Week 4 Tutorial

DNF, CNF, Resoltuion, and Natural Deduction

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Prepared based off of the notes of CS245 Instructors, past and present.

27 January 2017

Plan

1. Review
2. Propositional Equivalence
   - Review
   - CNF Conversion
   - DNF Conversion
3. Resolution
   - Review
   - Resolution Example Problems
4. Natural Deduction
   - Review
   - Natural Deduction Examples
5. The End

Entailment

Entailment has the same logical semantics of implication, but the usage is a matter of context.

1. $\implies$ is a connective for a formula.
2. $\models$ compares a set of premises to a conclusion (a Theorem)

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Lots of Rules

- Commutativity
  1. $\alpha \wedge \beta \equiv \beta \wedge \alpha$
  2. $\alpha \vee \beta \equiv \beta \vee \alpha$
  3. $\alpha \iff \beta \equiv \beta \iff \alpha$
- Associativity
  1. $\alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma$
  2. $\alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma$
- Distributivity
  1. $\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
  2. $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
- Idempotence
  1. $\alpha \vee \alpha \equiv \alpha$
  2. $\alpha \wedge \alpha \equiv \alpha$
- Double Negation
  1. $\neg(\neg\alpha) \equiv \alpha$
- De Morgan's Laws
  1. $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$
  2. $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$
- Simplification 1 (Absorbtion)
  1. $\alpha \wedge \top \equiv \alpha$
  2. $\alpha \vee \top \equiv \top$
  3. $\alpha \wedge \bot \equiv \bot$
  4. $\alpha \vee \bot \equiv \alpha$
- Simplification 2
  1. $\alpha \vee (\alpha \wedge \beta) \equiv \alpha$
  2. $\alpha \wedge (\alpha \vee \beta) \equiv \alpha$
- Implication
  1. $\alpha \implies \beta \equiv \neg\alpha \vee \beta$
- Contrapositive
  1. $\alpha \implies \beta \equiv \neg\beta \implies \neg\alpha$
- Equivilance
  1. $\alpha \iff \beta \equiv (\alpha \implies \beta) \wedge (\beta \implies \alpha)$
- Excluded Middle
  1. $\alpha \vee \neg\alpha \equiv \top$
- Contradiction
  1. $\alpha \wedge \neg\alpha \equiv \bot$

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Conjunctive Normal Form

Definition

A literal is an atomic formula or its negation.

Definition

A clause is a disjunction of one or more literals.

Definition

A well-formed formula is considered to be in Conjunctive Normal Form (CNF) if and only if it is a conjunction of one or more clauses.

CNF - Informally

- A formula is in conjunctive normal form if its outter most binary connectives are all conjunctions ($\wedge$).
- We can write the formula ψ as $\psi = \bigwedge_i \psi_i$

The following are in CNF formula

- $((\neg A) \wedge (B \vee C))$
- $((A \vee B) \wedge ((\neg B) \vee C \vee (\neg D)) \wedge (D \vee (\neg E))$
- $(A \vee B)$
- $(A \wedge B)$

The following are not

- $\neg(B \vee C)$
- $(A \wedge B) \vee C$
- $A \wedge (B \vee (D \wedge E))$

Conjunction Normal Form Algorithm

An algorithm to convert a well formed formula to CNF

1. Eliminate implications and iffs
2. Apply De Morgan when applicable, and eliminate double negations.
3. Transform in to a conjunction with distributivity
4. Simplify using idemopotence, contradiction, excluded middle, and simplification rules.

1. Any wff can be converted to CNF!
2. Computers can convert formulas very quickly (using a different algorithm)

Example

Convert the following formula to CNF: $\left((\neg p) \implies (\neg(q \vee r))\right)$

Example

Convert the following formula to CNF: $\left((\neg p) \implies (\neg(q \vee r))\right)$

1. Eliminate implication and equivalence.

$$\left((\neg(\neg p)) \vee (\neg(q \vee r))\right)$$

Example

Convert the following formula to CNF: $\Big((\neg p) \implies (\neg(q \vee r))\Big)$

1. Eliminate implication and equivalence.

$$\Big((\neg(\neg p)) \vee (\neg(q \vee r))\Big)$$

2. Apply De Morgan 's and double-negation laws as often as possible.

$$\Big(p \vee ((\neg q) \wedge (\neg r))\Big)$$

Example

Convert the following formula to CNF: $\left((\neg p) \implies (\neg(q \vee r))\right)$

1. Eliminate implication and equivalence.

$$\left((\neg(\neg p)) \vee (\neg(q \vee r))\right)$$

2. Apply De Morgan 's and double-negation laws as often as possible.

$$\left(p \vee ((\neg q) \wedge (\neg r))\right)$$

3. Transform into a conjunction of clauses using distributivity

$$\left((p \vee (\neg q)) \wedge (p \vee (\neg r))\right)$$

Example

Convert the following formula to CNF: $\Big((\neg p) \implies (\neg(q \vee r))\Big)$

1. Eliminate implication and equivalence.

$$\Big((\neg(\neg p)) \vee (\neg(q \vee r))\Big)$$

2. Apply De Morgan 's and double-negation laws as often as possible.

$$\Big(p \vee ((\neg q) \wedge (\neg r))\Big)$$

3. Transform into a conjunction of clauses using distributivity

$$\Big((p \vee (\neg q)) \wedge (p \vee (\neg r))\Big)$$

4. Simplify using idempotence, contradiction, excluded middle and Simplification I & II.
Not Needed.

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5. The End

Disjunctive Normal Form

Definition

A literal is an atomic formula or its negation.

Definition

A well-formed formula is considered to be in Disjunctive Normal Form (DNF) if and only if it is a disjunction of one or more conjunctions of one or more literals.

DNF - Informally

- A formula is in disjunctive normal form if its outer most binary connectives are all disjunction ($\vee$).
- We can write the formula ψ as $\psi = \bigvee_i \psi_i$

The following are in DNF formula

- $(A \wedge \neg B \wedge \neg C) \vee (\neg D \wedge E \wedge F)$
- $(A \wedge B) \vee C$
- $A \wedge B$
- A

The following are not

- $\neg(A \vee B)$
- $A \vee (B \wedge (C \vee D))$

Disjunctive Normal Form Algorithm

An algorithm to convert a well formed formula to DNF

1. Eliminate implications and iffs
2. Apply De Morgan when applicable, and eliminate double negations.
3. Transform in to a disjunction with distributivity
4. Simplify using idemopotence, contradiction, excluded middle, and simplification rules.

1. Any wff can be converted to DNF
2. Computers can not convert these quickly

DNF Conversion Example

Convert the following formula to DNF:
$((p \wedge q) \implies r) \wedge (\neg(p \wedge q) \implies r)$

DNF Conversion Example

Convert the following formula to DNF:

$((p \wedge q) \implies r) \wedge (\neg(p \wedge q) \implies r)$

- $(\neg(p \wedge q) \vee r) \wedge ((p \wedge q) \vee r)$
- $((\neg p \vee \neg q) \vee r) \wedge ((p \wedge q) \vee r)$
- $(\neg p \vee \neg q \vee r) \wedge ((p \wedge q) \vee r)$
- $((\neg p \wedge p \wedge q) \vee (\neg q \wedge p \wedge q) \vee (r \wedge p \wedge q) \vee (\neg p \wedge r) \vee (\neg q \wedge r) \vee (r \wedge r)$
- $(r \wedge p \wedge q) \vee (\neg p \wedge r) \vee (\neg q \wedge r) \vee r$

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The resolution proof system

The resolution proof system had only ONE rule.

$$\frac{\alpha \vee b, \neg b \vee \beta}{\alpha \vee \beta}$$

- The resolution proof system was sound. (Everything proved is logically valid)
- The resolution system was complete (Everything logically valid is provable).

The resolution proof system

Resolution is a refutation system, with the following inference rule:

$$\frac{(\alpha \vee p), ((\neg p) \vee \beta)}{(\alpha \vee \beta)}$$

We also include the following special cases

$$\frac{(\alpha \vee p), (\neg p)}{\alpha} \text{ Unit resolution}$$

$$\frac{p, (\neg p)}{\perp} \text{ Contradiction}$$

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Example 1

Problem: Convert to CNF, draw a proof graph demonstrating how a contradiction is obtained.

$$\{((p \vee q) \to r)\} \vdash_{Res} ((p \wedge q) \to r).$$

We have $\{((p \vee q) \to r)\}$

and $((p \wedge q) \to r)$

Example 1

Problem: Convert to CNF, draw a proof graph demonstrating how a contradiction is obtained.

$$\{((p \vee q) \to r)\} \vdash_{Res} ((p \wedge q) \to r).$$

We have $\{((p \vee q) \to r)\}$

1. $((p \vee q) \to r) \equiv ((\neg(p \vee q)) \vee r)$

and $((p \wedge q) \to r)$

Example 1

Problem: Convert to CNF, draw a proof graph demonstrating how a contradiction is obtained.

$$\{((p \vee q) \rightarrow r)\} \vdash_{Res} ((p \wedge q) \rightarrow r).$$

We have $\{((p \vee q) \rightarrow r)\}$

1. $((p \vee q) \rightarrow r) \equiv ((\neg(p \vee q)) \vee r)$
2. $\equiv (((\neg p) \wedge (\neg q)) \vee r)$

and $((p \wedge q) \rightarrow r)$

Example 1

Problem: Convert to CNF, draw a proof graph demonstrating how a contradiction is obtained.

$$\{((p \vee q) \to r)\} \vdash_{Res} ((p \wedge q) \to r).$$

We have $\{((p \vee q) \to r)\}$

1. $((p \vee q) \to r) \equiv ((\neg(p \vee q)) \vee r)$
2. $\equiv (((\neg p) \wedge (\neg q)) \vee r)$
3. $\equiv (((\neg p) \vee r) \wedge ((\neg q) \vee r))$

and $((p \wedge q) \to r)$

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We have $\{((p \vee q) \to r)\}$

1. $((p \vee q) \to r) \equiv ((\neg(p \vee q)) \vee r)$
2. $\equiv (((\neg p) \wedge (\neg q)) \vee r)$
3. $\equiv (((\neg p) \vee r) \wedge ((\neg q) \vee r))$

and $((p \wedge q) \to r)$

1. $(\neg((p \wedge q) \to r)) \equiv (\neg((\neg(p \wedge q)) \vee r))$

Example 1

Problem: Convert to CNF, draw a proof graph demonstrating how a contradiction is obtained.

$$\{((p \vee q) \rightarrow r)\} \vdash_{Res} ((p \wedge q) \rightarrow r).$$

We have $\{((p \vee q) \rightarrow r)\}$

1. $((p \vee q) \rightarrow r) \equiv ((\neg(p \vee q)) \vee r)$
2. $\equiv (((\neg p) \wedge (\neg q)) \vee r)$
3. $\equiv (((\neg p) \vee r) \wedge ((\neg q) \vee r))$

and $((p \wedge q) \rightarrow r)$

1. $(\neg((p \wedge q) \rightarrow r)) \equiv (\neg((\neg(p \wedge q)) \vee r))$
2. $\equiv ((p \wedge q) \wedge (\neg r)$

Example 1

Problem: Convert to CNF, draw a proof graph demonstrating how a contradiction is obtained.

$$\{((p \vee q) \to r)\} \vdash_{Res} ((p \wedge q) \to r).$$

We have $\{((p \vee q) \to r)\}$

1. $((p \vee q) \to r) \equiv ((\neg(p \vee q)) \vee r)$
2. $\equiv (((\neg p) \wedge (\neg q)) \vee r)$
3. $\equiv (((\neg p) \vee r) \wedge ((\neg q) \vee r))$

and $((p \wedge q) \to r)$

1. $(\neg((p \wedge q) \to r)) \equiv (\neg((\neg(p \wedge q)) \vee r))$
2. $\equiv ((p \wedge q) \wedge (\neg r)$

- This yields
$$\{((\neg p) \vee r), ((\neg q) \vee r), p, q, (\neg r)\}.$$

Drawing of Example 1

Problem: Prove with resolution system. Draw a figure representing the proof.

$$\{((p \vee q) \rightarrow r)\} \vdash_{Res} ((p \wedge q) \rightarrow r).$$

This yields

$$\{((\neg p) \vee r), ((\neg q) \vee r), p, q, (\neg r)\}.$$

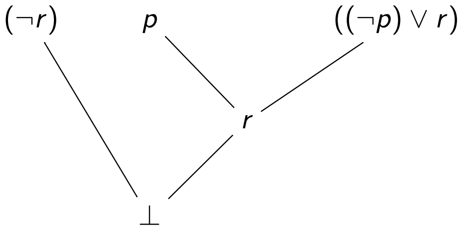
Drawing of Example 1

Problem: Prove with resolution system. Draw a figure representing the proof.

$$\{((p \vee q) \to r)\} \vdash_{Res} ((p \wedge q) \to r).$$

This yields

$$\{((\neg p) \vee r), ((\neg q) \vee r), p, q, (\neg r)\}.$$



Example 2

Consider the set of propositional formulas:
$\{a \implies b, (b \wedge c) \implies d, (d \wedge (e \vee f)) \implies g, a, c, \neg e\}$
Convert the formulas to conjunctive normal form and for each of the following queries, either prove the query using resolution refutation or show that the query does not logically follow.

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

Conversion of each formula into conjunctive normal form.

The hardest CNF of that set

1. $(d \wedge (e \vee f)) \implies g$

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1. $(d \wedge (e \vee f)) \implies g$
2. $\neg(d \wedge (e \vee f)) \vee g$

The hardest CNF of that set

1. $(d \wedge (e \vee f)) \implies g$
2. $\neg(d \wedge (e \vee f)) \vee g$
3. $(\neg d \vee \neg(e \vee f)) \vee g$
4. $(\neg d \vee (\neg e \wedge \neg f))) \vee g$

The hardest CNF of that set

1. $(d \wedge (e \vee f)) \implies g$
2. $\neg(d \wedge (e \vee f)) \vee g$
3. $(\neg d \vee \neg(e \vee f)) \vee g$
4. $(\neg d \vee (\neg e \wedge \neg f)) \vee g$
5. $((\neg d \vee \neg e) \wedge (\neg d \wedge \neg f)) \vee g$

The hardest CNF of that set

1. $(d \wedge (e \vee f)) \implies g$
2. $\neg(d \wedge (e \vee f)) \vee g$
3. $(\neg d \vee \neg(e \vee f)) \vee g$
4. $(\neg d \vee (\neg e \wedge \neg f))) \vee g$
5. $((\neg d \vee \neg e) \wedge (\neg d \wedge \neg f)) \vee g$
6. $(g \vee \neg d \vee \neg f) \wedge (g \wedge \neg d \wedge \neg e)$

Example 2

Consider the set of propositional formulas:
$\{a \implies b, (b \wedge c) \implies d, (d \wedge (e \vee f)) \implies g, a, c, \neg e\}$
Convert the formulas to conjunctive normal form and for each of the following queries, either prove the query using resolution refutation or show that the query does not logically follow.

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

Conversion of each formula into conjunctive normal form.

1.	$\neg a \vee b$
2.	$\neg b \vee \neg c \vee d$
3a.	$\neg d \vee g \vee \neg e$
3b.	$\neg d \vee g \vee \neg f$
4.	a
5.	c
6.	$\neg e$

Example 2, Q1

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption

Example 2, Q1

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \lor b$	assumption
2.	$\neg b \lor \neg c \lor d$	assumption
3a.	$\neg d \lor g \lor \neg e$	assumption
3b.	$\neg d \lor g \lor \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	$\neg d$	negated query

Example 2, Q1

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	$\neg d$	negated query
8.	$\neg b \vee \neg c$	2, 7

Example 2, Q1

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	$\neg d$	negated query
8.	$\neg b \vee \neg c$	2, 7
9.	$\neg b$	5, 8

Example 2, Q1

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	$\neg d$	negated query
8.	$\neg b \vee \neg c$	2, 7
9.	$\neg b$	5, 8
10.	$\neg a$	1, 9

Example 2, Q1

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	$\neg d$	negated query
8.	$\neg b \vee \neg c$	2, 7
9.	$\neg b$	5, 8
10.	$\neg a$	1, 9
11.	$\perp$	4, 10

Example 2, Q2

$Q1.\ d$

$Q2.\ f \implies g$

$Q3.\ g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption

Example 2, Q2

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \lor b$	assumption
2.	$\neg b \lor \neg c \lor d$	assumption
3a.	$\neg d \lor g \lor \neg e$	assumption
3b.	$\neg d \lor g \lor \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	$\neg g$	negated query

Example 2, Q2

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	$\neg g$	negated query
9.	$\neg d \vee g$	3b, 7

Example 2, Q2

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	$\neg g$	negated query
9.	$\neg d \vee g$	3b, 7
10.	$\neg d$	8, 9

Example 2, Q2

$Q1.\ d$
$Q2.\ f \implies g$
$Q3.\ g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	$\neg g$	negated query
9.	$\neg d \vee g$	3b, 7
10.	$\neg d$	8, 9
11.	$\neg b \vee \neg c$	2, 10

Example 2, Q2

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	$\neg g$	negated query
9.	$\neg d \vee g$	3b, 7
10.	$\neg d$	8, 9
11.	$\neg b \vee \neg c$	2, 10
12.	$\neg b$	5, 11

Example 2, Q2

$Q1.\ d$
$Q2.\ f \implies g$
$Q3.\ g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	$\neg g$	negated query
9.	$\neg d \vee g$	3b, 7
10.	$\neg d$	8, 9
11.	$\neg b \vee \neg c$	2, 10
12.	$\neg b$	5, 11
13.	$\neg a$	1, 12

Example 2, Q2

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	$\neg g$	negated query
9.	$\neg d \vee g$	3b, 7
10.	$\neg d$	8, 9
11.	$\neg b \vee \neg c$	2, 10
12.	$\neg b$	5, 11
13.	$\neg a$	1, 12
14.	$\perp$	4, 13

Example 2, Q3

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption

Example 2, Q3

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	g	negated query

Example 2, Q3

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	g	negated query
9.	.....	3b, 7

Example 2, Q3

Q1. d
Q2. $f \implies g$
Q3. $g \implies \neg f$

1.	$\neg a \vee b$	assumption
2.	$\neg b \vee \neg c \vee d$	assumption
3a.	$\neg d \vee g \vee \neg e$	assumption
3b.	$\neg d \vee g \vee \neg f$	assumption
4.	a	assumption
5.	c	assumption
6.	$\neg e$	assumption
7.	f	negated query
8.	g	negated query
9.	.....	3b, 7

Q3 does not logically follow. One can show that there is no proof; i.e., resolving all possible clauses together does not lead to the empty clause. One can also show a counter-example: where the assumptions are all true but the query $g \implies \neg f$ is false.

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   - Review
   - Natural Deduction Examples
5. The End

Outline

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Natural Deduction - Yet Another Proof System

- Natural Deduction is a sound and complete proof system.
- Allows both direct and refutation styled proofs.
- Inference rules are intuitive. Easier to put into words.

Natural Deduction Rules

1. $\phi \vdash \phi$ (Reflexive)
2. Introduction

$\phi \vdash \phi \lor \psi$	$(\lor_i)$
$\phi, \psi \vdash \phi \land \psi$	$(\land_i)$
$(\phi \vdash \bot) \vdash \neg\phi$	$\neg_i$
$(\phi \vdash \psi) \vdash \phi \to \psi$	$\to_i$
$(\phi \to \psi), (\psi \to \phi) \vdash \phi \leftrightarrow \psi$	$\leftrightarrow_i$
$(\phi, \neg\phi) \vdash \bot$	$\bot_i$

3. Elimination

$(\phi \to \eta, \psi \to \eta, \phi \lor \psi) \vdash \eta$	$\lor_e$
$(\phi \land \psi) \vdash \phi$	$\land_e$
$(\neg(\neg\phi)) \vdash \phi$	$\neg_e$
$(\phi \to \psi, \phi) \vdash \psi$	$\to_e$
$(\phi \leftrightarrow \psi) \vdash \phi \to \psi$	$\leftrightarrow_e$
$\bot \vdash \phi$	$\bot_e$

Tips

- Make subproofs for rules like (e.g $\rightarrow_i$, $\neg_i$) clear. Indent each of them (like nested loops in code)
- The instructors recommend the following
- Write down the premises and conclusion.
- Consider eliminations from premises.
- Work backwards, see if you can use introduction while going backwards.
- Repeat this process in subproofs.

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Example 1

Problem

Prove $p \to (q \to r) \vdash (p \wedge q) \to r$

Example 1

Problem

Prove $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$

1	$p \rightarrow (q \rightarrow r)$	Premise

Example 1

Problem

Prove $p \to (q \to r) \vdash (p \wedge q) \to r$

1	$p \to (q \to r)$	Premise
2	$p \wedge q$	Assumption

Example 1

Problem

Prove $p \to (q \to r) \vdash (p \wedge q) \to r$

1	$p \to (q \to r)$	Premise
2	$p \wedge q$	Assumption
3	p	$\wedge_e$: 2

Example 1

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1	$p \to (q \to r)$	Premise
2	$p \land q$	Assumption
3	p	$\land_e$: 2
4	q	$\land_e$: 2

Example 1

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1	$p \to (q \to r)$	Premise
2	$p \land q$	Assumption
3	p	$\land_e$: 2
4	q	$\land_e$: 2
5	$(q \to r)$	$\to_e$: 3, 1

Example 1

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1	$p \to (q \to r)$	Premise
2	$p \land q$	Assumption
3	p	$\land_e$: 2
4	q	$\land_e$: 2
5	$(q \to r)$	$\to_e$: 3, 1
6	r	$\to_e$: 4,5

Example 1

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1	$p \to (q \to r)$	Premise
2	$p \land q$	Assumption
3	p	$\land_e$: 2
4	q	$\land_e$: 2
5	$(q \to r)$	$\to_e$: 3, 1
6	r	$\to_e$: 4,5
7	$(p \land q) \to r$	$\to_i$: $2 - 6$

Example 1

Problem

Prove $p \to (q \to r) \vdash (p \land q) \to r$

1	$p \to (q \to r)$	Premise
2	$p \land q$	Assumption
3	p	$\land_e$: 2
4	q	$\land_e$: 2
5	$(q \to r)$	$\to_e$: 3, 1
6	r	$\to_e$: 4,5
7	$(p \land q) \to r$	$\to_i$: $2 - 6$

Example 2

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

Example 2

Problem

Prove $\psi \rightarrow \beta \vdash \neg\beta \rightarrow \neg\psi$

1	$\psi \rightarrow \beta$	(Assumption)

Example 2

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1	$\psi \to \beta$	(Assumption)
2	$(\neg\beta)$	(Assumption)

Example 2

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1	$\psi \to \beta$	(Assumption)
2	$(\neg\beta)$	(Assumption)
3	ψ	(Assumption)

Example 2

Problem

Prove $\psi \rightarrow \beta \vdash \neg\beta \rightarrow \neg\psi$

1	$\psi \rightarrow \beta$	(Assumption)
2	$(\neg\beta)$	(Assumption)
3	ψ	(Assumption)
4	β	$(\rightarrow_e : 1,3)$

Example 2

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1	$\psi \to \beta$	(Assumption)
2	$(\neg\beta)$	(Assumption)
3	ψ	(Assumption)
4	β	$(\to_e : 1,3)$
5	$\perp$	$(\perp_i : 2,4)$

Example 2

Problem

Prove $\psi \rightarrow \beta \vdash \neg\beta \rightarrow \neg\psi$

1	$\psi \rightarrow \beta$	(Assumption)
2	$(\neg\beta)$	(Assumption)
3	ψ	(Assumption)
4	β	$(\rightarrow_e$: 1,3)
5	$\perp$	$(\perp_i$: 2,4)
6	$\neg\psi$	$(\neg_i$ 3-5)

Example 2

Problem

Prove $\psi \to \beta \vdash \neg\beta \to \neg\psi$

1	$\psi \to \beta$	(Assumption)
2	$(\neg\beta)$	(Assumption)
3	ψ	(Assumption)
4	β	($\to_e$: 1,3)
5	$\perp$	($\perp_i$: 2,4)
6	$\neg\psi$	($\neg_i$ 3-5)
7	$(\neg\beta) \to (\neg\psi)$	($\to_i$: 2-6)

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The end

Thats it folks. Feel free to hang out and ask questions.

These slides are based off of the tutorial notes and lecture slides provided to you online.
If you want a copy feel free to email me. The are also available on my personal website joe-scott.net

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