CS 245

Notes

Lemma: Any non-empty proper initial segment of a formula of L^P has more occurrences of left than right parentheses.

Proof: By induction on the structure of the formulae in L^p.

Base case: Suppose $A \in Form(L^p)$ and $A \in Atom(L^p)$. Then A is a single symbol and has no non-empty proper initial segments.

I.H. Assume that if B and C are in Form (LP) then any non-empty proper initial segment of either B or C has more left than right parentheses.

Show that if $A = (\neg B)$ then any non-empty proper initial segment of $(\neg B)$ has more left than right parentheses.

First, recall from the previous lemma, that all formulae of L^P have the same number of left and right parentheses.

So both B and C, each have equal numbers of left and right parentheses.

Consider $A = (\neg B)$. The non-empty proper initial segments of A are the following:

$($

$(\neg$

$(\neg B$

$(\neg B'$

where $B = B'B''$.

In the two cases, $($ and $(\neg$, it is clear that these segments have more left than right parens

In the case $(\neg B$, recall that B, a formula, has an equal number of left and right parentheses, so that $(\neg B$ has more left than right parentheses.

Since in the last case, $(\neg B'$, where $B = B'B''$, we use the I.H.

B' is a non-empty proper initial segment of B. Therefore B' has more left than right parentheses, by assumption.

Hence $(\neg B'$ must have more left than right parentheses.

In case $A = (B * C)$ where $*$ is one of $\wedge, \vee, \rightarrow$ or $\leftrightarrow$, then the non-empty proper initial segments of A are

$($

$(B'$ where $B = B'B''$

$(B$

$(B *$

$(B * C'$ where $C = C'C''$

$(B * C$

(clearly has more left than right parentheses.

B and C have equal numbers of parentheses, so

(B

(B*

and

(B*C

all have more left than right parentheses.

Since $B = B'B''$ and $C = C'C''$ where B and C have equal numbers of parentheses, then B' and C' have more left than right parentheses. (B' and C' are non-empty proper initial segments of B and C respectively.)

Therefore $(B'$

and $(B * C'$

each have more left than right parentheses

(End proof of lemma.)

Now consider the previous Theorem, every formula in Form(L^P) is of exactly one of the six forms.

We can show that any two of the six forms must differ.

An atom, say p, is a single symbol and so differs from a formula of the form $(\neg A)$ or $(B * C)$.

Consider a formula $(\neg A)$.

Suppose $(\neg A) = (B * C)$.

By removing the left parentheses of the two equal formulae we must have that

$$\neg A) = B * C)$$

However, by assumption B is a formula in $\mathrm{Form}(L^P)$ and no formula in $\mathrm{Form}(L^P)$ has a $\neg$ symbol as its initial symbol.

Hence a formula of the form $(\neg A)$ cannot also be of the form $(B * C)$.

Suppose $(A \vee B) = (C \wedge D)$.

This implies that

$$A \vee B) = C \wedge D).$$

Since, by assumption, A, B, C and D are all formulae in $\mathrm{Form}(L^P)$, neither A is a proper initial segment of C nor C is a proper initial segment of A.

Since the formulae were assumed to be equal, they have the same occurrences of symbols, we must have that $A = C$.

But then '$\wedge$' must equal '$\vee$' and by similar reasoning $B = D$.

Hence the different forms produce different formulae.

Now similar arguments show that if $(\neg A) = (\neg B)$ then $A = B$ and if $(A * B) = (C * D)$ then $A = C$ and $B = D$.

In the first case, suppose that

$(\neg A) = (\neg B)$.

It immediately follows that

$A) = B)$.

Hence $A = B$ is immediate.

Similarly, if $(A * B) = (C * D)$ then $A * B) = C * D)$. Again neither A can be a non-empty proper initial segment of C nor C a non-empty proper initial segment of A.

So $A = C$ and $B = D$.

Putting together all the facts about how formulae are syntactically constructed is useful when giving semantics to the formulae.

Semantic meaning is defined through the syntacic construction.

Because the syntacic construction is a well defined, essentially unique process, there will be a well defined semantic interpretation for each formula of $\mathrm{Form}(\mathcal{L}^{\varphi})$.

Recall that a truth valuation, t, is a function that maps $Atom(\mathcal{L}^p)$ into $\{0, 1\}$.

That is $t : Atom(\mathcal{L}^p) \longrightarrow \{0, 1\}$.

Given atom, p, in $Atom(L^P)$ and truth valuation, t, we typically write, p^t, rather than $t(p)$, to denote the application of valuation function t to the atom p.

Def. Values of formulae.

For $p \in \mathrm{Atom}(L^p)$: $p^t \in \{0, \underline{1}\}$.

$$(\neg A)^t = \begin{cases} \underline{1} & \text{if } A^t = 0. \\ 0 & \text{if } A^t = \underline{1}. \end{cases}$$

$$(A \wedge B)^t = \begin{cases} \underline{1} & \text{if } A^t = \underline{1} \text{ and } B^t = \underline{1}. \\ 0 & \text{otherwise.} \end{cases}$$

$$(A \vee B)^t = \begin{cases} \underline{1} & \text{if } A^t = \underline{1} \text{ or } B^t = \underline{1} \\ 0 & \text{otherwise} \end{cases}$$

$$(A \rightarrow B)^t = \begin{cases} \underline{1} & \text{if } A^t = 0 \text{ or } B^t = \underline{1}, \\ 0 & \text{otherwise} \end{cases}$$

$$(A \longleftrightarrow B)^t = \begin{cases} \underline{1} & \text{if } A^t = B^t. \\ 0 & \text{otherwise} \end{cases}$$

Fact: For all $A \in \mathrm{Form}(\mathcal{L}^p)$ and any truth valuation, t, it is the case that $A^t \in \{0, 1\}$.

The proof of the fact follows by structural induction on the formulae in $\mathrm{Form}(\mathcal{L}^p)$.

Notice that formulae can become over populated with parentheses.

To alleviate this problem we can do several things.

We can omit the outer most, or top level parentheses. So

$$(((p \wedge q) \rightarrow (p \vee r)) \leftrightarrow (\neg q))$$

becomes

$$((p \wedge q) \rightarrow (p \vee r)) \longleftrightarrow (\neg q)$$

We can use alternative forms of parentheses.

For instance $[\quad]$

or $\{\quad\}$

For example,

$$((p \wedge q) \rightarrow (p \vee r)) \leftrightarrow (\neg q)$$

becomes

$$[(p \wedge q) \rightarrow (p \vee r)] \leftrightarrow (\neg q)$$

We can also simply omit parenthesis by following a set of given precedence or priorities.

Here we use the following priorities:

$\neg$
$\wedge$
$\vee$
$\rightarrow$
$\leftrightarrow$

listed from highest to lowest priority.

So $\neg p \to p \wedge \neg q \vee r \leftrightarrow q$

is a short hand for

$$\left[(\neg p) \to \left(\left(p \wedge (\neg q)\right) \vee r\right)\right] \leftrightarrow q$$

Now consider the formula:

$$p \vee q \to q \wedge r$$

This is a shorthand for:

$$(p \vee q) \to (q \wedge r)$$

And the valuation t where

$p^t = \underline{1}$, $q^t = \underline{1}$, and $r^t = \underline{1}$.

Using the definition of the value of a formula:

$$(q \wedge r)^t = \underline{1}$$

$$(p \vee q)^t = \underline{1}$$

$$((p \vee q) \rightarrow (q \wedge r))^t = \underline{1}.$$

Using the valuation

$p^{t'} = 0, \ q^{t'} = 0, \ r^{t'} = 0$

$(q \wedge r)^{t'} = 0$

$(p \vee q)^{t'} = 0$

$((p \vee q) \rightarrow (q \wedge r))^{t'} = \underline{1}$

Is there a valuation, t'', under which

$$(p \vee q \rightarrow q \wedge r)^{t''} = 0 ?$$

Consider t'' where

$$t''(p) = \underline{1}, \quad t''(q) = 0, \quad t''(r) = \underline{1}$$

then $(q \wedge r)^{t''} = 0$

$$(p \vee q)^{t''} = \underline{1}$$

$$(p \vee q \rightarrow q \wedge r)^{t''} = 0.$$

Let Σ denote some subset of formulae.

So $\Sigma \subseteq \mathrm{Form}(L^P)$.

And suppose that t is a valuation applicable to all propositions that appear in the formulae in Σ.

Then $$\Sigma^t = \begin{cases} 1 & \text{if for all } B \in \Sigma \\ & B^t = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $\Sigma^t = 0$ will hold as long as there is a single $B \in \Sigma$ such that $B^t = 0$.

Def. Let $A \in Form(L^p)$, then A is satisfiable if there is a valuation, t, to the atoms of A such that $A^t = 1$.

So $p \vee q \to q \wedge r$ is a satisfiable formula.

For set $\Sigma \subseteq Form(L^P)$, we say that Σ is satisfiable if there is a valuation, t, to the atoms in Σ such that

$\Sigma^t = \underline{1}$.

That is, for all $B \in \Sigma$, $B^t = \underline{1}$.

A formula, B, is satisfiable if there is some valuation t such that $B^t = \underline{1}$.

There are two related notions that are also important, tautology and contradiction.

Formula $B \in Form(L^p)$ is a tautology if for all valuations, t, it is the case that $B^t = 1$.

Formula $B \in Form(L^p)$ is a contradiction if for all valuations, t, it is the case that $B^t = 0$.

Example

p	q	r	$(p \vee q$	$\to r)$	$\leftrightarrow$	$(p \to r)$	$\wedge$	$(q \to r)$
1	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	0	0
1	0	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
0	1	1	1	1	1	1	1	1
0	1	0	1	0	1	1	0	0
0	0	1	0	1	1	1	1	1
0	0	0	0	1	1	1	1	1

To determine whether a formula, $A \in Form(L^P)$, is satisfiable, a tautology or a contradiction can be decided by looking up the information in a truth table.

The table contains all possible valuations to the propositions that are contained in A.

When A is comprised of n different propositions then there are 2^n different truth functions given in a truth table for A.

One can also 'calculate' using 'expressions' such as $A \wedge 0$ or $\underline{1} \vee A$.

$\neg \underline{1}$	0
$\neg 0$	$\underline{1}$
$A \wedge \underline{1}$	A
$\underline{1} \wedge A$	A
$A \wedge 0$	0
$0 \wedge A$	0

$A \vee \underline{1}$	$\underline{1}$
$\underline{1} \vee A$	$\underline{1}$
$A \vee 0$	A
$0 \vee A$	A
$A \rightarrow \underline{1}$	$\underline{1}$
$\underline{1} \rightarrow A$	A
$A \rightarrow 0$	$\neg A$
$0 \rightarrow A$	$\underline{1}$

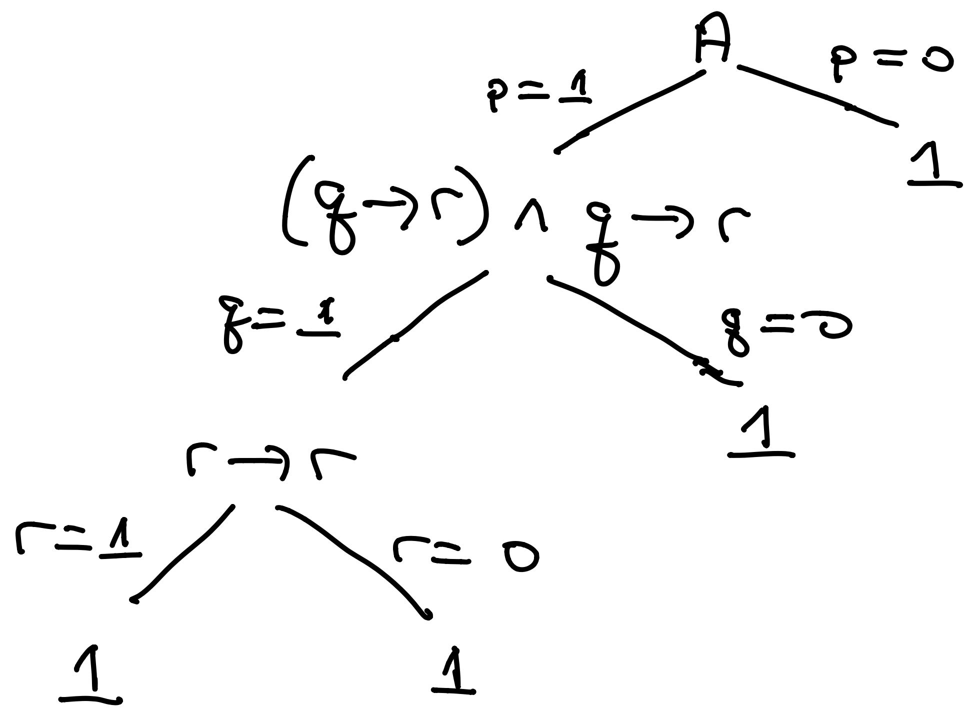
$A \longleftrightarrow \underline{1} \qquad A$

$\underline{1} \longleftrightarrow A \qquad A$

$A \longleftrightarrow 0 \qquad \neg A$

$0 \longleftrightarrow A \qquad \neg A$

Suppose $A = (p \wedge q \rightarrow r) \wedge (p \rightarrow q) \rightarrow (p \rightarrow r)$



This shows that A is a tautology.