STAT 230

PROBABILITY

(Chapter 1 and 2)

Introduction to Probability

Chapter one objectives

- Definitions of Probability

  - Understand the classical definition of probability.

  - Understand the relative-frequency definition of probability.

  - Understand the subjective (personal) definition of probability.

Thought Question

Here are two very different queries about probability:

(a) If you flip a fair coin once and do it fairly, what is the probability that it will land heads up?

http://us.123rf.com/400wm/400/400/scol22/scol220904/scol22090400006/4634226-macro-of-a-canadian-1-cent-coin-isolated-on-white-background.jpg

(b) What is the probability that you will eventually own a home?

http://www.hlg.co.nz/bits/stock/house-home.jpg

Two distinct interpretations:

- For the probability of a coin land heads up -- we can quantify the chances exactly.

- For the probability that we will eventually buy a home – we are basing our assessment on personal beliefs about how life will evolve for us.

What about Probability?

The foundation of probability theory lies in problems associated with gambling and games of chance.

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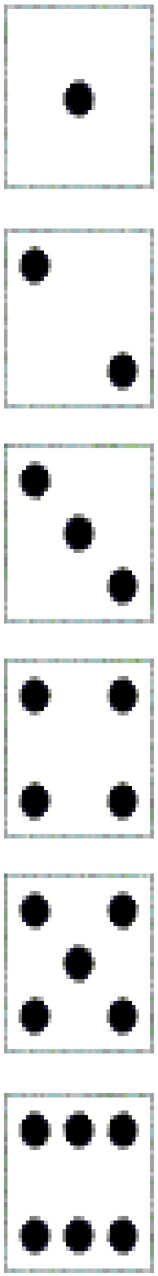
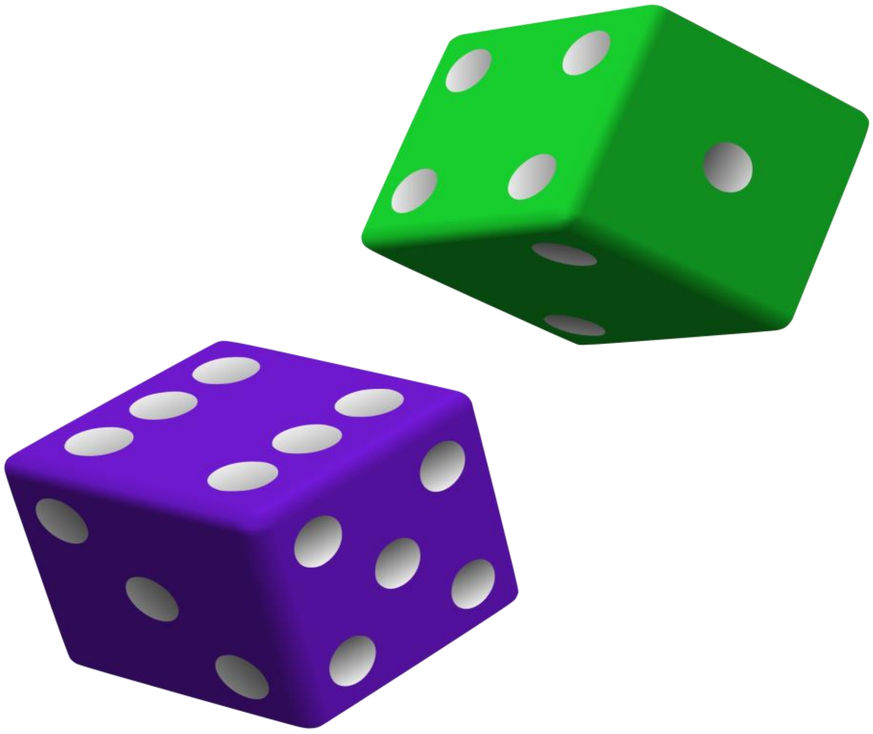


Dice



Dice as we know them were invented around 300 BC.

What does the word probability mean?

- The term ***probability*** refers to the study of randomness (variability of results).

- Probability is used to quantify the likelihood, or chance, that an outcome of random experiment will occur.

- By variability, we mean successive observations of a system or phenomenon do *not* produce exactly the same result.

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Example: Gasoline mileage performance of your car

Do you always get exactly the same mileage performance on every tank of fuel?

No it depends on many factors:

- Type of driving (city vs. highway).
- Changing in the condition of the car.
- The weather conditions.
- ………………………

- Variability is usually due to some factors

- Variability in populations.

  e.g., People vary in size, weight, blood type,…

- Variability in processes or phenomena.

  e.g., The random selection of 6 numbers from 49 in a lottery draw.

What is “Probability”?

Three ways to think about probability:

- The classical definition.

- Relative-frequency definition.

- The subjective probability/ personal-probability definition (probability as a measure of belief).

The Classical Interpretation

- The probability of some events

$$\frac{\text{Number of ways the event can occur}}{\text{The total number of outcomes}}$$

Note: All outcomes are "equally likely".

Example

What is the probability of rolling a die once and observing whether we get a “2”?

- We can assume that the die is fair and our roll is random.
- This means we have an equal probability “equally likely” to get each of the 6 sides.
- Therefore, the probability of getting a “2” from a rolled die is 1/6.

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Example

Lottery draws (assume every combination is equally likely).

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Relative-Frequency Interpretation

- When something happens (or can happen) over and over again, we can apply a relative-frequency interpretation.

- Probability of a specific outcome is defined as the **proportion of times it occurs over the long run**.

Example

- What is the probability of rolling a fair die and observing whether we get a “2”?

  Roll a die **many** times and see what proportion of the time we get a “2”.

Example

- What is the probability of flipping a fair coin and observing if we get heads up?

Flip a coin **many** times and see what proportion of the time we get a “H”.

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Example

When a small plastic cone was tossed into the air 300 times it fell on its side 203 times and on its base 97 times. We say:

- The relative frequency of side is

base

203 /300= 0.677

- The relative frequency of base is

97 /300 = 0.323

side

- This does not apply to situation where

  - We can not repeat an experiment or process indefinitely.

    e.g; The probability of rain today.

Because today can not be repeated again under identical conditions.

  - We can not repeat an experiment or process due to time, cost or other limitations.

Subjective /Personal-Probability Interpretation

- The degree to which a given individual believes the event will happen.

- Subjective Probability, since it is personal, there is no "single correct answer".

For example, what is the probability of…

- The defendant in a trial being guilty?
- Passing this course?
- Finding a parking space downtown Toronto on this Saturday.

Coherent: means your personal probability of one event does not contradict your personal probability of another.

Example

If you thought that the probability of finding a parking space downtown Toronto on this Saturday is 0.20, to be coherent, you must believe that the probability of not finding one is 0.80.

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Note

The difficulties in producing a satisfactory definition can be overcome by treating probability as a mathematical system defined by a set of axioms.

Chapter Two

Mathematical Probability Models.

Please Do Problems 2.1 - 2.5 Course Notes.

Chapter two objectives.

- Understand some probability concepts:
  - Experiment.
  - Sample Space.
  - Event.

Sample Spaces and Events

- Experiment is any action, phenomenon or process that can be infinitely repeated, at least in theory.

- An experiment is said to be *random* if it has more than one possible outcome, and *deterministic* if it has only one.

  - Tossing a coin once or several times.
  - Obtaining blood types from a group of individuals.

- Trial is a single repetition of the experiment.

- Sample Space of an experiment denoted by **S**, is the set of all possible distinct outcomes of that experiment.

Notes

- In a single trial, one and only one of the outcomes in S can occurs.

- Use { } to indicate the elements of a set.

Example

If the outcome of an experiment consists in the determination of the gender of a newborn child (in general), then

$$S = \{ M, F\}$$

where M: Male, F: Female.

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Example

If you flipped a coin once, the sample space **S** would be given by:

$$S = \{Head, Tail\}, \text{ or } \{H, T\}, \text{ or } \{0, 1\},$$

Example

If you flipped a coin twice, the sample space **S** would be given by:

$$S=\{ HH, TH, HT, TT\}$$

$2^2 = 4$ possible outcomes.

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Example

What is the sample space of rolling a fair die

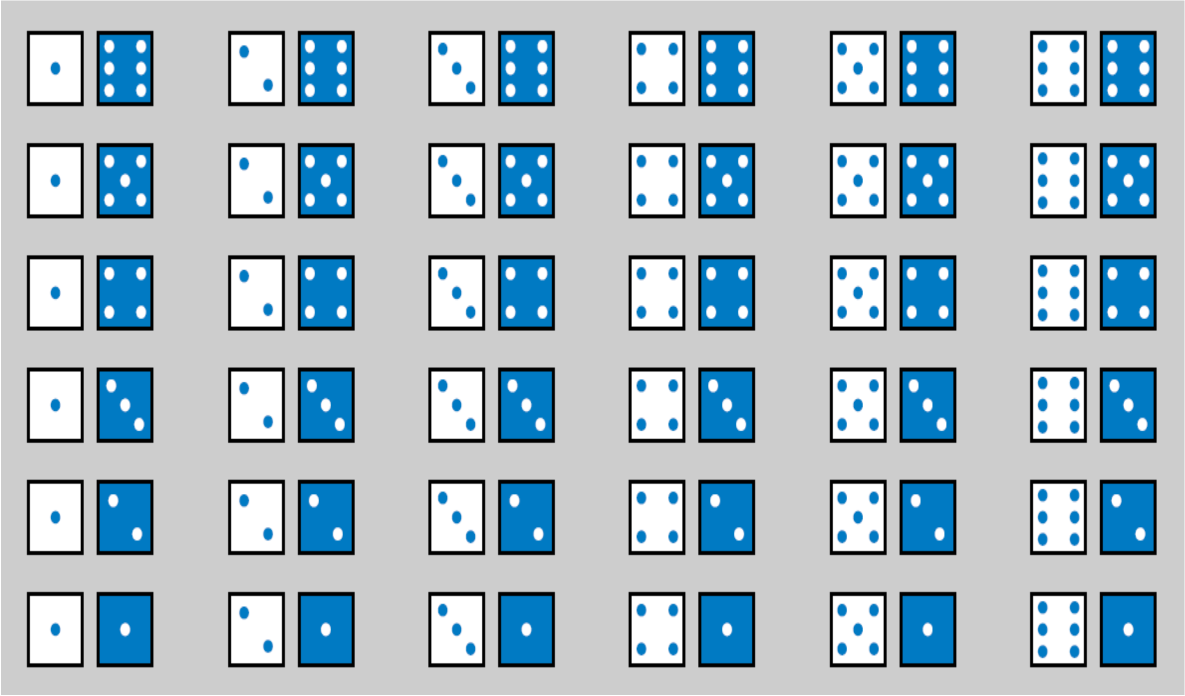
a) once?

b) twice? OR two different color die.

a) Once

S= { 1 , 2, 3, 4, 5, 6}

b) Twice; $6^2 = 36$



White / Blue	1	2	3	4	5	6
1	(1,1)	(1, 2)	(1, 3)	-	-	(1, 6)
2	(2, 1)	(2, 2)	-	-	-	(2, 6)
3	-	-	(3, 3)	-	-	-
4	-	-	-	(4, 4)	-	-
5	-	-	-	-	(5, 5)	-
6	(6, 1)	-	-	-	-	(6, 6)

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Note

- Sample space is often defined based on the objective of the analysis.

Example

What is the sample space of rolling a die once?

S= {1, 2, 3, 4, 5, 6}

S={E, O}

E: the event that an even number turns up.

O: the event that an odd number turns up.

Example

Testing each type D-flashlight battery as it comes off an assembly line until we first observe a success (S)

- F: Failure (if the voltage outside certain limits).
- S: Success (if the voltage within certain limits).

https://www.acklandsgrainger.com/

The sample space S={ S, FS, FFS, FFFS,…..}

Which contains an infinite number of possible outcomes.

Example

Two gas stations are located at a certain intersection. Each one has six gas pumps. Consider the experiment in which the number of pumps in use at a particular time of a day in each gas station. What is the sample space?

http://www.tampareads.com/phonics/singleletters/letter-g/graphics/gas.gif

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	0	1	2	3	4	5	6
0	(0,0)	(0,1)	(0, 2)	-	-	-	(0,6)
1	(1,0)	(1,1)	(1,2)	(1,3)	-	-	(1,6)
2	-	(2,1)	(2,2)	-	-	-	(2,6)
3	-	-	-	(3,3)	-	-	-
4	-	-	-	-	(4,4)	-	-
5	-	-	-	-	-	(5,5)	-
6	(6,0)	(6,1)	-	-	-	-	(6,6)

- The sample space may be either discrete or non discrete (continuous).

- A sample space is discrete if it consists of a finite or countably infinite set of simple events.

$$S= \{a_1, a_2, a_3, \dots\}$$

$S= \{1, 2, 3, 4,\dots\}$ All positive integers.

$S= \{ 1/2, 1/3, 1/4, 1/5\dots\}$ All rational numbers a/b where a = 1, and b is a positive integer other than zero.

- We say a set is discrete if the elements in it are 'separated'.

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- A sample space is continuous if it contains an interval (either finite or infinite) of real numbers.

Example

What is the sample space if the experiment consist s of measuring (in hours) the life time of a transistor?

$S = \{ x: 0 \leq x \leq \infty \}$

$S = R^{+} = \{x: x \geq 0\}$

The sample space consist of all nonnegative real numbers.

Example

What is the sample space?

(a) Select a connector, and measure its thickness (x).

$$S = R^{+} = \{x: x > 0\}$$

All positive real line, S is continuous.

(b) Do the connector conform to the manufacturing specifications?

$S = \{y, n\}$, S is discrete

- An event is any collection (subset) of outcomes contained in the sample space S.

  - Simple if it consists of exactly one outcome (point).

  - Compound it consists of more than one outcome.

Example

Suppose we toss a coin 3 times

a) What is the sample space?

b) How many simple event we have?

c) Is the event that exactly one of the 3 toss are H compound or simple?

d) Is the event that all of the 3 toss are the same compound or simple ?

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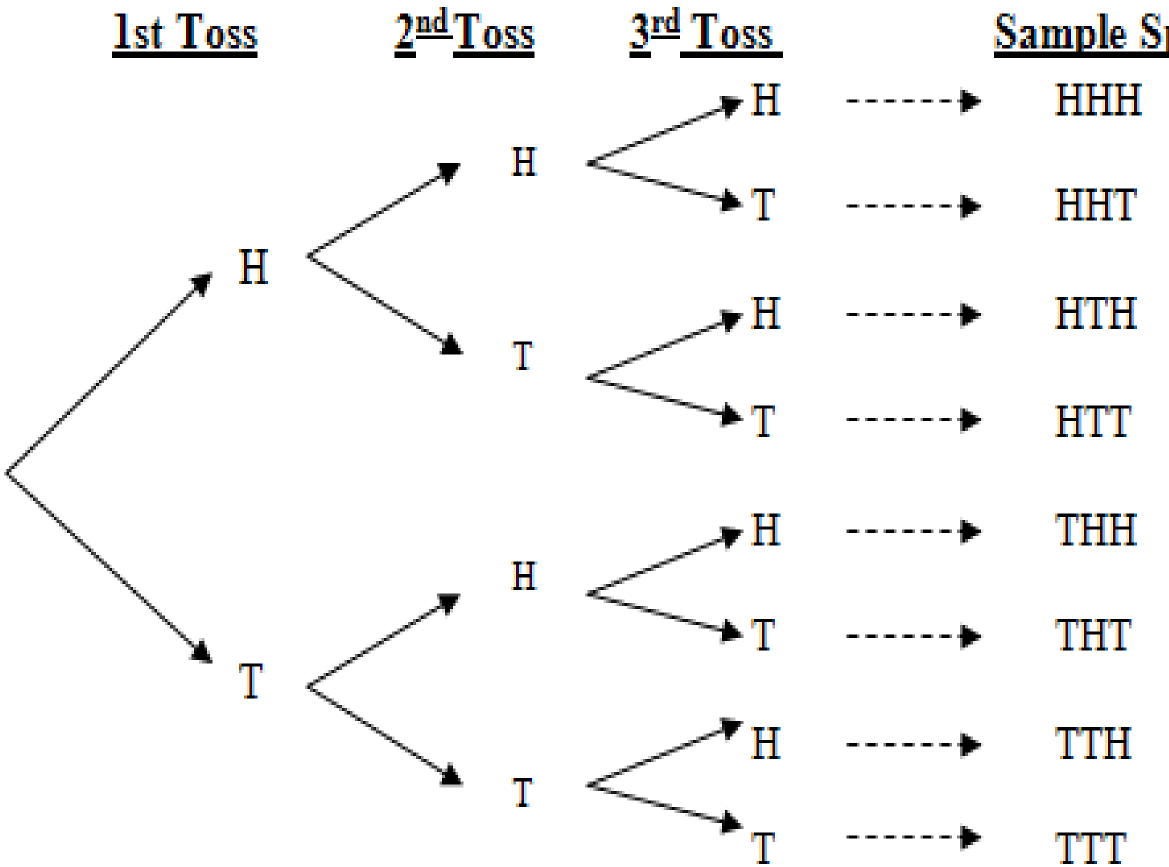
a) The number of possible outcomes in the sample space is $2^3 = 8$.

S= {HHH, TTT, HTT, THT, TTH, THH, HTH, HHT}



b) E_1= { HHH }, E_2={ TTT } ,------, E_8 = { HHT }

So we have 8 simple events.

c) A = { HTT, TTH, THT} is compound event.

d) B = { HHH, TTT} is compound event.

Probability Notation

- We use the notation P(event) = p to denote the probability of an event occurring and (1-p) it will not occur .

- For example
  - Probability of heads occurring is P(heads)
  - P(heads) = 0.4 indicates an unfair coin that turns up head 40% of the time.

- For the event A, event B, …
  - We will denote the probability of these happening as P(A), P(B), …

Definition

The odds in favour of an event A is defined by

$$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$$

That is, the odds of an event A tell how much more likely it is that the event A occurs than it is that it does not occur.

Example

If $P(A) = 2 / 3$ what are the odds of A (odds in favour of A)?

$1 - P(A) = 1/3$

$$\text{Odds }(A) = \frac{P(A)}{1 - P(A)} = 2:1$$

Roll Up the Rim, the Odds of Winning (1:6)

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Note

The odds against the event A is defined by

$$\frac{P(A^c)}{P(A)} = \frac{1 - P(A)}{P(A)}$$

Example

If the odds against a given horse winning a race are 20 to 1 or (20:1), what is the probability that the horse will win the race?

$$P(A) = 1/\ 21$$

- Let $S = \{ a_1, a_2, \ldots a_n \}$ be a sample space.
- Let $P(a_i)$, $i=1, 2, \ldots, n$ be the probabilities to the a_i's.
- The probability of an arbitrary compound event A can be determined by summing the probabilities of simple events in A.
- $A = \{ a_1, a_2, \ldots a_k \}$

If each simple event has probability 1/n (i.e. "equally likely").

$$P(A) = k / n$$

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Example

Suppose a 6-sided fair die is rolled, what is the probability of getting an even number ?

Let A = “even number"

$A = \{2, 4, 6\}$

$P(A) = P(2) + P(4) + P(6) = 1/2.$

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Rule of Probability

- Rule 1

$$\sum_{\text{all } i} P(a_i) = 1, \; P(S) = 1$$

The set of probabilities $P(a_i)$, $i=1,2\ldots$ is called probability distribution on S.

Proof:

$$P(S) = \sum_{a \in S} P(a) = \sum_{\text{all } a} P(a) = 1$$

- Rule 2

For any event A, $0 \leq P(A) \leq 1$

Probabilities are always between 0 and 1

0: event never happens,

1: event always happens.

Proof:

$$P(A) = \sum_{a \in A} P(a) \leq \sum_{a \in S} P(a) = 1$$ and since each

$P(a) \geq 0$, we have $0 \leq P(A) \leq 1$

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- Rule 3

If A and B are two events with $A \subseteq B$ (that is, all of the points in A are also in B), then $P(A) \leq P(B)$

Proof:

$$P(A) = \sum_{a \in A} P(a) \leq \sum_{a \in B} P(a) = P(B) \quad \text{so } P(A) \leq P(B)$$

Example

The probability of a random student in our class being under the age of 20 must be smaller than or equal to the probability of a random student being under the age of 25.

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Mutually Exclusive or Disjoint Events

- Two events are mutually exclusive
  - If they cannot happen simultaneously.
  - If they cannot occur at the same time.
  - When they have no outcomes in common .
  - $A \cap B = \emptyset$
- Another word that means **mutually exclusive** is **disjoint**.

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Example

Tossing a coin once, which can result in either heads or tails, but not both.

Example

Rolling a die ones:

- Rolling a 3 and rolling a 5 is mutually exclusive.
- Rolling a 3 and rolling an odd number is **not** mutually exclusive.

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