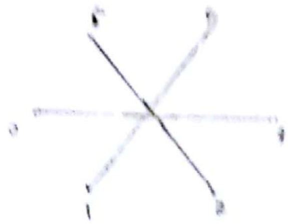
1. Let $n \geq 3$. Define G_n to be the graph where $V(G) = \{0, 1, \ldots, n-1\}$, and two vertices a, b are adjacent if and only if $a \pm 3 \equiv b \pmod{n}$.

(a) {2 marks} Draw G_5 and G_6.



(b) {6 marks} Prove that G_n is connected if and only if n is not a multiple of 3.

"$\Leftarrow$" If n is a multiple of 3.

$n = 3x$ for $x > 0$, x is an integer.

$a \pm 3 \equiv b \pmod{n}$

$a \pm 3 - b = nk$ for k is an integer.

$a \pm 3 - b = 3xk$

$a - b = 3(xk \pm 1)$

The difference of a and b is the multiple of 3, which means, $\{3k \mid k \in N\}$, $\{3k+1 \mid k \in N\}$, $\{3k+2 \mid k \in N\}$, in same set the vertices is adjacent, but the vertices in different set is not adjacent, contradict with assumption.

Thus, if n is not a multiple of 3, G_n is connected.

"$\Rightarrow$" If G is connected.

$a \pm 3 \equiv b \pmod{n}$

$a \pm 3 - b = nk \quad k \in N$

$a - b = nk \pm 3$

If n is a multiple of 3, $n = 3x$, $x \in N$.

$a - b = 3(xk \pm 1)$,

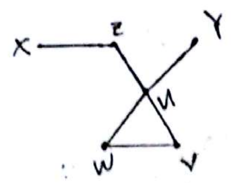
Then G is not connected.

Contradiction

Thus, If G is connected, then n is not a multiple of 3.

2. {6 marks} Each of the following statements is false. Give a counterexample and a brief explanation.

(a) If there is a walk containing vertices u, v, w, then there is a path containing u, v, w.

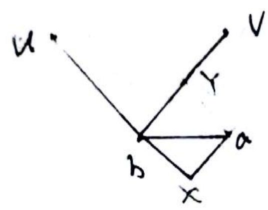


From x to Y:

Walk : x, {x,z}, z, {z,u}, u, {u,v}, v, {v,w}, w, {w,u}, u, {u,Y}, Y.

Path : x, {x,z}, z, {z,u}, u, {u,Y} (doesn't contain w and v)

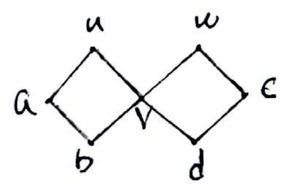
(b) If there is a u, v-walk of even length, then there is a u, v-path ~~of even~~ length.



Walk: u-b-x-a-b-y-v (length 6)

Path : u-b-y-v (length 3) (not even length)

(c) If there exist a cycle containing vertices u, v and a cycle containing vertices v, w, then there exists a cycle containing vertices u, w.



a-b-v-u is a cycle, contains u and v

c-d-v-w is a cycle, contains w and v.

There doesn't exist a cycle containing vertices u,w.

3. {6 marks} The *complement* of a graph G is the graph $\overline{G}$ where $V(G) = V(\overline{G})$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

(a) {4 marks} Prove that if G is disconnected, then $\overline{G}$ is connected.

Assume G is disconnected.

Let $a, b \in V(\overline{G})$. ~~We will show there is a (a-b) path in~~ $\overline{G}$. Since G is not connected. There exists $x, y \in V(G)$ such that there is no $(x-y)$ path in G.

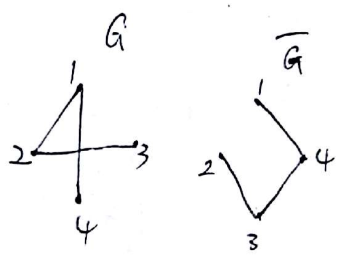
Case 1: If a and b are in the same component as x in G, Neither a nor b is conneted to y in G. Thus a and b is connected to y in $\overline{G}$. $\bar{a}, \bar{b}$ are connected.

Case 2: If neither a nor b is in the same component as x in G, neither a nor b is connected to x in G, which means a and b are connected to x in $\overline{G}$. Then, a, b are connected in $\overline{G}$.

Case 3: one of a or b is in the same component as x in G, but the other is not, Suppose a is not in the same component as x in G, b is not in the same component as y in $\overline{G}$. So in $\overline{G}$, a is connected to x, x is connected to y. b is conected to y. a and b are connected.

Thus, $\overline{G}$ is connected.

(b) {2 marks} Give an example where both G and $\overline{G}$ are connected.

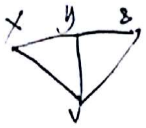


4. {4 marks} Let G be a graph where every vertex has degree at least 3. Prove that G contains a cycle of even length. (Hint: Start with a longest path.)

Let P be a maximal path in G with v is an endpoint of P.
Then, v has at least 3 adjacent vertices on P.

Let x, y, z be the 3 adjacent vertices of v in order on P.

Consider 3 v-y paths: $v-y$, $v-x-y$, $v-z-y$



$v-x-y-v$, $v-y-z-v$, $v-x-y-z-v$ are cycles.

For, $x-y-v$, $y-z-v$, $v-y$, two of these path have the same parity (odd or even).

Thus the union of two paths is an even cycle.

Initials:

ID:

5. {6 marks} Let G be a connected graph. Let P_1, P_2 be two paths in G.

(a) {3 marks} Prove that if P_1 and P_2 have no vertex in common, then there exists a path P_3 with its first vertex in P_1, its last vertex in P_2 and any remaining vertices not in $V(P_1) \cup V(P_2)$.

Since G is a connected graph There always exist a Path P_4 that one endpoint (v_1) is on P_1. the other (v_2) is on P_2.

If none of P_4's remaining vertices is on either P_1 or P_2, there exists P_3

If there are some remaining vertices on ether P_1 or P_2, there always exists a Path that $v_1 - (a_1) \cdots (b_1) - v_2$, we can find a_1 is on Path 1, b_1 is on Path 2. $a_1 - b_1$ is a path such that one endpoint is on P_1 the other is on P_2. Then check if $a_1 - b_1$ has some remaining vertices on ether P_1 or P_2. Finally there exist P_3 such that no remaing vertex is in $V(P_1) \cup V(P_2)$.

(b) {3 marks} Prove that if P_1, P_2 are two longest paths of G, then they have a vertex in common. You may assume part (a).

(Hint: Suppose for a contradiction that they do not have a vertex in common. Use the path P_3 from part (a) to find a path longer than either P_1 or P_2.)

Suppose P_1, P_2 have no common vertex

By (a), we have P_3 which has no remaining vertices in $V(P_1) \cup V(P_2)$ and P_3 is the shortest Path. P_1 has length a. P_2 has length b

P_3 has length c.

Assume P_1 is second-longest Path, P_2 is the longest Path.

There always exist $\frac{b}{2} + c + \frac{a}{2} > a$, and half of vertices of a are in common

Thus, If P_1, P_2 are two longest path of G, they have a vertex in common