1. {6 marks} Let a_n denote the number of compositions of n in which each part is an even number greater than 5. (Note the number of parts is not fixed.) Find the generating series $\sum_{n\geq 0} a_n x^n$, expressed as a *rational* expression (a quotient of two polynomials). You must define a suitable set S and weight function w on S, and indicate wherever you use results proved in class.

let $S = \{6, 8, 10 \ldots\}$

Assume k is the number of part of compositions of n, $0 \leq k \leq n$.

$w(\sigma) = i \ . \ i \in S.$

$$\sum_{n\geq 0} a_n x^n = \sum_{k=0}^{n} (\phi_S(x))^k = \sum_{k=0}^{n} (x^6 + x^8 + x^{10} + \ldots)^k$$

$$= \sum_{k=0}^{n} (x^6 (1 + (x^2)^1 + (x^2)^2 + (x^2)^3 \ldots))^k$$

$$= \sum_{k=0}^{n} \left(x^6 \left(\frac{1}{1-x^2}\right)\right)^k \quad \text{By Geometric series.}$$

$$= \sum_{k=0}^{n} \left(\frac{x^6}{1-x^2}\right)^k$$

$$= \frac{1}{1 - \frac{x^6}{1-x^2}}$$

$$= \frac{1-x^2}{1-x^2-x^6}$$

2. {6 marks} The generating series for compositions of n in which each part is at least 3 is

$$\frac{1-x}{1-x-x^3}.$$

(a) {4 marks} Find a recurrence relation for the sequence $\{a_n\}_{n\geq 0}$, together with initial conditions that uniquely specify the sequence, where a_n is the number of compositions of n in which each part is at least 3.

$a_n = [x^n]\frac{1-x}{1-x-x^3}$.

Assume $a_n = b_n$ such that $\sum_{n\geq 0} b_n x^n = \frac{1-x}{1-x-x^3}$

$(1-x-x^3)\sum_{n\geq 0} b_n x^n = 1-x$

$(1-x-x^3)\sum_{n\geq 0} b_n x^n = b_0 + (b_1-b_0)x + (b_2-b_1)x^2 + (b_3-b_2-b_0)x^3$

$+ (b_4-b_3-b_1)x^4 + \cdots$

$= b_0 + (b_1-b_0)x + (b_2-b_1)x^2 + \sum_{n\geq 3}(b_n - b_{n-1} - b_{n-3})x^n$

$1-x = b_0 + (b_1-b_0)x + (b_2-b_1)x^2 + \sum_{n\geq 3}(b_n - b_{n-1} - b_{n-3})x^n$

$b_0 = 1 \quad b_1 - b_0 = -1 \quad b_2 - b_1 = 0.$

$$\begin{cases} b_0 = 1 \\ b_1 = 0 \\ b_2 = 0 \end{cases}$$

Thus $\begin{cases} a_0 = 1,\ a_1 = 0,\ a_2 = 0 \\ a_n = a_{n-1} + a_{n-3} \end{cases}$

(b) {2 marks} Find the number of compositions of 10 in which each part is at least 3.

$a_0 = 1$

$a_1 = 0$

$a_2 = 0$

$a_3 = a_2 + a_0 = 1$

$a_4 = a_3 + a_1 = 1 + 0 = 1$

$a_5 = a_4 + a_2 = 1 + 0 = 1$

$a_6 = a_5 + a_3 = 1 + 1 = 2$

$a_7 = a_6 + a_4 = 2 + 1 = 3$

$a_8 = a_7 + a_5 = 3 + 1 = 4$

$a_9 = a_8 + a_6 = 4 + 2 = 6$

$a_{10} = a_9 + a_7 = 6 + 3 = 9.$

3. {6 marks} Let $A = \{00, 101, 110, 0001\}$ and $B = \{0110, 10, 111\}$.

(a) {4 marks} Determine whether AB is unambiguous and whether BA is unambiguous. Prove your conclusion in each case.

For AB: $AB = \{000110, 0010, 00111, 1010110, 10110, 101111, 1100110, 11010, 110111, 00010110, 0001111\}$

$A\times B = \{(00,0110), (00,10), (00,111), (101,0110), (101,10), (101,111), (110,0110), (110,10), (110,111), (0001,0110), (0001,10), (0001,111)\}$

$|AB| = 11$

$|A\times B| = 12$ AB is ambiguous.

For BA $BA = \{011000, 0110101, 0110110, 01100001, 1000, 10101, 10110, 100001, 11100, 111101, 111110, 1110001\}$

$B\times A = \{(0110,00), (0110,101), (0110,110), (0110,0001), (10,00), (10,101), (10,110), (10,0001), (111,00), (111,101), (111,110), (111,0001)\}$

$|BA| = |B\times A| = 12$.

BA is unambiguous.

(b) {2 marks} Find the generating series with respect to length for each of AB and BA.

By a).

$\phi_{AB} = x^4 + 3x^5 + 3x^6 + 3x^7 + x^8$

$\phi_{BA} = x^4 + 3x^5 + 4x^6 + 3x^7 + x^8$

4. [6 marks] Find the generating series (with respect to length) for the set of all binary strings in which each block of 1's (i.e. each maximal substring consisting entirely of 1's) has length divisible by 3. Write your answer as a rational expression. Your solution should include an unambiguous decomposition for S and a justification for why it is unambiguous. Indicate wherever you use results proved in class.

By question, $S = \{0\}^* \{\{111\}\{111\}^*\{0\}\{0\}^*\}^* \{111\}^*$.

$$\Phi_S(x) = \Phi_{\{0\}^*}(x)\, \Phi_{\{\{111\}\{111\}^*\{0\}\{0\}^*\}^*}(x)\, \Phi_{\{0\}^*}(x)$$

$$= (\Phi_{\{\varepsilon\}}(x) + \Phi_{\{0\}}(x) + \cdots)(\Phi_{\{\varepsilon\}}(x) + \Phi_{\{\{111\}\{111\}^*\{0\}\{0\}^*\}^1}(x) + \cdots)(\Phi_{\{\varepsilon\}}(x) + \Phi_{\{111\}}(x) + \cdots)$$

$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\Phi_{\{\{111\}\{111\}^*\{0\}\{0\}^*\}}(x)}\right)\left(\frac{1}{1-x^3}\right)$$

$$= \frac{1}{(1-x)(1-x^3)} \cdot \frac{1}{1-\Phi_{\{111\}} \cdot \Phi_{\{0\}} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x}}$$

$$= \frac{1}{(1-x)(1-x^3)} \cdot \frac{1}{1-\frac{x^3}{1-x^3} \cdot \frac{x}{1-x}}$$

$$= \frac{1}{1-x-x^3}$$