1. {6 marks} Fix $k \geq 1$. Find the generating series for binary strings with no 01^k0 substring. Justify your decomposition and write your generating series as a ratio of polynomials.

Solution.

Let B be the set of binary strings with no 01^k0 substring (where 1^k means k copies of 1). Let C be the set of binary strings with exactly one 01^k0 substring at the end. Then

$$\{\epsilon\} \cup B\{0,1\} = B \cup C$$

since adding a bit to the end of a string in B either stays in B or builds exactly one new 01^k0 at the end and, in the other direction, given a nonempty string in B or C, removing the last bit gives a string in B.

Consider $c \in B\{01^k0\}$. Either c is an element of C or at least two 01^k0 substrings exist in c. In the second case the two 01^k0 must overlap (or one of them would be in the initial substring of c which is in B. The only way this can happen is 01^k01^k0 and every string ending with this is also in $B\{01^k0\}$. Then

$$B\{01^k0\} = C \cup C\{1^k0\}.$$

Translating the two equations into generating series we get

$$\begin{aligned} 1 + \Phi_B(x)(2x) &= \Phi_B(x) + \Phi_C(x) \\ \Phi_B(x)x^{k+2} &= \Phi_C(x) + \Phi_C(x)x^{k+1} \end{aligned}$$

Substituting we get

$$1 + \Phi_B(x)(2x) = \Phi_B(x) + \Phi_B \frac{x^{k+2}}{1+x^{k+1}}$$

Solving for $\Phi_B(x)$ we get

$$\Phi_B(x) = \frac{1}{1 - 2x + \frac{x^{k+2}}{1+x^{k+1}}} = \frac{1+x^{k+1}}{1-2x+x^{k+1}-2x^{k+2}+x^{k+2}} = \frac{1+x^{k+1}}{1-2x+x^{k+1}-x^{k+2}}$$

2. {6 marks} Consider the set C of binary strings which includes the empty string and for which every nonempty element w of C, the first bit of w is 0, the last bit of w is 1, and the rest of w consists of a concatenation of zero of more NON-EMPTY elements of C.

Use the recursive decomposition technique to find an equation which the generating series of C satisfies. You do not need to solve your equation for $\Phi_C(x)$.

Solution.

We just need to rewrite the opening paragraph in our binary string decomposisition language. Let D be the set of nonempty elements of C, so

$$\begin{aligned} C &= \{\epsilon\} \cup D \\ D &= \{0\}D^*\{1\} \end{aligned}$$

This is unambiguous because any concatenation of elements of D can be reversed because every element of D has the form $0w1$ where any initial substring of w has at least as many 0s as 1s; thus there is no way to rewrite $0w1$ as $0w_110w_21$, which makes D^* unambiguous. Finally the $\cup$ is also unambiguous since D does not contain ϵ by construction.

Then translate into generating series

$$\begin{aligned} \Phi_C(x) &= 1 + \Phi_D(x) \\ \Phi_D(x) &= x\Phi_{D^*}(x)x = \frac{x^2}{1-\Phi_D(x)} \end{aligned}$$

Clearning denominators we get

$$\begin{aligned} \Phi_C(x) &= 1 + \Phi_D(x) \\ \Phi_D(x) - \Phi_D(x)^2 &= x^2 \end{aligned}$$

You could now solve the second equation by the quadratic formula and expand as in the rooted tree example, then add 1 to get the expansion for $\Phi_C(x)$ but the question didn't ask for that, instead it asked for a single equation involving only $\Phi_C(x)$, so substitute to get

$$\Phi_C(x) - 1 - (\Phi_C(x) - 1)^2 = x^2$$

so

$$\Phi_C(x)^2 - 3\Phi_C(x) + x^2 + 2 = 0$$

Which is what the question asks for, and again you could solve it by the quadratic formula and expand as in the rooted tree example if you wanted to get a closed form for the coefficients.

3. {5 marks} Solve the recurrence $a_n = -a_{n-1} + 2a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 2$, $a_1 = 3$.

Solution.

Rewriting we have $a_n + a_{n-1} - 2a_{n-2} = 0$ The characteristic polynomial is $x^2 + x - 2 = (x+2)(x-1)$, so the roots are 1 and -2 each with multiplicity 1. Therefore the general solution is

$$a_n = A(-2)^n + B$$

Now use the initial conditions to solve for A and B:

$$\begin{aligned} 2 &= a_0 = A + B \\ 3 &= a_1 = -2A + B \end{aligned}$$

Solving this system we get $A = -\frac{1}{3}$ and so $B = \frac{7}{3}$. Thus the solution is

$$a_n = -\frac{(-2)^n}{3} + \frac{7}{3}$$

4. {5 marks} Solve the recurrence $b_n = -3b_{n-1} + 4b_{n-3}$ for $n \geq 3$ with initial conditions $b_0 = 9$, $b_1 = -9$, $b_2 = 18$.

Solution.

Rewriting the recurrence we have $b_n + 3b_{n-1} - 4b_{n-3}$. The characteristic equation is $x^3 + 3x^2 - 4 = (x+2)^2(x-1)$, so the general solution is

$$b_n = c_1 1^n + c_2(-2)^n + nc_3(-2)^n$$

now we can solve for the c_i using the initial conditions

$$\begin{aligned} 9 &= c_1 + c_2 \\ -9 &= c_1 - 2c_2 - 2c_3 \\ 18 &= c_1 + 4c_2 + 8c_3 \end{aligned}$$

which has solution $c_1 = 2, c_2 = 7, c_3 = -3/2$ so

$$b_n = 2 + 7(-2)^n + 3n(-2)^{n-1}$$