Syntax and Semantics of Propositional Logic

Week 2 Tutorial

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Prepared based off of the notes of CS245 Instructors, past and present.

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Outline

Plan

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1. Review of Induction
   - Mathematical Induction
   - Structural Induction
2. Semantics of Propositional Logic

Induction

Problem: Prove, for all positive integers $n \geq 1$, that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

Induction

Solution: It is an exercise to verify the algebraic identity

$$\frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

The rest of the proof is by induction on $n \geq 1$.

Induction

Base ($n = 1$): We note that

$$\begin{aligned}\sum_{i=1}^{1} i^2 &= 1^2 \\ &= 1, \text{ and} \\ \frac{1^3}{3} + \frac{1^2}{2} + \frac{1}{6} &= \frac{2+3+1}{6} \\ &= 1.\end{aligned}$$

Induction

<u>Induction ($n > 1$):</u> The induction hypothesis is

$$\sum_{i=1}^{n-1} i^2 = \frac{(n-1)^3}{3} + \frac{(n-1)^2}{2} + \frac{(n-1)}{6}.$$

Then we compute

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= \sum_{i=1}^{n-1} i^2 + n^2 \\ &\underbrace{=}_{\text{I.H.}} \frac{(n-1)^3}{3} + \frac{(n-1)^2}{2} + \frac{(n-1)}{6} + n^2\end{aligned}$$

Induction

$$
\begin{aligned}
\underbrace{=}_{\text{I.H.}} \quad & \frac{(n-1)^3}{3} + \frac{(n-1)^2}{2} + \frac{(n-1)}{6} + n^2 \\
= \quad & \frac{n^3 - 3n^2 + 3n - 1}{3} + \frac{n^2 - 2n + 1}{2} + \frac{n-1}{6} + n^2 \\
= \quad & \left(\frac{1}{3}\right) n^3 + \left(-1 + \frac{1}{2} + 1\right) n^2 + \left(1 + -1 + \frac{1}{6}\right) n + \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{6}\right) \\
= \quad & \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.
\end{aligned}
$$

This completes the induction, and the proof.

Induction

Problem: Prove by induction, for all positive integers $n \geq 2$, that

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

Induction

Base ($n = 2$): We note that

$$\begin{aligned} 1 - \frac{1}{2^2} &= \frac{3}{4} \\ \frac{2+1}{4} &= \frac{3}{4} \end{aligned}$$

So the equation is true for $n = 2$.

Induction

Induction ($k \geq 2$): The induction hypothesis is as follows for $n = k$:

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

Now consider $k+1$:

$$\begin{aligned}
\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) &= \prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) \\
&\underbrace{=}_{\text{I.H.}} \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right)
\end{aligned}$$

Induction

$$
\begin{aligned}
\underbrace{=}_{\text{I.H.}} \quad & \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right) \\
= \quad & \frac{k+1}{2k} \cdot \frac{(k+1)^2 - 1}{(k+1)^2} \\
= \quad & \frac{k^2 + 2k}{2k(k+1)} \\
= \quad & \frac{k+2}{2(k+1)}
\end{aligned}
$$

This completes the induction, and the proof.

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1. Review of Induction
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Example Problem

Problem: Use structural induction to prove the following.

> *Let φ be a well-formed formula. Let m be the number of atoms in φ. Let n be the number of occurrences of the binary connectives $\wedge$, $\vee$, $\rightarrow, \leftrightarrow$ in φ. Then $m = n + 1$.*

Base Case

Base Case, ($\varphi = p$, for some propositional variable p): For this formula $\varphi = p$, we have $m = 1$ and $n = 0$. Therefore,

$$m = 1 = 0 + 1 = n + 1,$$

as required.

Inductive Step - Unary Operator

$m - \#$ of ap
$n - \#$ of binary connectives.

Inductive Step, Case if Unary , ($\varphi = (\neg\alpha)$, for some well-formed formula α): Let m_φ, m_α denote the number of atoms in φ, α, respectively. Let n_φ, n_α denote the number of binary connectives in φ, α, respectively. The inductive hypothesis is that $m_\alpha = n_\alpha + 1$. Then we have

$$m_\varphi \underbrace{=}_{\text{by construction}} m_\alpha \underbrace{=}_{\text{I.H.}} n_\alpha + 1 \underbrace{=}_{\text{by construction}} n_\varphi + 1,$$

as required.

Inductive Step - Binary Operator

Inductive Step, Case if Binary, ($\varphi = (\alpha \star \beta)$, for some well-formed formulæ α, β and some binary connective $\star$): Let $m_\varphi, m_\alpha, m_\beta$ denote the number of atoms in φ, α, β, respectively. Let $n_\varphi, n_\alpha, n_\beta$ denote the number of binary connectives in φ, α, β, respectively. The inductive hypothesis is that $m_\alpha = n_\alpha + 1$ and $m_\beta = n_\beta + 1$. Then we have

$$\begin{aligned}
m_\varphi &\underbrace{=}_{\text{by construction}} m_\alpha + m_\beta \\
&\underbrace{=}_{\text{I.H.}} (n_\alpha + 1) + (n_\beta + 1) \\
&\underbrace{=}_{\text{by construction, } n_\varphi = n_\alpha + n_\beta + 1} n_\varphi + 1,
\end{aligned}$$

as required.

Plan

1. Review of Induction
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   - Structural Induction

2. Semantics of Propositional Logic

- Syntax is the grammar. It describes the way to construct a correct sentence. For example, this water is triangular is syntactically correct.
- Semantics relates to the meaning. this water is triangular does not mean anything, though the grammar is ok.

Definition of a Formula

Let $\mathcal{P}$ be a set of propositional variables. We define the set of formulas over $\mathcal{P}$ inductively as follows.

1. An expression consisting of a single symbol of $\mathcal{P}$ is a formula.
2. if ψ is a formula, then $(\neg\psi)$ is a formula.
3. if ψ is a formula and η is a formula then the following are formula
   - $(\psi \wedge \eta)$
   - $(\psi \vee \eta)$
   - $(\psi \implies \eta)$
   - $(\psi \iff \eta)$

Truth Tables

Problem: Give a truth table for the following formula.

$$((p \to q) \wedge r)$$

Solution:

p	q	r	$(p \to q)$	$((p \to q) \wedge r)$
F	F	F	T	F
F	F	T	T	T
F	T	F	T	F
F	T	T	T	T
T	F	F	F	F
T	F	T	F	F
T	T	F	T	F
T	T	T	T	T

Truth Tables

Problem: Give a truth table for the following formula.

$$((p \wedge r) \vee ((\neg r) \to q))$$

Solution:

p	q	r	$(p \wedge r)$	$((\neg r) \to q)$	$((p \wedge r) \vee ((\neg r) \to q))$

Truth Tables

Problem: Give a truth table for the following formula.

$$((p \wedge r) \vee ((\neg r) \rightarrow q))$$

Solution:

p	q	r	$(p \wedge r)$	$((\neg r) \rightarrow q)$	$((p \wedge r) \vee ((\neg r) \rightarrow q))$
F	F	F	F	F	F
F	F	T	F	T	T
F	T	F	F	T	T
F	T	T	F	T	T
T	F	F	F	F	F
T	F	T	T	T	T
T	T	F	F	T	T
T	T	T	T	T	T

Example 1

Do the premises semantically entail (logically imply) the conclusion? Answer this question using a truth table. Explain.

$$\{(p \vee (\neg q)), (p \implies r), ((\neg q) \implies r)\} \models r$$

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$\top$						
$\top$						
$\top$						
$\top$						
$\bot$						
$\bot$						
$\bot$						
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$\top$	$\top$	$\bot$				
$\top$	$\bot$	$\top$				
$\top$	$\bot$	$\bot$				
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$\top$	$\bot$	$\top$	$\top$			
$\top$	$\bot$	$\bot$	$\top$			
$\bot$	$\top$	$\top$	$\bot$			
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$\top$	$\top$	$\top$	$\top$	$\top$		
$\top$	$\top$	$\perp$	$\top$	$\perp$		
$\top$	$\perp$	$\top$	$\top$	$\top$		
$\top$	$\perp$	$\perp$	$\top$	$\perp$		
$\perp$	$\top$	$\top$	$\perp$	$\top$		
$\perp$	$\top$	$\perp$	$\perp$	$\top$		
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$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	
$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	

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$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$
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$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\lhd$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	
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$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\lhd$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\lhd$
$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	

The definition of semantically entails states that the premises semantically entail the conclusion iff every truth valuation which satisfies the premises also satisfies the conclusion.

Example 2

Do the premises semantically entail (logically imply) the conclusion? Answer this question using a truth table. Explain.

$$\{((\neg p) \implies ((\neg q) \vee r)), (p \implies r), (\neg r)\} \models r$$

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$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	
$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	
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$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	
$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	
$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	
$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\triangleleft$