1. {6 marks} Let G be a bipartite graph with bipartition (A, B).

(a) {2 marks} Prove that

$$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v).$$

For each edge in bipartite graph, one end is in A, and the other is in B. Thus the number of edge is equal to $\sum_{v \in A} \deg(v)$ and $\sum_{v \in B} \deg(v)$. Thus, $\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$

(b) {2 marks} Let a, b be the number of odd-degree vertices in A, B respectively. Prove that $a \equiv b \pmod 2$.

From the corollary, we know "The number of vertices of odd degree in a graph is even." So the sum of $a+b$ must be even. Thus both of a and b should be either odd or even. The difference between two different odd/even integers is the multiple of 2, thus $a \equiv b \pmod 2$.

(c) {2 marks} Let $k \geq 1$ be an integer. Prove that if G is k-regular, then $|A| = |B|$.

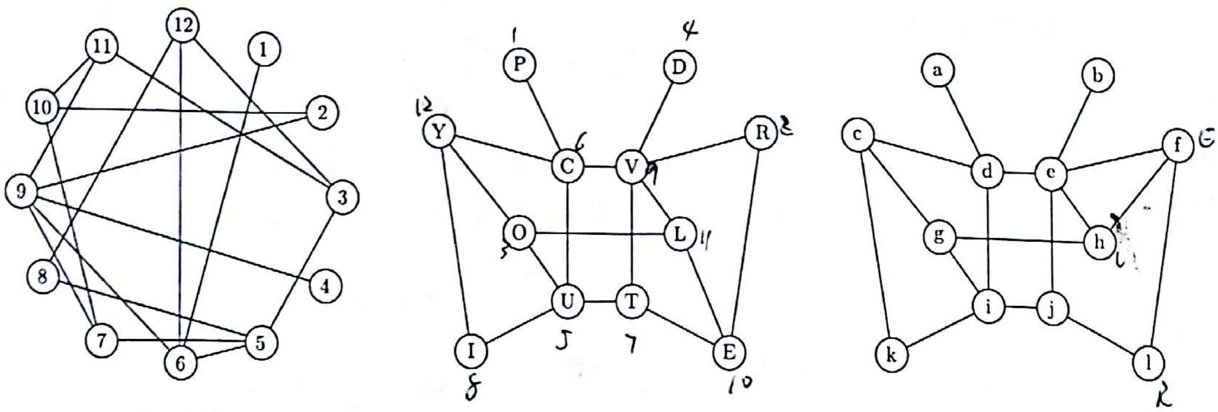
From (a), we know that $\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$.

Since G is k-regular, $\deg(v) = k$ $v \in A$, $\deg(u) = k$ $u \in B$.

Thus $\sum_{v \in A} k = \sum_{u \in B} k$

$|A| = |B|$.

2. {6 marks} Consider the following three graphs.



Graph G Graph H Graph J

(a) {3 marks} Graphs G and H are isomorphic. Provide an isomorphism. (You do not need to prove that your mapping is an isomorphism.)

$f(1)=P$ Thus G and H are isomorphic
$f(2)=R$
$f(3)=O$
$f(4)=D$
$f(5)=U$
$f(6)=C$
$f(7)=T$
$f(8)=I$
$f(9)=V$
$f(10)=E$
$f(11)=L$
$f(12)=Y$

(b) {3 marks} Graphs H and J are not isomorphic. Explain why.

The difference between H and J is that $\{L,E\}$ is deleted and $\{L,R\}$ is added in H in order to get J

Thus $\deg(E)=3$
$\deg(R)=2$
$\deg(f)=3$
$\deg(l)=2$

And we assume $g(E)=f$, $g(R)=l$ $g(U)=e$ $g(T)=j$ .... (just switch E and R, other vertices keep the same)

For e which is adjacent to f, $\deg(e)=5$ is different to $\deg(T)=3$
For j which is adjacent to l, $\deg(j)=3$ is different to $\deg(U)=5$

Thus, H and J are not isomorphic.

3. {4 marks} Prove that any graph with at least 2 vertices contains two vertices of the same degree. (Hint: Prove by contradiction.)

The possible degrees in a graph with n vertices are 0, 1, 2 .... n-1, and no graph with n vertices can contain both 0 degree and n-1 degrees, so in each case there are only n-1 possible degrees for n vertices. Thus, any graph with at least 2 vertices contains two vertices of the same degree.

4. {5 marks} Let G be a graph where the degrees of the vertices are either 1 or 3.

(a) {2 marks} Prove that G has even number of vertices.

Handshaking Lemma: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

$2|E(G)|$ is an even number. and the degrees of the vertices are odd. thus the number of vertices is even.

(b) {3 marks} Prove that if the number of vertices is equal to the number of edges in G, then the number of vertices of degree 1 is equal to the number of vertices of degree 3 in G.

Assume $|V(G)| = |E(G)|$

According to the corollary. the average degree of a vertex in graph G is $\frac{2|E(G)|}{|V(G)|} = 2$

x vertices' degree is 1
y vertices' degrees are 3.

$x + 3y = 2(x+y)$

$x = y$

Thus, the number of vertices of degree 1 is equal to that of degree 3 in G.