1. {8 marks} Let G be a 4-regular connected planar graph with a planar embedding where each face has degree either 3 or 4.

   (a) {5 marks} Determine the exact number of faces of degree 3.

   Solution. Suppose G has n vertices, m edges, s_3 faces of degree 3 and s_4 faces of degree 4. From the handshaking lemma, we get $2m = 4n$, which implies that $n = \frac{1}{2}m$. From the handshaking lemma for faces, we get $2m = 3s_3 + 4s_4$. From Euler's formula, we get $n - m + s_3 + s_4 = 2$, or $s_3 + s_4 = 2 - n + m = 2 - \frac{1}{2}m + m = 2 + \frac{1}{2}m$. This implies that $4s_3 + 4s_4 = 8 + 2m$. So then the equation from the handshaking lemma for faces become

   $$2m = 3s_3 + 4s_4 = (4s_3 + 4s_4) - s_3 = 8 + 2m - s_3.$$

   This gives $s_3 = 8$, so there are 8 faces of degree 3.

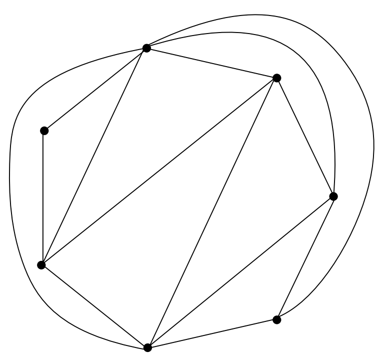
   (b) {3 marks} Suppose in addition, every edge has a face of degree 3 on one side, and a face of degree 4 on the other side. Determine the number of vertices, edges, and faces of degree 4 in G.

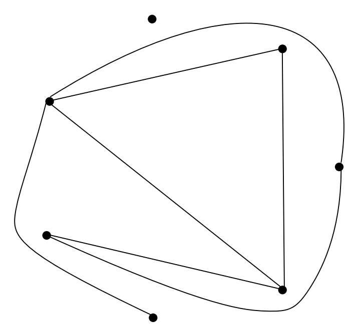
   Solution. Let F_3, F_4 be the sets of all faces of degree 3, 4, respectively. Consider the two sums $\sum_{f \in F_3} \deg(f) = 3f_3$ and $\sum_{f \in F_4} \deg(f) = 4f_4$. The criteria given in this question means that each edge contributes 1 to each of these sums. Therefore, $3f_3 = 4f_4$. Since $f_3 = 8$ from part (a), $f_4 = 6$. This means that $m = \frac{3}{2}s_3 + 2s_4 = 24$, and $n = \frac{1}{2}m = 12$.

2. {7 marks}Let G be a graph. The complement of G, denoted $\overline{G}$, is the graph where $V(\overline{G}) = V(G)$, and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. (In other words, non-adjacent pairs of vertices in G become edges in $\overline{G}$, and vice versa.)

   (a) {2 marks} Find a graph G on 7 vertices such that both G and $\overline{G}$ are planar. Draw planar embeddings of both your G and $\overline{G}$.

   Solution. This is one example.





   (b) {5 marks} Prove that if G has at least 11 vertices, then at least one of G and $\overline{G}$ must be nonplanar.

   Solution. Suppose G has $n \geq 11$ vertices. Suppose by way of contradiction that both G and $\overline{G}$ are planar. Then $|E(G)| \leq 3n - 6$ and $|E(\overline{G})| \leq 3n - 6$. This means that $|E(G)| + |E(\overline{G})| \leq 6n - 12$. Since there are $\binom{n}{2}$ edges in K_n, we have $|E(G)| + |E(\overline{G})| = \binom{n}{2} = \frac{1}{2}n(n-1)$. This implies that $\frac{1}{2}(n^2 - n) \leq 6n - 12$. This simplifies to $n^2 - 13n + 24 \leq 0$. Using the quadratic formula, we can factor this to $\left(n - \frac{13+\sqrt{73}}{2}\right)\left(n - \frac{13-\sqrt{73}}{2}\right) \leq 0$. Note that $\frac{13+\sqrt{73}}{2} \sim 10.7 < 11$. Since $n \geq 11$, both binomials in the inequality are positive, which contradicts the fact that the product is non-positive.

3. {5 marks} Prove that any planar embedding of a simple connected planar graph contains a vertex of degree at most 3 or a face of degree at most 3.

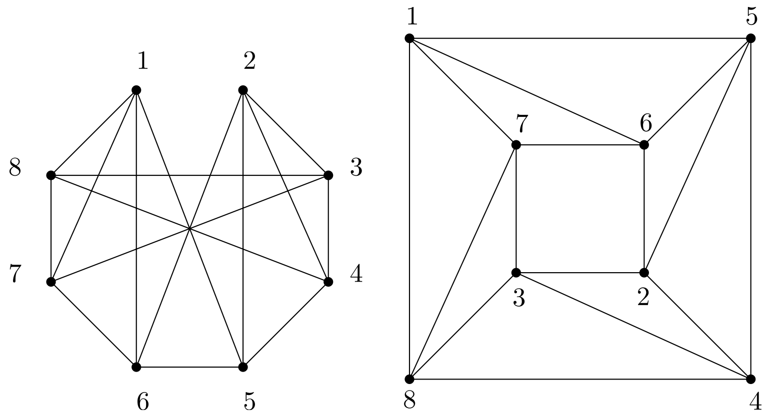
   Solution. Suppose by way of contradiction that there is a planar embedding of G where every vertex has degree at least 4 and every face has degree at least 4. Suppose G has n vertices, m edges and s faces. By the handshaking lemma, $2m \geq 4n$, so $n \leq m/2$. By the handshaking lemma for faces, $2m \geq 4s$, so $s \leq m/2$. By Euler's formula,

   $$2 = n - m + s \leq m/2 - m + m/2 = 0.$$

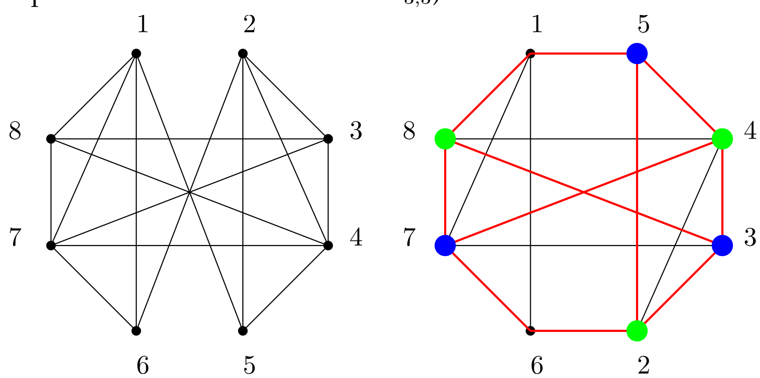
   This is a contradiction.

4. {9 marks} For each of the following graphs, determine whether it is planar or not. Prove your assertions.

(a) This graph is planar. We draw a planar embedding here.



(b) This graph is not planar. It contains an edge-subdivision of $K_{3,3}$ shown below (green vertices and blue vertices represent the 6 main vertices in $K_{3,3}$).



(c) This graph is planar. We draw a planar embedding here.