Week 5 Tutorial

Assignment Preparation; Natural Deduction

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Prepared based off of the notes of CS245 Instructors, past and present.

27 January 2017

Plan

Outline

Natural Deduction - Yet Another Proof System

- Natural Deduction is a sound and complete proof system.
- Allows both direct and refutation styled proofs.
- Inference rules are intuitive. Easier to put into words.

Natural Deduction Rules

1. $\phi \vdash \phi$ (Reflexive)
2. Introduction

$$
\begin{array}{ll}
\phi \vdash \phi \vee \psi & (\vee_i) \\
\phi, \psi \vdash \phi \wedge \psi & (\wedge_i) \\
(\phi \vdash \bot) \vdash \neg\phi & \neg_i \\
(\phi \vdash \psi) \vdash \phi \rightarrow \psi & \rightarrow_i \\
(\phi \rightarrow \psi), (\psi \rightarrow \phi) \vdash \phi \leftrightarrow \psi & \leftrightarrow_i \\
(\phi, \neg\phi) \vdash \bot & \bot_i
\end{array}
$$

3. Elimination

$$
\begin{array}{ll}
(\phi \rightarrow \eta, \psi \rightarrow \eta, \phi \vee \psi) \vdash \eta & \vee_e \\
(\phi \wedge \psi) \vdash \phi & \wedge_e \\
(\neg(\neg\phi)) \vdash \phi & \neg_e \\
(\phi \rightarrow \psi, \phi) \vdash \psi & \rightarrow_e \\
(\phi \leftrightarrow \psi) \vdash \phi \rightarrow \psi & \leftrightarrow_e \\
\bot \vdash \phi & \bot_e
\end{array}
$$

Tips

- Make subproofs for rules like (e.g $\rightarrow_i$, $\neg_i$) clear. Indent each of them (like nested loops in code)
- The instructors recommend the following
- Write down the premises and conclusion.
- Consider eliminations from premises.
- Work backwards, see if you can use introduction while going backwards.
- Repeat this process in subproofs.

Outline

Example 1

Problem

Prove $r \vee (\neg s) \vdash (s \to r)$

Example 1

Problem

Prove $r \vee (\neg s) \vdash (s \rightarrow r)$

1	$(r \vee (\neg s))$	Premise

Example 1

Problem

Prove $r \lor (\neg s) \vdash (s \to r)$

1	$(r \lor (\neg s))$		Premise
2		r	Assumption

Example 1

Problem

Prove $r \vee (\neg s) \vdash (s \rightarrow r)$

1	$(r \vee (\neg s))$			Premise
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3			s	Assumption

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4			r	Reflexivity: 2

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5		$(s \rightarrow r)$		$\rightarrow_i$: $3-4$

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3	s	Assumption
4	r	Reflexivity: 2
5	$(s \to r)$	$\to_i$: $3 - 4$
6	$(\neg s)$	Assumption
7	s	Assumption
8	$\perp$	$\perp_i : 6, 7$

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10		$s \to r$		$\to_i$ $7-9$

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8		$\perp$	$\perp_i : 6, 7$
9		r	$\perp_e : 8$
10		$s \rightarrow r$	$\rightarrow_i 7 - 9$
11	$s \rightarrow r$		$\vee_e : 2 - 5, 6 - 10$

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Example 2

Problem

Let Σ be any set of formulas. Suppose that $\Sigma \vdash (\neg\psi)$ for some ψ. Prove that $\Sigma \vdash \psi \to p$

Let F be the proof of $\Sigma \vdash (\neg\psi)$.

0	Σ	Premise

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Let Σ be any set of formulas. Suppose that $\Sigma \vdash (\neg\psi)$ for some ψ. Prove that $\Sigma \vdash \psi \rightarrow p$

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$n+1$		$\perp$	$\perp_i : 1, n$
$n+2$		p	$\perp_e$: n+1
$n+3$	$\psi \rightarrow p$		$\rightarrow_i$: 1-(n+2)

Example 3

Problem

Suppose that $\Sigma \vdash \psi$ *and* $\Delta \vdash \psi$ *for some sets of* Σ *and* Δ *and formula* ψ*. Prove or Disprove* $\Sigma \cap \Delta \vdash \psi$*.*

Example 3

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No. Suppose $\Sigma = \{p\}, \delta = \{\neg(\neg p)\}$. Then $\varnothing \vdash \psi$, as ψ could be anything (i.e the world is flat).

Plan

1. Natural Deduction
   - Review
   - Natural Deduction Examples
2. Assignment 2 Preparation
3. The End

Double Entailment

How to prove double entailments?

1. Truth Table
2. Axioms

How to disprove double entailment

1. Counter example of a single direction
2. Truth Table (but this is the same as a counter example).

Sound Introduction/Elimination Rules

As an example, we will prove that $\rightarrow i$ is sound.

$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \rightarrow B}$$

- We assume that the derivation of B from A is sound (i.e. if $A = T$, then $B = T$.)

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  - Case 2: $A = T$. By assumption, in this case, $B = T$, so $(A \rightarrow B) = T$

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As an example, we will prove that $\to i$ is sound.

$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \to B}$$

- We assume that the derivation of B from A is sound (i.e. if $A = T$, then $B = T$.)
- We prove that $(A \to B) = T$
  - Case 1: $A = F$; clear.
  - Case 2: $A = T$. By assumption, in this case, $B = T$, so $(A \to B) = T$

Adequate Sets I

An adequate set of connectives is a set of connectives with the capability to express all truth tables.

Adequate Sets II

Theorem

$\{\neg, \wedge, \vee\}$ *is an adequate set.*

Proof.

For any formulas A, B,

$$\neg A \quad \models\!\!\!= \quad \neg A$$

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A \to B & \vDash\Dashv & \neg A \vee B
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A \vee B & \models\!\!\!\dashv & A \vee B \\
A \rightarrow B & \models\!\!\!\dashv & \neg A \vee B \\
A \leftrightarrow B & \models\!\!\!\dashv & (\neg A \vee B) \wedge (\neg B \vee A)
\end{array}$$

$\square$

A Resolution Example

Prove that for any $0 \leq i \leq n$ there exists a Resolution refutation proof to witness

$$\{p_i \vdash_{Res} (p_0 \vee \ldots \vee p_n)\}$$

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- Our set of clauses then becomes $\{(\neg p_0), \ldots, p_i, (\neg p_i), \ldots, (\neg p_n)\}$

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- By our inference rules we achieve a $\perp$

Plan

The end

Thats it folks. Feel free to hang out and ask questions.

These slides are based off of the tutorial notes and lecture slides provided to you online.
If you want a copy feel free to email me. The are also available on my personal website joe-scott.net

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