Week 10 Tutorial

Predicate Logic Interpretations and Natural Deduction

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Prepared based off of the notes of CS245 Instructors, past and present.

10 March 2017

Plan

Lecture Refresher

Definition

Fix a set $\mathcal{L}$ of constant symbols, function symbols, and relation symbols. An interpretation $\mathcal{I}$ (for the set $\mathcal{L}$) consists of

- A non-empty set $dom(\mathcal{I})$, called the domain (or universe) of I.
- For each constant symbol c, a member $c^{\mathcal{I}}$ of $dom(\mathcal{I})$.
- For each function symbol $f^{(i)}$, an i–ary function $f^{\mathcal{I}}$.
- For each relation symbol $R^{(i)}$, an i-ary relation $R^{\mathcal{I}}$.

Definition

An interpretation $\mathcal{I}$ and environment E satisfy a formula α, denoted as $\mathcal{I} \vDash_E \alpha$, if $\alpha^{(\mathcal{I},E)} = \top$.

A formula α is said to be

1. valid if every interpretation and environment satisfy α.
2. satisfiable if there exists an interpretation and environment that satisfy α.
3. unsatisfiable if no interpretation and environment satisfy α.

Interpretation Problem

Consider

$$(\forall x((P(x) \wedge Q(x, a)) \implies Q(x, b)))$$

Find an interpretation $\mathcal{I}_1$ that is satisfies α and $\mathcal{I}_2$ that is not satisfied, (under any environment).

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1. $dom(\mathcal{I}_1) = \mathbb{N}$.

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1. $dom(\mathcal{I}_1) = \mathbb{N}$.
2. $P^{\mathcal{I}_1}(x) : x$ is positive

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This translates to:

for all $x \in \mathbb{N}$, if x is positive and $x \geq 5$ then $x \geq 0$.

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This translates to:

for all $x \in \mathbb{N}$, if x is positive and $x \geq 0$ then $x \geq 5$.

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This translates to:

for all $x \in \mathbb{N}$, if x is positive and $x \geq 0$ then $x \geq 5$.

$$\mathcal{I}_2 \not\models (\forall x((P(x) \land Q(x,a)) \implies Q(x,b)))$$

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This translates to:

$$\text{for all } x \in \mathbb{N}, \text{ if } x \text{ is positive and } x \geq 0 \text{ then } x \geq 5.$$

$$\mathcal{I}_2 \nvDash (\forall x((P(x) \wedge Q(x, a)) \implies Q(x, b)))$$

Plan

(Basic) New Rules for Natural Deduction

- $(\approx -)$ If $\Sigma \vdash A(\ldots, t_1, \ldots)$ and $t_1 \approx t_2$ then $\Sigma \vdash A(\ldots, t_2, \ldots)$ where $A(t_2)$ results from $A(t_1)$ by replacing some (but not necessarily all) occurrences of t_1 in $A(t_1)$ by t_2
- $(\approx +)$ $\vdash u \approx u$.

(Extended) New Rules for Natural Deduction

- $(\forall -)$ If $\Sigma \vdash \forall x_1, \ldots, x_n A(x_1, \ldots, x_n)$, then $\Sigma \vdash A(t_1, \ldots, t_n)$
  - t is some term, and replaces *all* occurrences of x in $A(x)$.
- $(\forall +)$ If $\Sigma \vdash A(u_1, \ldots, u_n)$ where $u_1, \ldots, u_n$ do not occur in Σ, then $\Sigma \vdash \forall x_1, \ldots, x_n A(x_1, \ldots, x_n)$.
  - u_i should be distinct!

(Extended) New Rules for Natural Deduction II

- ($\exists-$) If $\Sigma, A(u_1, \dots, u_n) \vdash B$ where $u_1, \dots, u_n$ do not occur in Σ or B, then $\Sigma, \exists x_1, \dots, x_n A(x_1, \dots, x_n) \vdash B$.
  - u_i should be distinct!
- ($\exists+$) If $\Sigma \vdash A(t_1, \dots, t_n)$ then $\Sigma \vdash \exists x_1, \dots, x_n A(x_1, \dots, x_n)$ where $A(x_1, \dots, x_n)$ results from simultaneously replacing some (not necessarily all) occurrences of t_i in $A(t_1, \dots, t_n)$ by x_i for $i \in [1..n]$.

Question 3 - Revisited

Problem

Show that $\forall x \forall y (S(y) \rightarrow F(x)) \vDash \exists y S(y) \rightarrow \forall x F(x)$.

1. $\forall x \forall y (S(y) \rightarrow F(x)), S(a) \vdash \forall x \forall y (S(y) \rightarrow F(x))$ ϵ

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2. $\forall x \forall y (S(y) \rightarrow F(x)), S(a) \vdash \forall y (S(y) \rightarrow F(u))$ $\forall -$ 1
3. $\forall x \forall y (S(y) \rightarrow F(x)), S(a) \vdash S(a) \rightarrow F(u)$ $\forall -$ 2

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3. $\forall x \forall y (S(y) \to F(x)), S(a) \vdash S(a) \to F(u)$ $\forall -$ 2
4. $\forall x \forall y (S(y) \to F(x)), S(a) \vdash S(a)$ ϵ

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3. $\forall x \forall y(S(y) \to F(x)), S(a) \vdash S(a) \to F(u)$ $\forall -$ 2
4. $\forall x \forall y(S(y) \to F(x)), S(a) \vdash S(a)$ ϵ
5. $\forall x \forall y(S(y) \to F(x)), S(a) \vdash F(u)$ $\to -$ 4

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5. $\forall x \forall y(S(y) \to F(x)), S(a) \vdash F(u)$ $\to -$ 4
6. $\forall x \forall y(S(y) \to F(x)), S(a) \vdash \forall x F(x)$ $\forall +$ 5

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5. $\forall x \forall y(S(y) \to F(x)), S(a) \vdash F(u)$ $\to -$ 4
6. $\forall x \forall y(S(y) \to F(x)), S(a) \vdash \forall x F(x)$ $\forall +$ 5
7. $\forall x \forall y(S(y) \to F(x)) \vdash S(a) \to \forall x F(f)$ $\to +$ 6

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5. $\forall x \forall y(S(y) \to F(x)), S(a) \vdash F(u)$ $\to -$ 4
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7. $\forall x \forall y(S(y) \to F(x)) \vdash S(a) \to \forall x F(f)$ $\to +$ 6
8. $\forall x \forall y(S(y) \to F(x)) \vdash \exists y S(y) \to \forall x F(x)$ $\exists +$ 7

Plan

The end

Thats it folks. Feel free to hang out and ask questions.

These slides are based off of the tutorial notes and lecture slides provided to you online.
If you want a copy feel free to email me. The are also available on my personal website joe-scott.net

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Jan and Joe have an office hour Mondays at 3pm in the Tutorial Center in MC.

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