1. {6 marks} Fix $k \geq 1$. Find the generating series for binary strings with no 01^k0 substring (where 1^k means k copies of 1). Justify your decomposition and write your generating series as a ratio of polynomials.

Let S be the set of all binary strings with no 01^k0 substring

$\{0,1\}^* = \{1\}^*(\{0\}\{1\}^*)^*$ 0- decomposition.

$S = \{1\}^*(\{0\}\{0\}^*(\{1\}^* - \{\varepsilon \cup 1^k\}))^*\{0\}\{0\}^*\{1\}^* \cup \{1\}^*$

It is unambiguous since 0- decomposition is unambiguous.

Let $A = \{1\}^* - \{\varepsilon \cup 1^k\}$ By Sum lemma.

$\Phi_{\{1\}^*}(x) = \Phi_A(x) + \Phi_{\{\varepsilon \cup 1^k\}}$ By Star lemma

$$\Phi_A(x) = \Phi_{\{1\}^*}(x) - \Phi_{\{\varepsilon \cup 1^k\}}$$

$$= \frac{1}{1-x} - (1+x^k)$$

$$= \frac{1-(1-x)-x^k(1-x)}{1-x}$$

$$= \frac{x-x^k+x^{k+1}}{1-x}$$

$$\Phi_{\{0\}\{0\}^*A}(x) = \frac{x}{1-x} \cdot \frac{x-x^k+x^{k+1}}{1-x}$$

$$= \frac{x^2-x^{k+1}+x^{k+2}}{(1-x)^2}$$

$$\Phi_{(\{0\}\{0\}^*A)^*}(x) = \frac{1}{1-\frac{x^2-x^{k+1}+x^{k+2}}{(1-x)^2}} = \frac{(1-x)^2}{1-2x+x^{k+1}-x^{k+2}}$$

$$\Phi_S(x) = \frac{1}{1-x} \cdot \frac{(1-x)^2}{1-2x+x^{k+1}-x^{k+2}} \cdot x \cdot \frac{1}{1-x} \cdot \frac{1}{1-x} + \frac{1}{1-x}$$

$$= \frac{x}{(1-2x+x^{k+1}-x^{k+2})(1-x)} + \frac{1}{1-x}$$

2. {6 marks} Consider the set C of binary strings which includes the empty string and for which every nonempty element w of C, the first bit of w is 0, the last bit of w is 1, and the rest of w consists of a concatenation of zero ~~or more~~ NON-EMPTY elements of C.
Or

Use the recursive decomposition technique to find an equation which the generating series of C satisfies. You do not need to solve your equation for $\Phi_C(x)$.

By the discription.

$$C = \{\varepsilon\} \cup \{0\}\{C \setminus \{\varepsilon\}\}^*\{1\}$$

C is unambiguous, so

$$\phi_C(x) = 1 + x \cdot \frac{1}{1-(\phi_C(x)-1)} \cdot x$$ By Star lemma, Product lemma. Sum lemma.

$$\phi_C(x) = 1 + \frac{x^2}{2-\phi_C(x)}$$

$$(\phi_C(x)-1)(2-\phi_C(x)) = x^2$$

$$2\phi_C(x) - \phi_C^2(x) - 2 + \phi_C(x) = x^2$$

$$-\phi_C^2(x) + 3\phi_C(x) - x^2 - 2 = 0$$

3. {5 marks} Solve the recurrence $a_n = -a_{n-1} + 2a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 2$, $a_1 = 3$.

$a_n + a_{n-1} - 2a_{n-2} = 0$

The characteristic polynomial is:

$x^2 + x - 2 = (x-1)(x+2)$

It has root $x_1 = -2$ with multiplicity 1
$x_2 = 1$ with multiplicity 1.

$a_n = A(-2)^n + B(1)^n$

$$\begin{cases} a_0 = A(-2)^0 + B(1)^0 \\ a_1 = A(-2)^1 + B(1)^1 \end{cases}$$

Since $a_0 = 2$, $a_1 = 3$.

$$\Rightarrow \begin{cases} A = -1/3 \\ B = 7/3 \end{cases}$$

Thus $a_n = -\frac{1}{3}(-2)^n + \frac{7}{3}$

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4. {5 marks} Solve the recurrence $b_n = -3b_{n-1} + 4b_{n-3}$ for $n \geq 3$ with initial conditions $b_0 = 9$, $b_1 = -9$, $b_2 = 18$.

$b_n + 3b_{n-1} - 4b_{n-3} = 0$

The characteristic polynomial is

$x^3 + 3x^2 - 4 = (x-1)(x+2)^2$

The ~~root~~ has $x_1 = 1$ with multiplicity 1

$x_2 = -2$ with multiplicity 2

$b_n = (A + nB)(-2)^n + C(1)^n$

$$\left.\begin{array}{l} b_0 = (A + B\cdot 0)(-2)^0 + C(1)^0 = 9 \\ b_1 = (A + B\times 1)(-2)^1 + C(1)^1 = -9 \\ b_2 = (A + B\times 2)(-2)^2 + C(1)^2 = 18 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A = 7 \\ B = -3/2 \\ C = 2 \end{array}\right.$$

Thus $b_n = (7 - \frac{3n}{2})(-2)^n + 2$ for $n \geq 0$.

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