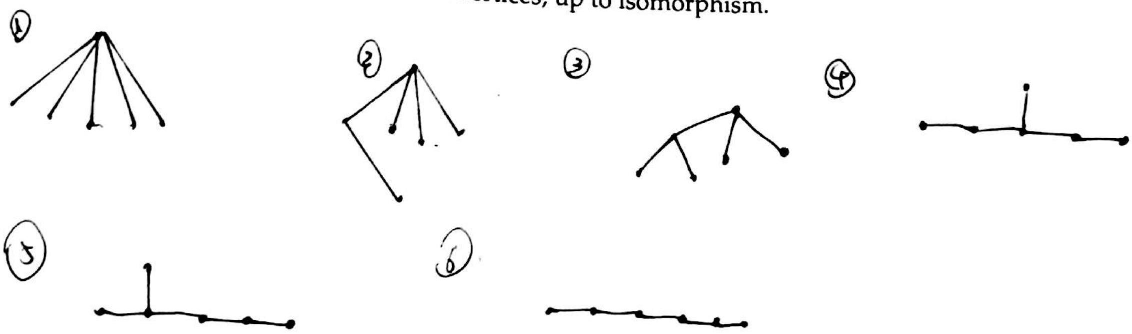
1. {8 marks}

(a) {3 marks} Draw all trees on six vertices, up to isomorphism.



(b) {3 marks} Find the smallest possible number n of vertices in a tree that has four vertices of degree 3, three vertices of degree 5 and two vertices of degree 7. Prove that your answer is correct.

Let T be a tree.

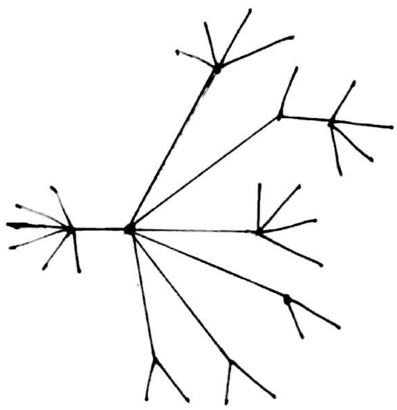
Recall: r_i = number of vertices in T with degree i

$$r_1 = 2 + \sum_{i=3}^{\infty}(i-2)r_i$$
$$\geq 2 + r_3 + 3r_5 + 5r_7$$
$$= 2 + 4 + 3\times3 + 5\times2$$
$$= 25$$

Thus, $r_1 \geq 25$, $n = r_1 + 4 + 3 + 2 = 34$.

We need at least 34 vertices.

(c) {2 marks} Draw an example of a tree with four vertices of degree 3, three vertices of degree 5 and two vertices of degree 7, that has exactly n vertices (where n is as in the previous part).



2. {5 marks} Let $p \geq 2$ be given. Suppose $d_1, d_2, \ldots, d_p$ is a sequence of p positive integers such that $\sum_{i=1}^{p} d_i = 2p-2$. Prove that there exists a tree with p vertices whose degrees are $d_1, d_2, \ldots, d_p$. (Hint: use induction on p.)

Base case: $p=1$, $d_1=0$

$\sum_{i=1}^{1} d_i = 0$

$2p-2 = 2\times1-2 = 0$

$\sum_{i=1}^{1} d_i = 2p-2$. True.

I.H. Assume $p=k$, $k \in N^+$, such that $\sum_{i=1}^{k} d_i = 2k-2$.

Conclusion: $p=k+1$. $k \in N^+$

d_{k+1} is the degree of the $k+1$th vertex, which has one and only one edge e between arbitrary $d_1 \cdots d_k$, otherwise the Graph will have a cycle or is not connected.

Thus the degree of d_{k+1} is 1 and the degree of the other end point will add 1.

The sum of the addition of the degrees is 2 compared to the k vertices

Thus, $\sum_{i=1}^{k+1} d_i = 2 + \sum_{i=1}^{k} d_i$

$= 2 + 2k - 2$

$= 2(k+1) - 2$.

The statement is true.

3. {5 marks} Let G be a graph and let H be a subgraph of G that does not contain a cycle.

(a) {3 marks} Suppose J is a subgraph of H and e is an edge of H that is not in $E(J)$. Prove that if T is a spanning tree of G that contains all the edges in $E(J)$, then there exists a spanning tree T' of G that contains $E(J) \cup \{e\}$.

$|E(J)| \in |E(T)|$, J is a subgraph of T.

J is a subgraph of H, H is a subgraph of T.

There exists edge e such that $\{e\} \in E(H)$ but $\{e\} \notin E(J)$

① if H is a subgraph of T, then $e \in E(H) \in E(T)$, and T is T' which contains $E(J) \cup \{e\}$. We are done.

② If H is not a subgraph of T.

Thus $T+e$ is a cycle of G. Let C be that cycle.

There must exist at least one edge f in C such that $f \in E(T)$ and $f \notin E(H)$, otherwise H is a cycle.

Thus by definition, $T-e+f$ is also a spanning tree. We are done.

(b) {2 marks} Use the previous part to prove that G has a spanning tree that contains all the edges in $E(H)$.

① If G is a tree.

G itself is a spanning tree that contains all the edges in $E(H)$.

② If G contain smallest cycles. one and only one.

For each cycle, If there exist $e \in E(H)$ but $e \notin E(T)$. There must also exist $e' \in E(T)$ but $e' \notin E(H)$. Thus let $T' = T - e' + e$ for each cycle. (If there exists two edge $\notin E(T)$, there will be one vertex $\notin V(T)$)

Then, we have T' conains all edges in $E(H)$.

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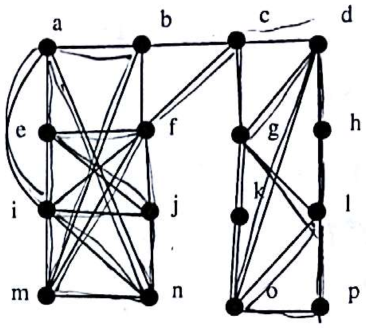
4. {6 marks}

(a) {3 marks} Find an Eulerian circuit in the graph shown. Make a list of the vertices in the order in which they appear on your circuit (note vertices may appear several times).



d, h, l, p, o, l, g, d, o, k, g, c, f, j, n, m, f, e, j, i, n, a, b, m, i, e, a, i, f, b, c.

(b) {3 marks} Prove or disprove the following statement (if false, provide a counterexample):

Let G be a graph that has an Eulerian circuit, and let e and f be edges of G that are incident to a common vertex v. Prove that G has an Eulerian circuit in which edge e is immediately followed by edge f.

Consider G has ~~two~~ closed paths. which their common vertex is v.

Path 1 is $v_0 e_1 \cdots e v f e_{k+1} \cdots v_n = v_0$

Path 2 is $v_0' e_1' \cdots e' v f' e_{k+1}' \cdots v_n' = v_0'$

The only way for Eulerian circuit is that $v_0 e_1 \cdots e v f e_{k+1}' \cdots v_n' \cdots e' v f' \cdots \cdots v_n = v_0$

If edge e is immediately followed by edge f, it couldn't pass all the vertices. Contradiction.

Thus, G doesn't has an Eulerian circuit in which edge e is immediately followed by edge f.