Math 239 Winter 2017 Assignment 1 Solutions

1. {6 marks} Let G be a bipartite graph with bipartition (A, B).

   (a) {2 marks} Prove that

   $$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v).$$

   Solution. Since G is bipartite, each edge uv has one end in A (say it is u) and one end in B (say it is v). Then uv contributes one to each side of the equation, 1 for $\deg(u)$ for the sum on the left hand side, and 1 for $\deg(v)$ for the sum on the right hand side. Hence the two sums are equal.

   (b) {2 marks} Let a, b be the number of odd-degree vertices in A, B respectively. Prove that $a \equiv b \pmod 2$.

   Solution. From the corollary of the handshaking lemma, we know that the number of odd-degree vertices is even. The number of odd-degree vertices in G is $a + b$, so $a + b \equiv 0 \pmod 2$. Hence $a \equiv -b \equiv b \pmod 2$ (in other words, they are both even or both odd).

   (c) {2 marks} Let $k \geq 1$ be an integer. Prove that if G is k-regular, then $|A| = |B|$.

   Solution. Since every vertex has degree k, the equation from part (a) gives us

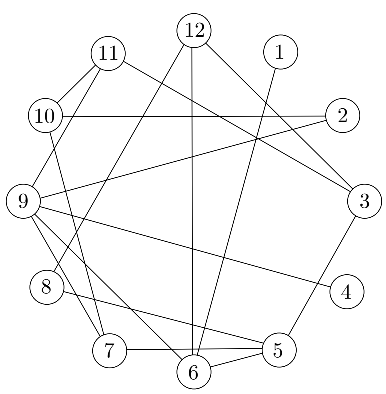
   $$\sum_{v \in A} k = \sum_{v \in B} k.$$

   This implies that

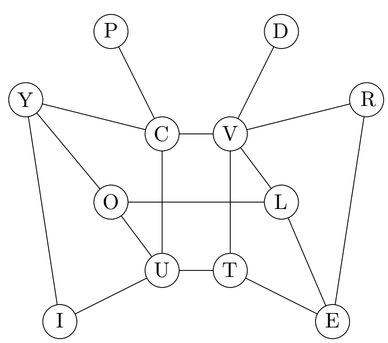
   $$|A|k = |B|k.$$

   Since $k \geq 1$, we can divide both sides by k to get $|A| = |B|$.

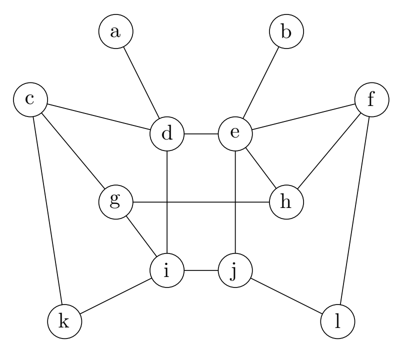
2. {6 marks} Consider the following three graphs.



Graph G



Graph H



Graph J

   (a) {3 marks} Graphs G and H are isomorphic. Provide an isomorphism. (You do not need to prove that your mapping is an isomorphism.)

   Solution. There is only one possible isomorphism $f : V(G) \to V(H)$, where

v	1	2	3	4	5	6	7	8	9	10	11	12
$f(v)$	P	R	O	D	U	C	T	I	V	E	L	Y

   One should verify that this is an isomorphism by checking each edge, ensuring that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.

(b) {3 marks} Graphs H and J are not isomorphic. Explain why.

Solution. Both graphs have only one vertex of degree 5, that is vertex V from graph H, and vertex e from graph J. So any isomorphism would map V to e. The vertex V has two neighbours with degree 3 in H, but vertex e has three neighbours with degree 3 in J. Hence no isomophism is possible.

Alternatively, we see that graph J contains 3 mutually adjacent vertices e, f, h (a cycle of length 3), but graph H does not contain such a structure. Hence they cannot be isomorphic.

3. {4 marks} Prove that any graph with at least 2 vertices contains two vertices of the same degree. (Hint: Prove by contradiction.)

Solution. Suppose by way of contradiction that there exists a graph G with n vertices where no two vertices have the same degree. The smallest and largest possible degrees are 0 and $n-1$ respectively, so the possibilities of vertex degrees in G are $0, 1, 2, \ldots, n-1$. There are n possibilities, but there are n vertices, so there must exist one vertex of each degree. In particular, there is one vertex of degree 0 and one vertex of degree $n-1$. This is not possible since the vertex of degree $n-1$ is adjacent to all the other vertices, including the vertex of degree 0. Therefore, there must exist two vertices of the same degree.

4. {5 marks} Let G be a graph where the degrees of the vertices are either 1 or 3.

(a) {2 marks} Prove that G has even number of vertices.

Solution. A result from class (proved via the handshaking lemma) is that any graph has even number of odd-degree vertices. Since every vertex in G has odd degree, this implies that G has even number of vertices.

(b) {3 marks} Prove that if the number of vertices is equal to the number of edges in G, then the number of vertices of degree 1 is equal to the number of vertices of degree 3 in G.

Solution. Suppose G has k vertices of degree 1, l vertices of degree 3, and m edges. Using the handshaking lemma, we get

$$2m = 1 \cdot k + 3 \cdot l.$$

We are given that $m = k + l$, so $2k + 2l = k + 3l$, which implies that $k = l$.