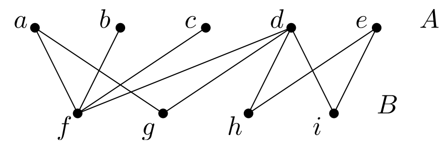
1. {8 marks} Run König's theorem on the following graph, starting with the empty matching.



Each time you change the matching show

- the augmenting path
- X and Y (either the full X and Y for this step, or if you use the shortcutted algorithm, the partial X and Y computed so far).
- the new matching

Solution. The algorithm as we stated it doesn't specify the order to take things and so the steps are not unique.

Step 1 begin with $M = \emptyset$
build X and Y until we get an unsaturated element of Y

- $X = \{a, b, c, d, e\}, Y = \emptyset$
- $X = \{a, b, c, d, e\}, Y = \{f\}$. f is unsaturated so we can stop without finishing building X and Y.

the augmenting path is f, a
now $M = \{af\}$

Step 2 begin with $M = \{af\}$
build X and Y until we get an unsaturated element of Y

- $X = \{b, c, d, e\}, Y = \emptyset$
- $X = \{b, c, d, e\}, Y = \{f\}$
- $X = \{b, c, d, e\}, Y = \{f, g\}$. g is unsaturated so we can stop without finishing building X and Y.

the augmenting path is g, d
now $M = \{af, gd\}$

Step 3 begin with $M = \{af, gd\}$
build X and Y until we get an unsaturated element of Y

- $X = \{b, c, e\}, Y = \emptyset$
- $X = \{b, c, e\}, Y = \{f\}$
- $X = \{b, c, d\}, Y = \{f, h\}$. h is unsaturated so we can stop without finishing building X and Y.

the augmenting path is h, e
now $M = \{af, gd, he\}$

Step 4 begin with $M = \{af, gd, he\}$
build X and Y until we get an unsaturated element of Y

- $X = \{b, c\}, Y = \emptyset$
- $X = \{b, c\}, Y = \{f\}$
- $X = \{b, c, a\}, Y = \{f\}$
- $X = \{b, c, a\}, Y = \{f, g\}$
- $X = \{b, c, a, d\}, Y = \{f, g\}$
- $X = \{b, c, a, d\}, Y = \{f, g, h\}$
- $X = \{b, c, a, d\}, Y = \{f, g, h, i\}$. i is unsaturated so we can stop without finishing building X and Y.

the augmenting path is i, d, g, a, f, b
now $M = \{he, id, ga, fb\}$

Step 4 begin with $M = \{he, id, ga, fb\}$
build X and Y

- $X = \{c\}, Y = \emptyset$
- $X = \{c\}, Y = \{f\}$
- $X = \{b, c\}, Y = \{f\}$

No changes can be made to M so we're done final answer: $M = \{he, id, ga, fb\}$ and $C = \{f, a, d, e\}$

2. $\{6$ marks$\}$

   (a) $\{4$ marks$\}$ Let G be a bipartite graph. Let Δ be the largest degree of any vertex of G. Prove that G has a matching with at least $|E(G)|/\Delta$ edges

   (b) $\{2$ marks$\}$ Give an example of a non-bipartite graph for which this lower bound on the size of a matching is not true.

Solution.

(a) Let S be any vertex cover of G. Each vertex of G is incident with at most Δ edges, so the total number of edges incident with at least one vertex in S is at most $\Delta|S|$. Since S is a vertex-cover, every edge of G is incident with some vertex in S, so $|E(G)| \leq \Delta|S|$. Therefore, for a minimum cardinality vertex-cover C we have $|C| \geq |E(G)|/\Delta$. By Königs Theorem, G has a matching M of cardinality $|C|$, so $|M| \geq |E(G)|/\Delta$ as well.

(b) The triangle K_3 has 3 vertices and $\Delta = 2$, but a maximum matching in K_3 has only one edge, and $1 \not\geq 3/2$.

3. $\{6$ marks$\}$ Let n be a positive integer and ℓ, k nonnegative integers with $\ell + k \leq n$. This question concerns the binomial identity

$$\binom{n}{\ell}\binom{n-\ell}{k} = \binom{n}{k}\binom{n-k}{\ell}$$

   (a) $\{4$ marks$\}$ Prove the identity combinatorially by interpreting both sides as the size of some set of combinatorial objects.

   (b) $\{2$ marks$\}$ Prove the identity algebraically.

Solution.

(a) Consider the set $\{1, 2, \ldots, n\}$. Let A be the set of pairs of subsets of $\{1, 2, \ldots, n\}$ where the first element of the pair has size ℓ and the second element of the pair has size k, and the two subsets are disjoint. Then $|A| = \binom{n}{\ell}\binom{n-\ell}{k}$ because there are $\binom{n}{\ell}$ ways to choose the first subset. This leaves a set of $n-\ell$ elements of which k must be chosen for the second subset There are $\binom{n-\ell}{k}$ ways to do this. This gives $|A| = \binom{n}{\ell}\binom{n-\ell}{k}$.
Another way we could count the size of A is to first choose k elements of $\{1, 2, \ldots, n\}$ to get the second subset (there are $\binom{n}{k}$ ways to do this) and then thre remains a set of $n-k$ elements of which ℓ must be chosen for the first subset. There are $\binom{n-k}{\ell}$ ways to do this giving that $|A| = \binom{n}{k}\binom{n-k}{\ell}$.
Therefore $\binom{n}{\ell}\binom{n-\ell}{k} = \binom{n}{k}\binom{n-k}{\ell}$.

(b) Expand using the definition

$$\begin{aligned}
\binom{n}{\ell}\binom{n-\ell}{k} &= \left(\frac{n(n-1)(n-2)\cdots(n-\ell+1)}{\ell!}\right)\left(\frac{(n-\ell)(n-\ell-1)\cdots(n-\ell-k+1)}{k!}\right) \\
&= \frac{n(n-1)(n-2)\cdots(n-\ell-k+1)}{\ell!k!} \\
&= \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)(n-k-1)\cdots(n-\ell-k+1)}{k!\ell!} \\
&= \left(\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\right)\left(\frac{(n-k)(n-k-1)\cdots(n-\ell-k+1)}{\ell!}\right) \\
&= \binom{n}{k}\binom{n-k}{\ell}
\end{aligned}$$

4. $\{5$ marks$\}$

Prove that the set of compositions of n with k parts is in bijection with the set of subsets of $\{1, 2, \ldots, n-1\}$ of size $k-1$ by describing the bijection and its inverse explicitly.

Solution. Let C be the set of compositions of n with k parts and let S be the set of subsets of $\{1, 2, \ldots, n-1\}$ of size $k-1$. Define the map $f : C \to S$ as follows:

$$f((m_1, m_2, \ldots, m_k)) = \{m_1, m_1 + m_2, m_1 + m_2 + m_3, \ldots, m_1 + m_2 + \cdots + m_{k-1}\}$$

This map is well defined because each $m_i \geq 1$ so the partial sums in $f((m_1, m_2, \ldots, m_k))$ are distinct and all lie in $\{1, 2, \ldots, n-1\}$.

Define the map $g : S \to C$ as follows: Given $s \in S$ write $s = \{s_1, s_2, \ldots, s_{k-1}\}$ with $s_1 < s_2 < \cdots < s_{k-1}$. Then

$$g(s) = (s_1, s_2 - s_1, s_3 - s_2, \ldots, s_{k-1} - s_{k-2}, n - s_{k-1})$$

This is well defined as the sum of the parts of $g(s)$ is $s_1 + s_2 - s_1 + s_3 - s_2 + \ldots + s_{k-1} - s_{k-2} + n - s_{k-1} = n$ (it is a telescoping sum).

Finally, f and g are mutually inverse because $m_1 + m_2 + \cdots m_i - (m_1 + m_2 + \cdots m_{i-1}) = m_i$ and $s_1 + (s_2 - s_1) + \cdots + (s_i - s_{i-1}) = s_i$ (again by telescoping).