CS 245

Notes

Tautological Consequence

In deductive logic we study the conditions and tools needed to deduce that the truth of a conclusion, A, is implied by the truth of proposition $A_1, \ldots, A_n$.

This may also be stated as, the truth of the propositions $A_1, \ldots, A_n$ implies the truth of A.

We can group the propositions together and ask when does the truth of $\{A_1, \ldots, A_n\}$ imply the truth of A?

Def. (Tautological consequence)

Suppose $\Sigma \subseteq Form(L^p)$ and $A \in Form(L^p)$ then A is a tautological consequence of Σ, written $\Sigma \models A$, if

for all truth valuations, t:

if $\Sigma^t = \underline{1}$ then $A^t = \underline{1}$.

Notice that '$\models$' is not a symbol in L^P.

This means that $\Sigma \models A$ is not a formula in $Form(L^P)$.

$\Sigma \models A$ is a statement in the 'metalanguage.'

That is, $\Sigma \models A$ is a statement about things we have said in the logic.

We write $\Gamma \not\models A$ as a shorthand for:

it is not the case that

$$\Gamma \models A.$$

(It is not the case that A is a tautological consequance of Γ.)

Consider the situation where $\Sigma = \emptyset$.

If we write, $\Sigma \models A$, this means that for all valuations, t, if $\Sigma^t = \underline{1}$ then $A^t = \underline{1}$.

If $\Sigma = \emptyset$ then

$\Sigma^t = \underline{1}$ if forall $B \in \Sigma$,
$B^t = \underline{1}$.

However, since $\Sigma = \emptyset$ there does not exist $B \in \Sigma$.
Hence for all $B \in \Sigma$ it is the case that $B^t = \underline{1}$.

So, when $\Sigma = \emptyset$ it follows that $\Sigma^t = \underline{1}$ for all t.

This implies that if $\Sigma = \emptyset$ and $\Sigma \models A$ then $A^t = \underline{1}$ for all t.

Therefore A is a tautology.

Let A and B be two formulae in $Form(L^P)$.

We write $A \models\!\!\dashv B$ to denote that $A \models B$ and $B \models A$.

$A \models\!\!\dashv B$ means that A and B are (tautologically) equivalent.

To show that $\Gamma \models A$ we must show that for all valuations, t, if $\Gamma^t = \underline{1}$ then $A^t = 1$.

Example: $A \to B,\ B \to C \models A \to C$.

Notice that $\Sigma = \{A \to B,\ B \to C\}$ is given as a list.

We need to show that if $t : \{A, B, C\} \to \{0, 1\}$ where $(A \to B)^t = \underline{1}$ and $(B \to C)^t = \underline{1}$ then it must be that $(A \to C)^t = \underline{1}$.

We prove this by showing that there is no valuation t where $(A \to B)^t = 1$, $(B \to C)^t = 1$ and $(A \to C)^t = 0$.

Suppose $(A \to C)^t = 0$.

This implies that $A^t = \underline{1}$

and $C^t = 0$.

If $A^t = \underline{1}$ and $(A \to B)^t = \underline{1}$

then it must be that

$B^t = \underline{1}$.

If $B^t = \underline{1}$ and $(B \to C)^t = \underline{1}$

then $C^t = \underline{1}$.

However this contradicts

the fact that $C^t = 0$.

Therefore there is no

valuation t where $(A \to B)^t = \underline{1}$,

$(B \to C)^t = \underline{1}$ and $(A \to C)^t = 0$.

So this was an example where $\mathfrak{A} \models A$ holds.

Now we look at an example where $\mathfrak{A} \not\models A$.

Example

$(A \to \neg B) \vee C,\ B \wedge \neg C,\ A \leftrightarrow C \not\models \neg A \wedge (B \to C)$

In order to show this example, we should find a valuation, t, where $((A \to \neg B) \vee C)^t = \underline{1}$

$$(B \wedge \neg C)^t = \underline{1}$$

$$(A \leftrightarrow C)^t = \underline{1}$$

and $(\neg A \wedge (B \to C))^t = 0$.

Consider the valuation t such that $A^t = 0$, $B^t = 1$ and $C^t = 0$.

This means that $(\neg B)^t = 0$, $(A \to \neg B)^t = \underline{1}$ and $((A \to \neg B) \vee C)^t = \underline{1}$.

Furthermore $(\neg C)^t = \underline{1}$ and $(B \wedge \neg C)^t = \underline{1}$.

In addition, $(A \leftrightarrow C)^t = \underline{1}$

$(B \to C)^t = 0$, $(\neg A)^t = \underline{1}$

and $(\neg A \land (B \to C))^t = 0$.

Thus $((A \to B) \lor C)^t = \underline{1}$

$(B \land \neg C)^t = \underline{1}$

$(A \leftrightarrow C)^t = \underline{1}$

and $(\neg A \land (B \to C))^t = 0$.

So

$(A \to B) \lor C, B \land \neg C, A \leftrightarrow C \not\models \neg A \land (B \to C)$

We can also show that the $\wedge$ operator and $\vee$ operator are both commutative and associative.

commutative: $A \wedge B \dashv\vdash B \wedge A$

$A \vee B \dashv\vdash B \vee A$

associative $A \wedge (B \wedge C) \dashv\vdash (A \wedge B) \wedge C$

$A \vee (B \vee C) \dashv\vdash (A \vee B) \vee C$

These facts allow us to simply remove parentheses from lists of $\wedge$ operators and from lists of $\vee$ operators.

So we can write:

$$A_1 \wedge \ldots \wedge A_n$$

$$A_1 \vee \ldots \vee A_n$$

Fact: $A_1, \ldots, A_n \models A$

iff $\models A_1 \wedge \ldots \wedge A_n \rightarrow A$

Fact: $A_1, \ldots, A_n \models A$

iff

$\models A_1 \rightarrow (\ldots(A_n \rightarrow A))$

The previous fact is interesting because it shows that tautalogical consequences can be rewritten as tautologies. And that tautologies of a certain form can be rewritten as tautological consequences.

For example:

$$A \models B \quad \text{iff} \quad \models A \to B$$

$$A, B \models C \quad \text{iff} \quad \models A \wedge B \to C$$

We can now make a distinction between equivalent formulas and equality of expressions.

So $A \wedge B$ is not the same expression as $B \wedge A$ but the two formulae are equivalent.

Fact: Suppose $A \mathrel{\vDash\!\dashv} A'$ and $B \mathrel{\vDash\!\dashv} B'$ then

[1] $\neg A \mathrel{\vDash\!\dashv} \neg A'$.

[2] $A \wedge B \mathrel{\vDash\!\dashv} A' \wedge B'$.

[3] $A \vee B \mathrel{\vDash\!\dashv} A' \vee B'$.

[4] $A \rightarrow B \mathrel{\vDash\!\dashv} A' \rightarrow B'$.

[5] $A \leftrightarrow B \mathrel{\vDash\!\dashv} A' \leftrightarrow B'$.

Fact: If $B \dashv\vdash C$ and A' results from A by replacing some (but not necessarilly all) occurrences of B in A by C, then $A \dashv\vdash A'$.

Suppose $B = A$.

Then $A' = C$.

Since $B \models\!\!\dashv C$ then

$A \models\!\!\dashv A'$.

For the general case we can proceed by induction.

Base case: A is an atom. Then $A = B$, therefore

Since $B \dashv\vdash C$ we have that $A' \dashv\vdash A$. (Here $A' = C$)

Show the theorem holds if A is one of the forms $(\neg A_1)$, $(A_1 \wedge A_2)$, $(A_1 \vee A_2)$, $(A_1 \rightarrow A_2)$, $(A_1 \leftrightarrow A_2)$.

Suppose $A = (\neg A_1)$.

By the inductive hypothesis if A_1' results from replacing some occurrences of $\mathcal{B}$ in A_1 by C then $A_1 \models\!\dashv A_1'$.

Then by the previous fact, it follows that

$$(\neg A_1) \models\!\dashv (\neg A_1').$$

Suppose $A = A_1 \wedge A_2$.

Let A_1' and A_2' result from substituting (respectively) some occurrences of B in A_1 by C and some occurrences of B in A_2 by C.

By the inductive hypothesis we have that $A_1 \mathrel{\vDash\!\dashv} A_1'$ and $A_2 \mathrel{\vDash\!\dashv} A_2'$.

Then by applying the previous lemma we have that

$$A_1 \wedge A_2 \mathrel{\models\!\!\!\dashv} A_1' \wedge A_2'$$

as required ($A' = A_1' \wedge A_2'$).

The cases where

$$A = A_1 \vee A_2$$
$$A = A_1 \rightarrow A_2$$
$$A = A_1 \leftrightarrow A_2$$

are similar.

Fact: Suppose A is in form(L^P) and composed of atoms and the connectives $\neg$, $\wedge$ and $\vee$.

If A' results from replacing in A $\wedge$ for $\vee$, $\vee$ for $\wedge$ and each atom for its negation then $A' \models\!\mid \neg A$.

A' is called the dual of A.