Math 239 Winter 2017 Assignment 8 Solutions

1. {6 marks} For a positive integer k, let S be the set of all subsets of $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ with k elements, where the weight w of a set is its largest element.

   (a) {3 marks} Determine the coefficient of x^n in the generating series $\Phi_S(x)$ with respect to w.

   Solution. The coefficient of x^n in $\Phi_S(x)$ is the number of k-subsets of $\mathbb{N}_0$ whose largest element is n (the weight of the set). Such a subset must consist of $\{n\}$ union with a $(k-1)$-subset of $\{0, 1, \ldots, n-1\}$. Therefore, this coefficient is $\binom{n}{k-1}$.

   (b) {3 marks} Prove that $\Phi_S(x) = \dfrac{x^{k-1}}{(1-x)^k}$.

   Solution. Part (a) tells us that $[x^n]\Phi_S(x) = \binom{n}{k-1}$.
   For the right hand side, we use the binomial theorem:

$$[x^n]\frac{x^{k-1}}{(1-x)^k} = [x^{n-(k-1)}]\frac{1}{(1-x)^k} = [x^{n-k+1}]\frac{1}{(1-x)^k} = \binom{(n-k+1)+k-1}{k-1} = \binom{n}{k-1} = [x^n]\Phi_S(x).$$

   Since $[x^n]\Phi_S(x) = [x^n]\frac{x^{k-1}}{(1-x)^k}$ for each n, we can conclude that $\Phi_S(x) = \frac{x^{k-1}}{(1-x)^k}$.

2. {7 marks} Let $n \in \mathbb{N}$. Define $\mathcal{E}_n$ to be the set of all subsets of $\{1, \ldots, n\}$ of even cardinality, and define $\mathcal{O}_n$ to be the set of all subsets of $\{1, \ldots, n\}$ of odd cardinality.

   (a) {5 marks} Define a bijection $f_n : \mathcal{E}_n \to \mathcal{O}_n$. Prove that for any $X \in \mathcal{E}_n$, $f_n(X) \in \mathcal{O}_n$. Provide the inverse of f_n.

   Solution. One mapping is $f_n : \mathcal{E}_n \to \mathcal{O}_n$ where for any $X \in \mathcal{E}_n$,

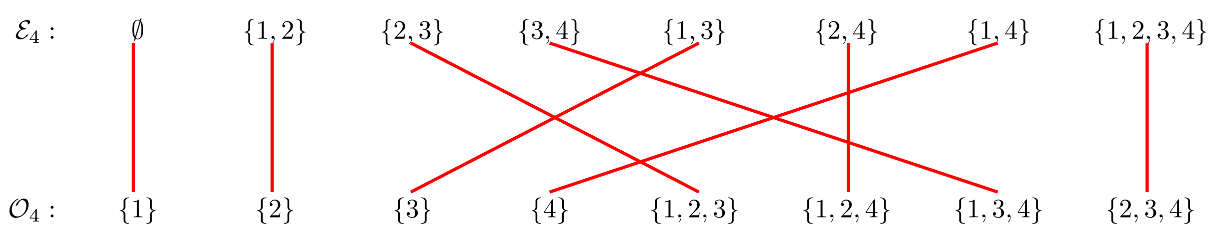
$$f_n(X) = \begin{cases} X \setminus \{1\} & \text{when } 1 \in X \\ X \cup \{1\} & \text{when } 1 \notin X \end{cases}$$

   We note that X has even size, and $f_n(X)$ either removes 1 element from X or adds 1 element to X. So $f_n(X)$ must have odd size, hence $f_n(X) \in \mathcal{O}_n$.
   The inverse function is $f_n^{-1} : \mathcal{O}_n \to \mathcal{E}_n$ where for any $Y \in \mathcal{O}_n$,

$$f_n^{-1}(Y) = \begin{cases} Y \setminus \{1\} & \text{when } 1 \in Y \\ Y \cup \{1\} & \text{when } 1 \notin Y \end{cases}$$

   (b) {2 marks} Illustrate your bijection by pairing up each element X of $\mathcal{E}_4$ with its image $f_4(X)$ of $\mathcal{O}_4$.

   Solution.



3. {6 marks} For each of the following, determine the generating series of the set with respect to the weight function. Simplify your expression.

   (a) {3 marks} Set: $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Weight function: $w(a) = \begin{cases} a & a \equiv 0 \pmod 3 \\ a+1 & a \equiv 1 \pmod 3 \\ 3a & a \equiv 2 \pmod 3 \end{cases}$.

   Solution. We partition $\mathbb{N}_0$ into three sets: A, B, C where

$$A = \{3k \mid k \in \mathbb{N}_0\}, B = \{3k+1 \mid k \in \mathbb{N}_0\}, C = \{3k+2 \mid k \in \mathbb{N}_0\}.$$

Using the given weight function,

$$\Phi_A(x) = \sum_{k\geq 0} x^{w(3k)} = \sum_{k\geq 0} x^{3k} = \frac{1}{1-x^3}$$

$$\Phi_B(x) = \sum_{k\geq 0} x^{w(3k+1)} = \sum_{k\geq 0} x^{3k+2} = x^2 \sum_{k\geq 0} x^{3k} = \frac{x^2}{1-x^3}$$

$$\Phi_C(x) = \sum_{k\geq 0} x^{w(3k+2)} = \sum_{k\geq 0} x^{9k+6} = x^6 \sum_{k\geq 0} x^{9k} = \frac{x^6}{1-x^9}$$

Since $\mathbb{N}_0 = A \cup B \cup C$ and these are disjoint sets, by sum lemma,

$$\Phi_{\mathbb{N}_0}(x) = \Phi_A(x) + \Phi_B(x) + \Phi_C(x) = \frac{1}{1-x^3} + \frac{x^2}{1-x^3} + \frac{x^6}{1-x^9} = \frac{(1+x^2)(1-x^9) + x^6(1-x^3)}{(1-x^3)(1-x^9)}.$$

(This can be simplified further to $\frac{1+x^2+x^6-x^{11}}{1-x^3-x^9+x^{12}}$.)

(b) {3 marks} Set: $S = \{1,2\} \times \{1,\ldots,314\} \times \mathbb{N}_0$. Weight function: $w(a,b,c) = a + 3b + 2c$.

Solution. Using the weight function $\alpha(a) = a$ for $\{1,2\}$, we have

$$\Phi_{\{1,2\}}(x) = x + x^2.$$

Using the weight function $\beta(b) = 3b$ for $\{1,\ldots,314\}$, we have

$$\Phi_{\{1,\ldots,314\}}(x) = x^3 + x^6 + x^9 + \cdots + x^{314\cdot 3} = \frac{x^3(1-x^{942})}{1-x^3}.$$

Using the weight function $\gamma(c) = 2c$ for $\mathbb{N}_0$, we have

$$\Phi_{\mathbb{N}_0}(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}.$$

Since $w(a,b,c) = \alpha(a) + \beta(b) + \gamma(c)$, we can apply the product lemma to get

$$\Phi_{\{1,2\}\times\{1,\ldots,314\}\times\mathbb{N}_0}(x) = (x+x^2)\cdot\frac{x^3(1-x^{942})}{1-x^3}\cdot\frac{1}{1-x^2} = \frac{x^4(1+x)(1-x^{942})}{(1-x^3)(1-x^2)}.$$

4. {5 marks}

(a) {3 marks} Determine the following coefficient.

$$[x^{15}]\frac{x^2}{(1-5x)^{10}}.$$

Solution. $\frac{1}{(1-5x)^{10}}$ is the composition of $A(x) = \frac{1}{(1-x)^{10}}$ and $B(x) = 5x$ which is well-defined since $b_0 = 0$. So

$$\frac{1}{(1-5x)^{10}} = \sum_{i\geq 0}\binom{i+10-1}{10-1}(5x)^i = \sum_{i\geq 0}\binom{i+9}{9}5^i x^i.$$

Multiplying by x^2 gives: $\sum_{i\geq 0}\binom{i+9}{9}5^i x^{2+i}$. We want the coefficient of x^{15} and hence the coefficient when $i = 13$. Thus

$$[x^{15}]\frac{x^2}{(1-5x)^{10}} = \binom{13+9}{9}5^{13} = \binom{22}{9}5^{13}.$$

(b) {2 marks} Let $A(x) = \frac{x}{(1-x)^3}$ and $B(x) = \frac{1}{1-2x}$. Determine whether or not $A(B(x))$ is a power series, and explain why.

Solution. The constant term of $B(x) = \frac{1}{1-2x} = \sum_{n\geq 0} 2^n x^n$ is 1 meanwhile the power series $A(x) = \frac{x}{(1-x)^3} = x + x^4 + x^7 + \ldots$ has infinitely many non-zero terms. Hence $A(B(x))$ is not a formal power series since the terms are not well-defined.

5. {6 marks} Let $n \geq 0$. How many compositions of n consist of exactly 4 parts where each part is congruent to 2 modulo 3? You need to define a relevant set, a weight function, determine a generating series, and then find an explicit formula for the answer.

Solution. Let $A = \{2, 5, 8, 11, \ldots\}$ be the set of all positive integers congruent to 2 modulo 3. Then the set of all compositions with 4 parts where each part is congruent to 2 modulo 3 is $S = A^4$. Define the weight of a composition in S to be $w(a_1, \ldots, a_4) = a_1 + \cdots + a_4$. Using $\alpha(a) = a$ for A, we see that

$$\Phi_A(x) = x^2 + x^5 + x^8 + x^{11} + \cdots = \frac{x^2}{1 - x^3}.$$

Using the product lemma, we have

$$\Phi_S(x) = \Phi_{A^4}(x) = (\Phi_A(x))^4 = \left(\frac{x^2}{1 - x^3}\right)^4 = \frac{x^8}{(1 - x^3)^4}.$$

So the answer to our question is

$$[x^n]\frac{x^8}{(1 - x^3)^4} = [x^{n-8}]\frac{1}{(1 - x^3)^4} = [x^{n-8}]\sum_{i \geq 0} \binom{i + 3}{3} x^{3i}.$$

This requires $n \geq 8$, otherwise the coefficient is 0. We need $3i = n - 8$, so $i = \frac{n-8}{3}$ when $n - 8$ is divisible by 3. If $n - 8$ is not divisible by 3, then the coefficient is 0. So our answer is

$$[x^n]\frac{x^8}{(1 - x^3)^4} = \begin{cases} \binom{\frac{n-8}{3}+3}{3} & \text{if } 3 \mid (n - 8),\ n \geq 8 \\ 0 & \text{otherwise} \end{cases}$$