Week 13 Tutorial

Course Review

Joe Scott / Jan Gorzny

Prepared based off of the notes of CS245 Instructors, past and present.

31 March 2017

Plan

1. Course Review
   - Hoare Triples
   - Conversion to CNF
   - Resolution
   - Natural Deduction
   - Tautological Implication
   - Key Theorems
2. The End

Outline

1. Course Review
   - Hoare Triples
   - Conversion to CNF
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Problem 1

1. $\llparenthesis(((a = a_0) \wedge (a \geq 0)) \wedge (b > 0))\rrparenthesis$
2. while $(a - b \geq 0)$ {
3. $\quad a = a - b;$
4. }
5. r = a;
6. $\llparenthesis(\exists q((r = a - qb) \wedge (0 \leq r < b)))\rrparenthesis$

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Try $I := (\!|a \geq 0|\!)$

Problem 1 - cont.

1. $\llparenthesis(((a = a_0) \wedge (a \geq 0)) \wedge (b > 0))\rrparenthesis$ assumption
2. $\llparenthesis(a \geq 0)\rrparenthesis$ Implied(a)
3. while $(a - b \geq 0)$ {
4. $\llparenthesis((a \geq 0) \wedge (a - b \geq 0))\rrparenthesis$ partial-while
5. $\llparenthesis(a - b \geq 0)\rrparenthesis$ implied(b)
6. $a = a - b;$
7. $\llparenthesis a \geq 0)$ assignment
8. }
9. $\llparenthesis((a \geq 0) \wedge (\neg(a - b \geq 0)))\rrparenthesis$ partial while
10. $\llparenthesis(\exists q((a = a - qb) \wedge (0 \geq a < b)))\rrparenthesis$ implied(c)
11. r = a;
12. $\llparenthesis(\exists q((r = a - qb) \wedge (0 \leq r < b)))\rrparenthesis$ assignment

Problem 1 - impliesds

For implied(a) we want to show that

$$((((a = a_0) \wedge (a \geq 0)) \wedge (b > 0)) \rightarrow (a \geq 0))$$

This is obvious from $\wedge_e$.
for implied(b)

$$(((a \geq 0) \wedge (a - b \geq 0)) \rightarrow (a - b \geq 0))$$

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implied c

Implied c says

$$(((a \geq 0) \land (\neg(a - b \geq 0)) \rightarrow (\exists q((a = a - qb) \land (0 \leq a < b)))))$$

Firstly

$$(\neg(a - b \geq 0))$$

$$\iff a - b < 0$$

$$\iff a < b$$

As $0 \leq a$, $0 \leq a < b$. Clearly then given $q = 0$, $a = a - qb$, as needed.

Termination

We need to find a loop variant.

1. $\llparenthesis(((a = a_0) \wedge (a \geq 0)) \wedge (b > 0))\rrparenthesis$
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Consider the following $v = \lfloor \frac{a}{b} \rfloor - (\frac{a_0 - a}{b})$.
On initialization $v = \lfloor \frac{a}{b} \rfloor$ as $a = a_0$.
For each iteration, $\frac{a_0 - a}{b}$ takes the values $1, 2, 3, 4..$ as required. The loop terminates on $v = 0$.

Outline

CNF Conversion

Convert the following to CNF:

$$p \leftrightarrow (r \wedge s)$$

CNF Conversion

Convert the following to CNF:

$$p \leftrightarrow (r \land s)$$

$$\iff (p \to (r \land s)) \land ((r \land s) \to p) \ \leftrightarrow\text{-equiv}$$

CNF Conversion

Convert the following to CNF:

$$p \leftrightarrow (r \land s)$$

$$\iff (p \rightarrow (r \land s)) \land ((r \land s) \rightarrow p) \quad \leftrightarrow\text{-equiv}$$

$$\iff (\neg p \lor (r \land s)) \land (\neg(r \land s) \lor p) \quad \rightarrow\text{-equiv}$$

CNF Conversion

Convert the following to CNF:

$$p \leftrightarrow (r \wedge s)$$

$$\iff (p \rightarrow (r \wedge s)) \wedge ((r \wedge s) \rightarrow p) \text{ } \leftrightarrow\text{-equiv}$$

$$\iff (\neg p \vee (r \wedge s)) \wedge (\neg(r \wedge s) \vee p) \text{ } \rightarrow\text{-equiv}$$

$$\iff (\neg p \vee r) \wedge (\neg p \vee s) \wedge (\neg r \vee \neg s \vee p) \text{ Distributivity, DeMorgan's}$$

Outline

Resolution

Problem

Does $\{a, b, c, d\} \vdash_{Res} a \wedge b$*?*

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Step 2: Ensure everything is CNF. Need to convert negated conclusion:
$\neg(a \wedge b) \iff \neg a \vee \neg b$ by DeMorgan's

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Step 3: Resolution

- a premise
- b premise
- c premise
- d premise
- $\neg a \vee \neg b$ negated conclusion
- $\neg b$ resolution 1,5
- $\perp$ resolution 2,6

Done - the conclusion is implied by the premise.

Outline

Example

Problem

Prove $\neg(A \wedge B) \vdash A \rightarrow \neg B$

Example

Problem

Prove $\neg(A \wedge B) \vdash A \to \neg B$

1. $A, B \vdash A$ by ϵ
2. $A, B \vdash B$ by ϵ
3. $A, B \vdash A \wedge B$ by $\wedge+$ 1,2
4. $\neg(A \wedge B), A, B \vdash A \wedge B$ by $(+)$ 3
5. $\neg(A \wedge B) A, B \vdash \neg(A \wedge B)$ by ϵ
6. $\neg(A \wedge B), A \vdash \neg B$ by $\neg+$ 4,5
7. $\neg(A \wedge B) \vdash A \to \neg B$ by $\to +$ 6

Outline

Semantic Entailment

Do the premises semantically entail (logically imply) the conclusion? Answer this question using a truth table. Explain.

$$\{(\neg p \wedge (q \vee \neg r))\} \models \neg r \vee p$$

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p	q	r	$\neg p \wedge (q \vee \neg r)$	$\neg r \vee p$
1	1	1		
1	1	0		
1	0	1		
1	0	0		
0	1	1		
0	1	0		
0	0	1		
0	0	0		

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1	1	1	0	
1	1	0	0	
1	0	1	0	
1	0	0	0	
0	1	1	1	
0	1	0	1	
0	0	1	0	
0	0	0	1	

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1	1	1	0	1
1	1	0	0	1
1	0	1	0	1
1	0	0	0	1
0	1	1	1	0
0	1	0	1	1
0	0	1	0	0
0	0	0	1	1

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Interpretations

Definition

A sentence ψ is true in an interpretation $\mathcal{I}$, denoted $\mathcal{I} \vDash \psi$, if for every possible sequence of elements in the interpretation, substituting these elements into the variables present in ψ yields a true sentence. Such an interpretation $\mathcal{I}$ is called a satisfying interpretation.

Completeness

Theorem (Godel's Completeness Theorem)

Let Σ be a set of formulas. If Σ is consistent, then it has a satisfying interpretation.

Completeness

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Let Σ be a set of formulas. If Σ is consistent, then it has a satisfying interpretation.

Theorem (Godel's Completeness Theorem - Contrapositive)

If Σ does not have a satisfying interpretation, then Σ is not consistent.

Key Theorems

Theorem (Compactness Theorem)

A set of sentences Σ has a satisfying interpretation if and only if every finite set of Σ has a satisfying interpretation.

Theorem

Let A be a sentence of first-order logic such that for any $n \in \mathbb{N}$, $n \geq 1$, there is a domain D and valuation v with at least n elements in D, such that $A^v = 1$. Then A has a domain D' and a valuation v' such that $A^{v'} = 1$ and D' has an infinite number of elements.

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The end

That's it folks. Feel free to hang out and ask questions.

These slides are based off of the tutorial notes and lecture slides provided to you online.
If you want a copy feel free to email me. The are also available on my personal website joe-scott.net

IA Email:

IA	email
Jan Gorzny	jgorzny
Joe Scott	j29scott

Jan and Joe have an office hour Mondays at 3pm in the Tutorial Center in MC.

Instructor Office Hours:

Instructor	Time	Room	Email
Trefler	Tue, Thur 4:00pm	DC 2336	trefler
Rahkooy	Tue, Thur 4:00pm	DC 2302B	hamid.rahkooy