CS 245

Fact: If $\Sigma \vdash_{ND} A$ then $\Sigma \models A$.

Proof idea: proceed by induction on the ND proof.

The proofs steps are a combination of steps from propositional logic and predicate logic.

The steps for propositional logic are sound.

($+$)

($\neg -$)

($\to -$)

($\to +$)

($\wedge -$)

($\wedge +$)

($\vee -$)

($\vee +$)

($\leftrightarrow -$)

($\leftrightarrow +$)

Consider the steps for first order logic.

For example: $(\exists -)$

if $\Sigma, A(u) \vdash B$

where u does not occur in B or Σ

then $\Sigma, \exists x\, A(x) \vdash B$.

So suppose

$\Sigma, \exists x\, A(x) \vdash B$

has been derived from

$\Sigma, A(u) \vdash B.$

Assume that v is a valuation such that

$\Sigma^v = \underline{1}$

and $\exists x\, A(x)^v = \underline{1}.$

Then there is some $d \in D$ such that $A(u)^{v(u/d)} = \underline{1}$.

Since u does not appear in Σ or in B then

$$\Sigma^{v(u/d)} = \Sigma^{v}.$$

Since $\Sigma^{v} = \underline{1}$ then

$$\Sigma^{v(u/d)} = \underline{1}.$$

Since $\Sigma, A(u) \models B$

and $\Sigma^{v(u/d)} = \underline{1}$ and $A(u)^{v(u/d)} = \underline{1}$

then $B^{v(u/d)} = \underline{1}$.

Again u does not occur in B

hence $B^{v} = B^{v(u/d)}$, so $B^{v} = \underline{1}$.

Therefore if $\Sigma^{v} = \underline{1}$ and

$\exists x A(x)^{v} = \underline{1}$ then $B^{v} = \underline{1}$.

Hence $\Sigma, \exists x A(x) \models B$.

Def. (Consistency)

Let $\Sigma \subseteq \mathrm{Form}(L)$, Σ is consistent iff there is no $A \in \mathrm{Form}(L)$ such that $\Sigma \vdash A$ and $\Sigma \vdash \neg A$.

Fact: If Σ is satisfiable then Σ is consistent.

Proof idea: Suppose Σ is satisfiable and that Σ is not consistent.

Since Σ is not consistent then there is some $A \in \Sigma$ such that $\Sigma \vdash A$ and $\Sigma \vdash \neg A$.

Since the ND proof system is sound it must be that $\Sigma \models A$ and $\Sigma \models \neg A$.

As Σ is satisfiable

then there is some valuation, v, such that $\Sigma^v = \underline{1}$.

Since $\Sigma \vdash A$ then $A^v = \underline{1}$.

Since $\Sigma \vdash \neg A$ then $(\neg A)^v = \underline{1}$

and $A^v = 0$, a contradiction.

Therefore if Σ is satisfiable then Σ is consistent.

Fact: Let $A \in \mathrm{Form}(L)$.
Then A is satisfiable iff $\neg A$ is invalid.

Fact: A is valid iff $\neg A$ is unsatisfiable.

Fact: $A(u_1, \ldots, u_n)$ is satisfiable iff $\exists x_1 \ldots x_n\, A(x_1, \ldots, x_n)$ is satisfiable.

Proof idea:

If $A(u)$ is satisfiable then there is some domain D and valuation v such that $A(u)^v = 1$. This implies that $\exists x\, A(x)^v = 1$. Therefore $\exists x\, A(x)$ is satisfiable.

If $\exists x\, A(x)$ is satisfiable then there is some domain D and valuation v such that $\exists x\, A(x)^{v} = \underline{1}$.

Therefore for some $d \in D$, $A(u)^{v(u/d)} = \underline{1}$.

Fact: If $A \in Form(L)$ then

$A(u_1, \ldots u_n)$ is valid

iff $\forall x_1 \ldots x_n\, A(u_1, \ldots, u_n)$ is valid.

Proof idea:

$A(u)$ is valid

iff for all domains D

and all valuations, v,

$A(u)^v = \underline{1}$.

iff for all domains D,

and all valuations v,

$\neg A(u)^v = 0$.

iff

$\neg A(u)$ is unsatisfiable.

iff

$\exists x \neg A(x)$ is unsatisfiable.

iff

$\neg \forall x A(x)$ is unsatisfiable.

iff

$\forall x A(x)$ is valid.

Def. (Prenex normal form)

A formula of the form

$$Q_1 x_1 \ldots Q_n x_n B$$

where each Q_i is either $\forall$ or $\exists$ and B is quantifier free is said to be in prenex normal form.

Notice that a formula with no quantifiers is, trivially, in prenex normal form.

Fact: Suppose A' results from A by replacing in A some (not necessarily all) occurrences of $Qx\, B(x)$ by $Qy\, B(y)$.

Then $A \dashv\vdash A'$.

Proof idea: Proceed by induction on structure of A noting that $Qx\, B(x) \dashv\vdash Qy\, B(y)$.

Fact: Every formula in Form(L) is equivalent to some formula in prenex normal form.

Proof idea:

$$\neg \forall x\, A(x) \;\dashv\vdash\; \exists x\, \neg A(x)$$

$$\neg \exists x\, A(x) \;\dashv\vdash\; \forall x\, \neg A(x)$$

$$A \wedge Q x\, B(x) \;\dashv\vdash\; Q x\, (A \wedge B(x))$$

if x does not occur in A.

$$A \vee Qx\, B(x) \dashv\vdash Qx\,(A \vee B(x))$$

if x does not occur in A.

$$\forall x\, A(x) \wedge \forall x\, B(x) \dashv\vdash \forall x\,(A(x) \wedge B(x))$$

$$\exists x\, A(x) \vee \exists x\, B(x) \dashv\vdash \exists x\,(A(x) \vee B(x))$$

$$Q_1 x\, A(x) \wedge Q_2 y\, B(y) \dashv\vdash Q_1 x\, Q_2 y\,(A(x) \wedge B(y))$$

if x does not occur in $B(y)$
and y does not occur in $A(x)$.

$$Q_1 x\, A(x) \vee Q_2 y\, B(y) \;\vDash\!\!\!\dashv\; Q_1 x\, Q_2 y\, (A(x) \vee B(y))$$

if x does not occur in $B(y)$

and y does not occur in $A(x)$.