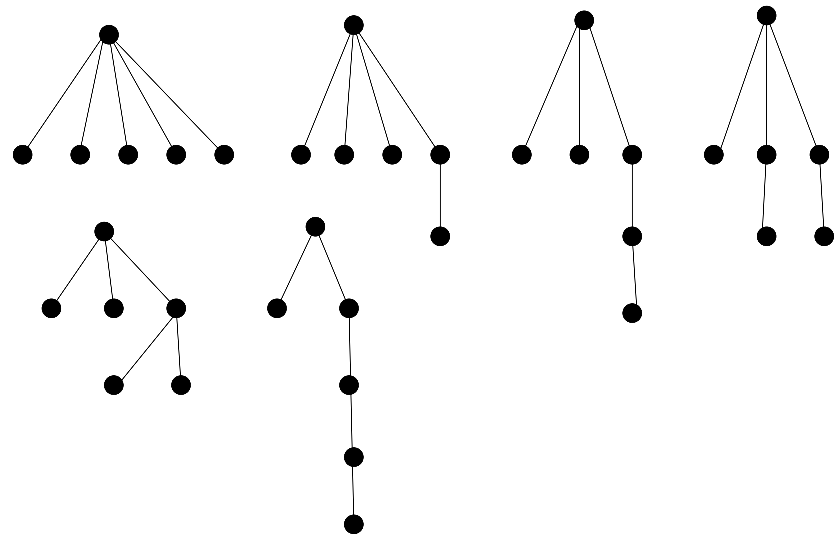
Math 239 Winter 2017 Assignment 3 Solutions

1. {8 marks}

   (a) {3 marks} Draw all trees on six vertices, up to isomorphism.

   Solution.



   (b) {3 marks} Find the smallest possible number n of vertices in a tree that has four vertices of degree 3, three vertices of degree 5 and two vertices of degree 7. Prove that your answer is correct.

   Solution. Let T be a tree with the given properties. Let n_i denote the number of vertices of T of degree i, so in particular n_1 is the number of leaves in T. As proved in class (see Math239 Notes, Alternate Proof of Theorem 5.1.4) we know

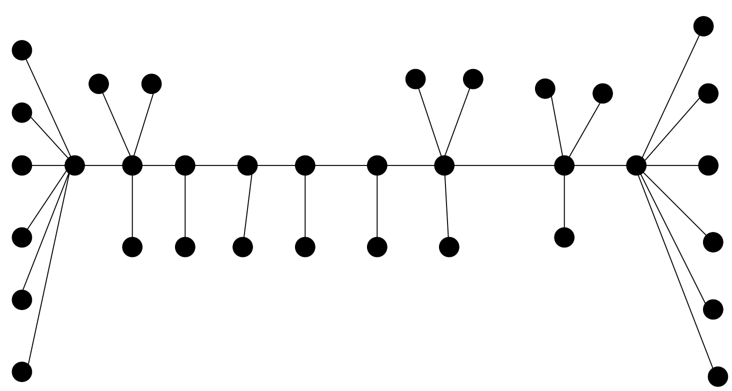
$$n_1 = 2 + \sum_{i \geq 3} (i-2) n_i.$$

   Since $n_3 = 4$, $n_5 = 3$ and $n_7 = 2$, we find that $n_1 \geq 2 + (1)4 + (3)3 + (5)2 = 25$. Therefore in total the number n of vertices of T is at least

$$n_1 + n_3 + n_5 + n_7 \geq 25 + 4 + 3 + 2 = 34.$$

   (c) {2 marks} Draw an example of a tree with four vertices of degree 3, three vertices of degree 5 and two vertices of degree 7, that has exactly n vertices (where n is as in the previous part).

   Solution.



2. {5 marks} Let $p \geq 2$ be given. Suppose $d_1, d_2, \ldots, d_p$ is a sequence of p positive integers such that $\sum_{i=1}^{p} d_i = 2p - 2$. Prove that there exists a tree with p vertices whose degrees are $d_1, d_2, \ldots, d_p$. (Hint: use induction on p.)

Solution. As in the hint, we prove the statement by induction on p.

Base case: $p = 2$. The only possible sequence of 2 positive integers summing to $2p - 2 = 2$ is $d_1 = d_2 = 1$. The tree consisting of two vertices and one edge has degree sequence $(1, 1)$ which satisfies the conditions.

Induction hypothesis: Assume that $p \geq 3$ and every sequence $c_1, c_2, \ldots, c_{p-1}$ of $p - 1$ positive integers satisfying $\sum_{i=1}^{p-1} c_i = 2(p-1) - 2$ is the degree sequence of a tree. Let a sequence $d_1, d_2, \ldots, d_p$ of p positive integers satisfying $\sum_{i=1}^{p} d_i = 2p - 2$ be given. Since $\sum_{i=1}^{p} 2 = 2p > 2p - 2$ and each d_i is positive, we see that some $d_i = 1$. By renaming the d_i if necessary we may assume $d_p = 1$. Also, since $\sum_{i=1}^{p} 1 = p < 2p - 2$ for $p \geq 3$, we see that some $d_i > 1$. By renaming the d_i if necessary we may assume $d_{p-1} > 1$.

Now create a new sequence $c_1, c_2, \ldots, c_{p-1}$ of $p - 1$ positive integers by setting

- $c_i = d_i$ for $1 \leq i \leq p - 2$,
- $c_{p-1} = d_{p-1} - 1$.

Then $\sum_{i=1}^{p-1} c_i = 2p - 4$, so by the Induction Hypothesis there exists a tree T with degree sequence $c_1, c_2, \ldots, c_{p-1}$. Construct the tree T' by adding one new vertex v and joining it by an edge to the vertex w of T that has degree c_{p-1}. Then by definition of the c_i we see that T has degree sequence $d_1, d_2, \ldots, d_p$ as required.

Hence by induction the statement is true for all p.

3. {5 marks} Let G be a connected graph and let H be a subgraph of G that does not contain a cycle.

   (a) {3 marks} Suppose J is a subgraph of H and e is an edge of H that is not in $E(J)$. Prove that if T is a spanning tree of G that contains all the edges in $E(J)$, then there exists a spanning tree T' of G that contains $E(J) \cup \{e\}$.

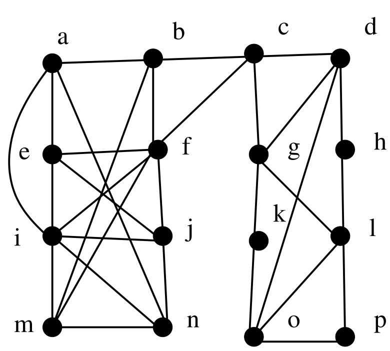
   Solution. Let T be a spanning tree of G that contains all the edges in $E(J)$. If T also contains e then T itself has the desired property, so we may assume that e is not an edge of T. As proved in class (Theorem 5.2.3 in Math239 Notes), we know that the graph $T + e$ formed by adding the edge e to T has exactly one cycle C. Since H does not contain any cycles, there exists an edge $f \neq e$ that is in $E(C)$ but not in $E(H)$ (and therefore not in $E(J)$). By the same theorem (second part), we know that $T' = T + e - f$ is also a spanning tree of G, which contains $E(J) \cup \{e\}$ by construction.

   (b) {2 marks} Use the previous part to prove that G has a spanning tree that contains all the edges in $E(H)$.

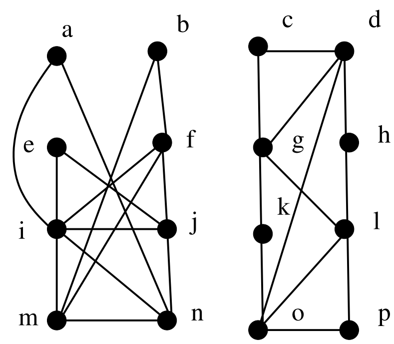
   Solution. Let m be the largest possible number of edges of H that can be contained in a spanning tree of G. We claim that $m = |E(H)|$ and therefore there exists a spanning tree of G that contains H.
   Suppose the claim is false. Let T be a spanning tree that contains a subgraph J of H where $|E(J)| = m$. Let e be an edge of H that is not an edge of J. Then by the previous part, there exists a spanning tree T' of G that contains $E(J) \cup \{e\}$, which contradicts the definition of m. Therefore the claim is true.

4. {6 marks}

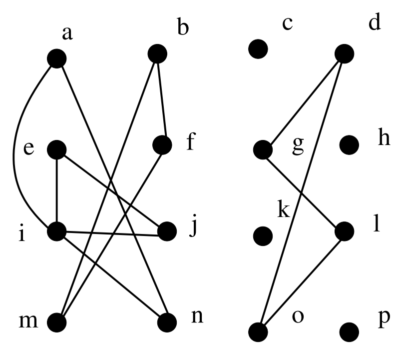
   (a) {3 marks} Find an Eulerian circuit in the graph shown. Make a list of the vertices in the order in which they appear on your circuit (note vertices may appear several times).



Solution. Following the algorithm given in class, we can remove cycle $abcfea$ to get the following graph.



Then we can remove cycle $fimnjf$ from the left component and $cgkoplhdc$ from the right component to get the following.



Now we can see the three remaining components have Eulerian circuits $ianieji$, $mbfm$ and $dglod$. Inserting these into the cycles found for the previous picture gives Eulerian circuits $fianiejimbfmnjf$ for the left component and $cgkoplhdglodc$ for the right component. Finally inserting these into the cycle $abcfea$ found at the beginning gives an Eulerian circuit $abcgkoplhdglodcfianiejimbfmnjfea$ as required.

(b) {3 marks} Prove or disprove the following statement.

Let G be a graph that has an Eulerian circuit, and let e and f be edges of G that are incident to a common vertex v. Then G has an Eulerian circuit in which edge e is immediately followed by edge f.

Solution. The statement is not true in general. For example, consider the even graph G consisting of two cycles of length 3, $vwxv$ and $vyzv$, that share exactly one vertex v. (Then v has degree 4 and all other vertices have degree 2, and G is connected.) Let $e = vw$ and $f = vx$. If an Eulerian circuit C contains f immediately after e then C must start at x or w, and cannot reach the edges vy, yz or zv without re-using e or f. So no such Eulerian circuit exists.