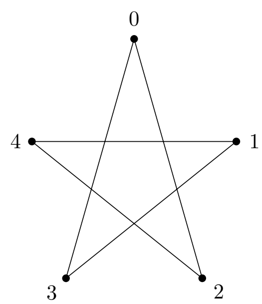
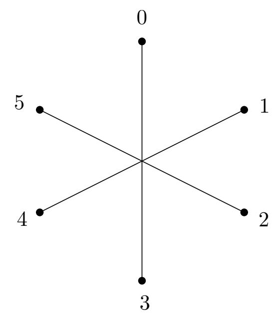
1. Let $n \geq 3$. Define G_n to be the graph where $V(G) = \{0, 1, \ldots, n-1\}$, and two vertices a, b are adjacent if and only if $a \pm 3 \equiv b \pmod{n}$.

   (a) {2 marks} Draw G_5 and G_6.

   Solution.



G_5



G_6

   (b) {6 marks} Prove that G_n is connected if and only if n is not a multiple of 3.

   Solution. ($\Rightarrow$) We prove the contrapositive. Suppose n is a multiple of 3. Let $A = \{0, 3, 6, \ldots, n-3\}$ be vertices that are divisible by 3. We will prove that the cut induced by A is empty, hence G_n is not connected. Suppose ab is an edge where $a \in A$. Then $a \pm 3 \equiv b \pmod{n}$. Since numbers in A are 3 apart, b is still in A, except the possibility of $a = n-3$ and $b = a+3$, or $a = 0$ and $b = a-3$. In the first case, $b \equiv a + 3 \equiv n \equiv 0 \pmod{n}$, so $b \in A$. In the second case, $b \equiv a - 3 \equiv -3 \equiv n - 3 \pmod{n}$, so $b \in A$. In all cases, $b \in A$, hence no edge is in the cut induced by A.

   ($\Leftarrow$) Suppose n is not a multiple of 3. We will show that there exists a $0, a$-path for all $a \in V(G_n)$. We can iteratively add 3 to 0 to get a path $0, 3, 6, 9, \ldots, 3(n-1)$. For a vertex a to be on this path, we need $3x \equiv a \pmod{n}$ for some integer x. Since n is not a multiple of 3, $\gcd(3, n) = 1$. So this linear congruence has an integer solution, hence a is on the path. Therefore, there is a $0, a$-path for all $a \in V(G)$, and G_n is connected.

2. {6 marks} Each of the following statements is false. Give a counterexample and a brief explanation.

   (a) If there is a walk containing vertices u, v, w, then there is a path containing u, v, w.

   Solution. Consider the following graph. There is a walk containing u, v, w, namely u, x, v, x, w. But there is no path containing u, v, w, since each vertex has degree 1 and if there's a path containing all of them, one of them has to have degree at least 2.

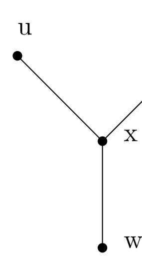
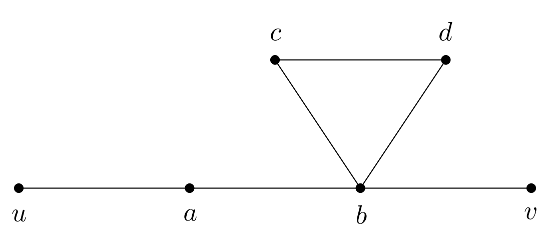


Figure 1: Flux capacitor.

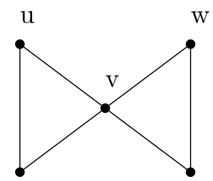
   (b) If there is a u, v-walk of even length, then there is a u, v-path of even length.

   Solution. Consider the following graph. There is a u, v-walk of even length: u, a, b, c, d, b, v. However, the only u, v-path is u, a, b, v, which has odd length.



(c) If there exist a cycle containing vertices u, v and a cycle containing vertices v, w, then there exists a cycle containing vertices u, w.

Solution. The following is a counterexample.



3. {6 marks} The *complement* of a graph G is the graph $\overline{G}$ where $V(G) = V(\overline{G})$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

(a) {4 marks} Prove that if G is disconnected, then $\overline{G}$ is connected.

Solution. Since G is disconnected, there exists a proper, non-empty subset X of $V(G)$ such that the cut induced by X is empty (by a theorem from notes/class). But then in $\overline{G}$, the cut induced by X has all possible edges, that is, for all $u \in X, v \notin X$, uv is an edge of $\overline{G}$.

Now let $x, y \in V(\overline{G})$. We claim that there is a path from x to y in $\overline{G}$ and hence G is connected as desired. If $x = y$, then x is a path as claimed. So we may suppose $x \neq y$. If $x \in X$ and $y \notin X$, then xy is an edge of $\overline{G}$ and hence is also a path from x to y as claimed. If $x, y \in X$, then let $z \notin X$ and note that xzy is a path as claimed. If $x \notin X$, then a symmetric argument (i.e. interchanging the role of X and $V(\overline{G}) - X$) proves the claim.

(b) {2 marks} Give an example where both G and $\overline{G}$ are connected.

Solution. Let $G = C_5$. Then $\overline{G} = C_5$ (its self-complementary). Now C_5 is connected and thus both G and $\overline{G}$ are connected.

4. {4 marks} Let G be a graph where every vertex has degree at least 3. Prove that G contains a cycle of even length. (Hint: Start with a longest path.)

Solution. Let $P = v_1, v_2, \ldots, v_k$ be a path of the longest length in G. The neighbours of v_1 must all be in P for otherwise we can extend P to a longer path. The vertex v_1 has at least 3 neighbours, one of them is v_2. Suppose two other neighbours are v_i, v_j where $i < j$. If i is even, then $v_1, \ldots, v_i, v_1$ is a cycle of length i, which is even. Similarly, if j is even, then $v_1, \ldots, v_j, v_1$ form a cycle of even length. Otherwise, both i, j are odd. Then $v_1, v_i, v_{i+1}, \ldots, v_j, v_1$ is a cycle of length $j - i + 2$, which is even. So an even cycle must exist.

5. {6 marks} Let G be a connected graph. Let P_1, P_2 be two paths in G.

(a) {3 marks} Prove that if P_1 and P_2 have no vertex in common, then there exists a path P_3 with its first vertex in P_1, its last vertex in P_2 and any remaining vertices not in $V(P_1) \cup V(P_2)$.

Solution. Let $x \in V(P_1)$, $y \in V(P_2)$. Since G is connected, there exists a path P from x to y in G. Let u be the last vertex (starting from x) of P that is also in $V(P_1)$ (note such a u exists since $x \in V(P_1)$). Then let v be the first vertex after u in P that is also in $V(P_2)$ (note such a v exists since $y \in V(P_2)$). Now let P_3 be the subpath of P from u to v.

Note that the first vertex of P_3 is u which in $V(P_1)$. The last vertex of P_3 is v which is in $V(P_2)$. Furthermore, we claim that any remaining vertex w of P_3 is not in $V(P_1) \cup V(P_2)$. For if $w \in V(P_1)$, then u is not the last vertex of P that is also in $V(P_1)$, contradicting the choice of u. Similarly if $w \in V(P_2)$, then v is not the first vertex of P after u that is also in $V(P_2)$, contraditing the choice of v. This proves the claim and hence P_3 is a path as desired.

(b) {3 marks} Prove that if P_1, P_2 are two longest paths of G, then they have a vertex in common. You may assume part (a).

(Hint: Suppose for a contradiction that they do not have a vertex in common. Use the path P_3 from part (a) to find a path longer than either P_1 or P_2.)

Solution. Suppose for a contradiction that they do not have a vertex in common. Let m be the length (i.e. the number of edges) of P_1 (and hence also of P_2 since both are longest).

Let P_3 be a path as in (a). Let u be the end of P_3 in $V(P_1)$ and v be the end of P_3 in $V(P_2)$.

Now let x_1, x_2 be the ends of P_1. Let P_1', P_1'' be the subpaths of P_1 from u to x_1, x_2 respectively. We may assume without loss of generality that $|E(P_1')| \geq |E(P_1'')|$ and hence $|E(P_1')| \geq m/2$.

Similarly let y_1, y_2 be the ends of P_2 and let P_2', P_2'' be the subpaths of P_2 from v to y_1, y_2 respectively. We may assume without loss of generality that $|E(P_2')| \geq |E(P_2'')|$ and hence $|E(P_2')| \geq m/2$.

Finally let P_4 be the path from x_1 to y_1 obtained by concatenating P_1', P_3, P_2' in that order. Now

$$|E(P_4)| = |E(P_1')| + |E(P_3)| + |E(P_2')| \geq m/2 + |E(P_3)| + m/2 = m + |E(P_3)|.$$

However, $|E(P_3)| \geq 1$ since u and v are distinct vertices. Thus $|E(P_4)| \geq m + 1$, contradicting that P_1, P_2 are longest paths.