Math 239 Winter 2017 Assignment 6 Solutions

1. {6 marks}

   (a) {5 marks} Let G be a bipartite graph and let H be a subdivision of G. Prove that H is 3-colourable.

   Solution. By definition of subdivision, the graph H is obtained from G by replacing each edge e by a path $P(e)$ of length at least 1, and all paths $P(e)$ are disjoint except for their endpoints. We can construct a 3-colouring f of H as follows. Since G is bipartite, say with vertex classes A and B, we can set $f(a) = 1$ for each $a \in A$ and $f(b) = 2$ for each $b \in B$. Now for each edge e of G, consider the path $P(e) = v_1 \dots v_k$, where $v_1 \in A$ and $v_k \in B$. If $k = 2$ then $P(e)$ is just a single edge and we have no more vertices we need to colour. Otherwise $k \geq 3$ and for $2 \leq i \leq k-1$ we set $f(v_i) = 1$ if i is odd and $f(v_i) = 3$ if i is even. Then since all $P(e)$ are disjoint except for their endpoints, and since each v_k is coloured 2, this is a proper colouring of H with 3 colours.

   (b) {1 marks} Give an example of a bipartite graph G and a subdivision H of G that is not 2-colourable (i.e. not bipartite).

   Solution. Let G be a cycle of length 4, and let H be obtained from G by replacing one edge by a path of length 2. Then G is bipartite, but H is a cycle of length 5 which is an odd cycle, and hence not bipartite.

2. {6 marks} Let G be a planar graph that does not contain any cycles of length three. Prove that G is 4-colourable. Do not assume the Four Colour Theorem. (Hint: use the result of Question 3 on Assignment 5.)

   Solution. We prove by induction on p that every planar graph with p vertices that does not contain a 3-cycle is 4-colourable. We may assume that our graphs are connected, since otherwise we can just apply the result to each component separately.

   Base case $p \leq 4$: every graph with at most 4 vertices is 4-colourable (just give each vertex a different colour).

   Induction Hypothesis: Assume that $p \geq 5$ and every planar graph with fewer than p vertices, that does not contain a 3-cycle, is 4-colourable.

   Let G be a planar graph with p vertices that does not contain a 3-cycle. On Assignment 5 it was shown that every connected planar embedding has a vertex of degree at most 3 or a face of degree at most 3. Since (as noted above) we may assume G is connected, and it has no 3-cycles and at least 5 vertices, no planar embedding of G may contain a face of degree at most 3. Therefore G has a vertex v with $deg(v) \leq 3$.

   Let $H = G - v$. Then H is a planar graph with $p-1$ vertices that does not contain a 3-cycle. Therefore by the induction hypothesis there exists a colouring f of H from the set of colours $\{1, 2, 3, 4\}$. The neighbours of v can be given in total at most 3 colours by f, and so there exists a colour c in $\{1, 2, 3, 4\}$ that is not used by f on any neightbour of v. Then we can extend the colouring f to the whole graph G by giving v colour c. Thus G is 4-colourable, and we have proved the result by induction.

3. {3 marks} Prove or disprove the following statement.

   Let G be a graph and let $H = G/e$ be the graph obtained from G by contracting an edge. Then G is planar if and only if H is planar.

   Solution. The statement is false. For example let G be the complete graph K_5. Contracting any edge of K_5 reults in the graph K_4. But we have shown that K_5 is not planar, and K_4 is planar.

4. {5 marks} Let G be a graph with $2k$ vertices. Suppose every vertex of G has degree at least k. Prove that G has a perfect matching.

   Solution. Let M be a maximum matching in G, and suppose on the contrary that the set $V(M)$ of vertices saturated by M is such that $V(M) \neq V(G)$. Then since $|V(G)|$ is even, there exist at least two unsaturated vertices x and y. Since M is a maximum matching, x and y are not adjacent and every edge incident to x or y must also be incident to a vertex of $V(M)$.

   Suppose that for each edge $uv \in M$, there are at most two edges from $\{u, v\}$ to $\{x, y\}$. Then altogether x and y are incident to at most $2|M| \leq 2(k-1)$ edges, which contradicts the fact that $deg(X) + deg(Y) \geq 2k$. Therefore there exists

$uv \in M$ with at least 3 edges from $\{u, v\}$ to $\{x, y\}$, which implies that two of these edges are disjoint. Let us assume without loss of generality that ux and vy are edges of G. But then $M \setminus \{uv\} \cup \{ux, vy\}$ is a matching of G of size $|M| + 1$, contradicting the choice of M.

Thus we conclude that M is a perfect matching of G.