STAT230

PROBABILITY

Chapter 8 (b)

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The Uniform Distribution from *a* to *b* or (Rectangular Distribution)

- A continuous random variable , X, is said to have a **Uniform** distribution from a to b on the interval [a, b], *U*(a, b), if all subintervals of a fixed length being equally likely

$$f(x)=\begin{cases}\dfrac{1}{b-a} & a\le x\le b\\ 0 & \text{otherwise}\end{cases}$$

$$\mu=E(X)=\frac{(a+b)}{2} \quad \text{and} \quad \sigma^2=V(X)=\frac{(b-a)^2}{12}$$

X ~ U(a, b)

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The Cumulative Distribution function, $F(x)$ (Uniform Distribution from a to b)

$$F(x) = P[X \le x] = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

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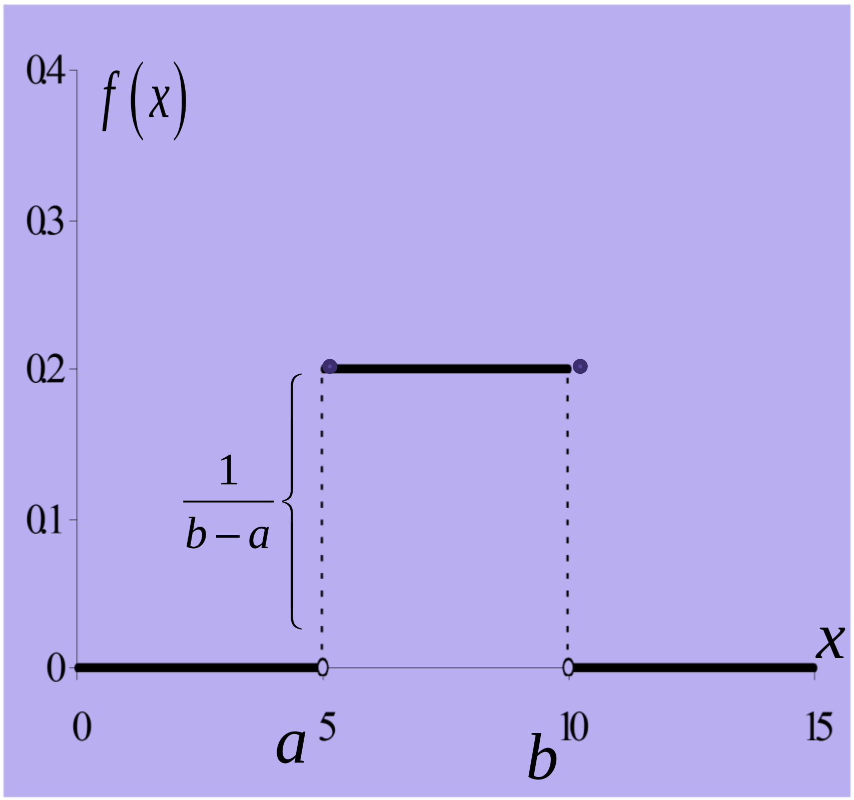
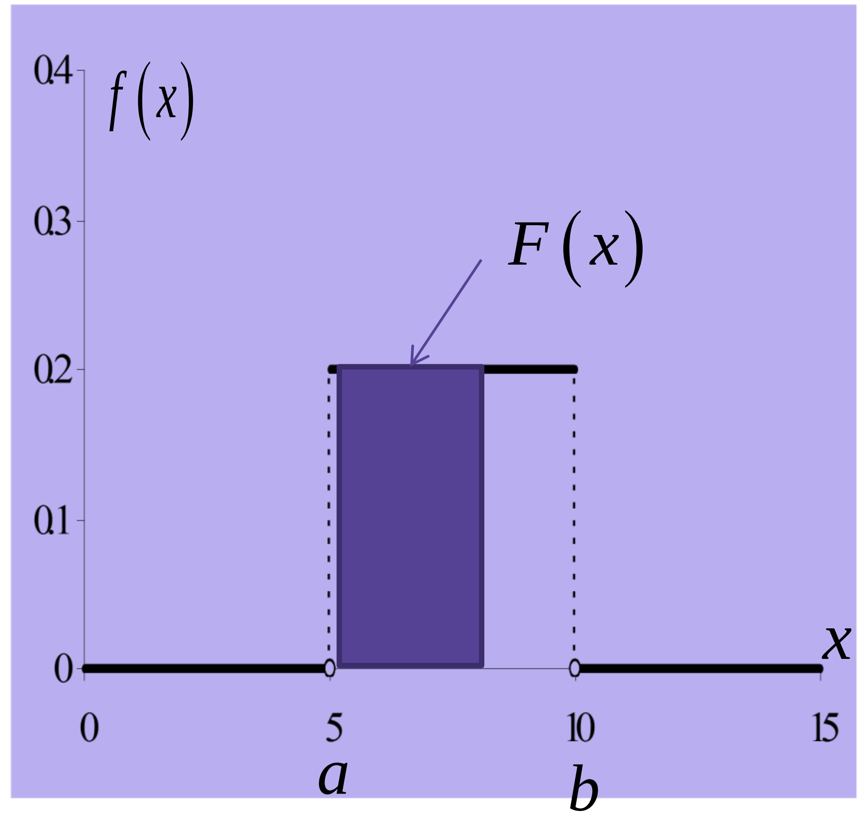
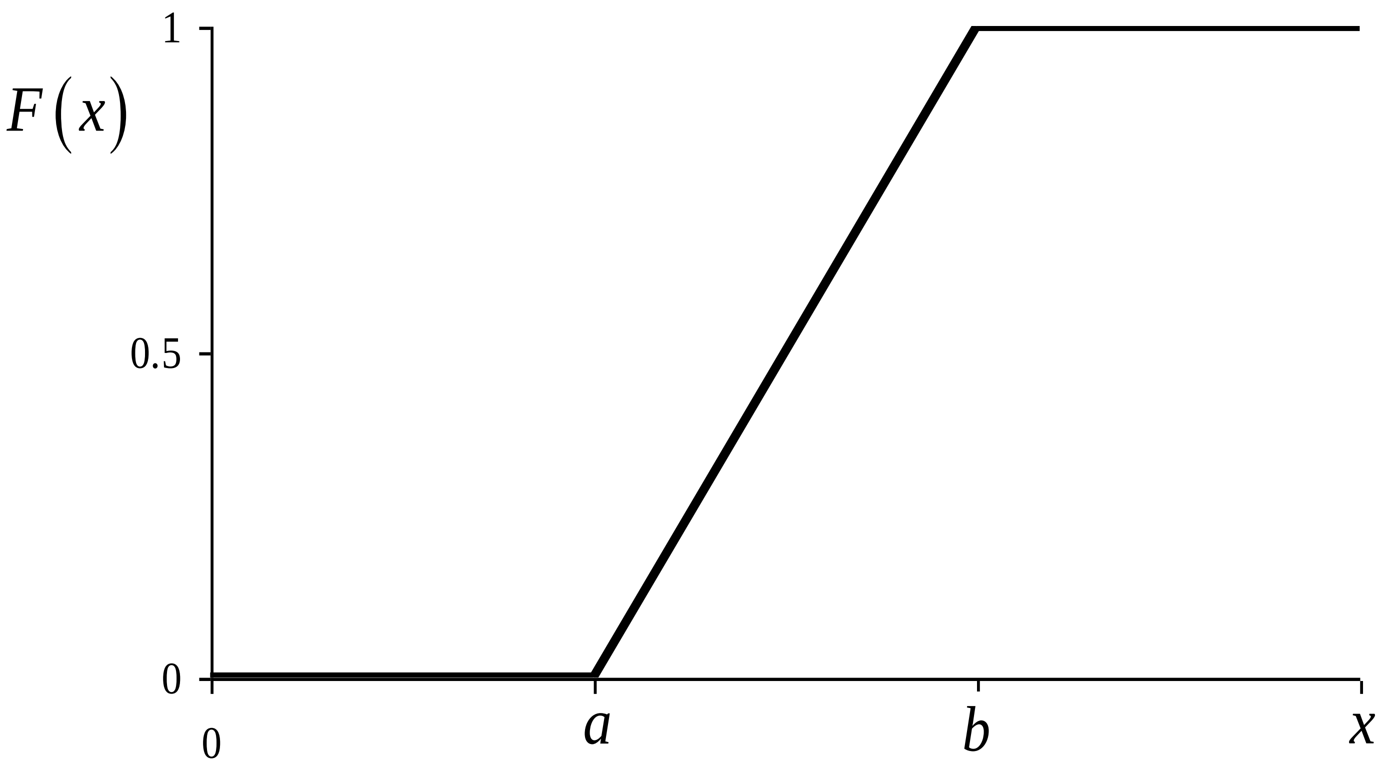
For each x, F(x) is the area under the density curve to the left of x.





Cumulative Distribution function, *F*(*x*)



Example

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes (mA) is Uniformly distributed . Assume that the range of X is $0 \leq x \leq 10$

- What is the probability that a current is less than 5mA?
- What is the cumulative distribution function?

$$f(x) = \frac{1}{b-a}$$

$$= \frac{1}{10-0}$$

$$f(x) = \begin{cases} 0.1\,0 & \text{for } 0 \le x \le 10 \\ 0 & \text{o/w} \end{cases}$$

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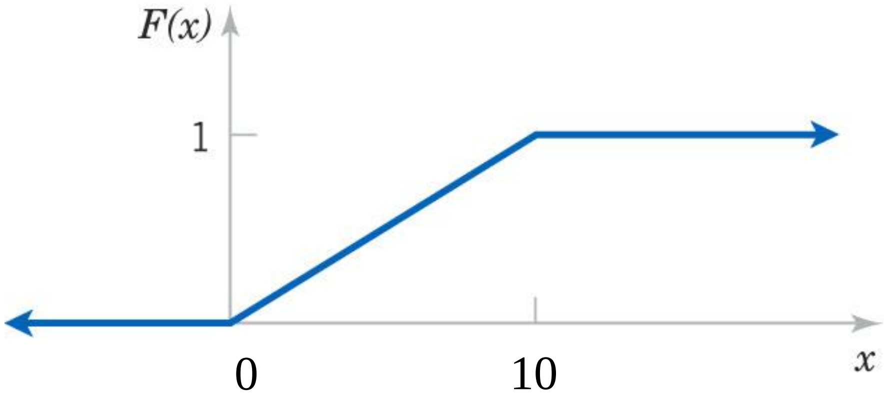
$$P(X < 5) = \int_0^5 f(x)\,dx = \int_0^5 0.1\,dx = [\,0.1\,x\,]_0^5 = 0.5$$

$$F(x) = \frac{x-a}{b-a} \qquad a \le x \le b$$

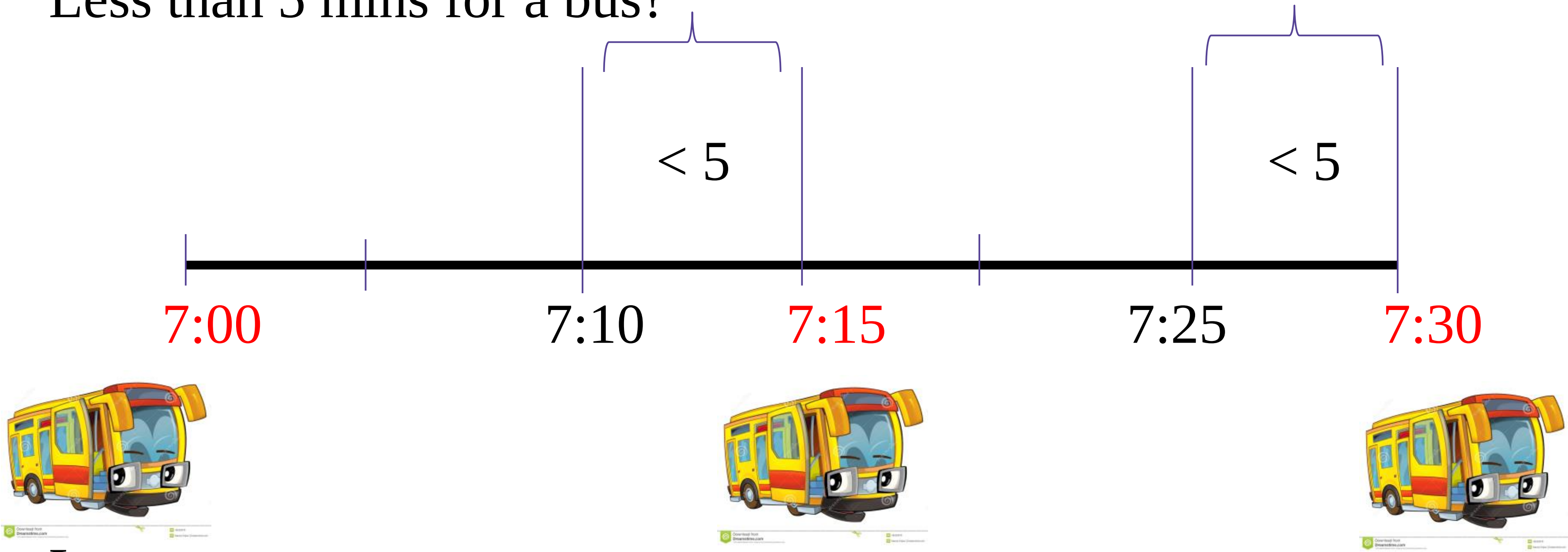
$$F(x) = \begin{cases} 0 & x < 0 \\ 0.1\,x & 0 \le x < 10 \\ 1 & 10 \le x \end{cases}$$



Example

Buses arrive at a specified stop at 15-min intervals starting at 7am. That is, they arrive at 7, 7:15, 7:30, 7:45 and so on. If a passenger arrives at the stop at a time that is Uniformly distributed between 7 and 7:30, find the probability that he waits less than 5 mins for a bus. Note: do not count before the 7:00am.

Less than 5 mins for a bus?



Let

X: number of minutes pass 7:00 am that a passenger arrive at the stop.

$X \sim \text{Uniform}(0, 30)$

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Example (8.3 Course Notes)

Let X have probability density function

$$f(x) = \begin{cases} \dfrac{1}{20} & -10 \le x \le 10 \\ 0 & otherwise \end{cases}$$

Show that Y = (X+10)/20 ~ U(0, 1)

$F_Y(\mathrm{y}) = \mathrm{G(y) = P(\ Y \le y)}$

$= \mathrm{P\ (\ (\ X{+}10)/20\ \ \le y)}$

$= \mathrm{P\ (\ X \le 20y\ {-}10\ )}$

$= \mathrm{F_x(\ 20y - 10)}$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = f(20y - 10)\,\frac{d}{dy}\,(20y - 10)$$

$$f_Y(y) = \left(\frac{1}{20}\right)20 = 1$$

Which is the p.d.f of a U(0, 1) random variable.

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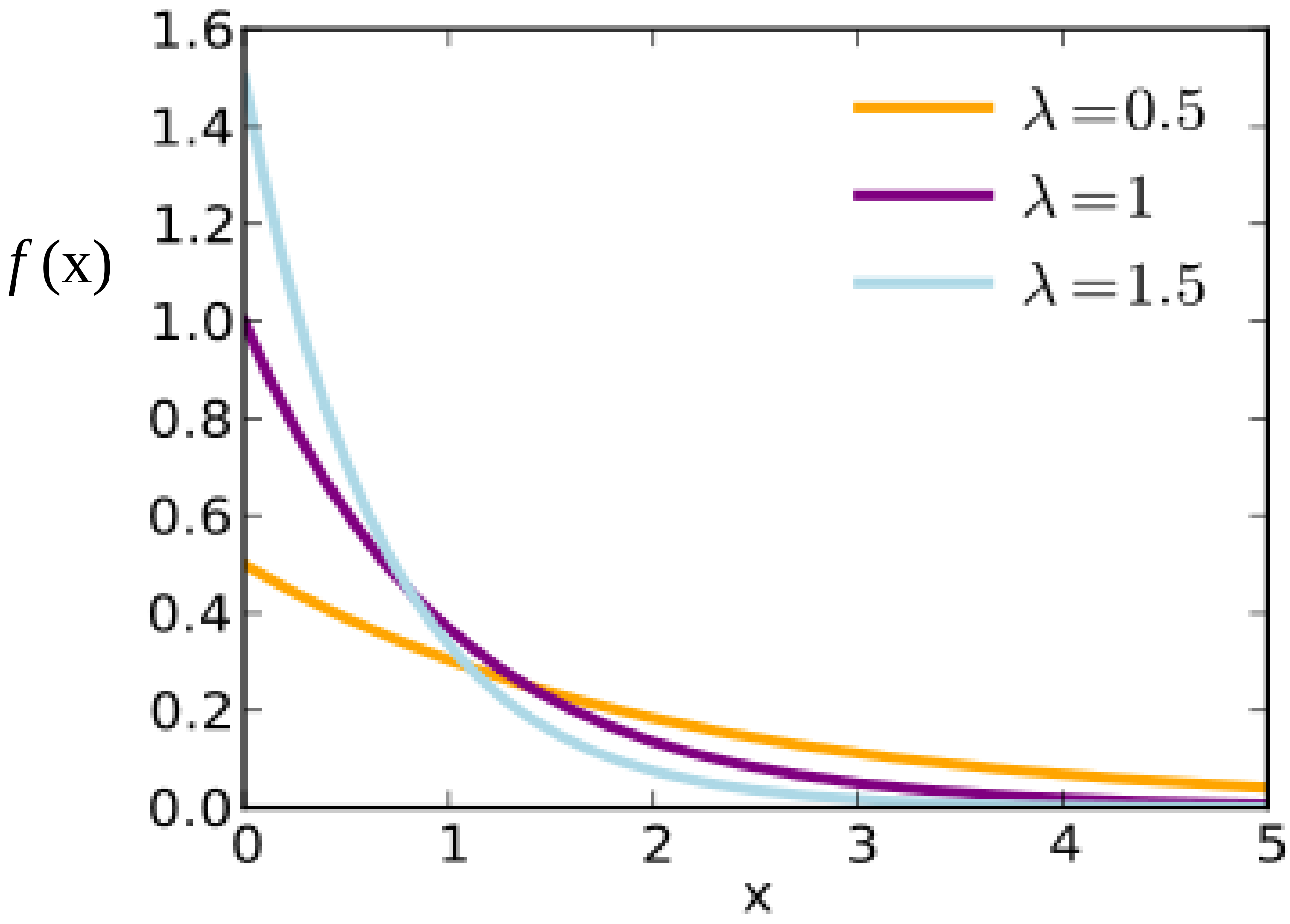
Exponential Distribution

- Widely used in engineering and science disciplines.
- X is said to have Exponential distribution if the p.d.f. of X is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

- Where $\lambda > 0$ is a real parameter value.



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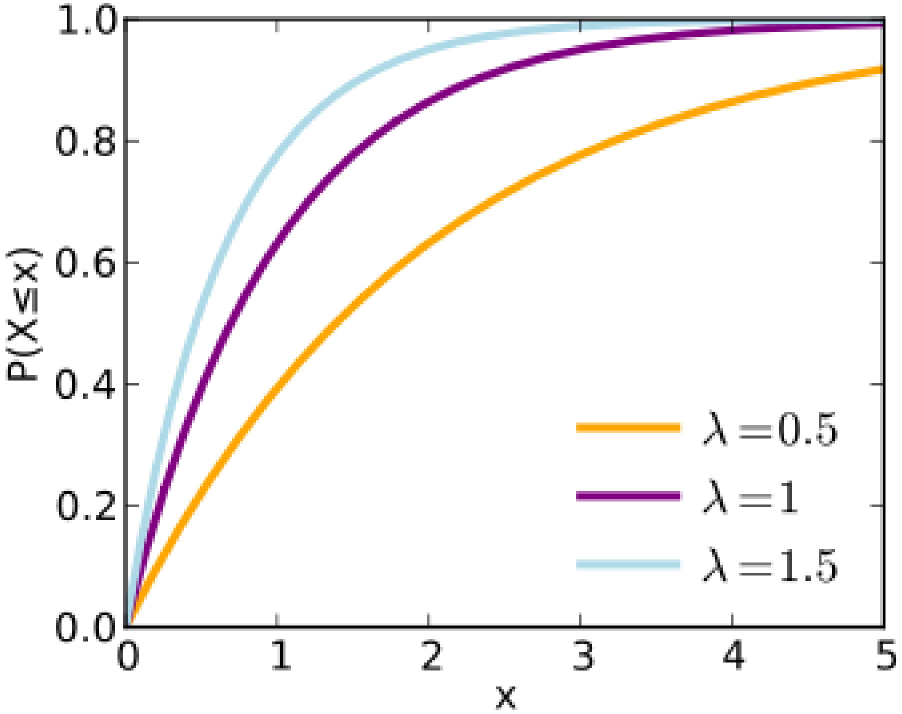
- The cumulative distribution function is given by

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$



In this case,

$$\lambda = \frac{1}{\mu}$$

- Exponential distribution is used to describe the time or distance until some event happens.

- In Poisson process for events in time, let X be the length of time we wait for the first event occurrence.

Applications of the Exponential distribution:

- The length of time between phone calls to a specific hospital (assuming calls follow a Poisson process).

- Waiting time for elevator to come.

Derivation of p.d.f. and c.d.f.

X = the waiting time until the 1st event in a Poisson process occurred.

$F(x) = P(X \le x) = 1 - P(X > x)$

$= 1 - P(\text{time to 1st occurrence is } > x)$

$= 1 - P(\text{no occurrences in the interval } (0, x))$

$$= 1 - \frac{(\lambda x)^0 e^{-\lambda x}}{0!}$$

$$= 1 - e^{-\lambda x} \quad \text{for } x > 0$$

The p.d.f. of X = waiting time until the next occurrence in a Poison process with rate λ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Mean and Variance

The mean or expected value of an exponentially distributed random variable *X* with rate parameter λ is given by

$$E(X) = \mu = 1 / \lambda$$

The variance of *X* is given by

$$Var(X) = 1 / \lambda^2$$

$$SD(X) = 1 / \lambda$$

Alternate Form

It is common to use the parameter $\theta = 1/\lambda$ in the Exponential distribution $X \sim \text{Exp}(\theta)$

$$E(X) = \mu = \theta$$

$$f_X(x) = \begin{cases} \dfrac{1}{\theta} e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Note

- If you are given the average rate of occurrence in a Poisson process, then this is the parameter λ.

- If you are given the average waiting time for an occurrence, then this is the parameter θ.

Example

The time between arrivals of cars at gas pump station follows an Exponential probability distribution with a mean(average) time between arrivals of 3 minutes.

What is the probability that the time between two successive arrivals will be 2 minutes or less.

$$P(X \leq 2) = ?$$

$$\theta = 3$$

$$F(x;\theta) = \begin{cases} 1 - e^{-x/\theta} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

$$\boxed{P(X \leq 2) = 1 - 2.71828^{-2/3} = 1 - .5134 = .4866}$$

Example

Suppose the response time X at a certain on-line computer terminal (the elapsed time between the end of a user's inquiry and the beginning of the system's response to that inquiry) has an Exponential distribution with expected response time equal to 5 sec.

(a) What is the probability that the response time is at most 10 seconds?

(b) What is the probability that the response time is between 5 and 10 seconds?

(c) What is the value of x for which the probability of exceeding that value is 1%?

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The $E(X) = \theta = 5$ so $\lambda = 0.2$.

a) The probability that the response time is at most 10 sec is:

$$\begin{aligned} P(X \leq 10) &= F(10, 0.2) \\ &= 1 - e^{-(.2)(10)} \\ &= 1 - 0.135 \\ &= 0.865 \end{aligned}$$

b)

$$
\begin{aligned}
P(5 \le X \le 10) &= F(10;0.2) - F(5;0.2) \\
&= (1 - e^{-2}) - (1 - e^{-1}) \\
&= 0.233
\end{aligned}
$$

c) What is the value of x for which the probability of exceeding that value is 1%?

$$P(X > x) = 0.01$$

$$P(X > x) = 1 - P(X \leq x) = 0.01$$

$$P(X \leq x) = 0.99$$

$$1 - e^{-\lambda x} = 0.99$$

$$e^{-\lambda x} = 0.01$$

$$-\lambda x = \ln(0.01)$$

$$x = \frac{4.605}{0.2}$$

$$x = 23.025\,\sec$$

Lack of Memory Property(Memoryless)

$$P(X > c + b \mid X > b) = P(X > c)$$

For a Poisson process, given that you have waited b unites of time for the next event, the probability you wait an additional c time does not depend on b but only depends on c.

Example

Suppose buses arrive at a bus stop according to a Poisson process with an average (rate) of 5 buses per hour. (i.e. λ= 5/hr.

So θ = 1 /5 hour or 12 minutes).

Find the probability

(a) You have to wait longer than 15 minutes for a bus.

(b) You have to wait more than 15 minutes longer, having already waited for 6 minutes.

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a) $P(X > 15) = 1 - P(X \leq 15)$

$= 1 - F(15)$

$= 1 - (1 - e^{-15/12})$

$= 0.2865$

b) If X is the total waiting time, the question asks for the probability

$$P(X > (15+6) \mid X > 6) = \frac{P(X > 21 \text{ and } X > 6)}{P(X > 6)}$$

$$= \frac{P(X > 21)}{P(X > 6)}$$

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$$P(X > 21|X > 6) = \frac{1 - (1 - e^{-21/12})}{1 - (1 - e^{-6/12})}$$

$$= \frac{e^{-21/12}}{e^{-6/12}}$$

$$= e^{-1.25}$$

$$= 0.2865$$

Does this surprise you?

The fact that you're already waited 6 minutes doesn't seem to matter.

$$P(X > c + b \mid X > b) = P(X > c)$$

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Example

Suppose $U \sim$ U(0,1).

Find the p.d.f. of Y = $-\theta$ ln(1-U).

OR

Show that Y~ Exponential (θ).

If $U \sim \mathrm{U}(0,1)$ then

$f(u) = 1$ if $0 \leq u \leq 1$ and $f(u) = 0$ otherwise.

The c.d.f. of $Y = -\theta \ln(1-U)$ is

$$
\begin{aligned}
G(y) = P(Y \le y) = P[-\theta \ln(1-U) \le y] \\
&= P[\ln(1-U) \ge -y/\theta] \\
&= P(1-U \ge e^{-y/\theta}) = P(-U \ge e^{-y/\theta} - 1) \\
&= P(U \le 1 - e^{-y/\theta}) \\
&= F_U(1 - e^{-y/\theta})
\end{aligned}
$$

where F is the c.d.f. of U.

The p.d.f. of Y is

$$g(y) = G'(y)$$

$$= f(1-e^{-y/\theta}) \cdot \frac{d}{dy}(1-e^{-y/\theta})$$

$$= (1)\left(\frac{1}{\theta}e^{-y/\theta}\right)$$

$$= \frac{1}{\theta}e^{-y/\theta} \quad \text{for } y > 0$$

and 0 otherwise which is the p.d.f. of an Exponential(θ) random variable. Thus $Y \sim$ Exponential(θ).

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Mean and Variance

- Finding μ and σ^2 directly involves integration by parts.

- An easier solution uses properties of Gamma functions.

Gamma Function

$\Gamma(\alpha)$, called the gamma function of α where $\alpha > 0$, is defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x}\, x^{\alpha-1}\, dx$$

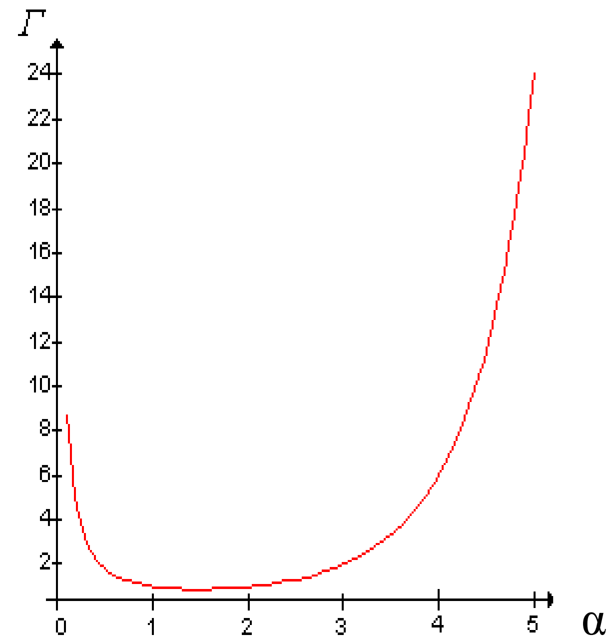
Integration for $\Gamma(\alpha)$ by parts yields

(1) $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$

(2) $\Gamma(\alpha) = (\alpha-1)!$ If α is a positive integer.

$\Gamma(1) = 1$

(3) $\Gamma(1/2) = \sqrt{\pi}$



Gamma function on the interval (0,5)

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Example

Show that the mean of the Exponential distribution is θ.

$$\mu = \int_{-\infty}^{\infty} x\, f(x) dx = \int_{0}^{\infty} x\, \frac{1}{\theta}\, e^{-x/\theta}\, dx$$

$$let\ y = \frac{x}{\theta}\ with\ dx = \ \theta\, dy$$

$$\mu = \int_{0}^{\infty} y\, e^{-y}\, \theta\, dy$$

$$= \theta \int_{0}^{\infty} y^{1}\, e^{-y}\, dy = \ \theta\Gamma(2) = \ \theta(1!)$$

$$= \theta$$

Gamma Distribution

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} \, x^{\alpha-1} \, e^{-\frac{x}{\beta}} & x \geq 0 \\ 0 & otherwise \end{cases}$$

With a shape parameter $\alpha > 0$ and a scale parameter $\beta > 0$.

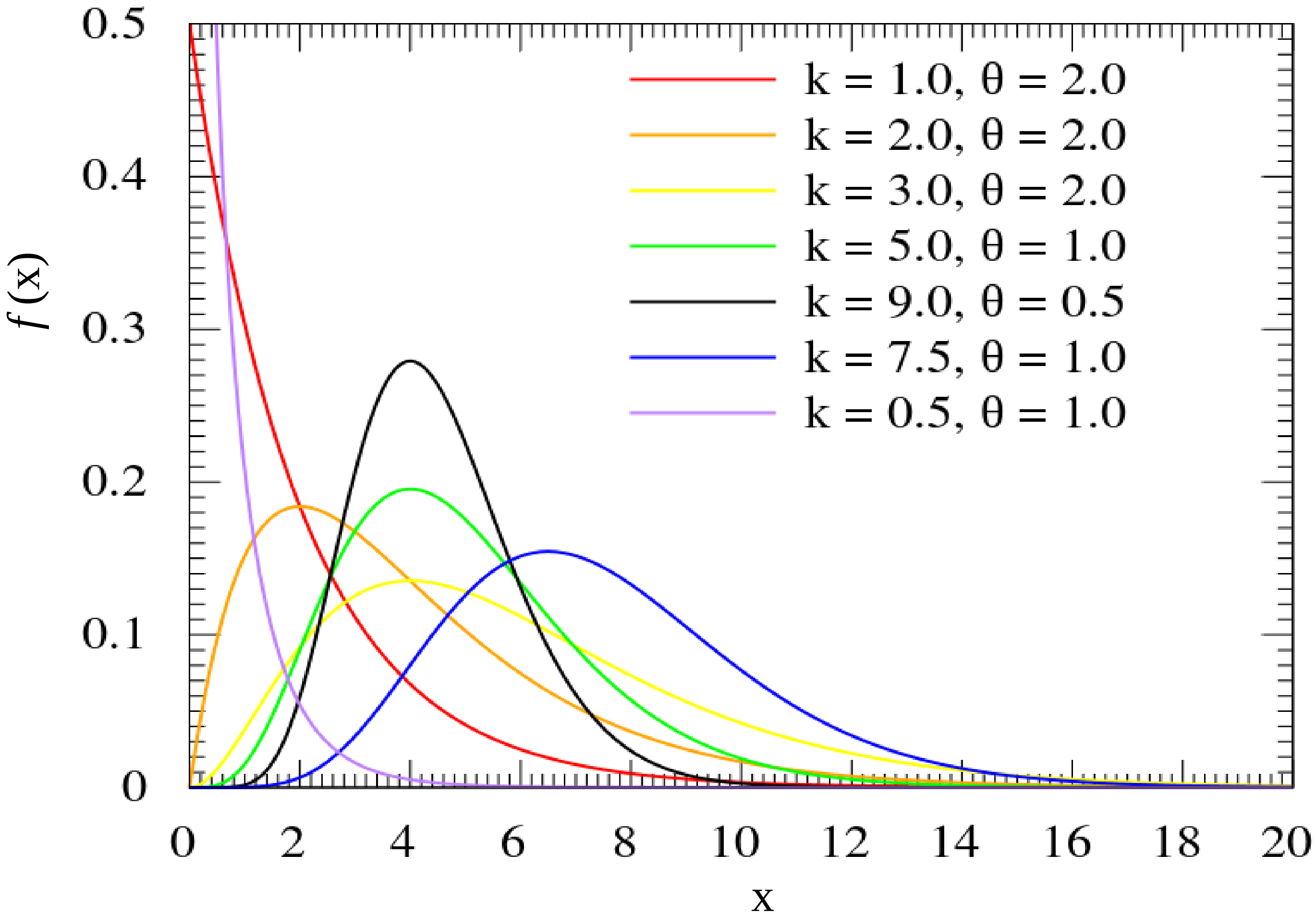
- The gamma distribution is a two-parameter family of continuous probability distributions.

- The common Exponential distribution and Chi-squared distribution are special cases of the Gamma distribution.

- $E(X) = \alpha\ \beta$

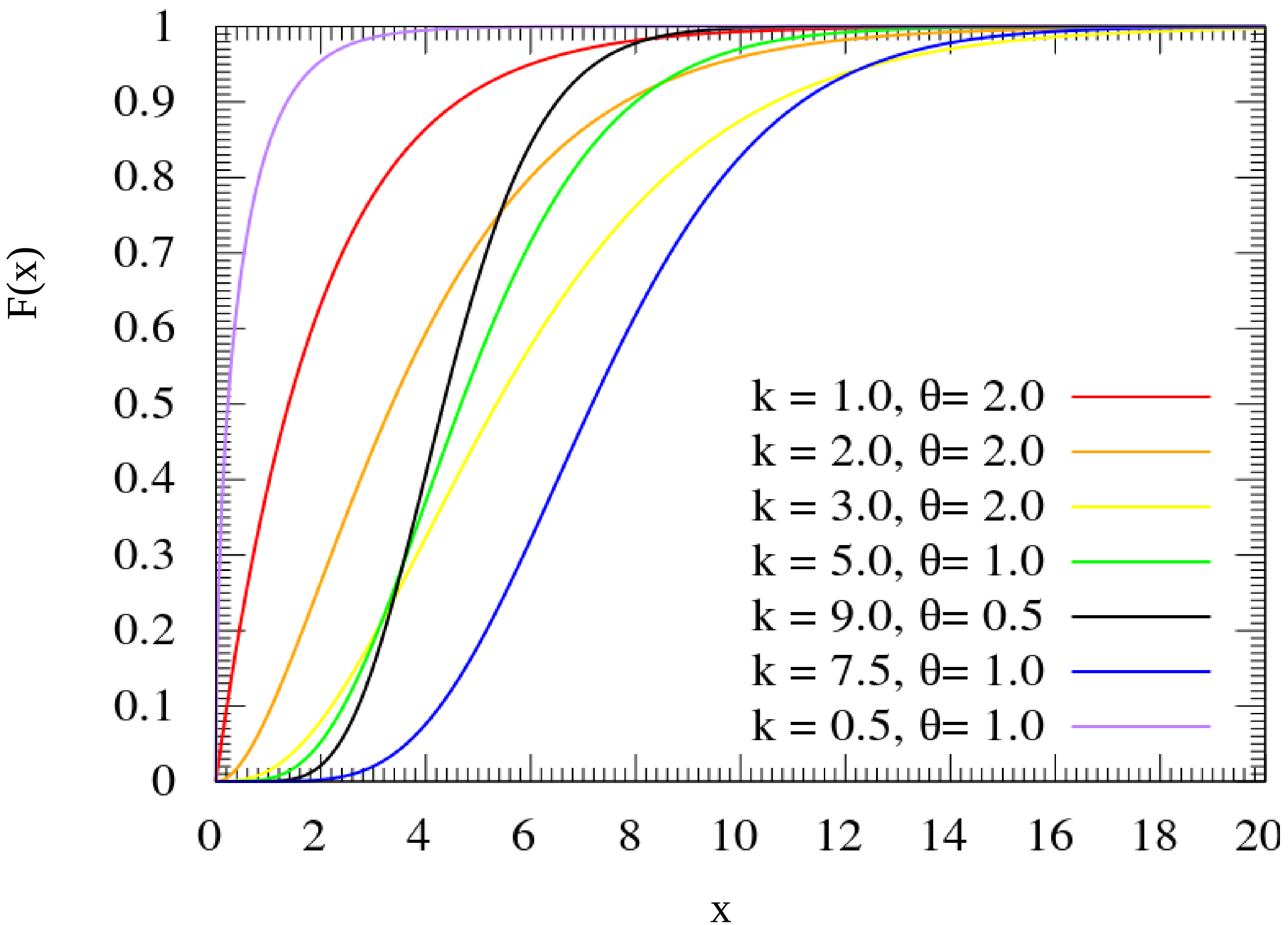
- $Var(\ X) = \alpha\ \beta^2$

Probability density plots of Gamma distributions



With a shape parameter $\alpha = k$ and an inverse scale parameter $\theta = 1/\beta$, called a rate parameter.

Cumulative distribution plots of Gamma distributions



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Applications

The gamma distribution can be used a range of disciplines including climatology, and financial services.

- In life testing, the waiting time until death.
- The amount of rainfall accumulated in a reservoir.
- The size of loan defaults or aggregate insurance claims.
- The flow of items through manufacturing and distribution processes.

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Omit 8.4

The Normal Distribution

A random variable, X, is said to have a Normal distribution with mean μ and variance σ^2, if X is a continuous random variable with probability density function $f(x)$:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\sigma > 0$$
$$-\infty < x < +\infty$$
$$-\infty < \mu < +\infty$$

The Normal distribution is also often denoted by

$$X \sim N(\mu, \sigma^2)$$

π =3.14159, e = 2.71828

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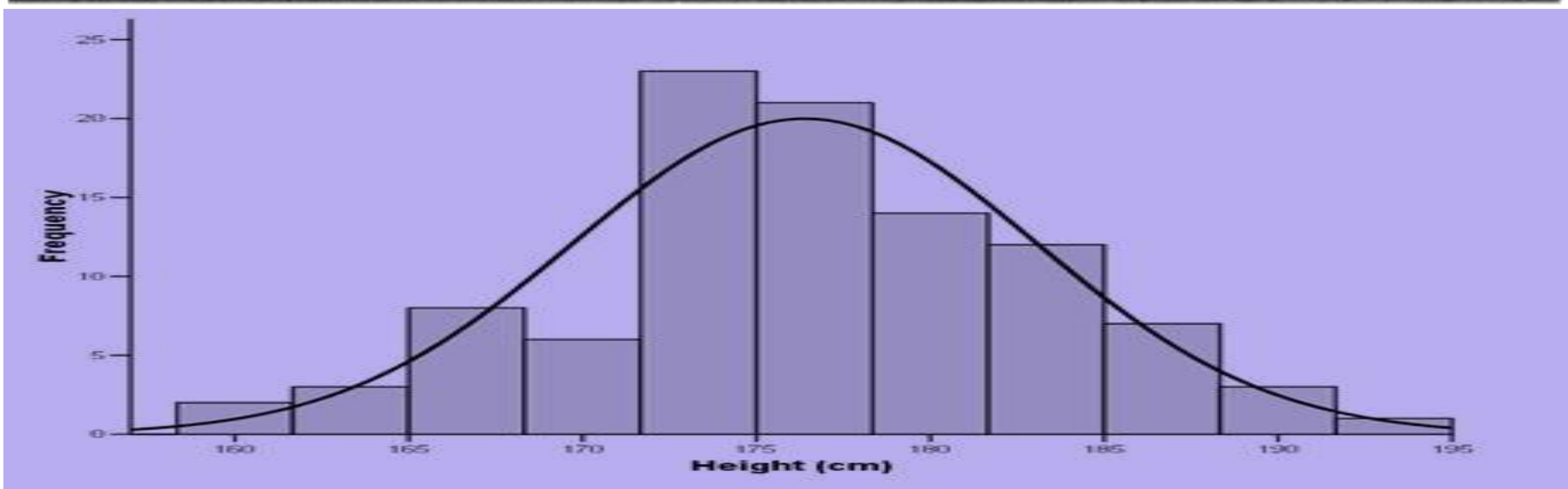
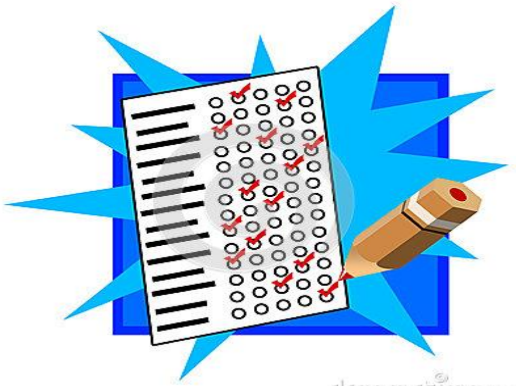
Examples of the Normal Distribution

Many populations of measurements follow approximately a Normal distribution:

Heights of People (males or females)



- IQ and test scores.



http://thumbs.dreamstime.com/x/test-scores-28179353.jpg

- Average rainfall per year.

http://www.clker.com/cliparts/a/a/6/4/12284285206675828 58sivvus_weather_symbols_4.svg.med.png

- Individual resting pulse rates.

https://myhealth.alberta.ca/HEALTH/_layouts/healthwise/media/medical/hw/hwkb17_071.jpg

- … and many other examples.

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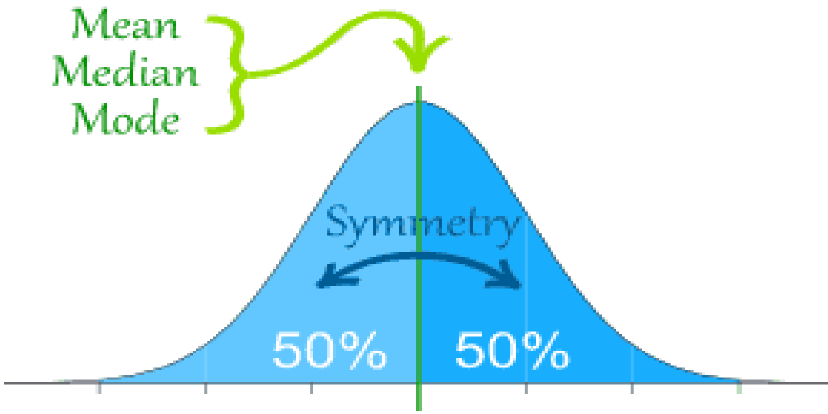
The Bell-Curve

- The bell-curve appears frequently in nature and everyday life.

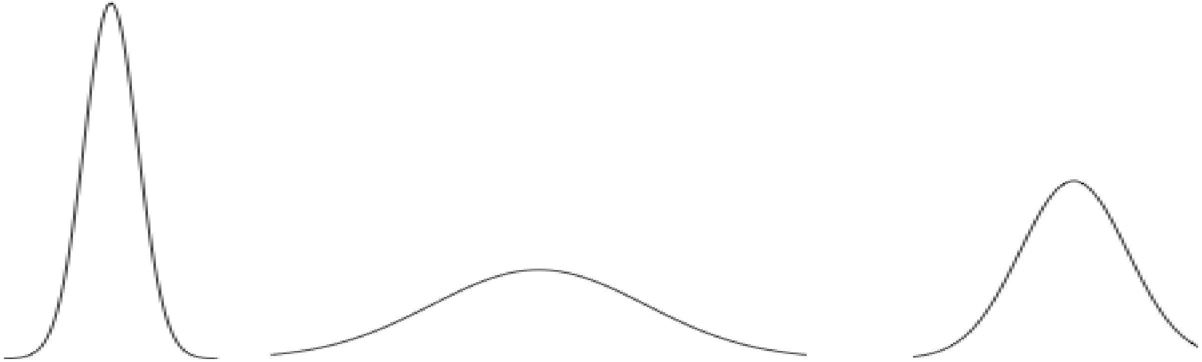
  We call it a

  - Normal curve.
  - Normal distribution

    "bell-curve", N(μ, σ^2).
  - Gaussian curve, G(μ, σ)



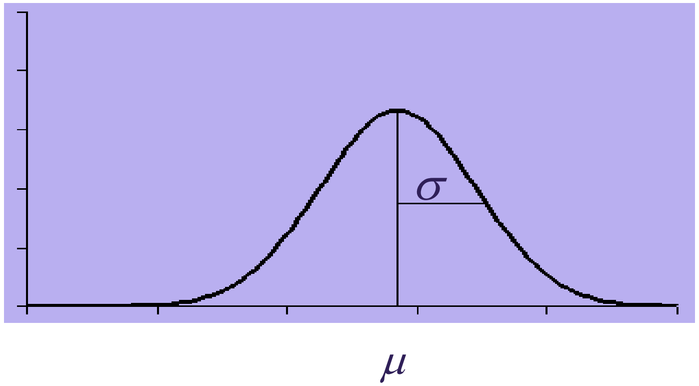
- The theoretical Normal curve comes in different shapes, but is always symmetric, unimodal, and unskewed.



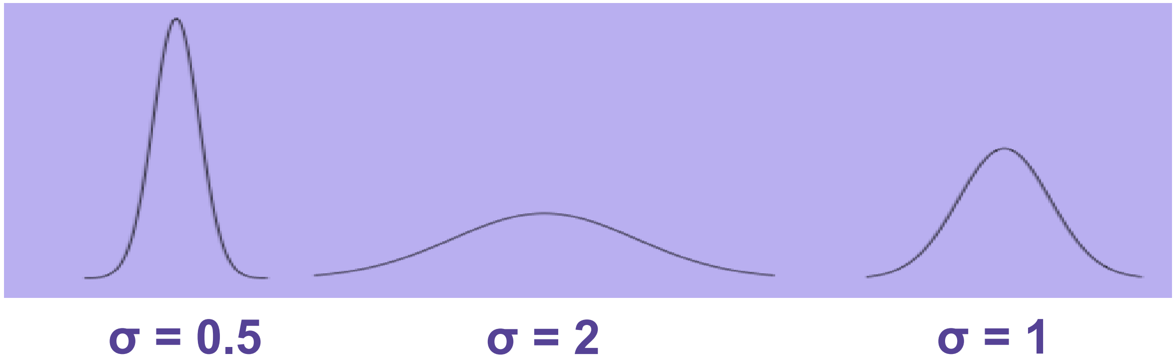
Describing the Normal Curve

We describe a Normal curve with two numbers:

- Mean (μ): the center of the curve.



- Standard deviation (σ): the “fatness” of the curve



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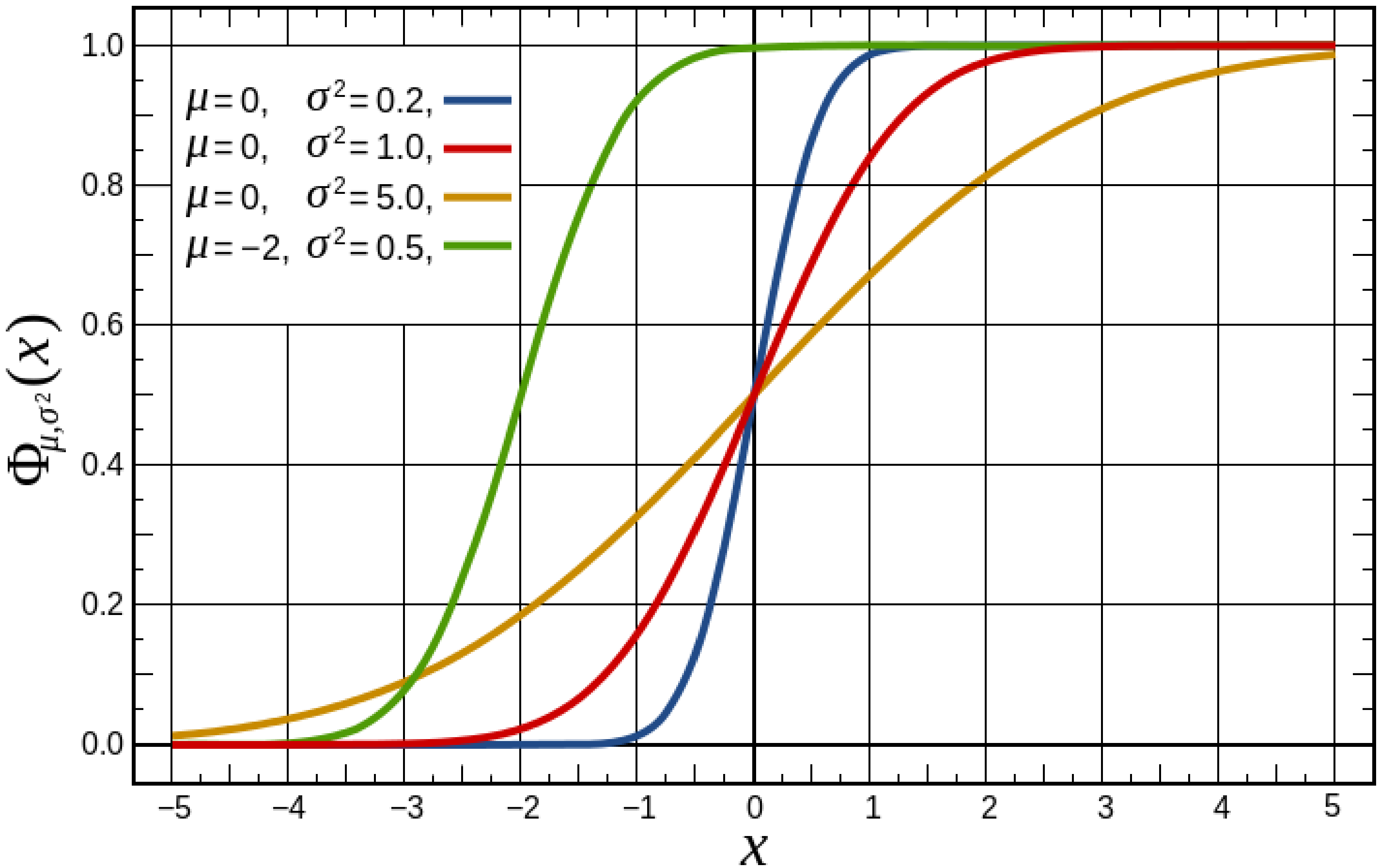
$$F(x) = \int_{-\infty}^{x} f(y)\, dy$$



The Standard Normal Distribution

- The Normal distribution with parameter values $\mu = 0$ and $\sigma = 1$ is called a standard Normal distribution.

- A r.v. has a standard normal distribution is called a standard Normal random variable and denoted by Z.

- $Z \sim N(0,1)$ where $\mu = 0$ and $\sigma = 1$ has probability density function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad for\ z \in R$$

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Standardized Score

- Also known as "standard score" or "z-score"

- The standardized score is a number that represents the number of standard deviations a data point is from the mean.

$$\text{z-score} = \frac{\text{observed value} - \text{mean}}{\text{standard deviation}}$$

$$z = \frac{x - \mu}{\sigma}$$

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Example

IQ scores have a **Normal distribution** with a **mean** of **100** and a **standard deviation** of **16.**

- Suppose your IQ **score** was **116.**

Standardized score =

http://www.mynamesnotmommy.com/wp-content/uploads/2013/05/question-mark.png

- Suppose your IQ **score was 84.**

Standardized score =

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- Suppose your IQ **score** was **116.**

Standardized score = (116 – 100)/16 = +1

Your IQ is 1 standard deviation ***above the mean*.**

- Suppose your IQ **score was 84.**

Standardized score = (84 – 100)/16 = –1

Your IQ is 1 standard deviation ***below the mean*.**

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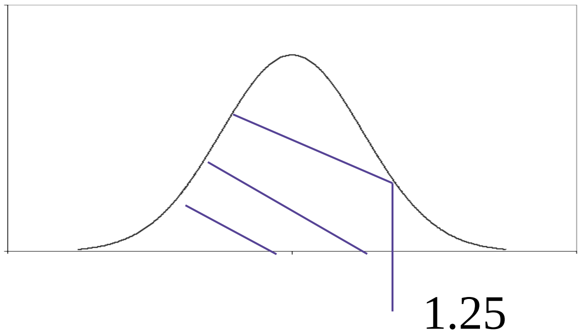
Example

Compute the following probabilities



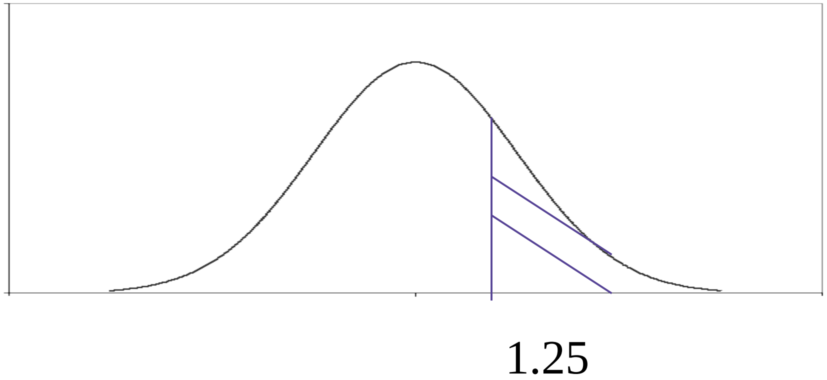
(a) $P(Z \leq 1.25) =$

$\Phi(1.25)=0.89435$

(b) $P(Z > 1.25) =$

$1- P(Z \leq 1.25) = 1 - \Phi(1.25) = 1- 0.89435$

$=0.10565$



Note

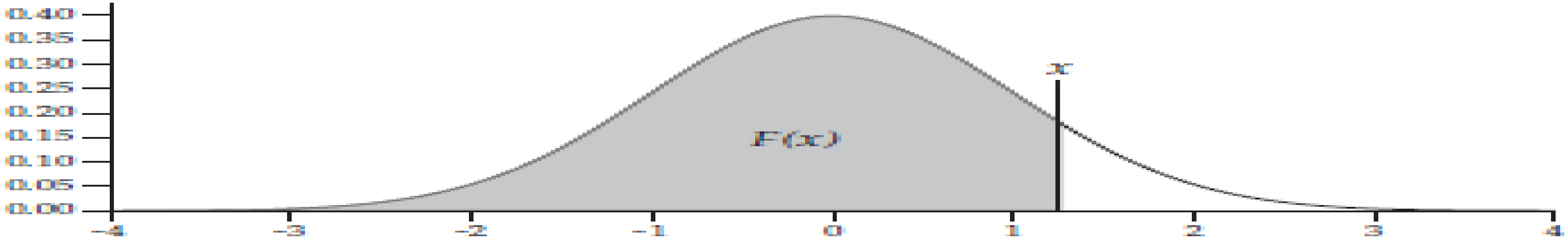
- $\Phi(1.25)$ is just a notation means get the value from the Table.

You can use $Z(1.25)$ as notation instead or

- $F(1.25)$ or

- just leave it as $P(Z \leq 1.25)$ and get the value from the table and this is what we have in our Course Notes.

Probabilities for Standard Normal $N(0,1)$ Distribution



This table gives the values of $F(x)$ for $x \geq 0$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56750	0.57142	0.57534
0.2	0.57926	0.58317	0.58706	0.59095	0.59484	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189

(c) $P(Z \le -1.25) =$

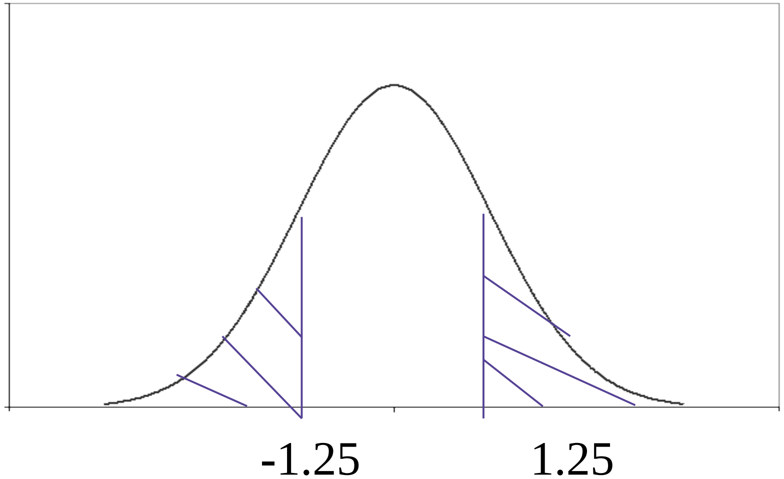
$P(Z > 1.25) = 1- P(Z < 1.25)$

(By symmetry of the Normal curve)

$= 1 - \Phi(1.25)$

$= \Phi(-1.25)$

$= 0.1056\ 5$



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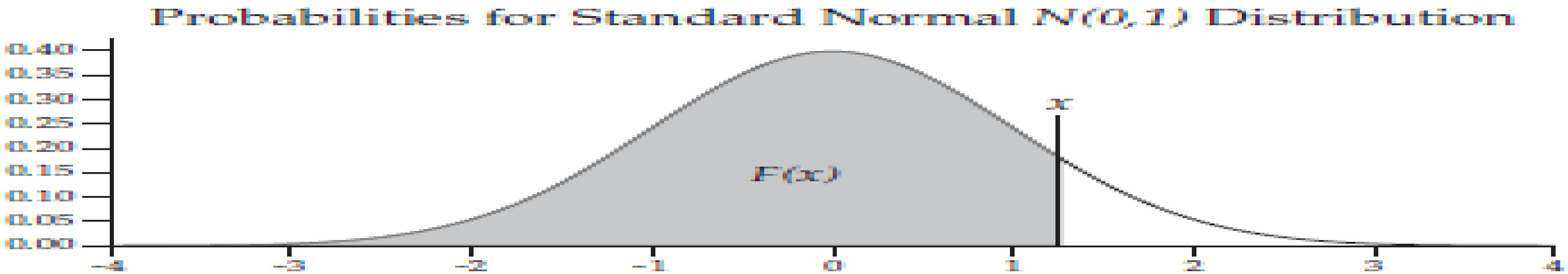

This table gives the values of $F(x)$ for $x \geq 0$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56750	0.57142	0.57534
0.2	0.57926	0.58317	0.58706	0.59095	0.59484	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189

(d)

$P(\ -0.38 \ \leq \ Z \ \leq \ 1.25\ )$

$= P(Z \leq 1.25) - P(Z \leq -0.38)$

$= P(Z \leq 1.25) - [\ 1 - P(Z \leq 0.38)\ ]$

$P(Z > 0.38) =$

$= 0.89435 - [1 - 0.64803]$

$= 0.89435 - 0.35197$

$= 0.54238$

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Nonstandard Normal Distribution

The lifetime of a battery is Normally distributed with a mean life of 40 hours and a standard deviation of 1.2 hours. Find the probability that a randomly selected battery

a) Lasts between 42 to 43 hours.

$$P(42 \leq X \leq 43) = P((42-40)/1.2 \leq Z \leq (43-40)/1.2)$$

$$P(1.67 \leq Z \leq 2.5) = \Phi(2.5) - \Phi(1.67)$$

$$= 0.99379 - 0.95254$$

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b) Lasts longer than 42 hours

$P(X > 42)$

$= 1 - p(X \leq 42)$

$= 1 - P(Z \leq (42-40)/1.2)$

$= 1 - P(Z \leq 1.67)$

$= 1 - \Phi(1.67)$

$= 1 - 0.95254$

$= 0.04746$

(approximately 4.8%)

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This table gives the values of $F(x)$ for $x \geq 0$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56750	0.57142	0.57534
0.2	0.57926	0.58317	0.58706	0.59095	0.59484	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899

Percentiles

- Percentile: the value for which a percentage of data falls below.

Examples:

- If you are in the 50^{th} percentile for height.
  - 50% of people are shorter than you.
  - 50% of people are taller than you.

- If you are in the 84^{th} percentile for height.
  - 84% of people are shorter than you.
  - 16% of people are taller than you.

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Percentiles and Standardized Scores

- **Finding percentiles for Normal curves requires:**

1- Your **own value.**

2- The **mean for the population of values.**

3- The **standard deviation for the population(**s.d)**.**

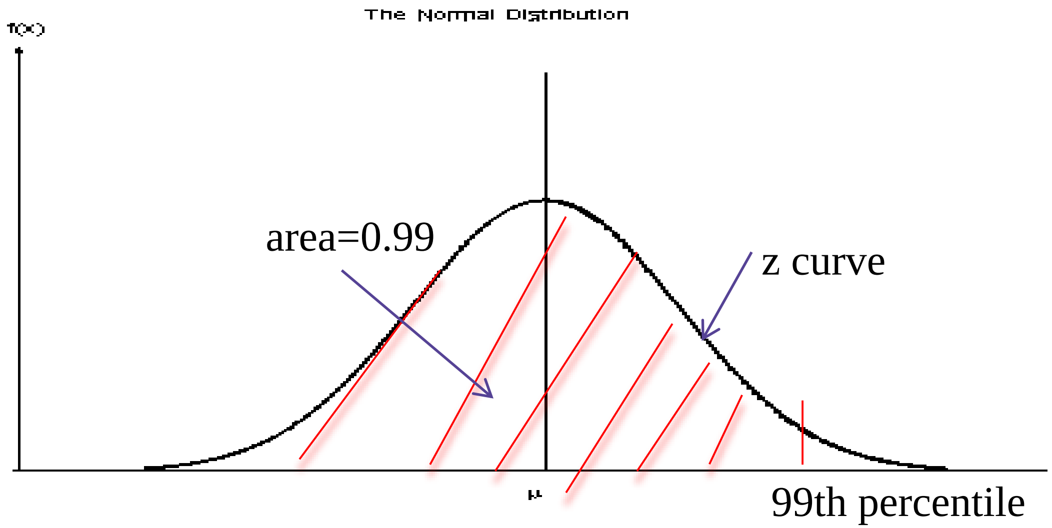
4- Find the standardized score

Note: keep the plus or minus sign.

5- Table can be used to find percentiles.

Percentile of the Standard Normal Distribution

- The 99th percentiles of the standard Normal distribution, is that value on the axis such that the area under the curve to the left of the value is 0.99.
- We have the area and we want the value of z.
- The Z table is used in inverse fashion.

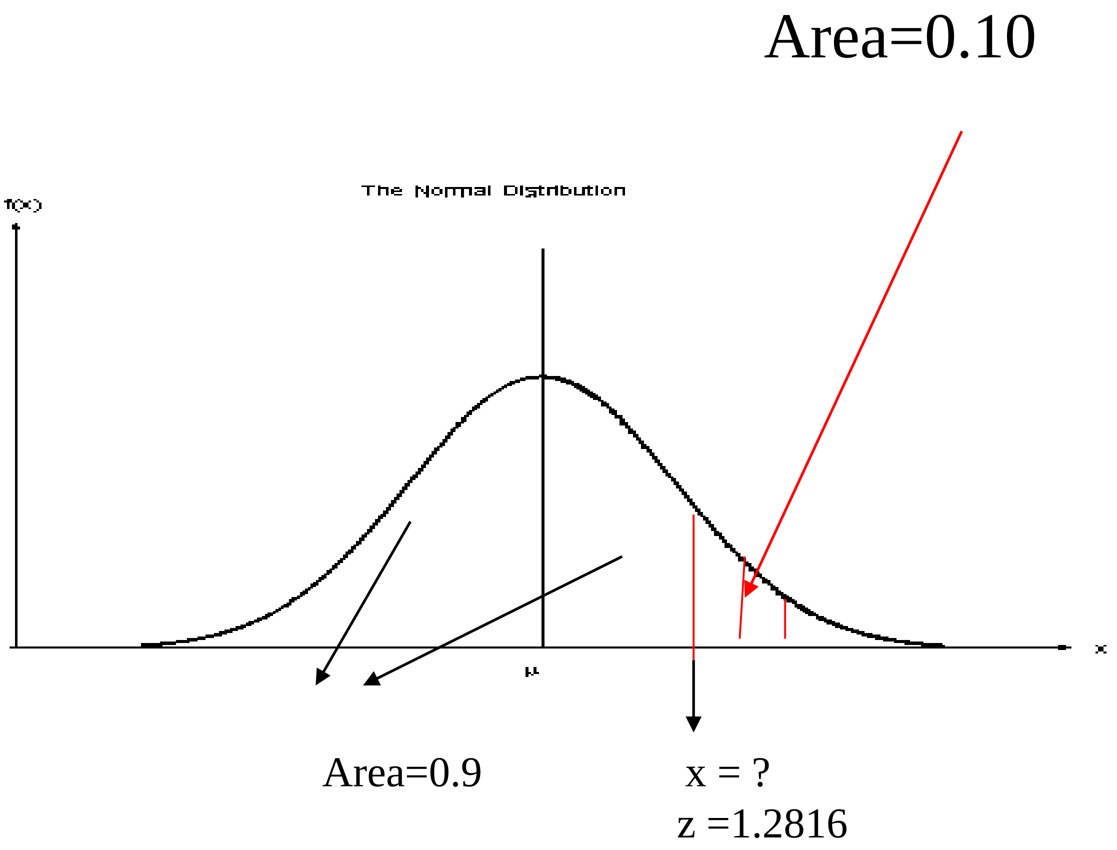


Example

Scores on the SAT Reading test in recent years follow approximately the G(504,111) distribution. How high does a student need to score in order to place in the top 10% of all students taking SAT Reading test?

Note: $G(\mu, \sigma)$

1) State the problem and draw a picture



- The x- value that put the student in the top 10% is the same as the x- value for which 90% of the area is to the left of x.

- Look in the body of the Z-table for the entry closest to 0.9.

- It is 0.8997. This is the entry corresponding to z =1.28

  (is the standardized value with area 0.9 to the left)

OR

- look in the **inverse tables for p ≥ 0.50**

  z =1.2816

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2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983

This table gives the values of $F^{-1}(p)$ for $p \geq 0.50$

p	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.5	0.0000	0.0251	0.0502	0.0753	0.1004	0.1257	0.1510	0.1764	0.2019	0.2275
0.6	0.2533	0.2793	0.3055	0.3319	0.3585	0.3853	0.4125	0.4399	0.4677	0.4959
0.7	0.5244	0.5534	0.5828	0.6128	0.6433	0.6745	0.7063	0.7388	0.7722	0.8064
0.8	0.8416	0.8779	0.9154	0.9542	0.9945	1.0364	1.0803	1.1264	1.1750	1.2265
0.9	1.2816	1.3408	1.4051	1.4758	1.5548	1.6449	1.7507	1.8808	2.0537	2.3263

This table gives the values of $F(x)$ for $x \geq 0$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
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0.2	0.57926	0.58317	0.58706	0.59095	0.59484	0.59871	0.60257	0.60642	0.61026	0.61409
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0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
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0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
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1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899

Unstandardize Transform z back to the original x scale.

$$z = \frac{x - \mu}{\sigma}$$

$$\Rightarrow \quad z\sigma = x - \mu \quad \Rightarrow \quad x = \mu + z\sigma$$

x= mean+ (1.2816)(standard deviation)

x= 504+(1.2816)(111)= 646.2576

- Student most score at least 647 to place in the highest 10%.

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Example

If X is a Normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$,

(a) $P(2 < X < 5)=$

$P(\ (2\text{-}3)/3 < Z < (5\text{-}3)/3) = \Phi(2/3) - \Phi(-1/3)$

$= \Phi(0.67) - \Phi(-0.33)$

$= 0.74857 - 0.3707$

$= 0.37787$

$\boxed{1 - \Phi(0.33) = 1 - 0.62930}$

(b) $P(X > 0) =$

$1 - P(X \le 0) = 1 - P(Z \le (0-3)/3) = 1 - \Phi(-1) = 1 - \{1 - \Phi(1)\}$

$= 1 - \{1 - 0.84134\}$

$= 1 - 0.15866$

$= 0.84134$

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(c) $P(|X - 3| > 6) = 1- P(|X - 3| \le 6)$

$= 1- P(-6 < X-3 < 6) = 1- P(-6/3 < Z < 6/3)$

$= 1- \{ P(Z < 2) – P(Z < -2) \}$

$=1- [\Phi(2) – \Phi(-2)]$

$=1- [\Phi(2) – \{ 1- \Phi(2) \}]$

Just a notation

$= 1- [2 \Phi(2) -1]$

you can use Z(2) or

$= 1- [2 (0.97725) -1]$

$= 1- 0.9545$

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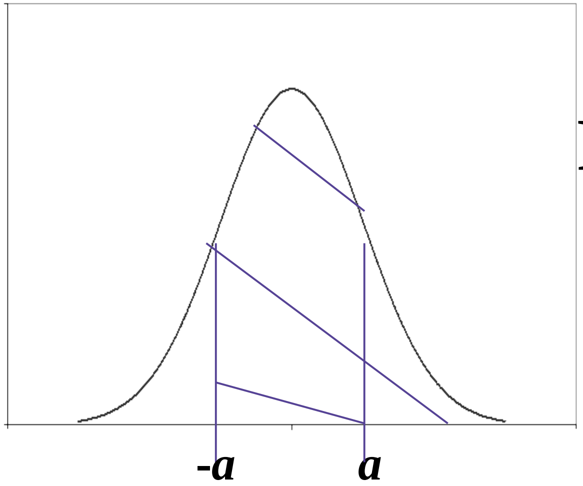
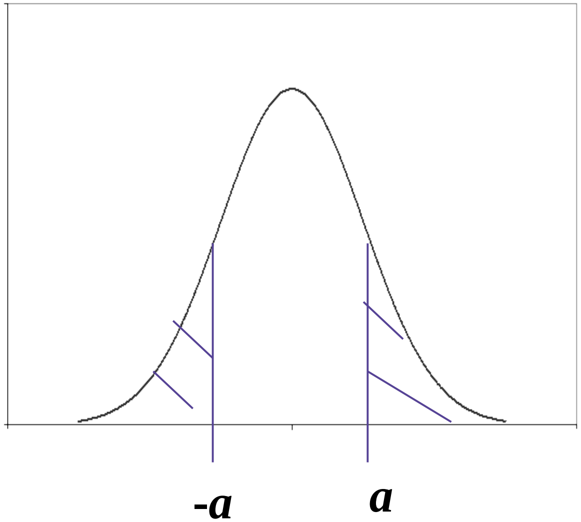
Extremely Useful Results

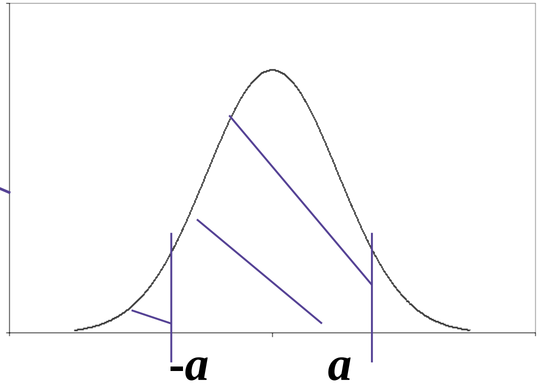
For $a > 0$



$$P(Z \leq -a) = 1 - P(Z \leq a)$$



$$P(Z > -a) = P(Z \leq a)$$



$$P(|Z| \leq a) = 2P(Z \leq a) - 1$$

Examples

Example

Find *a* such that

$$P(\,|Z| \le a\,) = P(-a \le Z \le a) = 0.95.$$

Since $\quad 0.95 = P(|Z| \le a) = 2\,P(Z \le a) - 1$

therefore $P(Z \le a) = (1 + 0.95)/2 = 0.975.$

The value p = 0.975 is not in the inverse tables.

However from the probability table we see that

$$P(Z \le 1.96) = 0.975$$

and so a = 1.96.

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2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983

This table gives the values of $F^{-1}(p)$ for $p \geq 0.50$

p	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.5	0.0000	0.0251	0.0502	0.0753	0.1004	0.1257	0.1510	0.1764	0.2019	0.2275
0.6	0.2533	0.2793	0.3055	0.3319	0.3585	0.3853	0.4125	0.4399	0.4677	0.4959
0.7	0.5244	0.5534	0.5828	0.6128	0.6433	0.6745	0.7063	0.7388	0.7722	0.8064
0.8	0.8416	0.8779	0.9154	0.9542	0.9945	1.0364	1.0803	1.1264	1.1750	1.2265
0.9	1.2816	1.3408	1.4051	1.4758	1.5548	1.6449	1.7507	1.8808	2.0537	2.3263

This table gives the values of $F(x)$ for $x \geq 0$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56750	0.57142	0.57534
0.2	0.57926	0.58317	0.58706	0.59095	0.59484	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899

Example

If X is a Normal random variable with parameters $\mu = 3$ and $\sigma^2 = 25$
Find the number c such that $P(X > c)=0.95$.

$$P (X > c) = P (Z > (c-3)/5) = 0.95$$

$$(c-3)/5= -1.6449$$

(from the table that gives the values of $F^{-1}(p)$ *for* $p \geq 0.50$)

$$c = -5.2245$$

Example : Find $P(|X-2| \leq 3)$ where X~N(2,9)

$$\text{Since } X \sim N(2,9) \quad \text{then } Z = \frac{X-2}{3} \sim \mathrm{N}(0,1).$$

$$P(|X-2| \leq 3) = P\left(\frac{|X-2|}{3} \leq \frac{3}{3}\right)$$

$$= P(|Z| \leq 1)$$

$$= 2P(Z \leq 1) - 1$$

$$= 2(0.84134) - 1$$

$$= 0.68268$$

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This table gives the values of $F(x)$ for $x \geq 0$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56750	0.57142	0.57534
0.2	0.57926	0.58317	0.58706	0.59095	0.59484	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
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0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
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1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
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1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
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1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
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2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899

Example : Find P(|X| > 2) where X~N(2,9)

$$\text{Since } X \sim N(2,9) \quad \text{then } Z = \frac{X-2}{3} \sim \mathrm{N}(0,1).$$

$$P(|X|>2) = 1 - P(|X| \le 2) = 1 - P(-2 \le X \le 2)$$

$$= 1 - P\left(\frac{-2-2}{3} \le Z \le \frac{2-2}{3}\right)$$

$$= 1 - P\left(\frac{-4}{3} \le Z \le 0\right)$$

$$= 1 - \left[P(Z \le 0) - P\left(Z \le \frac{-4}{3}\right)\right]$$

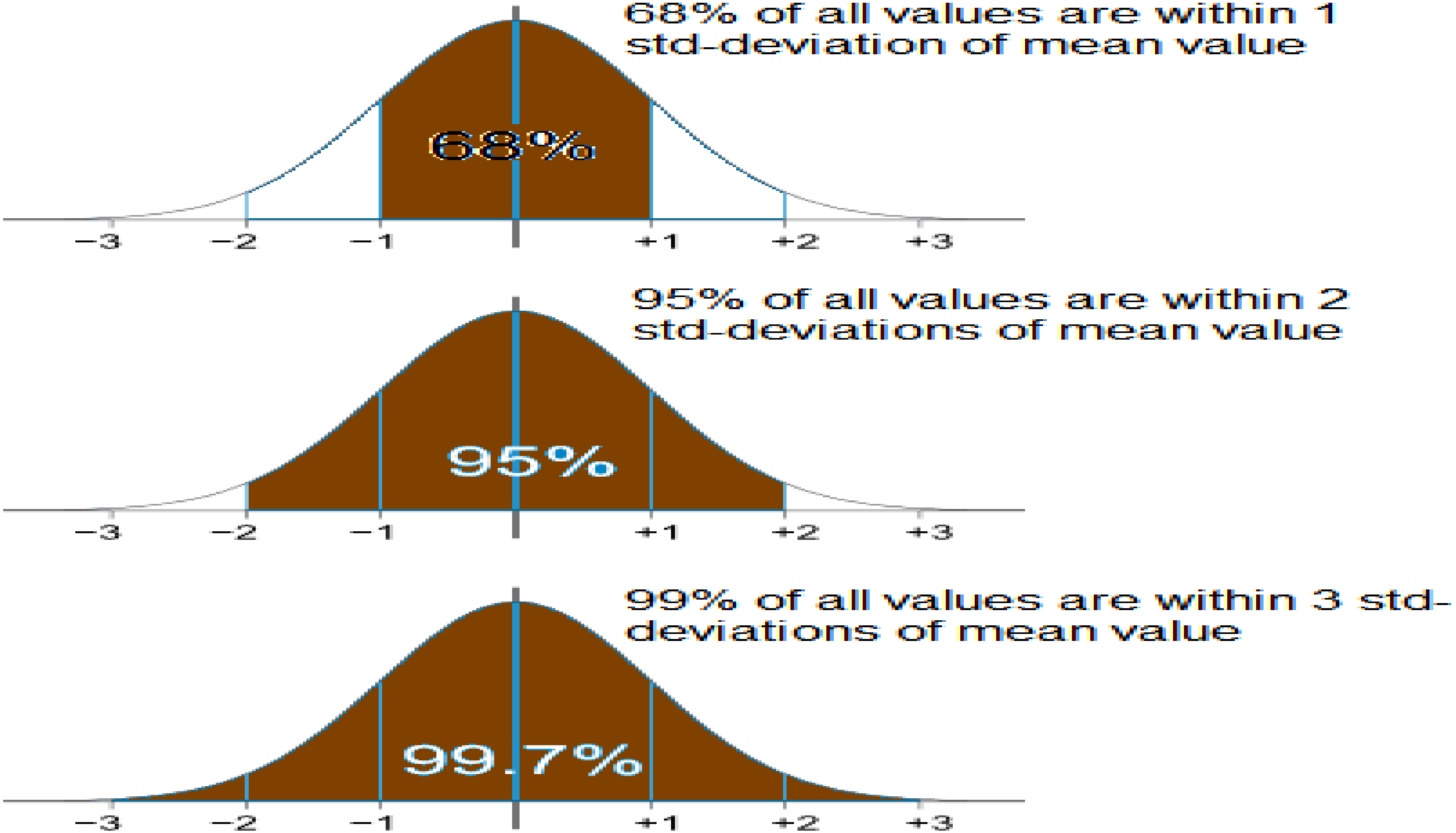
$$= 1 - P(Z \le 0) + \left[1 - P\left(Z \le \frac{4}{3}\right)\right]$$

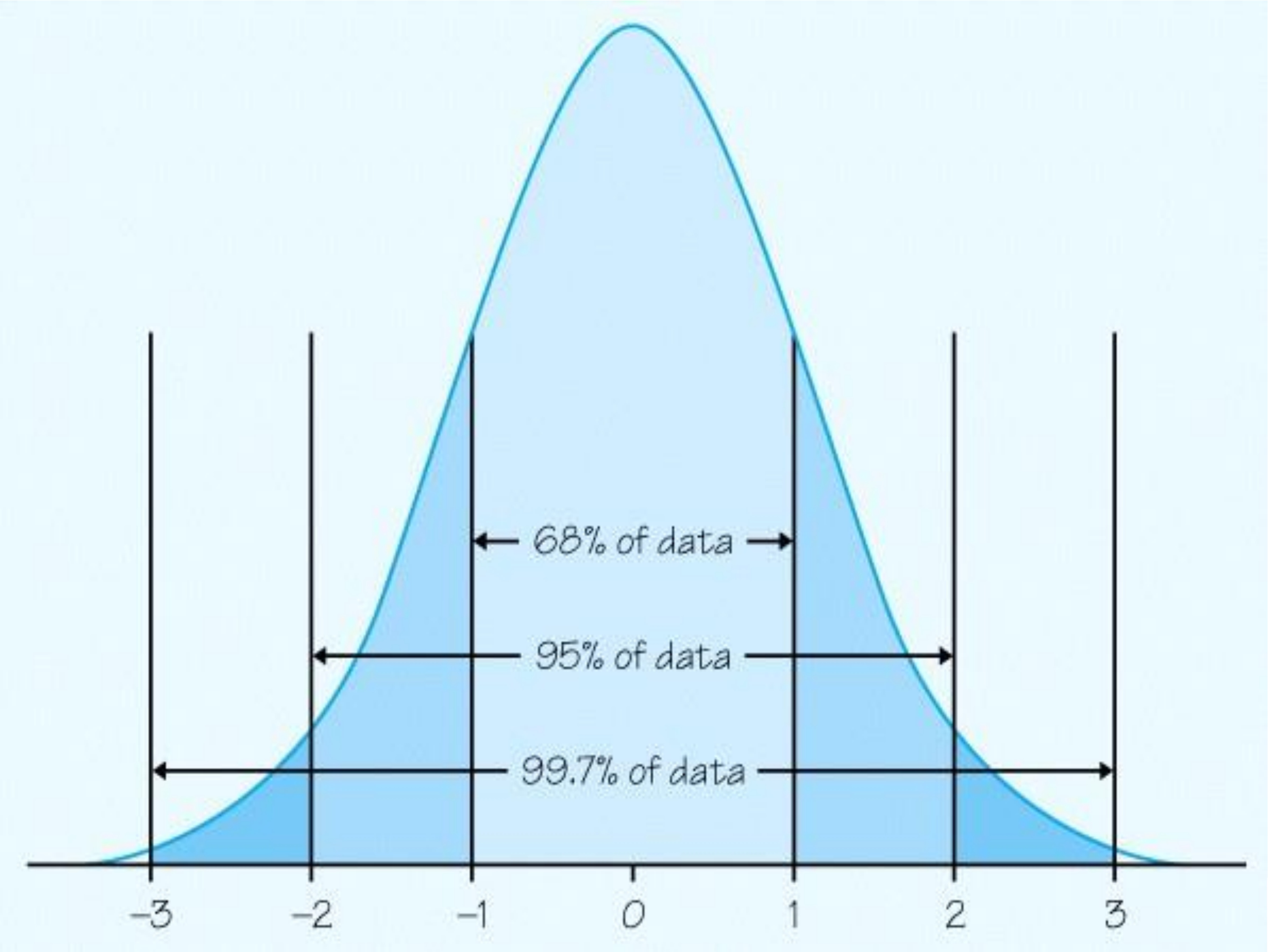
$$= 2 - P(Z \le 0) - P\left(Z \le \frac{4}{3}\right) \approx 2 - P(Z \le 0) - P(Z \le 1.33)$$

$$= 2 - 0.5 - 0.90824 = 0.59176$$

The Empirical Rule: “ 68-95-99.7” Rule

- If your data have a Normal distribution, then approximately:





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Example: The Empirical Rule

- It is known that height of adult men in Canada follow a Normal distribution. The mean height is 70 inches and the standard deviation is 3 inches.

- Applying the Empirical Rule, this means that:
  - 68% of adult men in Canada are between 67 and 73 inches.
  - 95% of adult men in Canada are between 64 and 76 inches.
  - 99.7% of adult men in Canada are between 61 and 79 inches.

- What percentage of adult men in the Canada are shorter than 73 inches?

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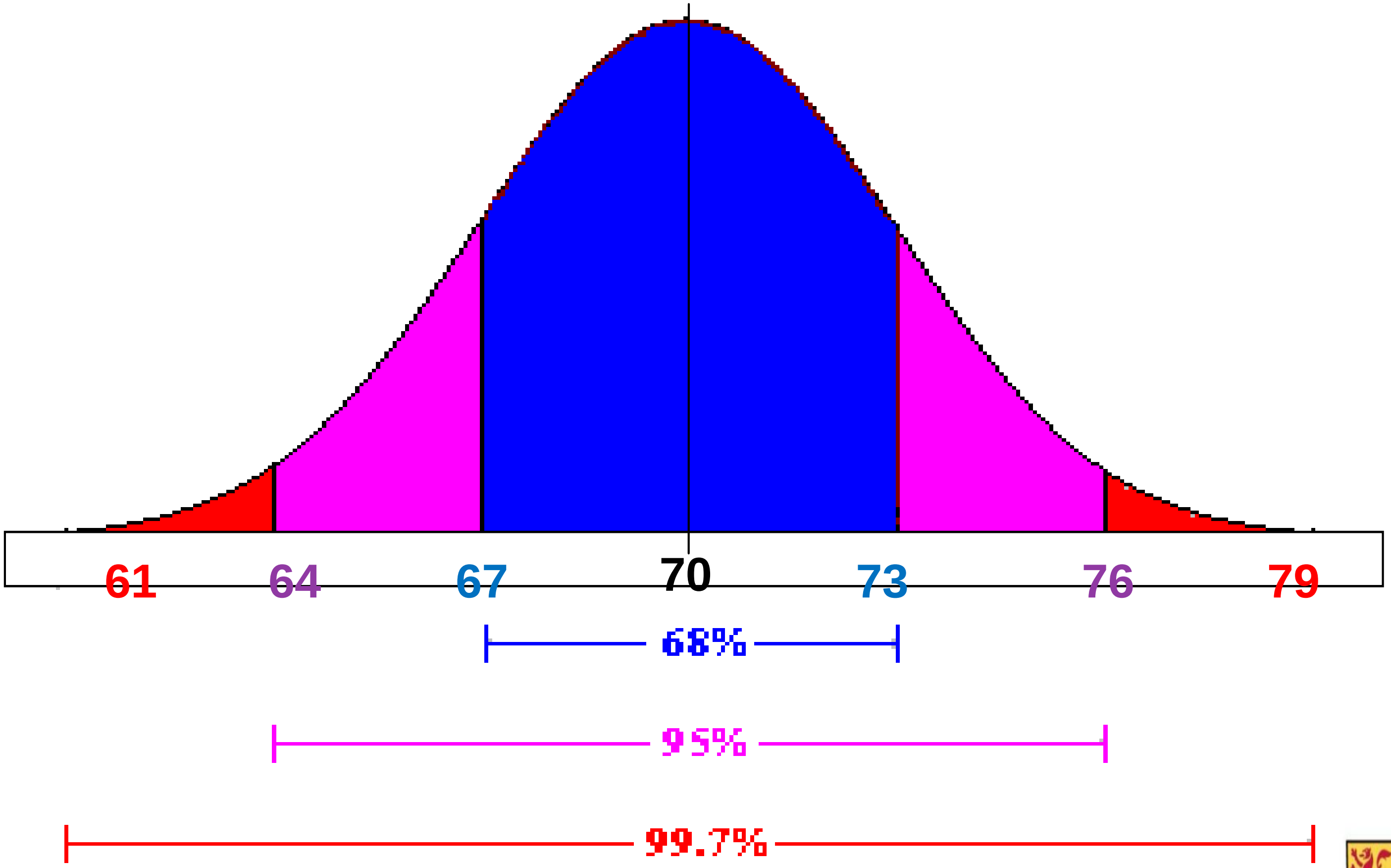

http://www.nku.edu/~statistics/212_Using_the_Empirical_Rule.htm

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