CS 245

When $A \in \mathrm{Form}(L^P)$ then

A has a top-level connective.

A is either an atom, a
$\neg$-formula, an $\wedge$-formula,
an $\vee$-formula, an $\rightarrow$-formula
or an $\leftrightarrow$-formula.

A similar characterization of formulae in Form($\mathcal{L}$) is possible, but we add the possibility that a formula is a universal formula or an existential formula.

If $B \in Form(L)$ is of the form $B = \forall x\, A(x)$ then B is a universal formula.

If $B = \exists x\, A(x)$ then B is an existential formula.

Let $\exists x\, A(x)$ and $\forall x\, B(x)$ be formulae in Form(L).

If $\exists x\, A(x)$ appears as a sub-formula in $C \in$ Form(L), then $A(x)$ is the scope in C of the '$\exists x$' quantifier to the left of $A(x)$.

If $\forall x\, B(x)$ appears as a subformula in C then $B(x)$ is the scope in C of the $\forall x$ quantifier that appears to the left of $B(x)$.

For formulae in Form(L^P) interpretations were constructed from interpretations of the atoms.

The atoms of Form(L), however, are relatively complex, so interpretations for Form(L) must take into account the meaning of terms in Term(L) and the meaning of n-ary relational symbols.

In studying the meaning of formulae in Form(L) we may have a particular structure in mind. That structure comes with a domain, designated individuals, relations and functions.

When L is not associated with a specific structure we assume there is an arbitrary, but non-empty, set D for the domain.

Then individual symbols, n-ary relation symbols, m-ary function symbols are interpreted as, respectively, individuals in D, n-ary truth functions on D, and m-ary total functions $D^m \to D$.

Again, it is stipulated that $\approx$ is always interpreted as $=$.

In addition to interpretations for individual symbols, function symbols and relational symbols, we must also asign meaning to free variable symbols.

We use the descriptor assignment to attach a value of the domain to a free variable.

A valuation combines an interpretation with an assignment to the free variables of (a set) of formulae.

Def. (Valuation)

Let Form(L) be the formulae of a first-order logic. Given domain, D, a valuation, v, to elements of L is defined as follows:

[1] $a^v \in D$

where a is an individual symbol.

$u^v \in D$

where u is a free variable symbol.

[2] $F^{v} \subseteq D^{n}$

where F is an n-ary relation symbol.

$$\approx^{v} = \{ \langle d, d \rangle \mid d \in D \}$$

[3] $f^{v} : D^{m} \to D$

where f is an m-ary function symbol.

It is required that f^{v} be a total function on D^{m}.

Notice that there is an intended distinction between:

a and a^{v}

u and u^{v}

F and F^{v}

f and f^{v}

$\approx$ and $=$

On the left are symbols from the logic, on the right their intended meaning.

In particular, if v and v' are two distinct valuations then, for example,

$$a^{v}$$

may differ from

$$a^{v'}$$

and

$$F^{v}$$

may differ from

$$F^{v'}$$

Although, by convention, for any particular domain, D,

$$\approx^{v}$$

is the same as

$$\approx^{v'}$$

Now that we have, v, a valuation to the elements of the logic we can give valuations to the terms and the formulae.

Def. (Value of a term)

Given domain D and valuation, v,

[1] $a^v \in D$

$u^v \in D$

[2] $f(t_1, \ldots, t_n)^v$

$= f^v(t_1^v, \ldots, t_n^v)$

Here $f(t_1, \ldots, t_n)$ is a term in L.

Then $f(t_1, \ldots, t_n)^v$ is the valuation, v, applied to $f(t_1, \ldots, t_n)$.

Fact: Let v be a valuation over domain D and $t \in \mathrm{Term}(L)$.

Then $t^v \in D$.

Suppose v is a valuation over D to the elements of L.

And let $d \in D$.

Then $v(u/d)$

is the valuation exactly the same as v

except that $u^{v(u/d)} = d$.

(Since v is a valuation over D, $u^v \in D$. It may be that $u^v = d$, but this is not required.)

This difference between

v

and

$v(u/d)$

appears only in the valuations to free variables.

$$a^{v(u/d)} = a^{v}$$

$$F^{v(u/d)} = F^{v}$$

$$f^{v(u/d)} = f^{v(u/d)}$$

$$w^{v(u/d)} = \begin{cases} d & \text{if } w = u \\ w^{v} & \text{otherwise} \end{cases}$$

Def. For formulae in Form(L) and valuation, v, over domain D, the truth value is defined as follows:

[1] $F(t_1, \ldots, t_n)^v = \begin{cases} 1 & \text{if } \langle t_1^v, \ldots, t_n^v \rangle \in F^v \\ 0 & \text{otherwise} \end{cases}$

$$(t_1 \approx t_2)^v = \begin{cases} 1 & t_1^v = t_2^v \\ 0 & \text{otherwise} \end{cases}$$

[2] $(\neg A)^v = \begin{cases} 1 & \text{if } A^v = 0 \\ 0 & \text{otherwise} \end{cases}$

[3] $(A \wedge B)^v = \begin{cases} \underline{1} & \text{if } A^v = \underline{1} \text{ and } B^v = \underline{1} \\ 0 & \text{otherwise} \end{cases}$

[4] $(A \vee B)^v = \begin{cases} \underline{1} & \text{if } A^v = \underline{1} \text{ or } B^v = \underline{1} \\ 0 & \text{otherwise} \end{cases}$

$$[5] \quad (A \rightarrow B)^v = \begin{cases} 1 & \text{if } A^v = 0 \text{ or } B^v = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$[6] \quad (A \leftrightarrow B)^v = \begin{cases} 1 & \text{if } A^v = B^v \\ 0 & \text{otherwise} \end{cases}$$

$$
[7]\ (\forall x\, A(x))^{v} = \begin{cases} 1 & \text{if } u \text{ does not appear in } A(x) \text{ and } A(u) \text{ is constructed from } A(x) \text{ then, for all } d \in D\text{: } A(u)^{v(u/d)} = 1 \\ 0 & \text{otherwise} \end{cases}
$$

[8] $$\exists x A(x)^v = \begin{cases} \underline{1} & \text{if } u \text{ does not appear in } A(x) \text{ and } A(u) \text{ is constructed from } A(x) \text{ then, there exists } d \in D: A(u)^{v(u/d)} = \underline{1} \\ 0 & \text{otherwise} \end{cases}$$

Notes: $\langle t_1^v, \ldots, t_n^v \rangle \in F^v$ means that the n-tuple of elements $i \in [1..n]$, $t_i^v \in D$, is in the relation F^v.

Fact: If v is a valuation over domain D and $A \in Form(L)$ then $A^v \in \{0, 1\}$.

Let D be a given domain.

Two valuations, v and v', agree on non-logical symbols a, or F, or f or free variable symbol u if:

$a^v = a^{v'}$

or

$F^v = F^{v'}$

or

$f^v = f^{v'}$

or

$u^f = u^{f'}$

Fact: Let v and v' be valuations over domain D.

If v and v' agree on the individual symbols, relation symbols, function symbols, and free variable symbols contained in term t and formula A then:

$$t^{v} = t^{v'}$$

and

$$A^{t} = A^{t'}.$$

Let $\Sigma \subseteq \text{Form}(L)$ and let v be a valuation over domain D.

Then $$\Sigma^v = \begin{cases} \underline{1} & \text{if for all } A \in \Sigma, \ A^v = \underline{1} \\ 0 & \text{otherwise} \end{cases}$$

Def. Let $\Sigma \subseteq \mathrm{Form}(L)$. Then Σ is satisfiable if there is some valuation, v, (over some domain D) such that $\Sigma^v = \underline{1}$.

Def. Let $A \in Form(L)$, then A is valid if for all valuations, v, $A^v = 1$.

Def. Logical consequence.

Let $\Sigma \subseteq \text{Form}(L)$ and $A \in \text{Form}(L)$.

A is a logical consequence of Σ, written $\Sigma \models A$, if for all valuations, v, if $\Sigma^v = \underline{1}$ then $A^v = \underline{1}$.

When we write: $\emptyset \models A$

or just: $\models A$

then A is valid.

Writing $\Sigma \not\models A$ means that it is not the case that A is a logical consequence of A.

Two formulae are equivalent if $A \dashv\vdash B$ holds.