1. {8 marks} Let G be a 4-regular connected planar graph with a planar embedding where each face has degree either 3 or 4.

(a) {5 marks} Determine the exact number of faces of degree 3.

Theorem: $\sum_{i=1}^{s} \deg(f_i) = 2|E(G)|$ Handshaking lemma for faces.

$2|E(G)| = \sum_{v_i \in V(G)} \deg(v_i)$

$|V(G)| - |E(G)| + f = 2$ Euler's Formula.

Let x be the number of faces of degree 3.

① $4|V(G)| = 2|E(G)|$

② $2|E(G)| = 3 \cdot x + (f-x) \cdot 4$

③ $|V(G)| - |E(G)| + f = 2$.

From ①.②.③ $x = 8$.

The exact number of faces of degree 3 is 8.

(b) {3 marks} Suppose in addition, every edge has a face of degree 3 on one side, and a face of degree 4 on the other side. Determine the number of vertices, edges, and faces of degree 4 in G.

From (a), we only have 8 faces of degree 3,

by question $|E(G)| = 8 \times 3$ Since each edge has a face of degree 3 on one side. and a face of degree on other side.

$|E(G)| = 24$ is the sum of the degree of a face of degree 4.

$f_4 = \frac{24}{4} = 6$, $e_4 = 14$. $f = 6 + 8 = 14$

$v - e + f = 2$

$v = 12$.

Thus, $|E(G)| = 24$

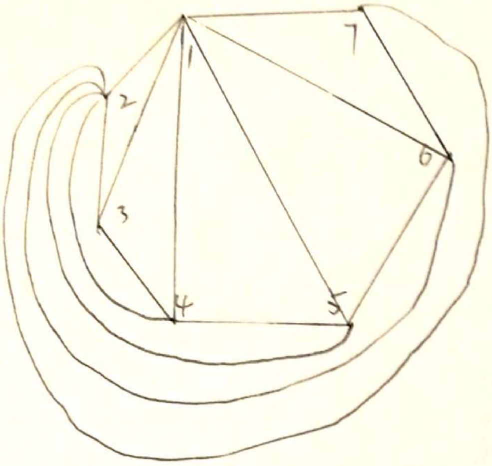
$|V(G)| = 12$

$f_4 = 6$

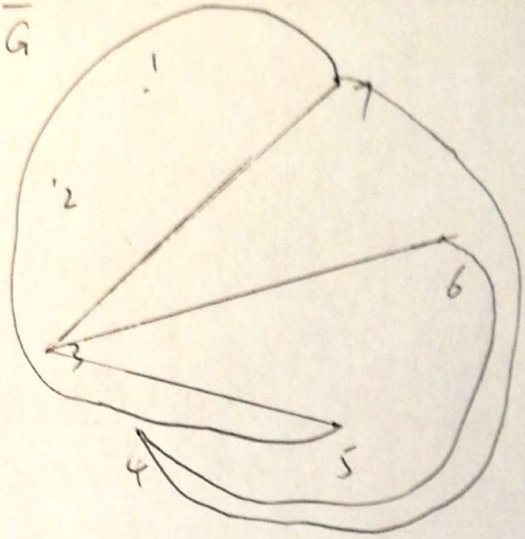
2. {7 marks}Let G be a graph. The complement of G, denoted $\overline{G}$, is the graph where $V(\overline{G}) = V(G)$, and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. (In other words, non-adjacent pairs of vertices in G become edges in $\overline{G}$, and vice versa.)

(a) {2 marks} Find a graph G on 7 vertices such that both G and $\overline{G}$ are planar. Draw planar embeddings of both your G and $\overline{G}$.

G



$\overline{G}$



(b) {5 marks} Prove that if G has at least 11 vertices, then at least one of G and $\overline{G}$ must be nonplanar.

Case 1: $|V(G)| > 11$, There must exist a vertex whose degree is greater than or equal to 6. By theorem, A planar graph has a vertex of degree at most five. Thus, G or $\overline{G}$ must be nonplanar. when $|V(G)| > 11$.

Case 2: $|V(G)| = 11$, the edges of K_{11} is $\binom{11}{2} = 55$.

In a planar graph.

$q \le 3p - 6$. (q: edges. p: vertices)

$q \le 27$.

but the edges of K_{11} is 55, so the edges of G or $\overline{G}$ is greater than 27.

Thus, if G has at least 11 vertices, then at least one of G or $\overline{G}$ must be nonplanar.

3. {5 marks} Prove that any planar embedding of a simple connected planar graph contains a vertex of degree at most 3 or a face of degree at most 3.

Suppose simple connected planar graph contains every vertex of degree at least 4 and every face of degree at least 4.

$2e = dv \geq 4v$ by Handshake lemma.

$2e = d^*f \geq 4f$ by Faceshake lemma.

Thus, $4e \geq 4v + 4f$

$e \geq v + f$

$0 \geq v - e + f$

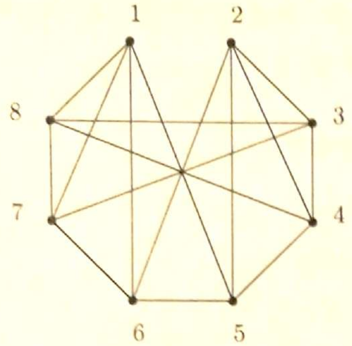
But by Euler's Formula.

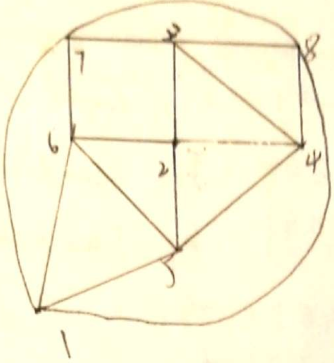
$v - e + f \geq 2$.

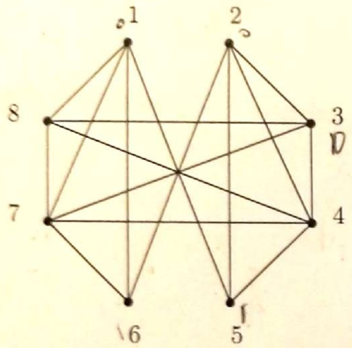
Contradiction.

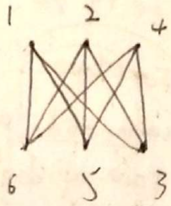
Thus, a simple connected planar graph contains a vertex of degree at most 3 or a face of degree at most 3.

4. {9 marks} For each of the following graphs, determine whether it is planar or not. Prove your assertions.

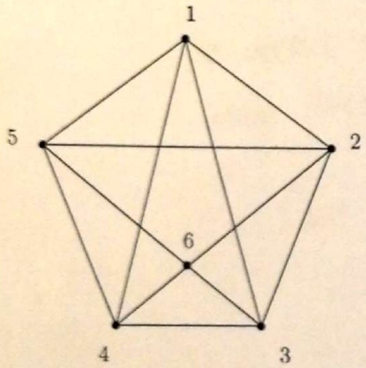


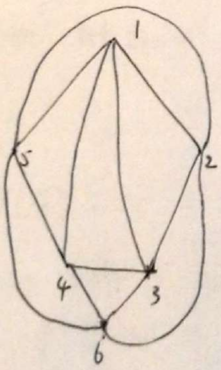






we find $K_{3,3}$ as a subgraph of G
by theorem, it's not planar





It's planar