Math 239 Winter 2017 Assignment 9 Solutions

1. {6 marks} Let a_n denote the number of compositions of n in which each part is an even number greater than 5. (Note the number of parts is not fixed.) Find the generating series $\sum_{n\geq 0} a_n x^n$, expressed as a *rational* expression (a quotient of two polynomials). You must define a suitable set S and weight function w on S, and indicate wherever you use results proved in class.

   Solution. Let $A = \{6, 8, 10, \ldots\}$ denote the set of all even integers greater than 5. Then

   $$S = \bigcup_{k\geq 0} A^k$$

   where as usual A^k denotes the Cartesian product of k copies of the set A. We define the weight function w on S by setting $w(t_1, \ldots, t_k) = t_1 + \ldots + t_k$.

   To find the generating series for S, we first note that the generating series for A with respect to the weight function $w_0(\sigma) = \sigma$ is

   $$\Phi_A(x) = x^6 + x^8 + x^{10} + \ldots = x^6(1 + x^2 + x^4 + \ldots) = \frac{x^6}{1-x^2}.$$

   Therefore by the Product Lemma (which is valid because the weight function w defined on S is the sum $w(t_1, \ldots, t_k) = w_0(t_1) + \ldots + w_0(t_k)$)

   $$\Phi_{A^k}(x) = \Phi_A(x)^k = \frac{x^{6k}}{(1-x^2)^k}.$$

   Now applying the Sum Lemma and noting that $\frac{x^6}{1-x^2}$ has constant coefficient zero we obtain

   $$\Phi_S(x) = \sum_{k\geq 0} \frac{x^{6k}}{(1-x^2)^k} = (1 - \frac{x^6}{1-x^2})^{-1}.$$

   Simplifying this expression we get

   $$\Phi_S(x) = \frac{1-x^2}{1-x^2-x^6}.$$

2. {6 marks} The generating series for compositions of n in which each part is at least 3 is

   $$\frac{1-x}{1-x-x^3}.$$

   (a) {4 marks} Find a recurrence relation for the sequence $\{a_n\}_{n\geq 0}$, together with initial conditions that uniquely specify the sequence, where a_n is the number of compositions of n in which each part is at least 3.

   Solution. The generating series for this set is

   $$\sum_{n\geq 0} a_n x^n = \frac{1-x}{1-x-x^3}.$$

   Multiplying through by the denominator and shifting summation indices gives

   $$\sum_{n\geq 0} a_n x^n - \sum_{n\geq 0} a_n x^{n+1} - \sum_{n\geq 0} a_n x^{n+3} = 1 - x$$
   $$\sum_{n\geq 0} a_n x^n - \sum_{n\geq 1} a_{n-1} x^n - \sum_{n\geq 3} a_{n-3} x^n = 1 - x.$$

   Comparing coefficients on both sides for x^0, x^1, x^2 and x^m for all $m \geq 3$ gives

   $$a_0 = 1$$

   $$a_1 - a_0 = -1$$

$$a_2 - a_1 = 0$$

$$a_m - a_{m-1} - a_{m-3} = 0.$$

We therefore obtain the recurrence relation $a_m = a_{m-1} + a_{m-3}$ for all $m \geq 3$, with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 0$.

(b) {2 marks} Find the number of compositions of 10 in which each part is at least 3.

Solution. Using the recurrence found in the previous part we get the following values: $a_3 = a_2 + a_0 = 1$, $a_4 = a_3 + a_1 = 1$, $a_5 = a_4 + a_2 = 1$, $a_6 = a_5 + a_3 = 2$, $a_7 = a_6 + a_4 = 3$, $a_8 = a_7 + a_5 = 4$, $a_9 = a_8 + a_6 = 6$, $a_{10} = a_9 + a_7 = 9$.
So there are 9 such decompositions of 10.

3. {6 marks} Let $A = \{00, 101, 110, 0001\}$ and $B = \{0110, 10, 111\}$.

(a) {4 marks} Determine whether AB is unambiguous and whether BA is unambiguous. Prove your conclusion in each case.

Solution. The concatenation set AB is ambiguous, since the string 000110 can be obtained in two different ways, namely (00)(0110) and (0001)(10).
The set BA is unambiguous. To see this, suppose that $ba = b'a'$ for some $a, a' \in A$ and $b, b' \in B$. Then in particular the first two digits of b must form the same substring as the first two digits in b'. But the three elements of B begin with the substrings $01, 10, 11$ which are all distinct, implying that $b = b'$. This in turn implies $a = a'$. This shows that BA is unambiguous.

(b) {2 marks} Find the generating series with respect to length for each of AB and BA.

Solution. We can write AB explicitly from the definition of concatenation:
$AB = \{000110, 1010110, 1100110, 00010110, 0010, 10110, 11010, 00111, 101111, 110111, 0001111\}$
Note that AB contains eleven binary strings: one of length 8, three of length 7, three of length 6, three of length 5, and one of length 4. Hence the generating series is

$$\Phi_{AB}(x) = x^4 + 3x^5 + 3x^6 + 3x^7 + x^8.$$

Since BA is unambiguous we can use the Product Lemma to conclude

$$\Phi_{BA}(x) = \Phi_B(x)\Phi_A(x) = (x^2 + x^3 + x^4)(x^2 + 2x^3 + x^4) = x^4 + 3x^5 + 4x^6 + 3x^7 + x^8.$$

4. {6 marks} Find the generating series (with respect to length) for the set of all binary strings in which each block of 1′s (i.e. each maximal substring consisting entirely of 1′s) has length divisible by 3. Write your answer as a rational expression. Your solution should include an unambiguous decomposition for S and a justification for why it is unambiguous. Indicate wherever you use results proved in class.

Solution. Here we use the 0-decomposition theorem from class:

$$\{0,1\}^* = \{1\}^*(\{0\}\{1\}^*)^*$$

and the RHS is an unambiguous expression. Modifying this so that every substring of 1′s has length divisible by 3 we get

$$S = \{111\}^*(\{0\}\{111\}^*)^*,$$

which is unambiguous because it is a subset of the 0-decomposition which we proved is unambiguous. Therefore we may apply the Product Lemma and "Star" Lemma.
The generating series with respect to length for $\{111\}$ is

$$\Phi_{\{111\}}(x) = x^3$$

and for $\{0\}$ is

$$\Phi_{\{0\}}(x) = x.$$

Thus by the Product Lemma and "Star" Lemma $\Phi_{\{0\}\{111\}^*}(x) = \frac{x}{1-x^3}$. Using the Product and Star Lemmas again and noting that both x^3 and $\frac{x}{1-x^3}$ have constant coefficient zero gives

$$\Phi_S(x) = \frac{1}{1-x^3}\frac{1}{1-\frac{x}{1-x^3}}.$$

Simplifying we obtain

$$\Phi_S(x) = \frac{1}{1-x-x^3}.$$

(Alternatively we could modify the Block Decomposition

$$\{0,1\}^* = \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$$

to obtain

$$S = \{111\}^*(\{0\}\{0\}^*\{111\}\{111\}^*)^*\{0\}^*.$$

Proceeding with this unambiguous expression gives the same generating series for S.)