Week 6 Tutorial

Natural Deduction Revisited

Joe Scott / Jan Gorzny

Prepared based off of the notes of CS245 Instructors, past and present.

10 February 2017

Plan

Outline

Natural Deduction - Yet Another Proof System

- Natural Deduction is a sound and complete proof system.
- Allows both direct and refutation styled proofs.
- Inference rules are intuitive. Easier to put into words.

Natural Deduction Rules - As Seen In Class

- (Ref) $A \vdash A$
- $(+)$ If $\Sigma \vdash A$, then $\Sigma, \Sigma' \vdash A$
- $(\neg-)$ If $\Sigma, \neg A \vdash B$, $\Sigma, \neg A \vdash \neg B$, then $\Sigma \vdash A$.
- $(\to-)$ If $\Sigma \vdash A \to B$, $\Sigma \vdash A$, then $\Sigma \vdash B$.
- $(\wedge-)$ If $\Sigma \vdash A \wedge B$, then $\Sigma \vdash A$, $\Sigma \vdash B$.
- $(\wedge+)$ If $\Sigma \vdash A$, $\Sigma \vdash B$, then $\Sigma \vdash A \wedge B$.
- $(\vee-)$ If $\Sigma, A \vdash C$, $\Sigma, B \vdash C$, then $\Sigma, A \vee B \vdash C$.
- $(\vee+)$ If $\Sigma \vdash A$, then $\Sigma \vdash A \vee B$, $\Sigma \vdash B \vee A$.
- $(\leftrightarrow-)$ If $\Sigma \vdash A \leftrightarrow B$, $\Sigma \vdash A$, then $\Sigma \vdash B$.
- $(\leftrightarrow-)$ If $\Sigma \vdash A \leftrightarrow B$, $\Sigma \vdash B$, then $\Sigma \vdash A$.
- $(\leftrightarrow+)$ If $\Sigma, A \vdash B$, $\Sigma, B \vdash A$, then $\Sigma \vdash A \leftrightarrow B$.

Outline

Example 1 - Week 5

Problem

Prove $r \vee (\neg s) \vdash (s \rightarrow r)$

Example 1 - Week 5

Problem

Prove $r \vee (\neg s) \vdash (s \rightarrow r)$

1	$(r \vee (\neg s))$			Premise
2		r		Assumption
3			s	Assumption
4			r	Reflexivity: 2
5		$(s \rightarrow r)$		$\rightarrow_i$: $3-4$
6		$(\neg s)$		Assumption
7			s	Assumption
8			$\perp$	$\perp_i : 6, 7$
9			r	$\perp_e : 8$
10		$s \rightarrow r$		$\rightarrow_i$ $7-9$
11	$s \rightarrow r$			$\vee_e : 2-5, 6-10$

Example 1 - Week 5 - Done Right

Problem

Prove $r \vee (\neg s) \vdash (s \to r)$

Example 1 - Week 5 - Done Right

Problem

Prove $r \vee (\neg s) \vdash (s \rightarrow r)$

1. $r, s \vdash r$ ϵ

Example 1 - Week 5 - Done Right

Problem

Prove $r \vee (\neg s) \vdash (s \to r)$

1.	$r, s \vdash r$	ϵ
2.	$r \vdash s \to r$	$\to$-introduction on 1

Example 1 - Week 5 - Done Right

Problem

Prove $r \vee (\neg s) \vdash (s \to r)$

1.	$r, s \vdash r$	ϵ
2.	$r \vdash s \to r$	$\to$-introduction on 1
3.	$\neg s, s \vdash r$	Theorem 2.6.5 [4]

Example 1 - Week 5 - Done Right

Problem

Prove $r \lor (\neg s) \vdash (s \to r)$

1.	$r, s \vdash r$	ϵ
2.	$r \vdash s \to r$	$\to$-introduction on 1
3.	$\neg s, s \vdash r$	Theorem 2.6.5 [4]
4.	$\neg s \vdash s \to r$	$\to$-introduction on 3

Example 1 - Week 5 - Done Right

Problem

Prove $r \vee (\neg s) \vdash (s \to r)$

1.	$r, s \vdash r$	ϵ
2.	$r \vdash s \to r$	$\to$-introduction on 1
3.	$\neg s, s \vdash r$	Theorem 2.6.5 [4]
4.	$\neg s \vdash s \to r$	$\to$-introduction on 3
5.	$r \vee \neg s \vdash s \to r$	$\vee$-elimination on 2,4

Theorem 2.6.5 [4]

Problem

Prove $X, \neg X \vdash Y$

(The variables have been renamed for clarity.)

Theorem 2.6.5 [4]

Problem

Prove $X, \neg X \vdash Y$

(The variables have been renamed for clarity.)

1. $\neg Y, X, \neg X \vdash X$ $\qquad \epsilon$

Theorem 2.6.5 [4]

Problem

Prove $X, \neg X \vdash Y$

(The variables have been renamed for clarity.)

1.	$\neg Y, X, \neg X \vdash X$	ϵ
2.	$\neg Y, X, \neg X \vdash \neg X$	ϵ

Theorem 2.6.5 [4]

Problem

Prove $X, \neg X \vdash Y$

(The variables have been renamed for clarity.)

1.	$\neg Y, X, \neg X \vdash X$	ϵ
2.	$\neg Y, X, \neg X \vdash \neg X$	ϵ
3.	$X, \neg X \vdash Y$	$\neg$-elimination 1,2

Note: to apply the proof rule, as it appears in the text book, take $A = Y$, $\Sigma = \{A, \neg A\}$, $B = X$, and $\neg B = \neg X$.

Example 1 - Week 4

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

Example 1 - Week 4

Problem

Prove $p \to (q \to r) \vdash (p \wedge q) \to r$

1	$p \to (q \to r)$	Premise
2	$p \wedge q$	Assumption
3	p	$\wedge_e$: 2
4	q	$\wedge_e$: 2
5	$(q \to r)$	$\to_e$: 3, 1
6	r	$\to_e$: 4,5
7	$(p \wedge q) \to r$	$\to_i$: 2 – 6

Example 1 - Week 4 - Done Right

Problem

Prove $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$

Example 1 - Week 4 - Done Right

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1. $p \to (q \to r), p \land q \vdash p \land q$ $\qquad \epsilon$

Example 1 - Week 4 - Done Right

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r), p \land q \vdash p \land q$	ϵ
2.	$p \to (q \to r), p \land q \vdash p$	$\land$-elimination 1

Example 1 - Week 4 - Done Right

Problem

Prove $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r), p \wedge q \vdash p \wedge q$	ϵ
2.	$p \to (q \to r), p \wedge q \vdash p$	$\wedge$-elimination 1
3.	$p \to (q \to r), p \wedge q \vdash q$	$\wedge$-elimination 1

Example 1 - Week 4 - Done Right

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r), p \land q \vdash p \land q$	ϵ
2.	$p \to (q \to r), p \land q \vdash p$	$\land$-elimination 1
3.	$p \to (q \to r), p \land q \vdash q$	$\land$-elimination 1
4.	$p \to (q \to r), p \land q \vdash p \to (q \to r)$	ϵ

Example 1 - Week 4 - Done Right

Problem

Prove $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r), p \wedge q \vdash p \wedge q$	ϵ
2.	$p \to (q \to r), p \wedge q \vdash p$	$\wedge$-elimination 1
3.	$p \to (q \to r), p \wedge q \vdash q$	$\wedge$-elimination 1
4.	$p \to (q \to r), p \wedge q \vdash p \to (q \to r)$	ϵ
5.	$p \to (q \to r), p \wedge q \vdash (q \to r)$	$\to$-elimination 2,4

Example 1 - Week 4 - Done Right

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r), p \land q \vdash p \land q$	ϵ
2.	$p \to (q \to r), p \land q \vdash p$	$\land$-elimination 1
3.	$p \to (q \to r), p \land q \vdash q$	$\land$-elimination 1
4.	$p \to (q \to r), p \land q \vdash p \to (q \to r)$	ϵ
5.	$p \to (q \to r), p \land q \vdash (q \to r)$	$\to$-elimination 2,4
6.	$p \to (q \to r), p \land q \vdash r$	$\to$-eliminiation 3,5

Example 1 - Week 4 - Done Right

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r), p \land q \vdash p \land q$	ϵ
2.	$p \to (q \to r), p \land q \vdash p$	$\land$-elimination 1
3.	$p \to (q \to r), p \land q \vdash q$	$\land$-elimination 1
4.	$p \to (q \to r), p \land q \vdash p \to (q \to r)$	ϵ
5.	$p \to (q \to r), p \land q \vdash (q \to r)$	$\to$-elimination 2,4
6.	$p \to (q \to r), p \land q \vdash r$	$\to$-eliminiation 3,5
7.	$p \to (q \to r) \vdash (p \land q) \to r$	$\to$-introduction 6

Example 2 - Week 4

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

Example 2 - Week 4

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1	$\psi \to \beta$			(Assumption)
2		$(\neg\beta)$		(Assumption)
3			ψ	(Assumption)
4			β	$(\to_e : 1,3)$
5			$\perp$	$(\perp_i : 2,4)$
6		$\neg\psi$		$(\neg_i$ 3-5)
7	$(\neg\beta) \to (\neg\psi)$			$(\to_i$: 2-6)

Example 2 - Week 4 - Done Right

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

Example 2 - Week 4 - Done Right

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1. $\psi \to \beta, \neg\beta, \psi \vdash \neg\beta$ $\qquad\qquad\qquad\qquad \epsilon$

Example 2 - Week 4 - Done Right

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1. $\psi \to \beta, \neg\beta, \psi \vdash \neg\beta$ $\qquad \epsilon$
2. $\psi \to \beta, \psi \vdash \beta$ $\qquad$ Theorem 2.6.4 [1]

Example 2 - Week 4 - Done Right

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1.	$\psi \to \beta, \neg\beta, \psi \vdash \neg\beta$	ϵ
2.	$\psi \to \beta, \psi \vdash \beta$	Theorem 2.6.4 [1]
3.	$\psi \to \beta, \neg\beta, \psi \vdash \beta$	+ 2

Example 2 - Week 4 - Done Right

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1.	$\psi \to \beta, \neg\beta, \psi \vdash \neg\beta$	ϵ
2.	$\psi \to \beta, \psi \vdash \beta$	Theorem 2.6.4 [1]
3.	$\psi \to \beta, \neg\beta, \psi \vdash \beta$	+ 2
4.	$\psi \to \beta, \neg\beta \vdash \neg\psi$	$\neg$-introduction 3,1

Example 2 - Week 4 - Done Right

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1.	$\psi \to \beta, \neg\beta, \psi \vdash \neg\beta$	ϵ
2.	$\psi \to \beta, \psi \vdash \beta$	Theorem 2.6.4 [1]
3.	$\psi \to \beta, \neg\beta, \psi \vdash \beta$	+ 2
4.	$\psi \to \beta, \neg\beta \vdash \neg\psi$	$\neg$-introduction 3,1
5.	$\psi \to \beta \vdash \neg\beta \to \neg\psi$	$\to$-introduction

Theorem 2.6.4 [1]

Problem

$A \to B, A \vdash B$

Theorem 2.6.4 [1]

Problem

$A \to B, A \vdash B$

1. $A \to B, A \vdash A \to B$ $\qquad\qquad \epsilon$

Theorem 2.6.4 [1]

Problem

$A \to B, A \vdash B$

1. $A \to B, A \vdash A \to B$ $\qquad \epsilon$
2. $A \to B, A \vdash A$ $\qquad \epsilon$

Theorem 2.6.4 [1]

Problem

$A \to B, A \vdash B$

1.	$A \to B, A \vdash A \to B$	ϵ
2.	$A \to B, A \vdash A$	ϵ
3.	$A \to B, A \vdash B$	$\to$-elimination 1,2

Plan

The end

Thats it folks. Feel free to hang out and ask questions.

These slides are based off of the tutorial notes and lecture slides provided to you online.

If you want a copy feel free to email me. The are also available on my personal website joe-scott.net

IA Email:

IA	email
Jan Gorzny	jgorzny
Joe Scott	j29scott

Jan and Joe have an office hour Mondays at 3pm in the Tutorial Center in MC.

Instructor Office Hours:

Instructor	Time	Room	Email
Trefler	Tue, Thur 4:00pm	DC 2336	trefler
Rahkooy	Tue, Thur 4:00pm	DC 2302B	hamid.rahkooy