CS 245

We have seen that proofs in ND are sound. That is if we show

$A_1, \ldots, A_n \vdash B$

then we also know

$A_1, \ldots, A_n \models B$.

It is also useful to know that the ND proof technique is complete for formulae in $Form(L^P)$.

That is if $A_1, \ldots, A_n \models B$ then there is an ND proof that $A_1, \ldots, A_n \vdash B$.

In what follows we give a proof outline of this fact.

The outline has three steps.

The goal is to show that if $A_1, \ldots, A_n \models B$ then there is an ND proof that $A_1, \ldots, A_n \vdash B$.

Step 1. If $A_1, \ldots, A_n \models B$

then $\models A_1 \to (\ldots(A_n \to B))$.

So step 1 involve converting a tautological consequence into a tautology.

(which we have seen before.)

Step 2. If $\models A_1 \to (\ldots(A_n \to B))$ then there is an ND proof $\vdash A_1 \to (\ldots(A_n \to B))$.

Step 2 says that for all tautologies of the form $A_1 \to (\ldots(A_n \to B))$ there is an ND proof $\vdash A_1 \to (\ldots(A_n \to B))$.

Step 3: If there is an ND proof that

$$\vdash A_1 \to (\ldots(A_n \to B))$$

then there is an ND proof

$$A_1, \ldots, A_n \vdash B.$$

This last step finshes the completeness argument.

$\longrightarrow$

That if $A_1, \ldots, A_n \models B$

then $A_1, \ldots, A_n \vdash B$ in ND.

Step 1:

Show that if

$$A_1, \ldots, A_n \models B$$

then

$$\models A_1 \to (\ldots (A_n \to B)).$$

(This fact was discussed earlier.)

If $A_1, \dots, A_n \models B$

then $\models A_1 \rightarrow (\dots (A_n \rightarrow B))$.

when $A_1^t = 0$ then

$(A_1 \rightarrow (\dots (A_n \rightarrow B)))^t = \underline{1}$.

when $A_1^t = \underline{1}$ and $A_2^t = 0$ then

$(A_1 \rightarrow (\dots (A_n \rightarrow B)))^t = 1$.

...

when $A_1^t = \underline{1}, \dots, A_n^t = \underline{1}$ then

since $A_1, \dots, A_n \models B$

it must be that $B^t = 1$.

therefore when

$A_1^{t} = \underline{1}$, ..., $A_n^{k} = \underline{1}$ then $B^{t} = \underline{1}$

and

$(A_1 \to (\ldots (A_n \to B)))^{t} = \underline{1}$.

Hence

$\models A_1 \to (\ldots (A_n \to B))$.

Step 2. Show that if

$\models C$ then $\vdash C$.

Suppose the C contains n atoms, $p_1, \dots, p_n$.

Since $\models C$ then for all valuations, t, it must be that $C^t = 1$.

Let A be a formula with n atoms, $p_1, \dots, p_n$.

There are 2^n truth valuations to the atoms.

Let t be one of the 2^n valuations. Create the literals $\hat{p}_1, \dots, \hat{p}_n$ as follows: if $p_i^t = \underline{1}$ then $\hat{p}_i = p_i$

else $p_i^t = 0$ and $\hat{p}_i = \neg p_i$.

Fact: Let A be a formula whose atoms are $p_1, \dots, p_n$. Let $\hat{p}_1, \dots, \hat{p}_n$ be the literals related to the truth valuation t.

Then $\hat{p}_1, \dots, \hat{p}_n \vdash A$ if $A^t = \underline{1}$

and $\hat{p}_1, \dots, \hat{p}_n \vdash \neg A$ if $A^t = 0$.

This fact follows by an induction on the structure of the formula A.

When A is an atom, p, then there are two cases.

If $A^t = \underline{1}$ show $p \vdash p$.

However $p \vdash p$ by (Ref).

If $A^2 = 0$ then show

$$\neg p \vdash \neg p.$$

However $\neg p \vdash \neg p$ also follows by (Ref).

Suppose A is of the form $\neg C$.

Either $(\neg C)^t = 1$ or $(\neg C)^t = 0$.

Suppose $(\neg C)^t = 1$. Then $C^t = 0$.

Since A has atoms $p_1, \ldots, p_n$ then C also has atoms $p_1, \ldots, p_n$.

By the IH: $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg C$ since $C^t = 0$. However since $A = \neg C$ then $\hat{p}_1, \ldots, \hat{p}_n \vdash A$.

Suppose $(\neg C)^t = 0$. Then $C^t = 1$.

By the I.H.: $\hat{p}_1, \ldots, \hat{p}_n \vdash C$.

Then $\hat{p}_1, \ldots, \hat{p}_n, \neg C \vdash C$ by $(+)$.

Hence $\hat{p}_1, \ldots, \hat{p}_n, \neg C \vdash \neg C$ by $(\in)$.

Therefore $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\neg C$ by $(\neg +)$.

Since $\neg\neg C = \neg A$ then

$\hat{p}_1, \ldots, \hat{p}_n \vdash \neg A$, as required.

Suppose A is of the form

$C_1 \to C_2$.

If $A^t = 0$ then $C_1^t = 1$ and $C_2^t = 0$.

Therefore by the I. H.

$\hat{q}_1, \ldots, \hat{q}_k \vdash C_1$

and

$\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg C_2$

where the q_i are the atoms of C_1 and r_i are the atoms of C_2.

This means that

$$\hat{q}_1, \ldots, \hat{q}_k, \hat{r}_1, \ldots, \hat{r}_l \vdash c_1$$

and

$$\hat{q}_1, \ldots, \hat{q}_k, \hat{r}_1, \ldots, \hat{r}_l \vdash \neg c_2$$

This implies that:

$$\hat{p}_1, \ldots, \hat{p}_n \vdash c_1 \wedge \neg c_2$$

Then it is possible to show that $c_1 \wedge \neg c_2 \vdash \neg(c_1 \rightarrow c_2)$, as required, since we assumed $A^t = 0$.

Suppose $A^t = 1$.

There are three possible cases, since $A = C_1 \rightarrow C_2$.

$C_1^t = 1$ and $C_2^t = 1$.

$C_1^t = 0$ and $C_2^t = 1$.

$C_1^t = 0$ and $C_2^t = 0$.

If $c_1^t = 1$ and $c_2^t = \underline{1}$

then by the I.H.

$\hat{q}_1, \ldots, \hat{q}_k \vdash c_1$

and

$\hat{r}_1, \ldots, \hat{r}_\ell \vdash c_2$.

Therefore

$\hat{q}_1, \ldots, \hat{q}_k, \hat{r}_1, \ldots, \hat{r}_\ell \vdash c_1$

and

$\hat{q}_1, \ldots, \hat{q}_k, \hat{r}_1, \ldots, \hat{r}_\ell \vdash c_2$.

This implies that

$$\hat{p}_1, \ldots, \hat{p}_n \vdash c_1 \wedge c_2 .$$

Therefore

$$\hat{p}_1, \ldots, \hat{p}_n \vdash c_2 .$$

This implies that

$$\hat{p}_1, \ldots, \hat{p}_n, c_1 \vdash c_2 .$$

Finally

$$\hat{p}_1, \ldots, \hat{p}_n \vdash c_1 \to c_2$$

as required.

If $c_1^* = 0$ and $c_2^* = 1$ then.

By the I.H.

$\hat{q}_1, \ldots, \hat{q}_k \vdash \neg c_1$

and

$\hat{r}_1, \ldots, \hat{r}_\ell \vdash c_2$.

Therefore

$\hat{q}_1, \ldots, \hat{q}_k, \hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg c_1$

and

$\hat{q}_1, \ldots, \hat{q}_k, \hat{r}_1, \ldots, \hat{r}_\ell \vdash c_2$.

This implies that

$$\hat{p}_1, \ldots, \hat{p}_k \vdash \neg C_1 \wedge C_2 .$$

Therefore

$$\hat{p}_1, \ldots, \hat{p}_k \vdash C_2$$

which implies that

$$\hat{p}_1, \ldots, \hat{p}_k, C_1 \vdash C_2$$

and then

$$\hat{p}_1, \ldots, \hat{p}_k \vdash C_1 \to C_2$$

as required.

If $c_1^t = 0$ and $c_2^t = 0$ then

$\hat{q}_1, \ldots, \hat{q}_k \vdash \neg c_1$ and

$\hat{r}_1, \ldots, \hat{r}_\ell \vdash \neg c_2$.

In which case

$\hat{p}_1, \ldots, \hat{p}_n \vdash \neg c_1 \wedge \neg c_2$.

This implies that

$\hat{p}_1, \ldots, \hat{p}_n, c_1, \neg c_2 \vdash \neg c_1 \wedge \neg c_2$.

This implies that

$\hat{p}_1, \ldots, \hat{p}_n, c_1, \neg c_2 \vdash \neg c_1$

In addition

$$\hat{p}_1, \ldots, \hat{p}_n, c_1, \neg c_2 \vdash c_1.$$

Therefore

$$\hat{p}_1, \ldots, \hat{p}_n, c_1 \vdash c_2$$

which leads to

$$\hat{p}_1, \ldots, \hat{p}_n \vdash c_1 \rightarrow c_2.$$

Since formulae of the form $F \leftrightarrow G$ can be written as $(F \rightarrow G) \wedge (G \rightarrow F)$ we can restrict attention to A is of the form $C_1 \wedge C_2$ and $C_1 \vee C_2$, but these cases are handled in an analogous manner.

For the formula

$$\models A_1 \to (A_2 \to \dots (A_n \to B))$$

we can organize the above method as follows.

Because the formula is a tautology with say m propositions, each of the 2^m valuations, t,

$$(A_1 \to (A_2 \to \dots (A_n \to B)))^t = \underline{1}.$$

From the previous fact we get 2^m proofs

$$\hat{p}_1, \ldots, \hat{p}_m \vdash A_1 \to (\ldots (A_n \to B)).$$

For the case when $m = 2$

where $C = A_1 \to (\ldots (A_n \to B))$

we have

$$p_1, p_2 \vdash C$$

$$p_1, \neg p_2 \vdash C$$

$$\neg p_1, p_2 \vdash C$$

$$\neg p_1, \neg p_2 \vdash C.$$

Using $(\vee -)$ we get

$$p_1, p_2 \vee \neg p_2 \vdash C$$

$$\neg p_1, p_2 \vee \neg p_2 \vdash C$$

But then we obtain

$$p_1 \vee \neg p_1, p_2 \vee \neg p_2 \vdash C.$$

We can also show that

$$\emptyset \vdash G \vee \neg G.$$

Therefore from

$$p_1 \vee \neg p_1, p_2 \vee \neg p_2 \vdash C$$

we obtain

$$\vdash C$$

or

$$\vdash A_1 \to (\dots (A_n \to B))$$

as needed.

Given the proof

$$\vdash A_1 \to (\dots (A_n \to B))$$

we can construct a proof of

$$A_1, \dots, A_n \vdash B.$$

From $\vdash A_1 \to (\ldots (A_n \to B))$

we obtain

$$A_1 \vdash A_1 \to (\ldots (A_n \to B))$$

and

$$A_1 \vdash A_1$$

this leads to

$$A_1 \vdash A_2 \to (\ldots (A_n \to B))$$

Similarly, from

$$A_1 \vdash A_2 \to (\ldots(A_n \to B))$$

we get

$$A_1, A_2 \vdash A_3 \to (\ldots(A_n \to B)).$$

And after n similar steps

$$A_1, A_2, \ldots, A_n \vdash B$$

as needed.

Fact: For formulae $A_1, \dots, A_n, B$ of $Form(L^P)$

$A_1, \dots, A_n \models B$ holds

iff

$A_1, \dots, A_n \vdash B$ is provable

in ND.

First-order Logic

(Predicate logic)

Example:

For any natural number, n, there is a prime number greater than n.

2^{100} is a natural number.

There is a prime number greater than 2^{100}.

We would like to be able to describe the previous argument in a logic similar to propositional logic.

In particular, it will be useful to describe individuals (or objects) and to speak about all objects (or some object) without having to enumerate all possible objects.

Example: Not all birds can fly.

This statement refers to all the objects in a group, that is, all the birds. However, the statement also talks about a property that each bird may have (a particular bird may fly, but not all birds do).

Example: Every student is younger than some instructor.

Notice that there are different kinds of objects (students, instrutors) and they may be in certain relationships with each other.

To understand the sentance we need a 'world' to give the sentence meaning.

Example world:

$\mathbb{N}$: the natural numbers

$<$: the binary predicate (or relation) lessthan

example statement:

for all natural numbers, x, there exists a natural number, y, such that $y < x$.

$$\forall x \, \exists y \, (y < x)$$