Math 239 Winter 2017 Assignment 0 Solutions

1. {4 marks}

   A *graph* is a set of elements called *vertices* and a set of pairs of distinct vertices called *edges*.

   (a) What is the largest number of edges that a graph on n vertices can have? Explain.

   (b) Suppose that the vertices can be partitioned into two sets A and B so that all edges have one end in A and the other in B. What is the largest number of edges the graph can have in terms of $|A|$ and $|B|$? Explain.

   Solution. For (a), the answer is the maximum number of pairs of distinct elements of an element set. This number is $\binom{n}{2} = \frac{n(n-1)}{2}$.

   For (b), the answer is $|A| \cdot |B|$ since there are $|A|$ choices for the end in A and $|B|$ choices for the end in B and these choices are independent.

2. {4 marks} A *binary string* is a sequence of digits where each digit is either 0 or 1. The *length* of a binary string is the number of digits in the sequence.

   (a) How many binary strings are there of length n? Explain.

   (b) How many binary strings are there of length n where the first two digits are distinct? Explain.

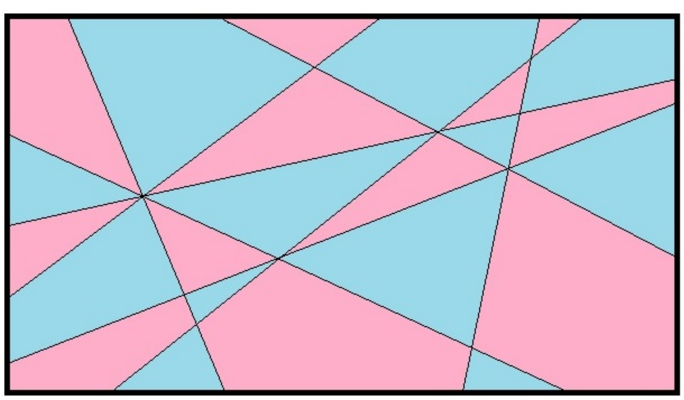
   (c) How many binary strings are there of length n where the first three digits are all distinct? Explain.

   Solution. For (a), the answer is 2^n since for each digit there are 2 choices, either 0 or 1, and these choices are independent.

   For (b), the answer is 2 for $n = 2$ because there are 2 strings 01 and 10. For $n \geq 2$, there are 2^{n-1} such strings because each string either starts with 01 or 10 and then has two choices for each of the remaining $n-2$ digits. For $n = 1$, the answer is 0 since there are no such strings.

   For (c), the answer is 0 for $n \geq 1$ since it is impossible to have all 3 first digits distinct if there are only two choices for each digit.

3. {4 marks} We are given a rectangle with several line segments, each joining border to border inside of the rectangle. This divides the rectangle into several regions. We wish to colour these regions with red and blue so that any two regions that border each other through some line segment receive different colours (two regions that touch each other at only a point may have the same colour). An example is given below.



   Prove (using induction on the number of line segments) that regardless of how we divide the rectangle, such a colouring is always possible.

   Solution. Let n denote the number of line segments in the rectangle R. We proceed by induction on n. When $n = 0$, the rectangle is just one region, and we can colour it with either red or blue. So we may assume that $n \geq 1$.

Take any one line and remove it. The resulting rectangle R' has $n-1$ line segments, and so by induction, there is a proper colouring of its regions using red and blue. We keep the colouring of R' and transfer it to R. Now we put the removed segment back into the rectangle. This line divides the rectangle into two parts, say P_1 and P_2. We now switch all the colours in P_1 (from red to blue, and from blue to red). In the new colouring, two regions that are adjacent inside P_1 or two regions that are adjacent inside P_2 still have different colours. Two regions that are adjacent via the line segment we removed are split from one region in R'. Because we have changed the colour of that part in P_1, these two regions now have different colours. So our colouring of R is valid.