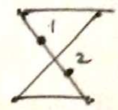
1. {6 marks}

(a) {5 marks} Let G be a bipartite graph and let H be a subdivision of G. Prove that H is 3-colourable.

Assume G is a bipartite, with bipartition {A,B}.
Then G is 2-colorable by the theorem
Let H be the subdivison of G.

Case 1: If we add even number of vertices.

Since G is a bipartite, we must add vertices on the edges which is incident between A, B.



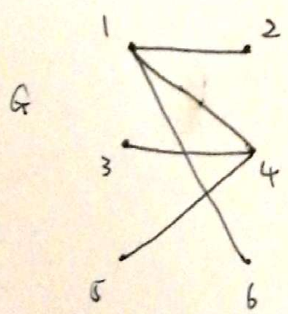
After we add even number of vertices, if the vertex's adjacent vertex is in A, then let the added vertex's color be the same as colour which B has; the other vertex's colour be the same as colour which A has. Then, we've done a pair of vertices. Since we add even vertices, G is 2-colourable, and is also 3-colourable.

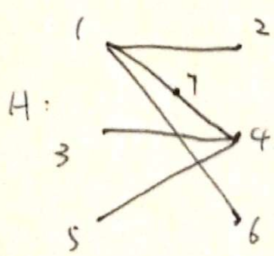
Case 2: if we add odd number of vertices,

Consider we add one vertex on the edge which is incident between A and B. Since, A has one colour and B has another colour. if the added vertex has either of two colours, its colour will be same to A or B. Thus, we need to add one colour. ~~And~~ And odd -1 is even, we can deal with the rest even number of vertices by using Case 1.

Thus, G is 3-colourable

(b) {1 marks} Give an example of a bipartite graph G and a subdivision H of G that is not 2-colourable (i.e. not bipartite).





2. {6 marks} Let G be a planar graph that does not contain any cycles of length three. Prove that G is 4-colourable. Do not assume the Four Colour Theorem. (Hint: use the result of Question 3 on Assignment 5.)

$v = \#$ of vertices

Prove by induction:

Base case: when $n=1$, G is 4-colourable. Thus, it's true.

I.H: Assume that every planar graph on $v \le k$ vertices and does not contain any cycle of length 3 is 4-colourable, for $k \geq 1$

Let G be a planar graph on $v = k+1$ vertices

The result of Question 3 in A5 is:

Any planar embedding of a simple connected planar graph contains a vertex of degree at most 3 or a face of degree at most 3.

Since G does not contain any cycles of ~~length~~ length 3, the G has a vertex x which has degree 3.

G is planar graph and the $G-x$ is a planar graph.

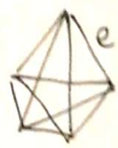
Then $G-x$ doesn't contain any cycle which has length 3.

Then by I.H. $G-x$ is 4-colourable. Let $g = \{1,2,3,4\}$ be the 4-coloring of $G-X$. We can extend g to be the 4-colouring of G. Let x chose the different number with its adjacent 3 vertices.

Then. Induction Complete.

3. {3 marks} Prove or disprove the following statement.
Let G be a graph and let $H = G/e$ be the graph obtained from G by contracting an edge. Then G is planar if and only if H is planar.

Consider G is K_5



$H = G/e$

Thus, when H is planar, G is not planar.

The statement is false.

4. {5 marks} Let G be a graph with $2k$ vertices. Suppose every vertex of G has degree at least k. Prove that G has a perfect matching.

Suppose the longest path has t vertices $x_1, \cdots x_t$. We want to show there is a Haniltotian cycle.

Suppose it does not contain Hamiltonian cycle.

All neighbors of x_1 and x_t must lie on path or else it is not longest.

The degree of each vertex is at least k.

And x_1 and x_t must have at least one common adjacent vertex v because the number of edge in the longest path is at most $2k-1$, and each of x_1 and x_t has at least k vertices on the longest path.

Thus we have a cycle $x_1, \cdots, x_t, v, x_1$

If the longest path is not the ful' $2k$ vertices, then the cycle we get missing some vertex x. Since every vertex of G has degree at least k, then the graph G is connected. So there is a path from x to C and gives a longer path than $x_1 \cdots x_t$. Contradiction

Thus, we have a Hamitonian cycle, the number of edge is $2k$, Choose the edge alternatively, and then G has a perfect matching.