CS 245

ND Rules — Basics

(Ref) $A \vdash A$

(+) If $\Sigma \vdash A$

then $\Sigma, \Sigma' \vdash A$

($\neg -$) If $\Sigma, \neg A \vdash B$

$\Sigma, \neg A \vdash \neg B$

then $\Sigma \vdash A$

$(\to -)$ If $\Sigma \vdash A \to B$

$\Sigma \vdash A$

then $\Sigma \vdash B$

$(\to +)$ If $\Sigma, A \vdash B$

then $\Sigma \vdash A \to B$

$(\wedge -)$ If $\Sigma \vdash A \wedge B$

then $\Sigma \vdash A$

$\Sigma \vdash B$

$(\wedge +)$ If $\Sigma \vdash A$

$\Sigma \vdash B$

then $\Sigma \vdash A \wedge B$

$(\vee -)$ If $\Sigma, A \vdash C$

$\Sigma, B \vdash C$

then $\Sigma, A \vee B \vdash C$

$(\vee +)$ If $\Sigma \vdash A$

then $\Sigma \vdash A \vee B$

$\Sigma \vdash B \vee A$

($\leftrightarrow$ $-$) If $\Sigma \vdash A \leftrightarrow B$

$\Sigma \vdash A$

then $\Sigma \vdash B$

If $\Sigma \vdash A \leftrightarrow B$

$\Sigma \vdash B$

then $\Sigma \vdash A$

($\leftrightarrow$ $+$) If $\Sigma, A \vdash B$

$\Sigma, B \vdash A$

then $\Sigma \vdash A \leftrightarrow B$

Derived Rules

($\in$) If $A \in \Sigma$

then $\Sigma \vdash A$

(Tr) If $\Sigma \vdash \Sigma'$

$\Sigma' \vdash A$

then $\Sigma \vdash A$

$(\neg +)$ If $\Sigma, A \vdash B$

(RAA) $\Sigma, A \vdash \neg B$

then $\Sigma \vdash \neg A$

The ND rules for $(\leftrightarrow -)$ are given as

If $\Sigma \vdash A \leftrightarrow B$

$\Sigma \vdash A$

then $\Sigma \vdash B$.

If $\Sigma \vdash A \leftrightarrow B$

$\Sigma \vdash B$

then $\Sigma \vdash A$.

This rule ($\leftrightarrow$ elimination) could also have been given as

($\leftrightarrow -$)

If $\Sigma \vdash A \leftrightarrow B$

then $\Sigma \vdash A \to B$

$\Sigma \vdash B \to A$.

Similarly $(\leftrightarrow +)$ is given as

If $\Sigma, A \vdash B$

$\Sigma, B \vdash A$

Then $\Sigma \vdash A \leftrightarrow B$.

So $(\leftrightarrow +)$ can be given as

If $\Sigma \vdash A \to B$

$\Sigma \vdash B \to A$

then $\Sigma \vdash A \leftrightarrow B$. $(\leftrightarrow +)$

Recall the earlier fact

Fact: If $A \mathrel{\text{⫤}}\!\!\vDash A'$ and $B \mathrel{\text{⫤}}\!\!\vDash B'$

then: [1] $\neg A \mathrel{\text{⫤}}\!\!\vDash \neg A'$

[2] $A \wedge B \mathrel{\text{⫤}}\!\!\vDash A' \wedge B'$

[3] $A \vee B \mathrel{\text{⫤}}\!\!\vDash A' \vee B'$

[4] $A \to B \mathrel{\text{⫤}}\!\!\vDash A' \to B'$

[5] $A \leftrightarrow B \mathrel{\text{⫤}}\!\!\vDash A' \leftrightarrow B'$.

In ND we can prove analogous facts.

Fact: If $A \dashv\vdash A'$ and $B \dashv\vdash B'$ then

[1] $\neg A \dashv\vdash \neg A'$

[2] $A \wedge B \dashv\vdash A' \wedge B'$

[3] $A \vee B \dashv\vdash A' \vee B'$

[4] $A \to B \dashv\vdash A' \to B'$

[5] $A \leftrightarrow B \dashv\vdash A' \leftrightarrow B'$.

This allows for the replacement of equivalent formulae.

Fact: If $B \dashv\vdash C$ and A' results from A by replacing some occurrences of B in A by C then $A \dashv\vdash A'$.

(Recall the analogous fact where $\dashv\vdash$ is exchanged for $\Dashv\vDash$.)

Similarly we have that:

$$A_1, \ldots, A_n \vdash A \text{ iff } \emptyset \vdash A_1 \wedge \ldots \wedge A_n \rightarrow A$$

and

$$A_1, \ldots, A_n \vdash A \text{ iff } \emptyset \vdash A_1 \rightarrow (\ldots (A_n \rightarrow A)).$$

If $A \in Form(L^p)$ and A is built from atoms and the connectives $\wedge$, $\vee$ and $\neg$ then A', the dual of A, is obtained by replacing each $\wedge$ in A by $\vee$, each $\vee$ in A by $\wedge$ and each atom, p, in A by $\neg p$.

Finally, we also have the facts regarding duality.

Fact: Suppose $A \in \mathrm{Form}(L^P)$ built of atoms and the connectives $\neg$, $\wedge$, and $\vee$, and that A' is the dual of A.

Then $A' \dashv\vdash \neg A$.

Fact: Soundness of ND proofs.

If $\Sigma \vdash A$ then $\Sigma \models A$.

Proof Outline: The proof procedes by an induction on the steps of the deduction $\Sigma \vdash A$.

The idea is to show that each application of an ND proof step, $\Sigma \vdash A$, is justified by showing that $\Sigma \models A$.

There are a total of 11 basic steps, we show the details for some of these.

Case of (Ref). The rule is $A \vdash A$.

However, notice that $A \models A$ must hold.

Case of (+). If $\Sigma \vdash A$

then $\Sigma, \Sigma' \vdash A$.

Suppose that $\Sigma \models A$. Then for all valuations, t, if $\Sigma^t = \underline{1}$ then $A^t = \underline{1}$.

To show that $\Sigma, \Sigma' \vdash A$ we must have that for all valuations, t_1, if $\Sigma^{t_1} = \underline{1}$ and $(\Sigma')^{t_1} = \underline{1}$ then $A^{t_1} = \underline{1}$.

However, since $\Sigma^{t_1} = \underline{1}$ then we must have $A^{t_1} = \underline{1}$, by the above supposition.

Case of $(\neg -)$. If $\Sigma, \neg A \vdash B$

$\Sigma, \neg A \vdash \neg B$

then $\Sigma \vdash A$.

Show that if $\Sigma, \neg A \models B$ and $\Sigma, \neg A \models \neg B$ then $\Sigma \models A$.

Suppose $\Sigma, \neg A \models B$ and $\Sigma, \neg A \models \neg B$.

Consider a valuation t such that $\Sigma^t = 1$ and $(\neg A)^t = 1$.

Since $\Sigma, \neg A \models B$ we must have that $B^t = 1$.

Since $\Sigma, \neg A \models \neg B$ we must have that $(\neg B)^t = 1$. This implies that $B^t = 0$. But this contradicts the fact that $B^t = 1$.

Therefore if $\Sigma^t = \underline{1}$ then $(\neg A)^t = 0$ implying that $A^t = \underline{1}$.

Hence $\Sigma \models A$.

Case of $(\vee -)$. If $\Sigma, A \vdash C$

$\Sigma, B \vdash C$

then $\Sigma, A \vee B \vdash C$.

So suppose that $\Sigma, A \models C$ and $\Sigma, B \models C$. Show that $\Sigma, A \vee B \models C$.

Consider a valuation t such that $\Sigma^t = 1$ and $(A \vee B)^t = 1$.

Since $(A \vee B)^t = 1$ then either $A^t = 1$ or $B^t = 1$.

Suppose $A^t = 1$. Since $\Sigma, A \models C$ then $C^t = 1$.

Suppose $B^t = 1$. Since $\Sigma, B \models C$ then $C^t = 1$.

Therefore if $\Sigma^t = \underline{1}$ and $(A \vee B)^t = \underline{1}$ then $C^t = \underline{1}$.

Hence $\Sigma, A \vee B \models C$.