1. {6 marks}

(a) {4 marks}Let $S_{n,k}$ be the subset graph as defined in the course notes in Example 4.1.5 (see course notes figure 4.5 for an example). Prove that for $n \geq 3$, $k \geq 2$, and $n > k$, $S_{n,k}$ is not bipartite.

- Assume $S_{n,k}$ is bipartite
  - vertex is $\{v_1, v_2 \cdots v_k\}$ $v_1 \neq v_2 \cdots \neq v_k$, $1 \leq v_1, v_2 \cdots v_k \leq n$
- By definition: Each vertex has the same degree, $l = n-k > k$.

let A, B be partites

- If $S_{n,k}$ is bipartite. $|V(A)| = |V(B)|$, since $|E(A)| = l \cdot |V(A)| = l\,|V(B)| = |E(B)|$
  - Thus, let the adjacent vertices in set B. the rest vertices in set A
  - Suppose $\{v_1, v_2, \cdots v_k\} \in A$, its adjacent vertices are in set B.
- let v_i ($i \in [1,k]$) be the only one different element between $\{v_1, \cdots v_k\}$ and its adjacent vertices. There exists $n-k-1$ verices in its adjacent
- vertices that only v_i is different, which means these vertices are adjacent. And all of these are in set B. Contradiction by the definition of bipartite.

(b) {2 marks}True or False: every bipartite graph contains an even cycle. If the statement is true give a proof and if the statement is false give a counterexample.

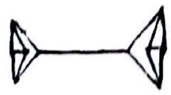
It's false.

Consider

, whis has no cycle.

2. {6 marks}

(a) {2 marks} Draw a 3-regular graph with at least one bridge.



(b) {4 marks} Prove that a 4-regular graph can not contain a bridge.

E is the number of edges
v is vertex.
Suppose 4-regular graph contain a bridge
if we romove that bridge, it would
create a component C which has k vertices
$|E(C)| = \frac{1}{2}(4(k-1)+3)$ by Handshaking Lemma.
Then $|E(C)|$ is not a integer. Contradiction.

3. {6 marks}

(a) {4 marks} Let C_n be the n-cube as defined in the course notes in Example 4.4.1 (see course notes figure 4.12 for an example). Prove that C_n does not have any bridges for $n \geq 2$.

and is n-regular

- By definition, C_n is bipartite, $V(C_n)=2^n$. let A, B be two parties.
  - Assume C_n has an bridge, $e=uv$, u in set A, v in set B.
  - Remove the bridge. let the component containing $u \in V(A)$ be K.

The degree of $V(C) \cap A$ is $n(|V(C) \cap A|-1)+(n-1)$, since $\deg(u)=n-1$

The degree of $V(C) \cap B$ is $n \cdot |V(C) \cap B|$

$$n(|V(C) \cap A|-1)+n-1 \neq n|V(C) \cap B|$$

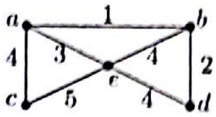
Contradiction.

(b) {2 marks} Suppose you have a graph G, a vertex v of G, and a spanning tree T of G. True or false: you can choose weights for the edges of G so that running Prim's algorithm on G beginning at vertex v gives T. If true describe the weights and explain why they work, if false give an example for which you prove that it is impossible.

It's true.

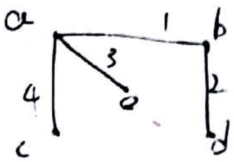
In graph G, let the weight of all the edges in the spaning tree T be 1, and the rest edges in G be 10. Then, if we begin at vertex v, we can just go through the edges whose weight is 1.

4. {5 marks} Find a minimal spanning tree for the graph G using Prim's algorithm starting at vertex a.

$G =$ 

Show both

- the spanning tree you construct on a drawing of the graph and
- a table showing which edges and vertices you add to the tree at each iteration of the loop.



Time through the loop.	V(T)	E(T)
1	{a}	$\emptyset$
2	{a, b}	{ab}
3	{a, b, d}	{ab, bd}
4	{a, b, d, e}	{ab, bd, ae}
5	{a, b, d, e, c}	{ab, bd, ae, ac}.