Tutorial 8: Polynomial-time reductions

1 Travelling Salesman

In the HamCycle problem, we are given a graph $G = (V, E)$ and we must determine whether there is a cycle in G that goes through every vertex in V.

In the decision version of the travelling salesman problem, that we will denote by TSP, we are given a weighted graph $G = (V, E)$ with positive integer weights $w : E \to \{1, 2, \ldots, L\}$ and a positive integer t; we need to determine whether there is a cycle that goes through every vertex in V and has total weight (= sum of the weights of the edges in the cycle) at most t.

Prove that HamCycle $\leq_{\mathbf{P}}$ TSP.

1.1 Solution

Transform the input graph $G = (V, E)$ into the triplet (G, w, n) with the weight function $w(e) = 1$ for each $e \in E$.

If G has Hamiltonian cycle then there exists a cycle in TSP with weight n, which is the Hamiltonian cycle itself.

If G, w has a tour of weight n then the tour is also a Hamiltonian cycle because every tour is a Hamiltonian cycle.

2 Longest Path

In the HamPath problem, we are given a graph $G = (V, E)$ and we must determine whether there is a path in G that goes through every vertex in V exactly once.

In the LongestPath problem, we are given a graph $G = (V, E)$ and a positive integer k, we must determine whether there is a simple path in G that goes through k edges.

Prove that HamPath $\leq_{\mathbf{P}}$ LongestPath.

2.1 Solution

Transform G into $(G, n - 1)$.

If G has a Hamiltonian path then there is a simple path in G that goes through $n - 1$ edges, which is the Hamiltinian path itself.

If G has a simple path of length $n - 1$ then the path would cover all n vertices, hence it is a Hamiltonian path as well.

3 Hamiltonian paths and cycles

Prove that HAMCYCLE $\leq_{\mathbf{P}}$ HAMPATH.

3.1 Solution

Let $G = (V, E)$ be the given graph. Construct a graph $G^{'}$ by picking an arbitrary vertex $X \in V$ and creating a duplicate vertex $X^{'}$ ($X^{'}$ is connected to all the vertices X is connected to in G). Create two more vertices t_1 and t_2 and join t_1 to X and t_2 to $X^{'}$. $G^{'}$ would be the input to the Hamiltonian path problem.

If G has a Hamiltonian cycle then $G^{'}$ has a Hamiltonian path. Let y_1 and y_2 be the vertices adjacent to X in the Hamiltonian cycle. The Hamiltonian path would start at t_1 go to X then y_1 and go along the Hamiltonian cycle till it reaches y_2 and then it goes to $X^{'}$ and t_2.

If $G^{'}$ has a Hamiltonian path then G has a Hamiltonian cycle. The Hamiltonian path has to start with t_1 and end with t_2 as both have degree one. The Hamiltonian path would look like $t_1, X, y_i, ..., y_j, X^{'}, t_2$. The Hamiltonian cycle in the original graph would be $X, y_i, ..., y_j, X$, since from the construction we know that y_j is adjacent to X.

4 Set packing

In the INDEPSET problem we are given a graph $G = (V, E)$ and a positive integer k and we must determine whether there is an independent set (=set of vertices that are all disconnected from each other) of size k in G.

In the SETPACKING problem, we are given a collection of subsets of $\{1, 2, \ldots, m\}$ and we must determine whether there are k sets $S_1, \ldots, S_k$ in the collection that are all disjoint from each other.

Prove that INDEPSET $\leq_{\mathbf{P}}$ SETPACKING.

4.1 Solution

Assign label from 1 to m to each edge in the graph. For each vertex $v \in V$, create the set $S_v = \{i \leq m : v \text{ incident to edge } i\}$.

If G has an independent set of size k then the sets corresponding to the k vertices would be disjoint, as two sets would have an intersection if and only if there is an edge between the corresponding vertices.

If there are k sets that are disjoint then the k vertices corresponding to sets are independent, as two sets would have an intersection if and only if there is an edge between the corresponding vertices.