CS 245

Lemma: Let $A \boxminus\!\!\dashv A'$, $B \boxminus\!\!\dashv B'$ and $C(u) \boxminus\!\!\dashv C'(u)$ then

[1] $\neg A \boxminus\!\!\dashv \neg A'$

[2] $A \wedge B \boxminus\!\!\dashv A' \wedge B'$

[3] $A \vee B \boxminus\!\!\dashv A' \vee B'$

[4] $A \to B \boxminus\!\!\dashv A' \to B'$

[5] $A \leftrightarrow B \boxminus\!\!\dashv A' \leftrightarrow B'$

[6] $\forall x\, C(x) \boxminus\!\!\dashv \forall x\, C'(x)$

[7] $\exists x\, C(x) \boxminus\!\!\dashv \exists x\, C'(x)$

Fact: Suppose $B \dashv\vdash C$ and A' results from A by replacing some but not necessarily all occurrences of B in A by C. Then $A \dashv\vdash A'$.

Proof: By induction on the structure of A.

Fact: Suppose $A \in \text{Form}(L)$ and A is composed of atoms, the connectives $\wedge$, $\vee$ and $\neg$, and the quantifiers $\forall$ and $\exists$. Then if A', the dual of A, results from A by exchanging $\wedge$ for $\vee$, $\vee$ for $\wedge$, $\forall$ for $\exists$, $\exists$ for $\forall$, and each atom for its negatron, then $A \mathrel{\#\!\!\!\!\!\dashv} A'$.

Natural Deduction for first-order logic.

To the ND rules for propositional logic add introduction and elimination rules for the universal quantifier $\forall$, the existential quantifier $\exists$, and for equality $\approx$.

$(\forall -)$ If $\Sigma \vdash \forall x\, A(x)$

then $\Sigma \vdash A(t)$

Notice that t is some term, in $Term(L)$, and t replaces all occurrences of x in $A(x)$.

($\forall$+) If $\Sigma \vdash A(u)$

where u does not occur in Σ,

then $\Sigma \vdash \forall x\, A(x)$.

Here, x must also not occur in $A(u)$.

($\exists$-) If $\Sigma, A(u) \vdash B$

and u does not occur in Σ

or B, then

$\Sigma, \exists x\, A(x) \vdash B$

($\exists$+) If $\Sigma \vdash A(t)$

then $\Sigma \vdash \exists x\, A(x)$

where $A(x)$ results by replacing some not necessarily all occurrences of t in $A(t)$ by x.

($\approx$ -) IF $\Sigma \vdash A(t_1)$

and $t_1 \approx t_2$

then $\Sigma \vdash A(t_2)$

where $A(t_2)$ results from $A(t_1)$ by replacing some but not necessarily all occurrences of t_1 in $A(t_1)$ by t_2.

$(\approx +)$ $\qquad \vdash u \approx u$

Def. (Formal deducibility)

Let $\Sigma \subseteq Form(L)$ and $A \in Form(L)$. Then A is formally deducible from Σ in first order logic if

$\Sigma \vdash A$

can be generated from the ND proof rules.

Suppose $x_1, \dots, x_n$ are variables then

$\forall x_1 \dots \forall x_n$

can be abbreviated as $\forall x_1 \dots x_n$.

Similarly

$\exists x_1 \dots \exists x_n$

can be abbreviated as $\exists x_1 \dots x_n$.

We can then extend the rules

$(\forall -)$

$(\forall +)$

$(\exists -)$

$(\exists +)$

as follows

$(\forall -)$ If $\Sigma \vdash \forall x_1 \ldots x_n\, A(x_1, \ldots, x_n)$

then $\Sigma \vdash A(t_1, \ldots, t_n)$.

($\forall +$) If $\Sigma \vdash A(u_1, \ldots, u_n)$

where $u_1, \ldots, u_n$ do not occur in Σ then

$\Sigma \vdash \forall x_1 \ldots x_n \, A(x_1, \ldots, x_n)$

($\exists -$) If $\Sigma, A(u_1, \ldots, u_n) \vdash B$

where $u_1, \ldots, u_n$ do not occur in B or Σ, then

$$\Sigma, \exists x_1 \ldots x_n \, A(x_1, \ldots, x_n) \vdash B$$

($\exists +$) If $\Sigma \vdash A(t_1, \ldots, t_n)$

then $\Sigma \vdash \exists x_1 \cdots x_n \, A(x_1, \ldots, x_n)$

where $A(x_1, \ldots, x_n)$ results from simultaneously replacing some, not necessarily all, occurrences of t_i in $A(t_1, \ldots, t_n)$ by x_i, for $i \in [1..n]$.

Notice that in

$$x_1, x_2, \ldots, x_n$$

the x_i should be distinct.

For instance

$$\forall x x F(x,x)$$

is not a well formed formula.

Neither is $\forall x \forall x F(x,x)$.

In the rules $(\forall +)$ and $(\exists -)$ the u_i should also be distinct.

Hower in $(\forall -)$ and $(\exists +)$ the t_i need not be distinct.