CS 245
Notes

Resolution

To show $\Sigma \vdash A$ do the following:

1. Convert each formula in $\Sigma \cup \{\neg A\}$ into CNF.
2. Split the CNF formulae into disjunctive clauses

$$(A_{1,1} \vee \ldots \vee A_{k_1,1}) \wedge \ldots \wedge (A_{1,n} \vee \ldots \vee A_{k_n,n})$$

3. Start with the above list of clauses.

Continue applying the resolution inference rule until:

- The emty clause $\perp$ results. This means that A is proven from Σ.

- The resolution rule does not produce any new formulae. The means that Σ does not prove A.

For disjunctive clauses where the literal A appears more than once rewrite using simplification so that each literal appears at most once:

$$p \vee p \vee q \equiv p \vee q$$

Clauses of the form

$$p \vee \neg p \vee q$$

can be simplified to $\underline{1}$ (and eliminated from the proof).

We now show that resolution is sound. That is, for formulae in $Form(L^p)$ if we show that $\Sigma \vdash A$ using resolution then we can conclude $\Sigma \models A$.

Fact: Suppose $\{A_1, \dots, A_n\} \vdash A$, meaning there is a refutation based proof using resolution for premises $A_1, \dots, A_n$ and conclusion A. Then it is the case that $A_1, \dots, A_n \models A$.

Fact: Let $\{B_1, \ldots, B_k\}$ be satisfiable. Let C be a formula obtained from $\{B_1, \ldots, B_k\}$ by the resolution rule applied to $\{B_1, \ldots, B_k\}$. Then $\{B_1, \ldots, B_k, C\}$ is satisfiable.

(Here, $C \neq \perp$.)

Resolution uses the inference rule

$$\frac{B \vee D \qquad \neg D \vee G}{B \vee G}$$

Here one of the B_i is of the form $B \vee D$, one of the B_j is of the form $\neg D \vee G$ and C is of the form $B \vee G$.

By assumption there is a valuation, t, such that $(B \vee D)^t = \underline{1}$ and $(\neg D \vee G)^t = \underline{1}$.

Since one of $D^t = 0$ or $(\neg D)^t = 0$ then $B^t = 1$ or $G^t = \underline{1}$.

Hence $(B \vee G)^t = \underline{1}$.

Since C is of the form $B \vee G$ then $C^t = \underline{1}$.

Thus since

$$\{B_1, \ldots, B_k\}^t = \underline{1},$$

by assumption, and the resolution proof rule is sound, then $\{B_1, \ldots, B_k, C\}^t = \underline{1}$.

Fact: Suppose that $\{B_1, \ldots, B_k\}$ is satisfiable.

Let $\{B_1, \ldots, B_k, C_1, \ldots, C_n\}$ result from n applications of the resolution proof rule. If $C_i \neq \bot$ for all $i \in [1..n]$ then $\{B_1, \ldots, B_k, C_1, \ldots, C_n\}$ is satisfiable.

Proof: By induction on n.

Base case: For the case where $\{B_1, \ldots, B_k, C_1\}$ is obtained from $\{B_1, \ldots, B_k\}$ by a single application of resolution then $\{B_1, \ldots, B_k, C_1\}$ is satisfiable if $\{B_1, \ldots, B_k\}$ is satisfiable. (See previous fact.)

I.H. Assume that

$\{B_1, \ldots, B_k, C_1, \ldots, C_n\}$ is satisfiable if $\{B_1, \ldots, B_k\}$ is satisfiable and C_1 to C_n are added by n applications of resolution.

So suppose $\{B_1, \ldots, B_k\}$ is satisfiable and $\{B_1, \ldots, B_k, C_1, \ldots, C_{n+1}\}$ result from $n+1$ applications of resolution.

By the I.H.

$\{B_1, \ldots, B_k, C_1, \ldots, C_n\}$ is satisfiable. Therefore since C_{n+1} is added by a step of resolution then

$\{B_1, \ldots, B_k, C_1, \ldots, C_n, C_{n+1}\}$

is satisfiable. (see previous fact).

The resolution proof system for propositional formulae in CNF is complete if, for all $\Sigma \subseteq Form(L^P)$ and $A \in Form(L^P)$, if the formulae in Σ and A are in CNF and $\Sigma \models A$ then there is a resolution proof that $\Sigma \vdash A$.

Fact: Resolution is complete for propositional formulae in CNF.

The consequence of this fact is that if there is no resolution proof that $\Sigma \vdash A$ then $\Sigma \cup \{\neg A\}$ is satisfiable.

Alternatively, if there is no resolution proof that $\Sigma \vdash \perp$ then Σ is satisfiable.

Fact: Consider a resolution proof for which $\perp$ has not been derived and no new applications of the resolution rule can be added. Then the set of formulae in the proof are satisfiable.

Consider an induction on the number of atoms in the proof.

Base case: there is one atom in the proof.

Since no new resolution step can be added, it must be that there are no two new formulae with p in one and $\neg p$ in the other.

Thus all the formulae must have p or all the formulae have $\neg p$.

In either case the entire set is satisfiable.

(If $\neg p$ appears in all the formulae then $p^t = 0$ is satisfying.
If p appears in all the formulae then $p^t = 1$ is satisfying.)

(If both p and $\neg p$ appear in the derived formulas then $\perp$ would be derivable.)

Suppose the fact holds for derivations with n atoms

Consider the case with $n+1$ atoms.

Suppose no new resolution steps are possible and $\perp$ has not been derived.

Choose some atom p that appears in the formulae.

Separate the formulae into three sets: S_p, $S_{\neg p}$, and R.

R is the set of formulae in which neither p nor $\neg p$ appear.

S_p is the set containing p but not containing $\neg p$.

$S_{\neg p}$ is the set containing $\neg p$ but not p.

Notice that R has at most n atoms. Therefore the I.H. applies and there is a valution t, such that $R^t = \underline{1}$.

Notice that each formula $A \in S_p$ is of the form $p \vee A'$.

If $(A')^t = \underline{1}$ for each $A \in S_p$ then set $p^t = 0$. Since $(A')^t = \underline{1}$ for each $A \in S_p$ then $S_p^t = \underline{1}$.

Also $S_{\neg p}^t = \underline{1}$.

If S_p has some clause $p \vee A'$ where $A' \overset{t}{=} 0$ then set $p^t = \underline{1}$.

Thus $S_p^t = \underline{1}$.

Consider some formula $\neg p \vee A''$ in $S_{\neg p}$. By resolution we must have $A' \vee A''$ in R. Since by assumption $R^t = \underline{1}$ and $A' \vee A'' \in R$ then $\longrightarrow$

$(A' \vee A'')^t = \underline{1}$.

Since $(A')^t = 0$ we have that $(A'')^t = \underline{1}$.

This implies that $(\neg p \vee A'')^t = \underline{1}$.

Therefore $S_p \cup S_{\neg p} \cup R$ is satisfiable.

Therefore, if the set of formulae cannot derive $\perp$ then the set of formulae given as premises are satisfiable.

In the case where $\Sigma \models A$ then it cannot hold that $\Sigma \cup \{\neg A\}$ is satisfiable.

For propositional logic formulae the resolution proof system provides an algorithm to show whether a set of formulae is satisfiable.

Convert each formula in the set to CNF.

Separate the CNF formulae into disjunctive clauses.

Simplify the clauses so they contain at most one copy of each literal.

Eliminate tautologies.

Apply the resolution steps until no more resolution steps are applicable.

So if $\perp$ is derived the input set of formulae are contradictory. Otherwise the input set of formulae is satisfiable.

It is a fact that resolution can be very slow.

Resolution in practice:

relatively straightforward to implement — one basic rule.

Resolution style provers used in software/hardware verification systems, planning, scheduling, test pattern generation, etc...