of the appropriate form.

Since it is also the case that each formula in $Form(L^P)$ can be built by the generating rules in, essentially, one way we can categorize the formulae in $Form(L^P)$ as follows:

$(\neg A)$ is a negation (formula).

$(A \wedge B)$ is a conjunction.

$(A \vee B)$ is a disjunction.

$(A \rightarrow B)$ is an implication.

$(A \leftrightarrow B)$ is an equivalence.

Recall that the natural numbers are given as

$0, 1, 2, 3, \ldots$

However they are also given by the following inductive definition of the set $\mathbb{N}$.

Def. [1] $0 \in N$.

[2] For all $n \in N$ it is the case that $n' \in N$.
(Here n' is the successor of n, or $n+1$).

[3] $n \in N$ only if it is generated by [1] and [2].

Using the natural numbers, N, allows us to apply the principle of mathematical induction.

For property R: if we establish that $R(0)$ holds and that for all natural numbers, n, if $R(n)$ holds then $R(n')$ holds. Then we may conclude that $R(n)$ holds for all $n \in N$.

Example: Show that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N}.$$

Base case, i.e. $R(0)$.

$$\sum_{i=0}^{0} i = 0 \qquad \frac{0(0+1)}{2} = 0$$

So $\displaystyle\sum_{i=0}^{0} i = \frac{0(0+1)}{2}$

Inductive step:

Assume $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Show that $\sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$

$$\sum_{i=0}^{n+1} = n+1 + \sum_{i=0}^{n} i$$

$$= n+1 + \frac{n(n+1)}{2}$$

$$= \frac{2(n+1)}{2} + \frac{n(n+1)}{2}$$

$$= \frac{2(n+1) + n(n+1)}{2}$$

$$= \frac{(n+1)(2+n)}{2}$$

$$= \frac{(n+1)(n+1+1)}{2}$$

Induction Proof:

For property R

If [1] $R(0)$ holds,

and [2] for all $n \in \mathbb{N}$

if $R(n)$ holds

then $R(n+1)$ holds.

Then $R(n)$ holds for all

$n \in \mathbb{N}$.

Course of values induction
(strong induction).

If [1] $R(0)$ holds, and

[2] for all $n \in N$

if $R(0), \dots, R(n)$ hold then

$R(n')$ holds.

Then $R(n)$ holds for all $n \in N$.

Induction on the structure of formulae (structural induction) is a special case of mathematical induction.

Fact: For all $A \in Form(L^p)$
A has an equal number of left and right parentheses.

Show by structural induction.

Base case: if $A \in Atom(L^p)$
then A has no parentheses, either left or right.

I. H. Suppose A and B each have equal numbers of left and right parentheses.

Then a formula of the form $(\neg A)$ has an equal number of left and right parentheses.

Similarly, any formula of the form $(A*B)$ has an equal number of left and right parentheses since both A and B each have equal numbers of left and right parentheses.