The Relational Model

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How do we ask Questions (and understand Answers)?

In the beginning ...

Set comprehension syntax for questions:

$$\{(x_1, \ldots, x_k) \mid \langle \mathit{condition} \rangle\}$$

Answers:

All *k-tuples* of *values* that satisfy $\langle \mathit{condition} \rangle$.

How do we ask Questions (and understand Answers)?

Find *all pairs of (natural) numbers that add to 5*!

Question: $\{(x, y) \mid x + y = 5\mathrm{PLUS}(x, y, 5)\}$

Answer: $\{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)\}$

... but but but why? (explain this to a 6 year old!)

because $(0, 5, 5)$, etc., appear in PLUS!

Find pairs of numbers that add to the same number as they subtract to (i.e., $x + y = x - y$)!

Question: $\{(x, y) \mid \exists z.\mathrm{PLUS}(x, y, z) \wedge \mathrm{PLUS}(z, y, x)\}$

Answer: $\{(0, 0), (1, 0), \ldots, (5, 5)?\}$

... answer depends on the content (instance) of PLUS!

Find the *neutral element* (of addition)!

Question: $\{(x) \mid \mathrm{PLUS}(x, x, x)\}$

Answer: $\{(0)\}$

Addition Table

PLUS		
0	0	0
0	1	1
1	0	1
0	2	2
2	0	2
	...	
0	5	5
	...	
1	4	5
	...	
2	3	5
	...	
	...	

How do we ask Questions about Employees?

Find *all employees who work for "Bob"*!

Question: $\{(x, y) \mid \text{EMP}(x, y, Bob)\}$

Answer: $\{(Sue, \text{CS}), (Bob, \text{CO})\}$

why? because (Sue, CS, Bob), etc., appear in EMP!

Find pairs of emp-s working for the same boss!

Q: $\{(x_1, x_2) \mid \exists y_1, y_2, z.\text{EMP}(x_1, y_1, z) \wedge \text{EMP}(x_2, y_2, z)\}$

A: $\{(Sue, Bob), (Fred, John), (Jim, Eve)\}$ $\leftarrow$ is that all?

Find employees who are their own bosses!

Q: $\{(x) \mid \exists y.\text{EMP}(x, y, x)\}$

A: $\{(Sue), (Bob)\}$

Employee Table

EMP

name	dept	boss
Sue	CS	*Bob*
Bob	CO	*Bob*
Fred	PM	*Mark*
John	PM	*Mark*
Jim	CS	*Fred*
Eve	CS	*Fred*
Sue	PM	*Sue*

The Relational Model

Idea

All information is organized in (a finite number of) relations.

Features:

- simple and clean data model
- powerful and declarative query/update languages
- semantic integrity constraints
- data independence

Relational Databases

Components:

Universe	- a set of *values* **D** with equality ($=$)
Relation	- predicate name R, and arity k of R (the number of columns) - instance: a relation $\mathbf{R} \subseteq \mathbf{D}^k$.
Database	- signature: finite set ρ of predicate names - instance: a relation $\mathbf{R_i}$ for each R_i

Notation

Signature: $\rho = (R_1, \ldots, R_n)$ *Instance:* $\mathbf{DB} = (\mathbf{D}, =, \mathbf{R_1}, \ldots, \mathbf{R_n})$

Examples of Relational Databases

- The integers, with addition and multiplication:

  $\rho = (\texttt{PLUS}, \texttt{TIMES})$ $\qquad$ $\mathbf{DB} = (\mathbf{Z}, =, \mathbf{PLUS}, \mathbf{TIMES})$

- A Bibliography Database (see following slides)

- ...

A Bibliography Relational Database Signature

Predicates (also called table headers):

```
AUTHOR(aid, name)
WROTE(author, publication)
PUBLICATION(pubid, title)
BOOK(pubid, publisher, year)
JOURNAL(pubid, volume, no, year)
PROCEEDINGS(pubid, year)
ARTICLE(pubid, crossref, startpage, endpage)
```

$\Rightarrow$ identifiers, called *attributes*, label columns (needed for SQL)

A Bibliography Relational Database Instance

Relations (also called tables):

$$
\begin{array}{rcll}
\texttt{AUTHOR} & = & \{ & (1, \text{John}), (2, \text{Sue}) \quad \} \\
\texttt{WROTE} & = & \{ & (1, 1), (1, 4), (2, 3) \quad \} \\
\texttt{PUBLICATION} & = & \{ & (1, \text{Mathematical Logic}), \\
 & & & (3, \text{Trans. Databases}), \\
 & & & (2, \text{Principles of DB Syst.}), \\
 & & & (4, \text{Query Languages}) \quad \} \\
\texttt{BOOK} & = & \{ & (1, \text{AMS}, 1990) \quad \} \\
\texttt{JOURNAL} & = & \{ & (3, 35, 1, 1990) \quad \} \\
\texttt{PROCEEDINGS} & = & \{ & (2, 1995) \quad \} \\
\texttt{ARTICLE} & = & \{ & (4, 2, 30, 41) \quad \}
\end{array}
$$

A Common Visualization for Relational Databases

AUTHOR

aid	name
1	John
2	Sue

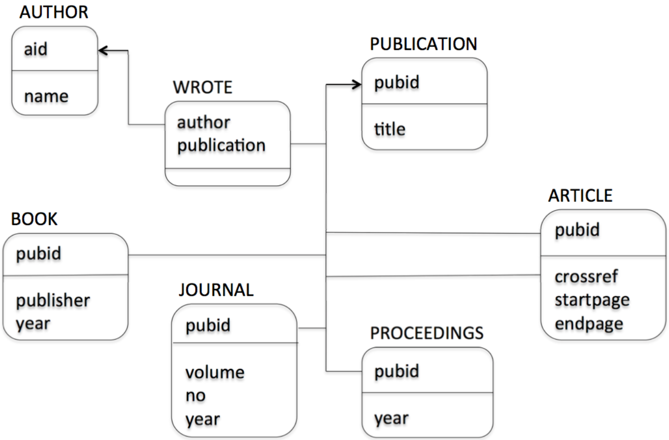
WROTE

author	publication
1	1
1	4
2	3

PUBLICATION

pubid	title
1	Mathematical Logic
3	Trans. Databases
2	Principles of DB Syst.
4	Query Languages

A Common Visualization for Relational Database Schemata†



†Relational database signatures plus *integrity constraints*.

Simple (Atomic) "Truth"

Idea

Relationships between objects (tuples) that are present in an instance are true, relationships absent are false.

In the sample *Bibliography* database instance

- "John" is an author with id "1": $(1, \text{John}) \in \textbf{AUTHOR};$
- "Mathematical Logic" is a publication: $(1, \text{Mathematical Logic}) \in \textbf{PUBLICATION};$
  Moreover, it is a book published by "AMS" in "1990": $(1, \text{AMS}, 1990) \in \textbf{BOOK};$
- "John" wrote "Mathematical Logic": $(1, 1) \in \textbf{WROTE};$
- "John" has **NOT** written "Trans. Databases": $(1, 3) \notin \textbf{WROTE};$
- etc.

Query Conditions

Idea1: use *variables* to generalize conditions

$\mathtt{AUTHOR}(x, y)$ will be true of any valuation $\{x \mapsto a, y \mapsto b, \ldots\}$ exactly when the pair $(a, b) \in$ **AUTHOR**

Idea2: build more complex conditions from simpler ones using ...

Logical connectives:

Conjunction (and):	$\mathtt{AUTHOR}(x, y) \wedge \mathtt{WROTE}(x, z)$
Disjunction (or):	$\mathtt{AUTHOR}(x, y) \vee \mathtt{PUBLICATION}(x, y)$
Negation (not):	$\neg \mathtt{AUTHOR}(x, y)$

Quantifiers:

Existential (there is...): $\exists x.\mathtt{author}(x, y)$

Conditions in the Relational Calculus

Idea

Conditions can be formulated using the language of first-order logic.

Definition (Syntax of Conditions)

Given a database schema $\rho = (R_1, \ldots, R_k)$ and a set of variable names $\{x_1, x_2, \ldots\}$, conditions are *formulas* defined by

$$\varphi ::= \underbrace{\underbrace{\underbrace{R_i(x_{i_1}, \ldots, x_{i_k}) \mid x_i = x_j \mid \varphi \land \varphi \mid \exists x_i.\varphi}_{\text{conjunctive formulas}} \mid \varphi \lor \varphi}_{\text{positive formulas}} \mid \neg\varphi}_{\text{first-order formulas}}$$

First-order Variables and Valuations

How do we *interpret* variables?

Definition (Valuation)

A valuation is a function θ that maps *variable names* to values in the universe:

$$\theta : \{x_1, x_2, \ldots\} \to \mathbf{D}.$$

To denote a modification to θ in which variable x is instead mapped to value v, one writes:

$$\theta[x \mapsto v].$$

Idea

Answers to queries $\Leftrightarrow$ *valuations to free variables that make the formula true with respect to a database.*

Complete Semantics for Conditions

Definition

The *truth* of a formula φ is defined with respect to

1. a **database instance DB** $= (\mathbf{D}, =, \mathbf{R}_1, \mathbf{R}_2, \ldots)$, and
2. a **valuation** $\theta : \{x_1, x_2, \ldots\} \to \mathbf{D}$

as follows:

$$
\begin{array}{ll}
\mathbf{DB}, \theta \models R(x_{i_1}, \ldots, x_{i_k}) & \text{if } R \in \rho, (\theta(x_{i_1}), \ldots, \theta(x_{i_k})) \in \mathbf{R} \\
\mathbf{DB}, \theta \models x_i = x_j & \text{if } \theta(x_i) = \theta(x_j) \\
\mathbf{DB}, \theta \models \varphi \wedge \psi & \text{if } \mathbf{DB}, \theta \models \varphi \text{ and } \mathbf{DB}, \theta \models \psi \\
\mathbf{DB}, \theta \models \neg\varphi & \text{if not } \mathbf{DB}, \theta \models \varphi \\
\mathbf{DB}, \theta \models \exists x_i.\varphi & \text{if } \mathbf{DB}, \theta[x_i \mapsto v] \models \varphi \text{ for some } v \in \mathbf{D}
\end{array}
$$

Equivalences and Syntactic Sugar

Boolean Equivalences

- $\neg(\neg\varphi_1) \equiv \varphi_1$
- $\varphi_1 \vee \varphi_2 \equiv \neg(\neg\varphi_1 \wedge \neg\varphi_2)$
- $\varphi_1 \rightarrow \varphi_2 \equiv \neg\varphi_1 \vee \varphi_2$
- $\varphi_1 \leftrightarrow \varphi_2 \equiv (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$
- $\ldots$

First-order Equivalences

- $\forall x.\varphi \equiv \neg\exists x.\neg\varphi$

Relational Calculus

Definition (Queries)

A *query* in the relational calculus is a set comprehension of the form

$$\{(x_1, \ldots, x_k) \mid \varphi\}.$$

Definition (Query Answers)

An *answer* to a *query* $\{(x_1, \ldots, x_k) \mid \varphi\}$ over **DB** is the **relation**

$$\{(\theta(x_1), \ldots, \theta(x_k)) \mid \mathbf{DB}, \theta \models \varphi\},$$

where $\{x_1, \ldots, x_k\} = FV(\varphi)^\dagger$.

†*FV* denotes the *free variables* of φ.

On Formulas

Definition (Free Variables)

The *free variables* of a formula φ, written $FV(\varphi)$, are defined as follows:

$$
\begin{array}{lcl}
FV(R(x_{i_1},\ldots,x_{i_k})) & \equiv & \{x_{i_1},\ldots,x_{i_k}\} \\
FV(x_i = x_j) & \equiv & \{x_i, x_j\} \\
FV(\varphi \wedge \psi) & \equiv & FV(\varphi) \cup FV(\psi) \\
FV(\neg\varphi) & \equiv & FV(\varphi) \\
FV(\exists x_i.\varphi) & \equiv & FV(\varphi) - \{x_i\}
\end{array}
$$

A formula that has no free variables expresses is called a *sentence*.

Example

Find pairs of emp-s working for the same boss!

Q: $\{(x_1, x_2) \mid \exists y_1, y_2, z.\text{EMP}(x_1, y_1, z) \wedge \text{EMP}(x_2, y_2, z)\}$
A: $\{(\mathit{Sue}, \mathit{Fred}), \ldots\}$

because:

1. $\textbf{EMP}, \{x_1 \mapsto \mathit{Sue}, y_1 \mapsto \text{CS}, z \mapsto \mathit{Bob}, \ldots\} \models \text{EMP}(x_1, y_1, z)$
2. $\textbf{EMP}, \{x_2 \mapsto \mathit{Fred}, y_2 \mapsto \text{CO}, z \mapsto \mathit{Bob}, \ldots\} \models \text{EMP}(x_2, y_2, z)$
3. $\textbf{EMP}, \{x_1 \mapsto \mathit{Sue}, y_1 \mapsto \text{CS}, x_2 \mapsto \mathit{Fred}, y_2 \mapsto \text{CO}, z \mapsto \mathit{Bob}, \ldots\}$
   $\models \text{EMP}(x_1, y_1, z) \wedge \text{EMP}(x_2, y_2, z)$
4. $\textbf{EMP}, \{x_1 \mapsto \mathit{Sue}, x_2 \mapsto \mathit{Fred}, \ldots\}$
   $\models \exists y_1, y_2, z.\text{EMP}(x_1, y_1, z) \wedge \text{EMP}(x_2, y_2, z)$

Emp Table

EMP

name	dept	boss
Sue	CS	*Bob*
Fred	CO	*Bob*
Bob	PM	*Mark*
John	PM	*Mark*
Jim	CS	*Fred*
Eve	CS	*Fred*
Sue	PM	*Sue*

Sample Queries

Over numbers (with addition and multiplication):

- list all composite numbers
- list all prime numbers

Over the bibliography database:

- list all publications
- list titles of all publications
- list titles of all books
- list all publications without authors
- list (pairs of) coauthor names
- list titles of publications written by a single author

How do we ask Questions (and understand Answers)?

Find the *neutral element* (of addition)!

Question: $\{(x) \mid \text{PLUS}(x, x, x)\}$

Answer: $\{(0)\}$

but shouldn't the query really be

$$\{(x) \mid \forall y.\text{PLUS}(x, y, y) \wedge \text{PLUS}(y, x, y)\} \qquad (*)$$

Idea

$(*)$ is the same as $\{(x) \mid \forall y.\text{PLUS}(x, y, y)\}$

because PLUS is *commutative*

is the same as $\{(x) \mid \text{PLUS}(x, x, x)\}$

because PLUS is *monotone*

$\Rightarrow$ Laws of Arithmetic for Natural Numbers

Addition Table

PLUS		
0	0	0
0	1	1
1	0	1
0	2	0
2	0	2
	...	
0	5	5
	...	
1	4	5
	...	
2	3	5
	...	
	...	

Laws a.k.a. Integrity Constraints

Idea

What must be always true for the natural numbers (i.e., for PLUS)?

- addition is commutative

$$\forall x, y, z.\text{PLUS}(x, y, z) \rightarrow \text{PLUS}(y, x, z)$$
$$(\neg\exists x, y, z.\text{PLUS}(x, y, z) \wedge \neg\text{PLUS}(y, x, z))$$

- addition is a (relational representation of a) binary function

$$\forall x, y, z_1, z_2.\text{PLUS}(x, y, z_1) \wedge \text{PLUS}(x, y, z_2) \rightarrow z_1 = z_2$$
$$(\neg\exists x, y, z_1, z_2.\text{PLUS}(x, y, z_1) \wedge \text{PLUS}(x, y, z_2) \wedge \neg(z_1 = z_2))$$

- addition is a total function

$$\forall x, y.\exists z.\text{PLUS}(x, y, z)$$

- addition is monotone in both arguments (harder), etc., etc.

Laws a.k.a. Integrity Constraints for Employees

Idea

Integrity constraints
$\Rightarrow$ *yes/no conditions that must be true in every valid database instance.*

- Every Boss is an Employee

$$\forall x, y, z.\mathrm{EMP}(x, y, z) \rightarrow \exists u, w.\mathrm{EMP}(z, u, w)$$

- Every Boss manages a unique Department

$$\forall x_1, x_2, y_1, y_2, z.\mathrm{EMP}(x_1, y_1, z) \wedge \mathrm{EMP}(x_2, y_2, z) \rightarrow y_1 = y_2$$

- No Boss cannot have another Employee serving as their Boss

$$\forall x, y, z.\mathrm{EMP}(x, y, z) \rightarrow \mathrm{EMP}(z, y, z)$$

Integrity Constraints

A relational *signature* captures only the structure of relations.

Idea

Valid database instances satisfy additional integrity constraints.

- Values of a particular attribute belong to a prescribed *data type*.
- Values of attributes are unique among tuples in a relation (*keys*).
- Values appearing in one relation must also appear in another relation (*referential integrity*).
- Values cannot appear simultaneously in certain relations (*disjointness*).
- Values in certain relation must appear in at least one of another set of relations (*coverage*).
- ...

Example Revisited (Bibliography)

Typing Constraints / Domain Contraints

- Author id's are integers.
- Author names are strings.

Uniqueness of Values / Identification (keys)

- Author id's are unique and determine author names.
- Publication id's are unique as well.
- Articles can be identified by their publication id.
- Articles can also be identified by the publication id of the collection they have appeared in and their starting page number.

Referential Integrity / Foreign Keys

- Books, journals, proceedings and articles are publications.
- The components of a WROTE tuple must be an author and a publication.

Example Revisited (cont.)

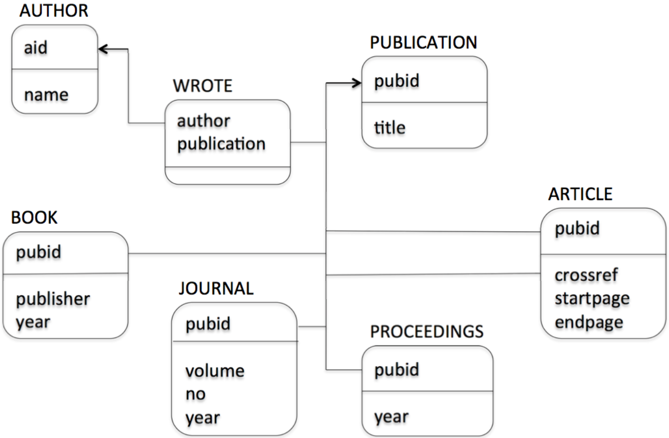
Disjointness

- Books are different from journals.
- Books are also different from proceedings.

Coverage

- Every publication is a book or a journal or a proceedings or an article.
- Every article appears in a journal or in a proceedings.

Example Revisited (cont.)



Views and Integrity Constraints

Idea

Answers to queries can be used to define derived relations (views)
$\Rightarrow$ *extension of a DB schema*

- subtraction, complement, ...
- *collection*-style publication, editor, ...

In general, a view is an integrity constraint of the form

$$\forall x_1, \ldots, x_k . R(x_1, \ldots, x_k) \leftrightarrow \varphi$$

for R a new relation name and $x_1, \ldots, x_k$ free variables of φ.

Database Instances and Integrity Constraints

Definition (Relational Database Schema)

A *relational database schema* is a signature ρ and a (finite) set of integrity constraints Σ over ρ.

Definition

A relational database instance **DB** (over a schema ρ) *conforms to a schema* Σ (written **DB** $\models \Sigma$) if and only if **DB**$, \theta \models \varphi$ for any integrity constraint $\varphi \in \Sigma$ and any valuation θ.

Story so far...

1. databases $\Leftrightarrow$ relational structures
2. queries $\Leftrightarrow$ set comprehensions with formulas in First-Order logic
3. integrity constraints $\Leftrightarrow$ closed formulas in FO logic

... so is there anything new here?

$\Rightarrow$YES: database instances must be finite

Unsafe Queries

- $\{(y) \mid \neg\exists x.\mathtt{author}(x, y)\}$
- $\{(x, y, z) \mid \mathtt{book}(x, y, z) \vee \mathtt{proceedings}(x, y)\}$
- $\{(x, y) \mid x = y\}$

$\Rightarrow$ we want only queries with finite answers (over finite databases).

Definition (Domain-independent Query)

A query $\{(x_1, \ldots, x_k) \mid \varphi\}$ is *domain-independent* if

$$\mathbf{DB}_1, \theta \models \varphi \iff \mathbf{DB}_2, \theta \models \varphi$$

for any pair of database instances $\mathbf{DB}_1 = (\mathbf{D_1}, =, \mathbf{R_1}, \ldots, \mathbf{R_k})$ and $\mathbf{DB}_2 = (\mathbf{D_2}, =, \mathbf{R_1}, \ldots, \mathbf{R_k})$ and all θ.

Theorem

Answers to domain-independent queries contain only values that exist in $\mathbf{R_1}, \ldots, \mathbf{R_k}$ (the active domain).

Domain-independent + finite database $\Rightarrow$ "safe"

Safety and Query Satisfiability

Theorem

Satisfiability[1] *of first-order formulas is undecidable;*

- *co-r.e. in general*
- *r.e for finite databases*

Proof.

Reduction from PCP (see Abiteboul *et. al.* book, p.122-126). □

Theorem

Domain-independence of first-order queries is undecidable.

Proof.

φ is satisfiable iff $\{(x, y) \mid (x = y) \wedge \varphi\}$ is not domain-independent. □

[1] Is there a database for which the answer is non-empty?

Range-restricted Queries

Definition (Range restricted formulas)

A formula φ is *range restricted* when, for φ_i that are also range restricted, φ has the form

$$
\begin{array}{ll}
R(x_{i_1}, \ldots, x_{i_k}), & \\
\varphi_1 \wedge \varphi_2\,, & \\
\varphi_1 \wedge (x_i = x_j) & (\{x_i, x_j\} \cap FV(\varphi_1) \neq \emptyset), \\
\exists x_i.\varphi_1 & (x_i \in FV(\varphi_1)), \\
\varphi_1 \vee \varphi_2 & (FV(\varphi_1) = FV(\varphi_2)), \text{ or} \\
\varphi_1 \wedge \neg\varphi_2 & (FV(\varphi_2) \subseteq FV(\varphi_1)).
\end{array}
$$

Theorem

Range-restricted $\Rightarrow$ *Domain-independent.*

Domain Independent v.s. Range-restricted

Do we lose expressiveness by restricting to Range-restricted queries?

Theorem

Every domain-independent query *can be written equivalently as a* range restricted *query.*

Proof.

1. restrict every variable in φ to *active domain*,
2. express the active domain using a *unary query* over the database instance.

$\square$

Computational Properties

- Evaluation of every query terminates
  $\Rightarrow$ relational calculus is not *Turing complete*
- **Data Complexity** in the size of the database, for a *fixed* query.
  $\Rightarrow$ in PTIME
  $\Rightarrow$ in LOGSPACE
  $\Rightarrow$ AC_0 (constant time on polynomially many CPUs in parallel)
- **Combined complexity**
  $\Rightarrow$ in PSPACE
  $\Rightarrow$ can express NP-hard problems (encode SAT)

Query Evaluation vs. Theorem Proving

Query Evaluation

Given a query $\{(x_1, \ldots, x_k) \mid \varphi\}$ and a finite database instance **DB** find all answers to the query.

Query Satisfiability

Given a query $\{(x_1, \ldots, x_k) \mid \varphi\}$ determine whether there is a (finite) database instance **DB** for which the answer is non-empty.

- much harder (undecidable) problem
- can be solved for fragments of the query language

Query Equivalence and DB Schema

Do we ever need the power of *theorem proving*?

Definition (Query Subsumption)

A query $\{(x_1, \ldots, x_k) \mid \varphi\}$ *subsumes* a query $\{(x_1, \ldots, x_k) \mid \psi\}$ with respect to a database schema Σ if

$$\{(\theta(x_1), \ldots, \theta(x_k)) \mid \mathbf{DB}, \theta \models \psi\} \subseteq \{(\theta(x_1), \ldots, \theta(x_k)) \mid \mathbf{DB}, \theta \models \varphi\}$$

for every database **DB** such that $\mathbf{DB} \models \Sigma$.

- *necessary* for query simplification
- equivalent to proving

$$\left(\bigwedge_{\phi_i \in \Sigma} \phi_i\right) \rightarrow (\forall x_1, \ldots x_k . \psi \rightarrow \varphi)$$

- undecidable in general; decidable for fragments of relational calculus

What queries cannot be expressed in RC?

Note

RC is not Turing-complete
$\Rightarrow$ *there must be computable queries that cannot be written in RC.*

Built-in Operations

- ordering, arithmetic, string operations, etc.

Counting/Aggregation

- cardinality of sets (*parity*)

Reachability/Connectivity/...

- *paths in a graph (binary relation)*

Model extensions: Incompleteness/Inconsistency

- tuples with *unknown* (but existing) values
- incomplete relations and *open world assumption*
- conflicting information (e.g., from different data sources)