1. {6 marks} For a positive integer k, let S be the set of all subsets of $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$ with k elements, where the weight w of a set is its largest element.

(a) {3 marks} Determine the coefficient of x^n in the generating series $\Phi_S(x)$ with respect to w.

(b) {3 marks} Prove that $\Phi_S(x) = \frac{x^{k-1}}{(1-x)^k}$.

a) the coefficient of x^n in $\phi S(x)$ is the number of k-subsets of N_0 whose largest element in n. The the subsets must contains $\{n\}$ and rest $k-1$ elements are from $\{1, \cdots n-1\}$. Thus, the coefficient is $\binom{n}{k-1}$

b) From a)

$[x^n]\phi_S(x) = \binom{n}{k-1}$

$[x^n]\frac{x^{k-1}}{(1-x)^k} = [x^{n-(k-1)}]\frac{1}{(1-x)^k} = \binom{n-k+1+k-1}{k-1} = \binom{n}{k-1} = [x^n]\phi_S(x)$.

Thus, we can conclude that $\phi_S(x) = \frac{x^{k-1}}{(1-x)^k}$

2. {7 marks} Let $n \in \mathbb{N}$. Define $\mathcal{E}_n$ to be the set of all subsets of $\{1, \ldots, n\}$ of even cardinality, and define $\mathcal{O}_n$ to be the set of all subsets of $\{1, \ldots, n\}$ of odd cardinality.

(a) {5 marks} Define a bijection $f_n : \mathcal{E}_n \to \mathcal{O}_n$. Prove that for any $X \in \mathcal{E}_n$, $f_n(X) \in \mathcal{O}_n$. Provide the inverse of f_n.

One mapping: $f_n : \mathcal{E}_n \to \mathcal{O}_n$ where for any $X \in \mathcal{E}_n$.

$$f_n(X) = \begin{cases} X \setminus \{1\} & \text{when } 1 \in X \\ X \cup \{1\} & \text{when } 1 \notin X. \end{cases}$$

The inverse: $f_n^{-1} : \mathcal{O}_n \to \mathcal{E}_n$ where for any $Y \in \mathcal{O}_n$.

$$f_n^{-1}(Y) = \begin{cases} Y \setminus \{1\} & \text{when } 1 \in Y \\ Y \cup \{1\} & \text{when } 1 \notin Y. \end{cases}$$

(b) {2 marks} Illustrate your bijection by pairing up each element X of $\mathcal{E}_4$ with its image $f_4(X)$ of $\mathcal{O}_4$.

$\mathcal{E}_4$: $\emptyset$ $\{1,2\}$ $\{2,3\}$ $\{3,4\}$ $\{1,3\}$ $\{2,4\}$ $\{1,4\}$ $\{1,2,3,4\}$

$\mathcal{O}_4$: $\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$ $\{1,2,3\}$ $\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$

3. {6 marks} For each of the following, determine the generating series of the set with respect to the weight function. Simplify your expression.

(a) {3 marks} Set: $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Weight function: $w(a) = \begin{cases} a & a \equiv 0 \pmod 3 \\ a+1 & a \equiv 1 \pmod 3 \\ 3a & a \equiv 2 \pmod 3 \end{cases}$.

Let $A = \{3k \mid k \in \mathbb{N}_0\}$, $B = \{3k+1 \mid k \in \mathbb{N}_0\}$, $C = \{3k+2 \mid k \in \mathbb{N}_0\}$

$\phi_A(x) = \sum_{k \geq 0} x^{w(3k)} = \sum_{k \geq 0} x^{3k} = \frac{1}{1-x^3}$

$\phi_B(x) = \sum_{k \geq 0} x^{w(3k+1)} = \sum_{k \geq 0} x^{3k+2} = x^2 \sum_{k \geq 0} x^{3k} = \frac{x^2}{1-x^3}$

$\phi_C(x) = \sum_{k \geq 0} x^{w(3k+2)} = \sum_{k \geq 0} x^{9k+6} = x^6 \sum_{k \geq 0} x^{9k} = \frac{x^6}{1-x^9}$

$$\phi_{N_0}(x) = \phi_A(x) + \phi_B(x) + \phi_C(x)$$

$$= \frac{x^2}{1-x^3} + \frac{1}{1-x^3} + \frac{x^6}{1-x^9}$$

$$= \frac{x^2(1-x^9) + 1 - x^9 + x^6(1-x^3)}{(1-x^3)(1-x^9)}$$

$$= \frac{1 + x^2 + x^6 - 2x^9 - x^{11}}{1 + x^{12} - x^3 - x^9}$$

(b) {3 marks} Set: $S = \{1, 2\} \times \{1, \ldots, 314\} \times \mathbb{N}_0$. Weight function: $w(a, b, c) = a + 3b + 2c$.

$\alpha(a) = a$ for $\{1,2\}$.

$\phi_{\{1,2\}}(x) = x + x^2$

$\beta(b) = 3b$ for $\{1, \cdots, 314\}$.

$\phi_{\{1, \cdots, 314\}}(x) = x^3 + x^6 + x^9 + \cdots x^{3 \times 314}$.

$r(c) = 2c$ for N_0

$\phi_{\{N_0\}}(x) = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$

Since $w(a,b,c) = \alpha(a) + \beta(b) + r(c)$.

$$\phi_{\{1,2\} \times \{1, \cdots 314\} \times N_0}(x) = (x + x^2) \cdot \frac{1}{1-x^2} \cdot \sum_{i \geq 0}^{314} x^{3i}$$

$$= \frac{x+x^2}{1-x^2} \cdot \frac{x^3(1-x^{942})}{1-x^3}$$

$$= \frac{x^4}{1-x} \cdot \frac{(1-x^{942})}{1-x^3}$$

4. {5 marks}

(a) {3 marks} Determine the following coefficient.

$$[x^{15}]\frac{x^2}{(1-5x)^{10}}.$$

(b) {2 marks} Let $A(x) = \frac{x}{(1-x)^3}$ and $B(x) = \frac{1}{1-2x}$. Determine whether or not $A(B(x))$ is a power series, and explain why.

a) $[x^{15}]\frac{x^2}{(1-5x)^{10}} = [x^{15-2}](1-5x)^{-10}$

$= [x^{13}](1-5x)^{-10}$

$= [x^{13}]\sum_{i\geq 0}\binom{i+10-1}{10-1}(5x)^i$

$= \binom{13+10-1}{10-1}\times 5^{13}$

$= \binom{22}{9}\times 5^{13}$

b). since $B = \frac{1}{1-2x}$

$= 1+2x+(2x)^2+(2x)^3+\cdots$

$b_0=1$, then $A(B(x))$ is not a power series unless $A(x)$ is finite.

$A(x) = \frac{x}{(1-x)^3} = x\cdot\sum_{n\geq 0}\binom{n+3-1}{3-1}\cdot x^n.$

$= x\sum_{n\geq 0}\binom{n+2}{2}x^n$

Thus $a_0=0$, then $A(x)$ is infinitely.

Thus, by theorem, $A(B(x))$ is not a power series.

5. {6 marks} Let $n \geq 0$. How many compositions of n consist of exactly 4 parts where each part is congruent to 2 modulo 3? You need to define a relevant set, a weight function, determine a generating series, and then find an explicit formula for the answer.

Let $A = \{2, 5, 8, \cdots\}$ be the set of all positive integers congruent to 2 modulo 3.

$$\Phi_A(x) = x^2 + x^5 + x^8 + \cdots$$
$$= x^2(1 + x^3 + x^6 + \cdots)$$
$$= x^2 \cdot \frac{1}{1-x^3}$$
$$= \frac{x^2}{1-x^3}$$

We have 4 parts.

$$\Phi_{A^4}(x) = \left(\frac{x^2}{1-x^3}\right)^4$$

$$[x^n]\left(\frac{x^2}{1-x^3}\right)^4 = [x^{n-8}]\frac{1}{(1-x^3)^4}$$
$$= [x^{n-8}]\sum_{i \geq 0}\binom{i+4-1}{4-1} \cdot x^{3i}$$
$$= [x^{n-8}]\sum_{i \geq 0}\binom{i+3}{3} \cdot x^{3i}$$

when $3i = n-8$

$i = \frac{n-8}{3}$ when $n-8$ is divisible by 3.

otherwise, coefficient is 0.

$$[x^n]\frac{x^8}{(1-x^3)^4} = \begin{cases} \binom{\frac{n-8}{3}+3}{3} & \text{when } 3 \mid n-8,\ n \geq 8. \\ 0 & \text{otherwise} \end{cases}$$