1. {6 marks} Let $G$ be a bipartite graph with bipartition $(A, B)$.

(a) {2 marks} Prove that

$$\sum_{v\in A} \deg(v) = \sum_{v\in B} \deg(v).$$

For each edge in bipartite graph, one end is in A, and the other is in B. Thus the number of edge is equal to $\sum_{v\in A} deg(v)$ and $\sum_{v\in B} deg(v)$. Thus, $\sum_{v\in A} deg(v) = \sum_{v\in B} deg(v)$

(b) {2 marks} Let $a, b$ be the number of odd-degree vertices in $A, B$ respectively. Prove that $a \equiv b \pmod 2$.

From the corollary, we know "The number of vertices of odd degree in a graph is even." So the sum of $a+b$ must be even.
Thus both of a and b should be either odd or even. The difference between two different odd/even integers is the multiple of 2, thus $a \equiv b$ (mod 2).

(c) {2 marks} Let $k \geq 1$ be an integer. Prove that if $G$ is $k$-regular, then $|A| = |B|$.

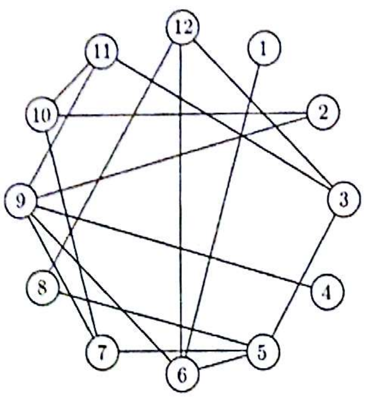
From (a), we know that $\sum_{v\in A} deg(v) = \sum_{v\in B} deg(v)$.

Since G is k-regular, $deg(v) = k$ $v\in A$, $deg(u) = k$ $u\in B$.

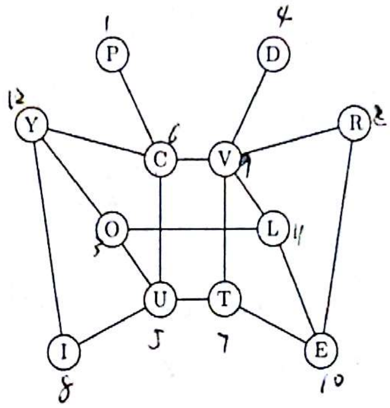
Thus $\sum_{v\in A} k = \sum_{u\in B} k$

$|A| = |B|$.

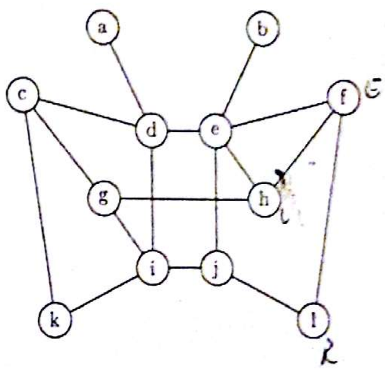
2. {6 marks} Consider the following three graphs.




Graph $G$




Graph $H$




Graph $J$

(a) {3 marks} Graphs $G$ and $H$ are isomorphic. Provide an isomorphism. (You do not need to prove that your mapping is an isomorphism.)

$f(1) = P$ Thus G and H are isomorphic

$f(2) = R$

$f(3) = O$

$f(4) = D$

$f(5) = U$

$f(6) = C$

$f(7) = T$

$f(8) = I$

$f(9) = V$

$f(10) = E$

$f(11) = L$

$f(12) = Y$

(b) {3 marks} Graphs $H$ and $J$ are not isomorphic. Explain why.

The difference between H and J is that $\{L, E\}$ is deleted and $\{L, R\}$ is added in H in order to get J

Thus $\deg(E) = 3$

$\deg(R) = 2$

$\deg(f) = 3$

$\deg(l) = 2$

And we assume $g(E) = f$, $g(I) = i$ $g(V) = e$ $g(T) = j$ .... ( just switch E and R. other vertices keep the same )

For e which is adjacent to f, $\deg(e) = 5$ is different to $\deg(T) = 3$

For j which is adjacent to l, $\deg(j) = 3$ is different to $\deg(V) = 5$

Thus, H and J are not isomorphic.

3. {4 marks} Prove that any graph with at least 2 vertices contains two vertices of the same degree. (Hint: Prove by contradiction.)

The possible degrees in a graph with $n$ vertices are $0, 1, 2, \ldots, n-1$, and no graph with $n$ vertices can contain both 0 degree and $n-1$ degrees, so in each case there are only $n-1$ possible degrees for $n$ vertices. Thus, any graph with at least 2 vertices contains two vertices of the same degrees.

ID:



Initials:

4. {5 marks} Let $G$ be a graph where the degrees of the vertices are either 1 or 3.

(a) {2 marks} Prove that $G$ has even number of vertices.

Handshaking Lemma: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

$2|E(G)|$ is an even number. and the degrees of the vertices are odd. thus the number of vertices is even.

(b) {3 marks} Prove that if the number of vertices is equal to the number of edges in $G$, then the number of vertices of degree 1 is equal to the number of vertices of degree 3 in $G$.

Assume $|V(G)| = |E(G)|$

According to the corollary. the average degree of a vertex in graph $G$ is $\frac{2|E(G)|}{|V(G)|} = 2$

$x$ vertices' degree is 1
$y$ vertices' degrees are 3.

$x + 3y = 2(x+y)$

$x = y$

Thus, the number of vertices of degree 1 is equal to that of degree 3 in $G$.