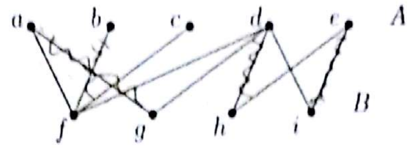
1. {8 marks} Run König's theorem on the following graph, starting with the empty matching.



Each time you change the matching show

- the augmenting path
- X and Y (either the full X and Y for this step, or if you use the shortcutted algorithm, the partial X and Y computed so far).
- the new matching

If your solution goes over the page you can upload a second page.

Time 1. $m = \phi$

$X_0 = \{a, b, c, d, e\}$

$X = \{a, b, c, d, e\}$

$Y = \{f, g, h, i\}$

choose $f \in Y$, $P(v) = \{a, f\}$

$m' = \{af\}$

Time 2: $X_0 = \{b, c, d, e\}$

$X = \{a, b, c, d, e\}$

$Y = \{f, g, h, i\}$

choose $g \in Y$ $P(v) = \{g, a, f, b\}$

$m'' = \{bf, ag\}$.

Time 3: $X_0 = \{d, c, e\}$

$X = \{a, b, c, d, e\}$

$Y = \{f, g, h, i\}$

choose $h \in Y$. $P(v) = \{d, h\}$

$m''' = \{bf, ag, dh\}$

Time 4: $X_0 = \{c, e\}$

$X = \{a, b, c, d, e\}$

$Y = \{f, g, h, i\}$

choose $i \in Y$. $P(v) = \{e, i\}$

$m'''' = \{bf, ag, dh, ei\}$

2. {6 marks}

(a) {4 marks} Let G be a bipartite graph. Let Δ be the largest degree of any vertex of G. Prove that G has a matching with at least $|E(G)|/\Delta$ edges

(b) {2 marks} Give an example of a non-bipartite graph for which this lower bound on the size of a matching is not true.

a) by König's Theorem, since G is a bipartite graph.

maximum matching M = minimum cover C

$|M_{max}| = |C_{min}|$

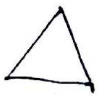
we need to show $|C_{min}| \geq \frac{|E(G)|}{\Delta}$,

since a single vertex cover at most Δ edges.

$|C_{min}| \geq \frac{E(G)}{\Delta}$

Thus, it is true.

b.



s: YF

3. {6 marks} Let n be a positive integer and ℓ, k nonnegative integers with $\ell + k \le n$. This question concerns the binomial identity

$$\binom{n}{\ell}\binom{n-\ell}{k} = \binom{n}{k}\binom{n-k}{\ell}$$

n-l n-k

(a) {4 marks} Prove the identity combinatorially by interpreting both sides as the size of some set of combinatorial objects.

(b) {2 marks} Prove the identity algebraically.

a). Imagine we want to choose k+l people from a group which contains n people. LHS is that firstly we choose l people from the group, $\binom{n}{l}$ then choose k people from the rest n-l people. $\binom{n-l}{k}$ RHS is that firstly we choose k people from the group, $\binom{n}{k}$ then we choose l people from the rest n-k people. $\binom{n-k}{l}$ In general, we totally choose k+l people from the group. thus its equal. $\binom{n}{l}\binom{n-l}{k} = \binom{n}{k}\binom{n-k}{l}$

b $\binom{n}{l}\cdot\binom{n-l}{k} = \frac{n!}{l!(n-l)!}\cdot\frac{(n-l)!}{k!(n-l-k)!}$

$= \frac{n!}{l!k!(n-l-k)!}$

$= \frac{n!(n-k)!}{(n-k)!k!l!(n-k-l)!}$

$= \frac{n!}{(n-k)!k!}\cdot\frac{(n-k)!}{l!(n-k-l)!}$

$= \binom{n}{k}\binom{n-k}{l}$

4. {5 marks}

Prove that the set of compositions of n with k parts is in bijection with the set of subsets of $\{1, 2, \ldots, n-1\}$ of size $k-1$ by describing the bijection and its inverse explicitly.

let S be the set of compositions of n with k points

let T be the set of subsets of $\{0, 1, 2, \ldots n-1\}$ of size $k-1$

① $f: S \to T$

let $A = \{a_1, a_2, a_3, \cdots a_k\} \in S$.

$f(A) = f(a_1, a_2, \cdots a_k) = \{b_1, b_2, \cdots b_{k-1}\}$.

Let $b_i = \sum_{j=0}^{i} a_j \quad k-1 \geq i \geq 1$ since $\sum_{j=0}^{k} a_j = n$,

Thus $\sum_{i=0}^{n-1} a_i \leq n-1$ and b_i are distinct for all i.

② $f^{-1}: T \to S$.

let $t = \{x_1, x_2, \cdots, x_{k-1}\} \in T$ and $x_i < x_{i+1}$ for $i \leq i \leq k-2$.

and $1 \leq x_i \leq n-1$

$f^{-1}(f(A)) = f^{-1}(t) = \{y_1, y_2, \cdots y_k\}$.

let $y_i = \begin{cases} x_1 & \text{if } i=1 \\ x_i - x_{i-1} & \text{if } 2 \leq i \leq k-1 \\ n - x_{k-1} & \text{if } i = k. \end{cases}$

Thus $\sum_{i=0}^{k} y_i = n$ because $\sum_{i=0}^{k} y_i = x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots$
$(n - x_{k-1})$
$= n$.

Thus, it is bijection.

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