Week 5 Tutorial

Assignment Review and Preparation; Natural Deduction

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Prepared based off of the notes of CS245 Instructors, past and present.

27 January 2017

Plan

1. Assignment 1 Review
2. Natural Deduction
   - Review
   - Natural Deduction Examples
3. Assignment 2 Preparation
4. The End

Problem 1

"If y is an integer then z is not real, provided that x is rational"

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1. a - y is an integer
2. b - z is real
3. c - x is rational

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1. a - y is an integer
2. b - z is real
3. c - x is rational

$$((a \to \neg b) \wedge c) \text{ or } (c \to (a \to \neg b))$$

- Negation must not be included in the definition of b, and must be present in the formula.

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$$\deg(\psi) = 0 = \mathrm{Con}(\psi)$$

$$\implies \deg(\psi) \leq \mathrm{Con}(\psi)$$

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$$\deg(\psi) = 0 = \mathrm{Con}(\psi)$$

$$\implies \deg(\psi) \leq \mathrm{Con}(\psi)$$

Let $\psi \coloneqq (\neg\alpha)$.

$\deg(\psi) = \deg(\alpha) + 1$	Construction
$\mathrm{Con}(\psi) = \mathrm{Con}(\alpha) + 1$	Construction
$\deg(\alpha) \leq \mathrm{Con}(\alpha)$	IH
$\deg(\alpha) + 1 \leq \mathrm{Con}(\alpha) + 1$	$\leq$ property
$\deg(\psi) \leq \mathrm{Con}(\psi)$	$\leq$ substitution

Problem 2 - Continued

Let $\psi \coloneqq (\alpha * \beta)$.

$\deg(\psi) = \deg(\alpha) + \deg(\beta) + 1$	Construction
$\mathrm{Con}(\psi) = \mathrm{Con}(\alpha) + \mathrm{Con}(\beta) + 1$	Construction
$\deg(\alpha) \leq \mathrm{Con}(\alpha)$	IH
$\deg(\beta) \leq \mathrm{Con}(\beta)$	IH
$\deg(\alpha) + \deg(\beta) \leq \mathrm{Con}(\alpha) + \mathrm{Con}(\beta)$	$\leq$ property
$\deg(\alpha) + \deg(\beta) + 1 \leq \mathrm{Con}(\alpha) + \mathrm{Con}(\beta) + 1$	$\leq$ property
$\deg(\psi) \leq \mathrm{Con}(\psi)$	$\leq$ substitution

Problem 3

A $r, p \to (r \to q) \vDash p \to (p \wedge r)$.

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A $r, p \to (r \to q) \vDash p \to (p \land r)$.
Claim holds, proof by truth table needed. Indication of rows where both premises are true must be indicated and related to the consequence.

B $p \to q, s \to t \vDash p \lor s \to q \land t$.

Problem 3

A $r, p \to (r \to q) \vDash p \to (p \wedge r)$.
Claim holds, proof by truth table needed. Indication of rows where both premises are true must be indicated and related to the consequence.

B $p \to q, s \to t \vDash p \vee s \to q \wedge t$.
Claim is false, consider $p = T, q = T, s = F, t = F$

$$T \to T, F \to F \vDash T \vee F \to t \wedge F$$

$$T, T \vDash F$$

Counter example or full truth table is fine.

Problem 4

Show that $(A \wedge B) \rightarrow C \models\!\!\!\!\dashv (A \rightarrow C) \vee (B \rightarrow C)$.

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Show that $(A \wedge B) \rightarrow C \models\mathrel{\reflectbox{$\models$}} (A \rightarrow C) \vee (B \rightarrow C)$.

The easiest solution is by applying the axioms.

$$
\begin{aligned}
& (A \wedge B) \rightarrow C \\
\iff{} & \neg(A \wedge B) \vee C \\
\iff{} & \neg A \vee \neg B \vee C \\
\iff{} & \neg A \vee \neg B \vee C \vee C \\
\iff{} & \neg A \vee C \vee \neg B \vee C \\
\iff{} & (A \rightarrow C) \vee (B \rightarrow C)
\end{aligned}
$$

Truth table proof is possible, but analysis needs to be provided for both directions.

Make sure you cite rules.

Problem 5

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The claim is false as for $n = 2$, and $A = F$ we would have:

$$(A \to A) \vDash A$$
$$\iff (F \to F) \vDash F$$
$$\iff T \vDash F$$

B $(\dots((A \to A) \to A)\dots) \vDash\Dashv \neg A$.

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The claim is false as for $n = 2$, and $A = F$ we would have:

$$
\begin{aligned}
&(A \to A) \vDash A \\
\iff &(F \to F) \vDash F \\
\iff &T \vDash F
\end{aligned}
$$

B $(\ldots((A \to A) \to A)\ldots) \vDash\Dashv \neg A$.

The claim is false as for $n = 1$ and $A = T$ we would have

$$
\begin{aligned}
&A \vDash \neg A \\
\iff &T \vDash F
\end{aligned}
$$

C $\emptyset \vDash (A \to B) \wedge (C \to B)$.

Problem 5

A $(\ldots((A \to A) \to A)\ldots) \models\Dashv A$.
The claim is false as for $n = 2$, and $A = F$ we would have:

$$(A \to A) \models A$$
$$\iff (F \to F) \models F$$
$$\iff T \models F$$

B $(\ldots((A \to A) \to A)\ldots) \models\Dashv \neg A$.
The claim is false as for $n = 1$ and $A = T$ we would have

$$A \models \neg A$$
$$\iff T \models F$$

C $\varnothing \models (A \to B) \wedge (C \to B)$.
The claim is false as for $A = T, B = F, C = T$

$$\varnothing \models (T \to F) \wedge (T \to F)$$

Problem 6

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This is false as if Σ is a contradiction than we would have $F \vDash \{T, F\}$ which always holds.

B If Σ is satisfiable and A is a tautology,then $\Sigma \vDash A$.

Problem 6

A If $\Sigma \models A$, then Σ is satisfiable.
  This is false as if Σ is a contradiction than we would have $F \models \{T, F\}$ which always holds.

B If Σ is satisfiable and A is a tautology,then $\Sigma \models A$.
  This is true as we would have $\{T, F\} \models T$ which always holds.

C If $B \in \Sigma$ and Σ is satisfiable, then B is a tautology.

Problem 6

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This is false as if Σ is a contradiction than we would have $F \models \{T, F\}$ which always holds.

B If Σ is satisfiable and A is a tautology,then $\Sigma \models A$.
This is true as we would have $\{T, F\} \models T$ which always holds.

C If $B \in \Sigma$ and Σ is satisfiable, then B is a tautology.
This is false. Consider $\Sigma = \{p, q\}$, with $p \in \Sigma$. p is satisfiable under $p = T$ but not a tautology as it is false under assignment $p = F$.

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Natural Deduction - Yet Another Proof System

- Natural Deduction is a sound and complete proof system.
- Allows both direct and refutation styled proofs.
- Inference rules are intuitive. Easier to put into words.

Natural Deduction Rules

1. $\phi \vdash \phi$ (Reflexive)
2. Introduction

$\phi \vdash \phi \vee \psi$	$(\vee_i)$
$\phi, \psi \vdash \phi \wedge \psi$	$(\wedge_i)$
$(\phi \vdash \bot) \vdash \neg\phi$	$\neg_i$
$(\phi \vdash \psi) \vdash \phi \to \psi$	$\to_i$
$(\phi \to \psi), (\psi \to \phi) \vdash \phi \leftrightarrow \psi$	$\leftrightarrow_i$
$(\phi, \neg\phi) \vdash \bot$	$\bot_i$

3. Elimination

$(\phi \to \eta, \psi \to \eta, \phi \vee \psi) \vdash \eta$	$\vee_e$
$(\phi \wedge \psi) \vdash \phi$	$\wedge_e$
$(\neg(\neg\phi)) \vdash \phi$	$\neg_e$
$(\phi \to \psi, \phi) \vdash \psi$	$\to_e$
$(\phi \leftrightarrow \psi) \vdash \phi \to \psi$	$\leftrightarrow_e$
$\bot \vdash \phi$	$\bot_e$

Tips

- Make subproofs for rules like (e.g $\rightarrow_i$, $\neg_i$) clear. Indent each of them (like nested loops in code)
- The instructors recommend the following
- Write down the premises and conclusion.
- Consider eliminations from premises.
- Work backwards, see if you can use introduction while going backwards.
- Repeat this process in subproofs.

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Example 1

Problem

Prove $r \vee (\neg s) \vdash (s \to r)$

Example 1

Problem

Prove $r \vee (\neg s) \vdash (s \rightarrow r)$

1	$(r \vee (\neg s))$	Premise

Example 1

Problem

Prove $r \vee (\neg s) \vdash (s \to r)$

1	$(r \vee (\neg s))$	Premise
2	r	Assumption

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3	s	Assumption

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1	$(r \vee (\neg s))$			Premise
2		r		Assumption
3			s	Assumption
4			r	Reflexivity: 2

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Prove $r \vee (\neg s) \vdash (s \rightarrow r)$

1	$(r \vee (\neg s))$	Premise
2	r	Assumption
3	s	Assumption
4	r	Reflexivity: 2
5	$(s \rightarrow r)$	$\rightarrow_i$: 3 – 4

Example 1

Problem

Prove $r \vee (\neg s) \vdash (s \rightarrow r)$

1	$(r \vee (\neg s))$			Premise
2		r		Assumption
3			s	Assumption
4			r	Reflexivity: 2
5		$(s \rightarrow r)$		$\rightarrow_i$: $3-4$
6		$(\neg s)$		Assumption
7			s	Assumption
8			$\perp$	$\perp_i : 6, 7$

Example 1

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6		$(\neg s)$		Assumption
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8			$\bot$	$\bot_i : 6, 7$
9			r	$\bot_e : 8$
10		$s \to r$		$\to_i$ $7-9$

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3			s	Assumption
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8			$\perp$	$\perp_i : 6, 7$
9			r	$\perp_e : 8$
10		$s \to r$		$\to_i$ $7-9$
11	$s \to r$			$\vee_e : 2-5, 6-10$

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Prove $r \vee (\neg s) \vdash (s \to r)$

1	$(r \vee (\neg s))$		Premise
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3		s	Assumption
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Example 2

Problem

Let Σ be any set of formulas. Suppose that $\Sigma \vdash (\neg\psi)$ for some ψ. Prove that $\Sigma \vdash \psi \to p$

Let F be the proof of $\Sigma \vdash (\neg\psi)$.

0	Σ	Premise

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n		$\neg\psi$	Since F exists

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$n+1$		$\perp$	$\perp_i : 1, n$

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$n+1$		$\perp$	$\perp_i : 1, n$
$n+2$		p	$\perp_e$: n+1

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		...	
n		$\neg\psi$	Since F exists
$n+1$		$\perp$	$\perp_i : 1, n$
$n+2$		p	$\perp_e$: n+1
$n+3$	$\psi \to p$		$\to_i$: 1-(n+2)

Example 3

Problem

Suppose that $\Sigma \vdash \psi$ *and* $\Delta \vdash \psi$ *for some sets of* Σ *and* Δ *and formula* ψ*. Prove or Disprove* $\Sigma \cap \Delta \vdash \psi$*.*

Example 3

Problem

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Prove or Disprove $\Sigma \cap \Delta \vdash \psi$*.*

No. Suppose $\Sigma = \{p\}, \delta = \{\neg(\neg p)\}$. Then $\varnothing \vdash \psi$, as ψ could be anything (i.e the world is flat).

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Double Entailment

How to prove double entailments?

1. Truth Table
2. Axioms

How to disprove double entailment

1. Counter example of a single direction
2. Truth Table (but this is the same as a counter example).

Sound Introduction/Elimination Rules

As an example, we will prove that $\rightarrow i$ is sound.

$$\dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \rightarrow B}$$

- We assume that the derivation of B from A is sound (i.e. if $A = T$, then $B = T$.)

Sound Introduction/Elimination Rules

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$$\frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \rightarrow B}$$

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- We prove that $(A \rightarrow B) = T$

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  - Case 1: $A = F$; clear.

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Adequate Sets I

An adequate set of connectives is a set of connectives with the capability to express all truth tables.

Adequate Sets II

Theorem

$\{\neg, \wedge, \vee\}$ *is an adequate set.*

Proof.

For any formulas A, B,

$$\neg A \quad \models\!\!\!\!\dashv \quad \neg A$$

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For any formulas A, B,

$$\begin{array}{rcl} \neg A & \models\!\!\!\dashv & \neg A \\ A \wedge B & \models\!\!\!\dashv & A \wedge B \end{array}$$

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For any formulas A, B,

$$\begin{array}{rcl} \neg A & \models\!\!\!\!\!\dashv & \neg A \\ A \wedge B & \models\!\!\!\!\!\dashv & A \wedge B \\ A \vee B & \models\!\!\!\!\!\dashv & A \vee B \\ A \rightarrow B & \models\!\!\!\!\!\dashv & \neg A \vee B \\ A \leftrightarrow B & \models\!\!\!\!\!\dashv & (\neg A \vee B) \wedge (\neg B \vee A) \end{array}$$

$\square$

A Resolution Example

Prove that for any $0 \leq i \leq n$ there exists a Resolution refutation proof to witness

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$$\{p_i \vdash_{Res} (p_0 \vee \ldots \vee p_n)\}$$

- Assume p_i and $\neg(p_0 \vee \ldots \vee p_n)$
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- By the lemma $\neg(p_0 \vee \ldots \vee p_n) \equiv (\neg p_0) \wedge \ldots \wedge (\neg p_n)$
- Our set of clauses then becomes $\{(\neg p_0), \ldots, p_i, (\neg p_i), \ldots, (\neg p_n)\}$

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- Our set of clauses then becomes $\{(\neg p_0), \ldots, p_i, (\neg p_i), \ldots, (\neg p_n)\}$
- By our inference rules we achieve a $\perp$

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The end

Thats it folks. Feel free to hang out and ask questions.

These slides are based off of the tutorial notes and lecture slides provided to you online.
If you want a copy feel free to email me. The are also available on my personal website joe-scott.net

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