STAT 230

PROBABILITY

(Chapter 4)

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Probability Rules and Conditional Probability
Chapter Four Objectives

- Be able to Apply the Basic Probability Rules when Applicable.
- Understand the Conditional Probabilities.
- Independent Events.
- Tree Diagrams.
- Bayes's Theorem.

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Some Relations From Set Theory

- Suppose *S* is a sample space for an experiment.
- Let A, B, C, A_1, A_2,… be events defined on *S*.

➢ The Union of two events A and B

A U B and read (A "or" B)

Is the set consisting of all outcomes that are

either in A or in B or in both events.

(At least one of A , B occurs.)

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- The intersection of two events A and B

$$A \cap B \text{ and read } (A \text{ and } B)$$

Is the set consisting of all outcomes that are *in both* A *and in* B.

- The complement of an event A denoted by $\bar{A}$, A^c, is the set of all outcomes in S that are not contained in A.

$$\overline{\left(\bar{A}\right)} = A$$

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- The empty event , or the null set Ø : is the set with no outcomes at all.

Note : $\emptyset = \overline{S}$

$P(\emptyset) = 0$

$P(S) = 1$

Example

If A is the event that the sum (total) of 2 dice is 7, and B is the event that the total is 6, what is the event of **A∩B (AB)** ?

http://cashback-online-casino.com/wp-content/uploads/2012/06/rolling-2-dice.png

B A

S={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

- All outcomes are equally probable with probability 1/36.

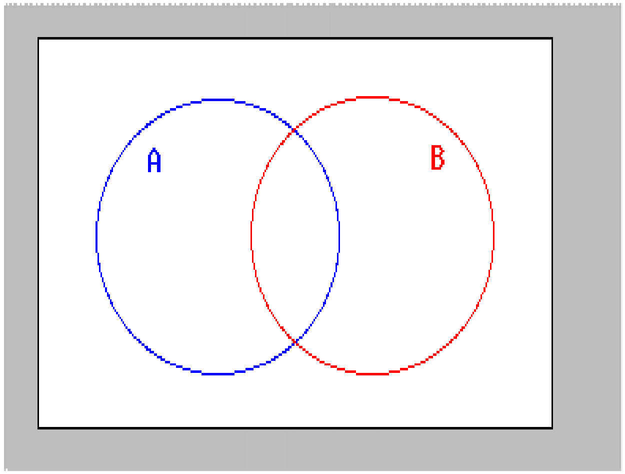
$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$B = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$

Then the event AB does not contain any outcomes and hence could not occur.

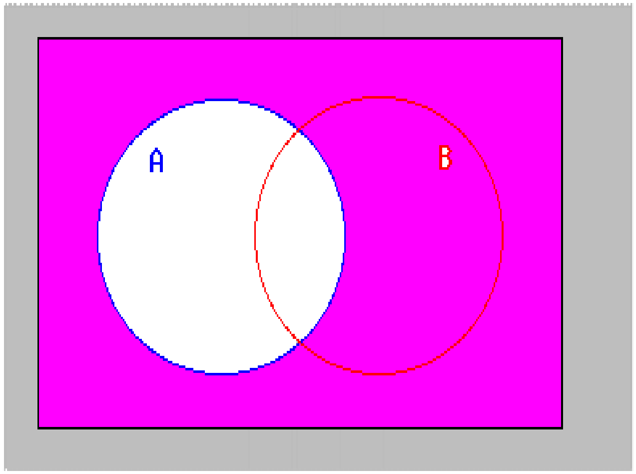
Venn Diagrams

S



(1)

S



(2) colored region:

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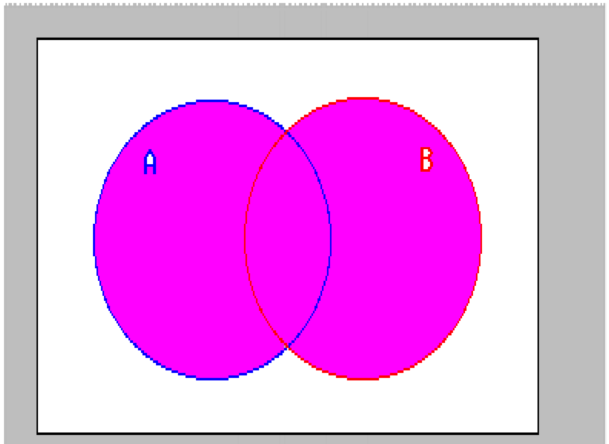

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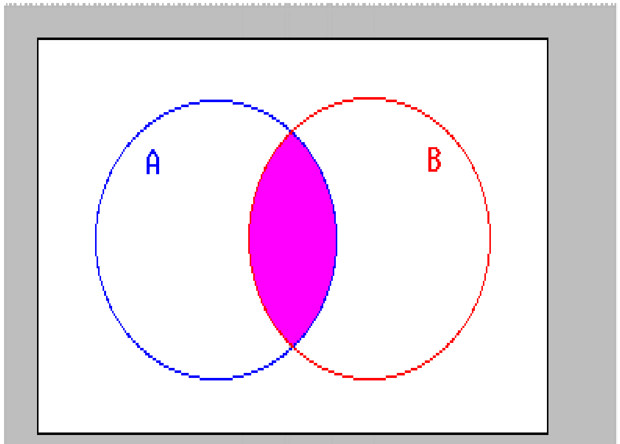

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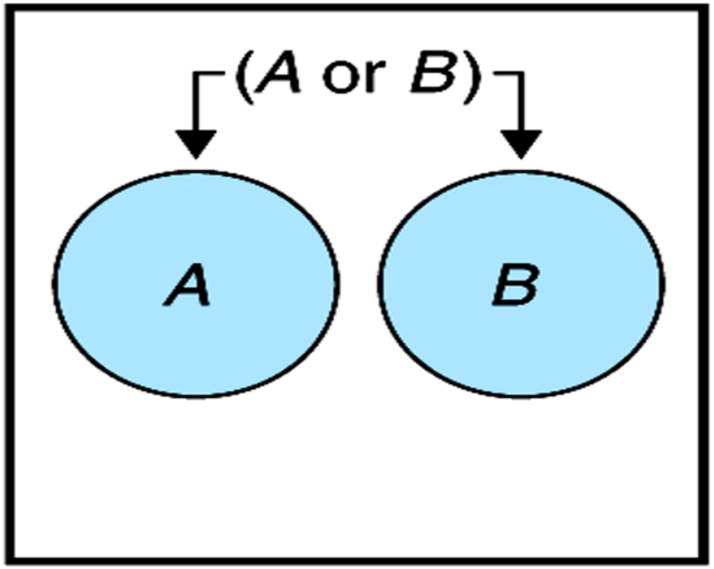
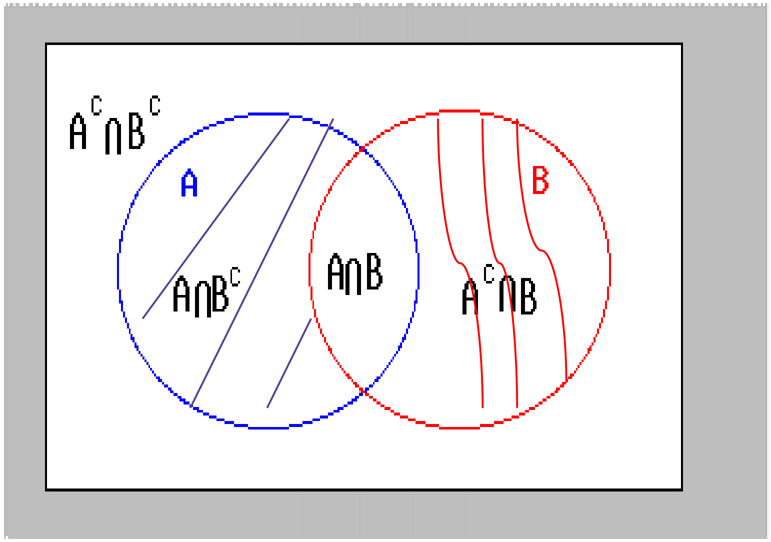


(5)

S



(6)

Note :
Suppose we denote set differences by
$A/B = A \cap B^c$

De Morgan's Laws

Union and intersection interchange under complementation

$$\boxed{\begin{array}{c} \overline{A \cup B} = \overline{A} \cap \overline{B} \\ \\ \overline{A \cap B} = \overline{A} \cup \overline{B} \end{array}}$$

More generally:

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}$$

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k}$$

Example

Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3\}$ and $B = \{3, 4, 5\}$.
Show that

a) $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

$A \cup B = \{2, 3, 4, 5\}$, $\overline{(A \cup B)} = \{1, 6\}$

$\overline{A} \cap \overline{B} = \{1, 4, 5, 6\} \cap \{1, 2, 6\} = \{1, 6\}$

b) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Please Do Example (page 40)

Suppose for students finishing second year Math that 22% have a math average greater than 80%, 24% have a STAT 230 mark greater than 80%, 20% have an overall average greater than 80%, 14% have both a math average and STAT 230 greater than 80%, 13% have both an overall average and STAT 230 greater than 80%, 10% have all 3 of these averages greater than 80%, and 67% have none of these 3 averages greater than 80%. Find the probability a randomly chosen math student finishing 2A

has math and overall averages both greater than 80% and STAT 230 less than or equal to 80%.

Rules for Unions of Events

Addition Law of Probability or the Sum Rule

1- Mutually Exclusive Events

- The probability of **either** event happening is the **sum** of their individual probabilities

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

If A and B are mutually exclusive

- If $A_1, A_2, \ldots, A_k$ is a finite collection of mutually exclusive events then

$$\boxed{P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k).}$$

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Example

Given P(A) = 0.20, P(B) = 0.70, A and B are mutually exclusive (disjoint). Find

http://www.mynamesnotmommy.com/wp-content/uploads/2013/05/question-mark.png

$P(A \cup B) =$

$= P(A) + P(B)$

$= 0.20 + 0.70$

$= 0.9$

$P(A \cap B) =$

$= 0$

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Example

If you think probability of getting an A in your statistics class is 50% and probability of getting a B is 30%, what is the probability of getting either an A or a B?

$$P(A \cup B) =$$
$$= P(A) + P(B)$$
$$= 0.5 + 0.3$$
$$= 0.8$$

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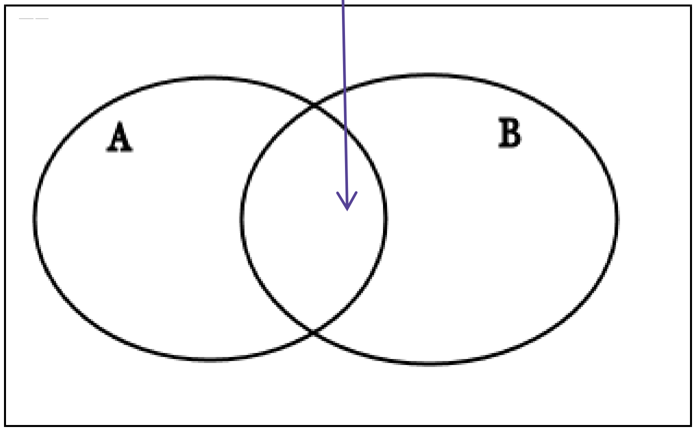
2- Non-Mutually Exclusive Events

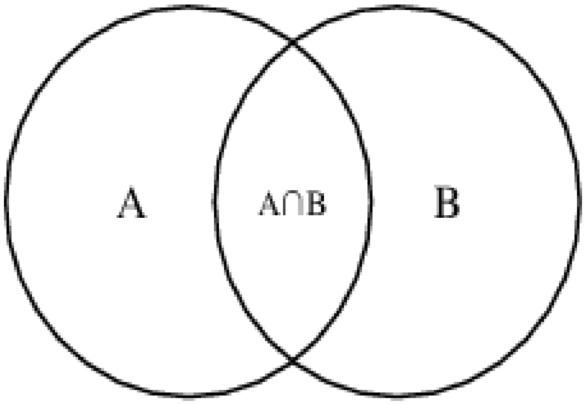
- In events which aren't mutually exclusive, there is some overlap.
- When P(A) and P(B) are added, the probability of the intersection (and) is added twice.
- To compensate for that double addition, the intersection needs to be subtracted

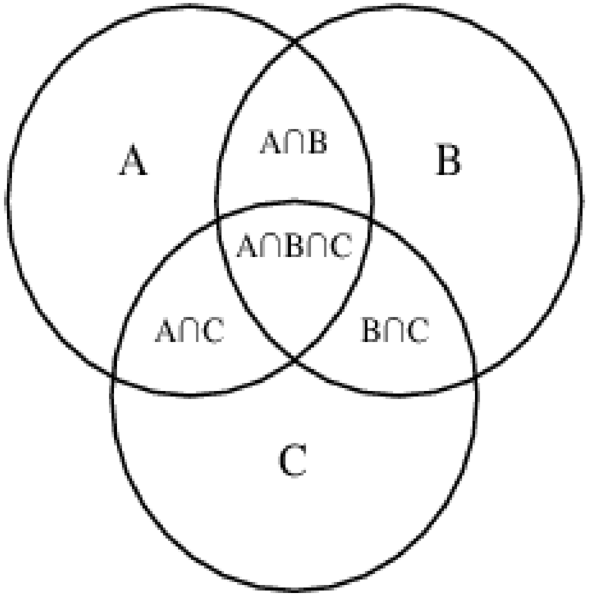


(a) $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

(b) **For any three events A, B and C**

$$\boxed{\begin{gathered} P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ - P(B \cap C) + P(A \cap B \cap C) \end{gathered}}$$





Example

Given P(A) = 0.20, P(B) = 0.70, P(A ∩ B) = 0.15, find the probability of

(a) A U B

(b) Not A and not B

(a) $P(A \cup B) = \mathbf{P(A) + P(B) - P(A \cap B)}$

$= 0.2 + 0.7 - 0.15$

$= 0.75$

(b) $\boxed{P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) \qquad \text{De Morgan's Laws}}$

$= 1 - P(A \cup B)$

$= 1 - 0.75$

$= 0.25$

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Example

Two fair dice are rolled. Find the probability that at least one of them turns up a 6?

Let $A = \{ \text{6 on the first die} \}$

$B = \{ \text{6 on the second die} \}$

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= 1/6 + 1/6 - 1/36 \\
&= 11/36
\end{aligned}
$$

Example

Roll a fair die 3 times. Find the probability of getting at least one 6?

Let A = { at least one die show 6}.

Then $\bar{A}$ = {no 6 on any die show}.

Using counting arguments, there are 6 outcomes on each roll, so S has $6 \times 6 \times 6 = 6^3 = 216$ points.

For $\bar{A}$ to occur we can't have a 6 on any roll.

Then $\bar{A}$ can occur in $5 \times 5 \times 5 = 125$ ways.

$P(\bar{A}) = 125/216$

$P(A) = 1 - 125/216 = 91/216$

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Example

If you think that the probability of getting an A in your Statistics class is 0.50 and probability of getting an A in your History class is 0.30.

a) What is the probability of getting an A in either class?

b) What is the probability of getting exactly one A?

Solution:

Let

A={ getting an A in your Statistics class},

B={ getting an A in your History class}

(a) $P(A \cup B) = P(A \text{ or } B) = P(A) + P(B) - P(A \cap B) \longrightarrow (AB)$

$P(A \cap B) = (0.5)(0.3) = 0.15$

$P(A \cup B) = 0.5 + 0.3 - 0.15 = 0.65$

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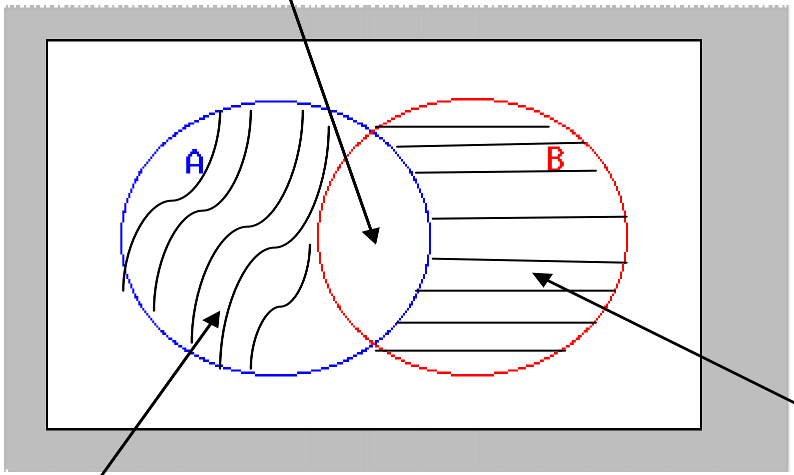
(b) $P(A \cap B) = 0.15$

Note $P(A \cup B) = P(A) + P(A' \cap B)$

Similarly $P(A \cup B) = P(B) + P(A \cap B')$



$P(A \cap B') = 0.65 - 0.3 = 0.35$

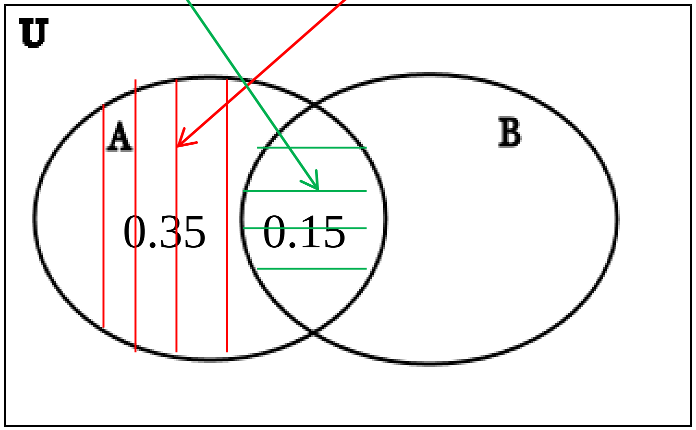
$P(A' \cap B) = 0.65 - 0.5 = 0.15$

P(exactly one)= $P(A \cap B') + P(A' \cap B) = 0.35 + 0.15 = 0.5$

Example

If you think that the probability of getting an A in your Statistics and History class is 0.15 and probability of getting an A in your Statistics class and not in History class is 35% what is the probability of getting an A in Statistics class?

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$



$P(A) = 0.35 + 0.15 = 0.5$

Intersection of Events and Independence

- Two events are independent if they do not influence each other.

- Knowing the outcome of one event does not help with knowing the outcome of the other event.

Examples:

- Coin flips are independent. Getting tails the first flip will have no affect on getting tails the second flip.

- Rolling two dice are independent.

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Definition

Events A and B are independent if and only if

$$P(A \cap B) = P(A)\,P(B)$$

If they are not independent, we call the events dependent.

Note

If events **A and B** are independent then

A^c and B

A and B^c

A^c and B^c

are independent.

Example

Roll a fair die once and let

A = {the number is even}

B = {number > 3}.

Are A and B independent?

http://www.psdgraphics.com/wp-content/uploads/2010/02/dice-icon.jpg

$P(A) = 1/2$

$P(B) = 1/2$

$P(AB) = P(4 \text{ or } 6 \text{ occurs}) = 2/6 \neq P(A)P(B)$

A and B are not independent.

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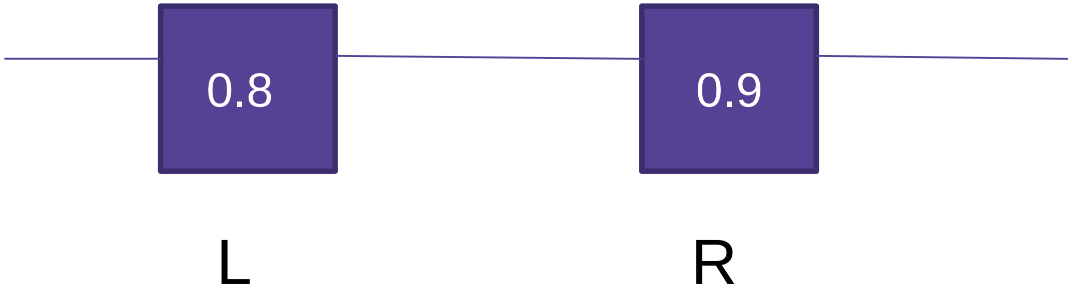
Example

The following circuit operates only if there is a path function device from left to right. The probability that each device functions is shown on the graph. Assume the devices fail independently. What is the probability that the circuit operates?



There is only a path if both operate. The probability the circuit operates is

$$
\begin{aligned}
P(L \text{ and } R) &= P\,(L \cap R) \\
&= P(L)\,P(R) \\
&= 0.8(0.9) \\
&= 0.72
\end{aligned}
$$

Example

Two fair coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands on heads and F the event that the second lands on tails.

Is the event E independent of the event F?

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$P(E) = P(\{(H,H), (H, T)\}) = 1 / 2$

$P(F) = P(\{(H, T), (T, T)\}) = 1 / 2$

$P(EF) = P(\{(H, T)\}) = 1 / 4$

$P(EF) = P(E)\, P(F)$

E and F are independent.

Exercise:

Suppose that we toss 2 fair dice. Let E_1 denote the event that the sum of the dice is 6 and F denote the event that the first die equals 4.

(a) Is E_1 independent of F?

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$$
\begin{aligned}
S=\{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\
&(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\
&(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\
&\boxed{(4,1),(\underline{\mathbf{4,2}}),(4,3),(4,4),(4,5),(4,6)}\\
&(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\
&(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}
\end{aligned}
$$

(a)

$P(E_1F) = P(\{(4, 2)\}) = 1 / 36$

Whereas

$P(E_1)P(F) = (5 / 36)(1 / 6)$

(b) suppose that we let E_2 be the event that the sum of the dice equals 7 and F denote the event that the first die equals 4.

Is E_2 independent of F?

$P(E_2F) = P(\{(4, 3)\}) = 1/36$

Whereas

$P(E_2)P(F) = (1/6)(1/6)$

$= 1/36$

E_2 is independent of F.

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Independence of Three Events

Three events, A, B and C are independent events if and only if all of the following statements hold:

$$P(A \cap B) = P(A)\,P(B)$$

$$P(A \cap C) = P(A)\,P(C)$$

$$P(B \cap C) = P(B)\,P(C)$$

and

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C).$$

Conditional Probability

The importance of this concept is:

We are often interested in calculating probabilities when some partial information concerning the result of an experiment is available.

For example, what is the probability a randomly selected person is over 6 feet tall, given that she is a female?

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Example

Suppose that we toss 2 fair dice, and suppose that each of the 36 possible outcomes is equally likely to occur. Suppose further that we observe that the first die is a 3. Then, given this information, what is the probability that the sum of the 2 dice equals 8?

Given that the initial die is a 3, there can be at most 6 possible outcomes of our experiment,

A= {(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)}

Desired probability will be 1/6

Conditional Probability

For any two events A and B with P(B) > 0, the conditional probability of A given B has occurred is defined by

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

Note that the conditional probability of B given A with P(A) > 0 is

$$\boxed{P(B \mid A) = \frac{P(A \cap B)}{P(A)}}$$

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$$P(A|B) + P(A^c|B) = 1$$

$$P(B|A) + P(B^c|A) = 1$$

The Multiplication Rule for P(A ∩ B)

The probability of the intersection of any two events A and B is

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

OR

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

Let A, B, C, D, ... Be arbitrary events in a sample space. Assume that $P(A) > 0$, $P(AB) > 0$, $P(ABC) > 0$. Then

$$P(ABC) = P(A)\,P(B \mid A)\,P(C \mid AB)$$

$$P(ABCD) = P(A)\,P(B \mid A)\,P(C \mid AB)\,P(D \mid ABC)$$

Independence

- For any two events A and B defined on S with P(B) > 0, P(A) > 0.
- Then A and B are independent if and only if either of the statements is true

$$P(A) = P(A \mid B)$$
$$P(B) = P(B \mid A)$$

Example

$P(A) = 0.20$, $P(B) = 0.70$, $P(B|A) = 0.40$, find $P(A \cap B)$ and $P(B^c|A)$?

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

(a) $P(A \cap B) = P(B|A)\ P(A)$

$= 0.40\ (0.20)$

(b) $P(B^c|A) = 1 - P(B|A)$

$= 1 - 0.4$

$= 0.6$

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Example

P(A) = 0.20, P(B) = 0.70, P(A|B) = 0.60, find P(A ∩ B) ?

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

P(A and B) = P(A|B) P(B)

= 0.70 x 0.60

Example

A coin is flipped twice. Assuming that all four points in the sample space S = {(H, H), (H, T), (T, H), (T, T)} are equally likely, what is the conditional probability that both flips land on heads, given that

(a) The first flip lands on heads?

(b) At least one flip lands on heads?

Let

$B = \{(H, H)\}$ be the event that both flips land on heads;

$F = \{(H, H), (H, T)\}$ be the event that the first flip lands on heads;

$A = \{(H, H), (H, T), (T, H)\}$ be the event that at least one flip lands on heads.

(a) $P(B|F) = P(BF) / P(F)$

$$= P(\{(H, H)\}) / P(\{(H, H), (H, T)\}) = (1/4) / (2/4) = 1/2$$

(b) $P(B|A) = P(BA)/ P(A)$

$$= P(\{(H, H)\}) / P(\{(H, H), (H, T), (T, H)\})$$

$$= (1/4) / (3/4) = 1/3$$

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Tree Diagrams

A tree diagram is useful for displaying all outcomes for a "multistage" experiment and determining their probabilities.

Example

A chain of video stores sells three different brands of TVs.

Of its TV sales, 50% are brand 1, 30% brand 2, 20% brand 3.

Each manufacturer offers a 1-year warranty on parts and labor.

It is known that 25% brand 1's TV's require warranty work, and 20%, 10% for brand 2, and 3 respectively.

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Let

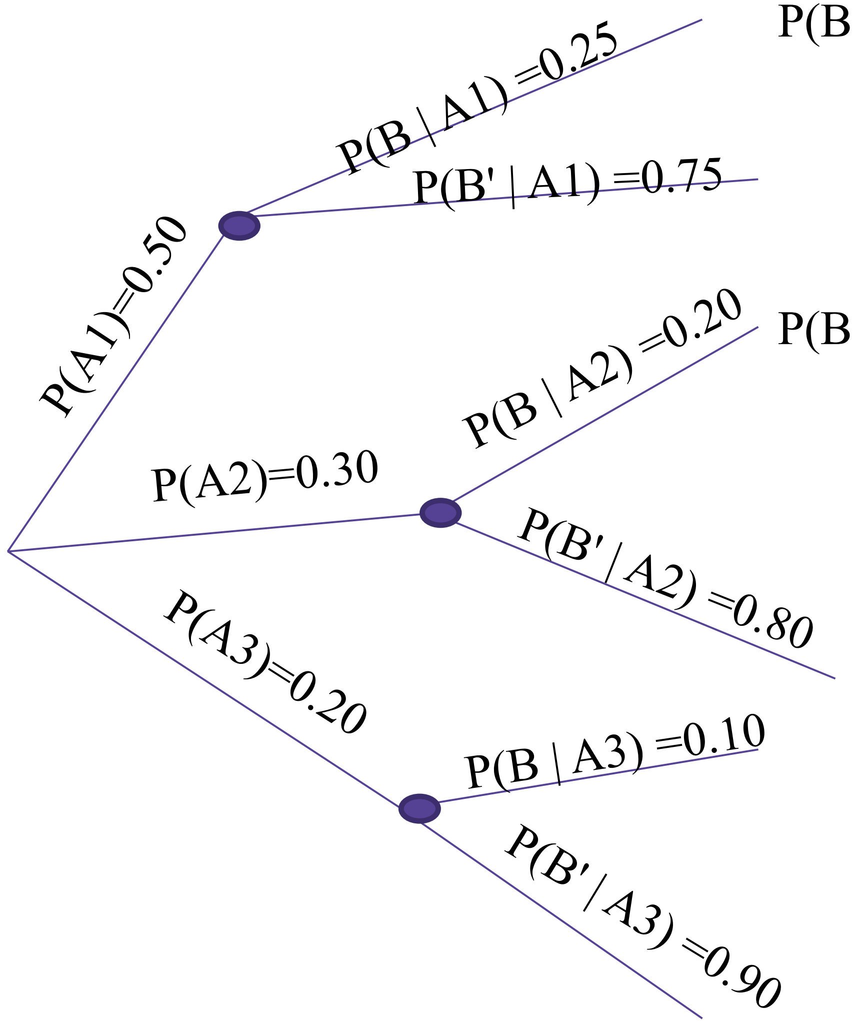
A_i ={ Brand i is purchased}, where i= 1, 2, 3

B= { need repair (warranty work)}

- What is the probability that a random selected purchaser has bought a brand 1 TV that will need repair while under warranty?

- What is the probability that a random selected purchaser has a TV that will need repair while under warranty?

- If a customer returns to the store with TV that needs warranty repair work, what is the probability that it is a brand 1 TV?



$P(B|A1).P(A1) = P(B \cap A1) = 0.125$

$P(B|A2).P(A2) = P(B \cap A2) = 0.060$

$P(B|A3).P(A3) = P(B \cap A3) = 0.020$

$P(B) = 0.205$

$$P(A1|B) = \frac{P(A1 \cap B)}{P(B)} = 0.125 / 0.205 = 0.61$$

The Law of Total Probability

- Let $A_1, A_2, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B

$$P(B)= P(B|A_1)\,P(A_1) + \ldots + P(B|A_k)P(A_k)$$

$$= \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

Note: The events are exhaustive if one A_i must occur, so that

$A_1 \cup \ldots \cup A_k = S$

A set of event is said to be exhaustive when at least one of the events compulsorily occurs.

Example

Tossing a fair coin once

A : The event of getting a H

B : The event of getting a T

- When we conduct the experiment, at least one of these will occur.
- The two events "A" and "B" together are called exhaustive events.

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Example

Throwing a fair dice once

- A : The event of getting 1
- B : The event of getting 2
- ...
- ...
- F : The event of getting 6

- One of these events will occur whenever the experiment is conducted.
- The six Events "A", "B", "C", "D", "E", "F“ together are called exhaustive events.

Terminology

- A false positive results when a test indicates a positive status when the true status is negative $(T|D^c)$

- A false negative results when a test indicates a negative status when the true status is positive $(T^c | D)$

- The Sensitivity(**true positive rate**)of a test is a probability of a positive test result given the presence of the disease $P(T|D)$.

- The Specificity **(true negative rate)** of a test is a probability of a negative test result given the absence of the disease $P(T^c | D^c)$

Notes

- **Sensitivity** is complementary to the **false negative rate.**

$$P(T \mid D) + (T^c \mid D) = 1$$

- **Specificity** is complementary to the **false positive rate.**

$$P(T^c \mid D^c) + (T \mid D^c) = 1$$

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Example: Testing for HIV

A fairly cheap blood test for the Human munodeficiency Virus (HIV) that causes AIDS (Acquired Immune Deficiency Syndrome) has the following characteristics:

- The false positive rate is 0.5%.
- The false negative rate is 2%
- It is assumed that around 0.04% of Canadian males are infected with HIV.

B

Find the probability that if a male tests positive for HIV, he actually has HIV.

A

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Suppose a male is randomly selected from the population, and define the events

A = {Person has HIV}

B = {Blood test is positive}

We are asked to find P (A|B).

From the information given we know that

$P(A) = 0.0004$, $P(\overline{A}) = 0.9996$

$P(B|A) = 0.98$, $P(B|\overline{A}) = 0.005$

Sensitivity =
1- False negative

False positive rate

Therefore we can find

$P(AB) = P(A)P(B|A) = 0.0004 \times 0.98 = 0.000392$

$P(\bar{A}B) = P(\bar{A})P(B|\bar{A}) = 0.9996 \times 0.005 = 0.004998$

Therefore $P(B) = P(AB) + P(\bar{A}B) = 0.00539$

And

$P(A|B) = P(AB)/P(B)$

$= 0.0727$

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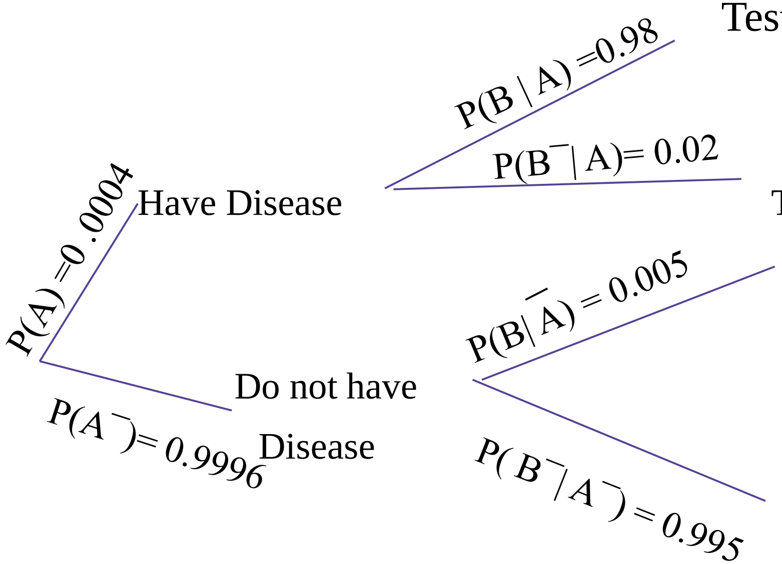

	Event	Probability
Test Positive	**True Positive**	$P(AB)=\underline{0.000392}$
Test Negative	False Negative	$P(A\bar{B})=$ 0.000008
Test Positive	**False Positive**	$P(\bar{A}B)=$ $\underline{0.004998}$
Test Negative	True Negative	$P(\bar{A}\bar{B})=$ 0.994602

The probability that a random chosen person will test positive is

$P(B) = P(AB) + P(\bar{A}B) = 0.00539$

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Example

Problem 4.3.2:

Suppose among UW students that 15% speaks French and 45% are women. Suppose also that 20% of women speak French. A committee of 10 students is formed by randomly selecting from UW students. What is the probability there will be at least 1 woman and at least 1 French student on the committee?

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Let

- F_i: French speaking i^{th} individual.

- W_i: Woman i^{th} individual.

We told that

- $P(F_i) = 0.15$
- $P(W_i) = 0.45$
- $P(F_i | W_i) = 0.2$

W = {at least 1 woman}

F= {at least 1 French student}

$P(W \cap F) = ?$

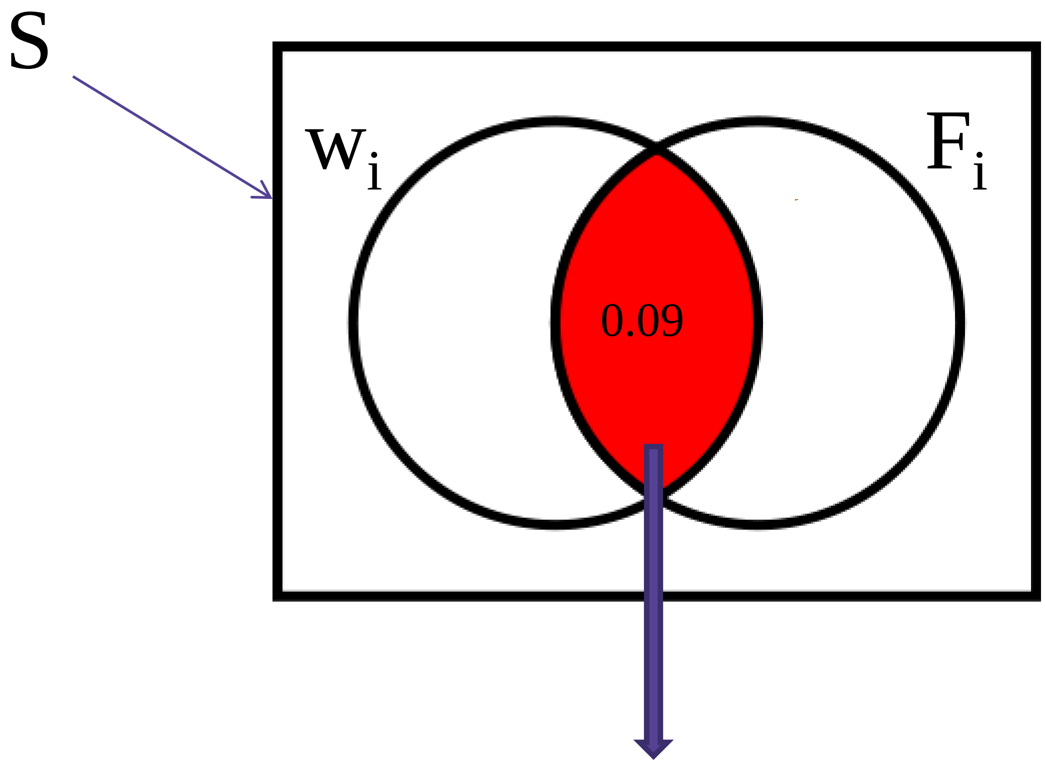
$$P(W \cap F) = 1 - P(\overline{W \cap F})$$

$$= 1 - P(\overline{W} \cup \overline{F})$$

De Morgan's Laws

$\overline{W}$ = {0 students are woman}

$\overline{F}$= {0 students are French}



$$P(F_i\, W_i) = P(W_i)\, P(F_i |\, W_i) = 0.45(0.2) = 0.09$$

Note

Although the selection is done without replacement, since the population is very large selection without replacement makes very little difference. Hence we can assume that they draws are independent.

$$P(\overline{W} \cup \overline{F}) = P(\overline{W}) + P(\overline{F}) - P(\overline{W} \cap \overline{F})$$

$$P(\overline{W}) = P\{0 \text{ students are woman}\}$$

$$= P(1^{st} M \cap 2^{nd} M \cap \ldots\ldots \cap 10^{th} M)$$

$$= P(1^{st} M)\, P(2^{nd} M) \ldots.P(10^{th} M)$$

$$= (1-0.45)\,(1-0.45)\ldots\,(1-0.45)$$

$$= (0.55)\,(0.55)\,\ldots\,(0.55)$$

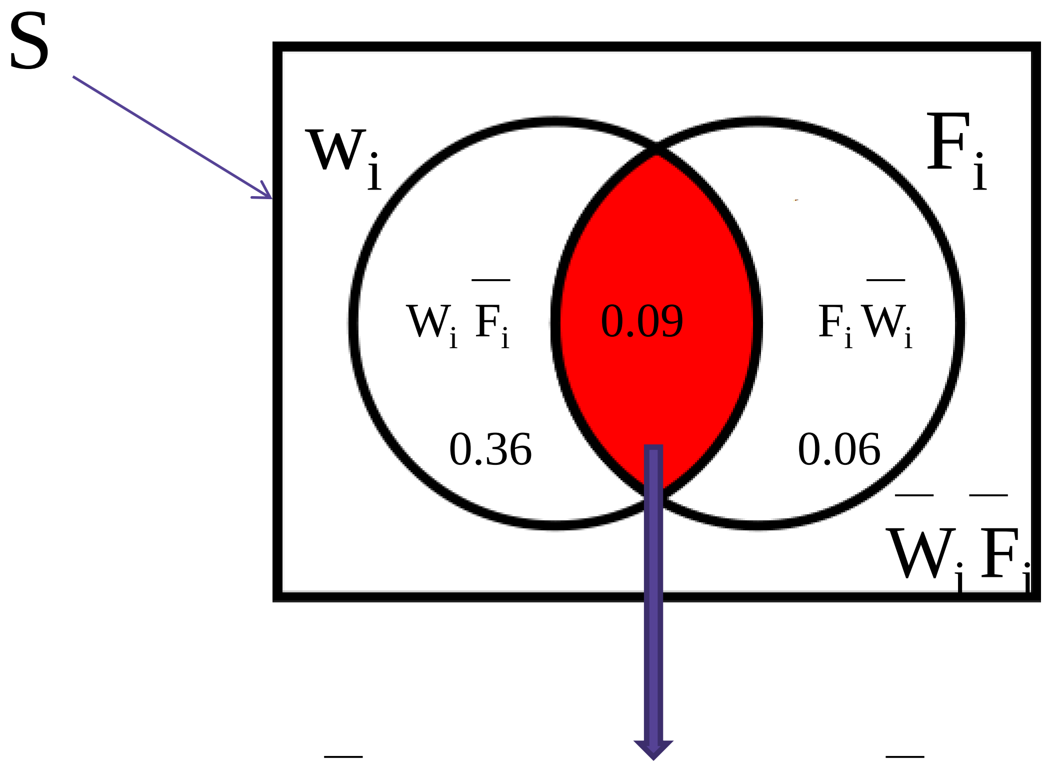
$$= 0.55^{10}$$

$$P(\overline{F}) = P\{0 \text{ students speak French}\}$$

$$= P(1^{st}\, \overline{F} \cap 2^{nd}\, \overline{F} \cap \ldots\ldots \cap 10^{th}\, \overline{F})$$

$$= P(1^{st}\, \overline{F})\, P(2^{nd}\, \overline{F}) \ldots.P(10^{th}\, \overline{F})$$

$$= (1-0.15)^{10} = 0.85^{10}$$



$$P(\overline{F_i}\,\overline{W_i}) = 1- \{ P(W_i\,\overline{F_i}) + P(F_i\,W_i) + P(F_i\,\overline{W_i})\}$$

$$P(W_i\,\overline{F_i}) = P(W_i) - P(W_i\,F_i) = 0.45- 0.09=0.36$$

$$P(F_i\,\overline{W_i}) = P(F_i) - P(W_i\,F_i) = 0.15- 0.09=0.06$$

$$P(\overline{W}_i\ \overline{F}_i) = 1- \{ 0.36 + 0.09+ 0.06) \} = 0.49$$

$$P(\ \overline{W}\ \overline{F}) = (0.49)^{10}$$

$$P(\ W \cap F) = 1- P(\ \overline{W} \cup \overline{F}\ )$$ De Morgan's Laws

$$= 1- \{P(\overline{W}) + P(\overline{F}) - P(\ \overline{W} \cap \overline{F}\ )\ \}$$

$$P(\ W \cap F) = 1-\{\ 0.55^{10} + 0.85^{10} - 0.49^{10}\}$$

$$= 0.8014$$

Exercise

In an insurance portfolio 10% of the policy holders are in Class A_1 (high risk), 40% are in Class A_2 (medium risk), and 50% are in Class A_3 (low risk). The probability there is a claim on a Class A_1 policy in a given year is 0.10; similar probabilities for Classes A_2 and A_3 are 0.05 and 0.02.

Find the probability that if a claim is made, it is made on a Class A_1 policy.

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Let

B = “policy has a claim”

A_i = “policy is of Class A_i”, i = 1; 2; 3.

We are asked to find $P(A_1| B)$?

$$P(A_1 \mid B) = \frac{P(A_1 B)}{P(B)}$$

$P(B) = P(A_1B) + P(A_2B) + P(A_3B)$

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$P(A_1) = 0.10$, $P(A_2) = 0.4$, $P(A_3) = 0.5$

$P(B|A_1) = 0.10$; $P(B|A_2) = 0.05$; $P(B|A_3) = 0.02$

$P(A_1B) = P(A_1)P(B|A_1) = 0.01$

$P(A_2B) = P(A_2)P(B|A_2) = 0.02$

$P(A_3B) = P(A_3)P(B|A_3) = 0.01$

This gives

$P(B) = P(A_1B) + P(A_2B) + P(A_3B)$

$P(B) = 0.01 + 0.02 + 0.01$

$= 0.04$

$P(A_1|B) = P(A_1B)/ P(B) = 0.01/0.04 = 0.25$

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Bayes's Theorem(Formula)

Reverend Thomas Bayes (1702-61)

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Baye's Theorem

- Let $A_1, A_2, \ldots, A_k$ be mutually exclusive and exhaustive events with prior probabilities $P(A_i)$ $i = 1, 2, \ldots, k$. Then for any other event B for which $P(B) > 0$ the posterior probability of A_j given that B has occurred is

$$P(A_j | B) = \frac{P(A_j \cap B)}{P(B)}$$

$$= \frac{P(B \mid A_j)\, P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i) P(A_i)} \qquad j=1,2,..k.$$

Example

A printer manufacturer obtained the following probabilities from a database of test results. Printer failures are associated with three types of problems: hardware(H), software(S), and other(O)(such as connectors), with probabilities 0.1, 0.6, and 0.3, respectively.

The probability of a printer failure(F) given a hardware problem is 0.9, given a software problem is 0.2, and given any other type of problem is 0.5.

If a customer enters the manufacturer's Web site to diagnose a printer failure, what is the probability that cause of the problem is

a) hardware problem.

b) software problem.

c) any other type of problem.

d) what is the <u>most likely cause of the problem?</u>

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$$P(H|F) = \frac{P(F|H)\,P(H)}{P(F)}$$

$$\begin{aligned} P(F) &= P(FH) + P(FS) + P(FO) \\ &= P(F|H)\,P(H) + P(F|S)\,P(S) + P(F|O)\,P(O) \\ &= 0.9(0.1) + 0.2(0.6) + 0.5(0.3) = 0.36 \end{aligned}$$

a) $P(H|F) = 0.9(0.1) / 0.36 = 0.25$

b) $P(S|F) = P(F|S)\,P(S) / P(F) = 0.2(0.6) / 0.36 = 0.333$

c) $P(O|F) = P(F|O)\,P(O) / P(F) = 0.5(0.3) / 0.36 = 0.417$

d)The most likely cause of the problem is the one with the largest of $P(H \mid F)$, $P(S \mid F)$, $P(O \mid F)$.

Note: You can use $P(HF)$, $P(SF)$, $P(OF)$ too.

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Examples on Chapter 4

4.5.1

If you take a bus to work in the morning there is a 20% chance you'll arrive late. When you go by bicycle there is a 10% chance you'll be late. 70% of the time you go by bike, and 30% by bus. Given that you arrive late, what is the probability you took the bus?

Let

$B=\{\text{Bus}\}$, $\bar{B} = \{\text{Bike}\}$

$L=\{\text{late}\}$

$P(L|B) = 0.20$

$P(L|\overline{B}) = 0.10$

$P(\overline{B}) = 0.70$

$P(B) = 0.30$

$P(B|L) = ????$

$$P(B|L) = \frac{P(LB)}{P(L)}$$

$$= \frac{P(L|B)P(B)}{P(L|B)P(B) + P(L|\bar{B})P(\bar{B})}$$

$$= \frac{(0.20)(0.30)}{(0.20)(0.30) + (0.10)(0.70)}$$

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4.5.2

A box contains 4 coins – 3 fair coins and 1 biased coin for which P(heads) = 0.8. A coin is picked at random and tossed 6 times. It shows 5 heads. Find the probability this coin is fair.

Let

$F = \{\text{fair}\}$, $\overline{F} = \{\text{ biased}\}$

$H = \{5 \text{ heads}\}$

$P(F|H) = ???$

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$$P(F|H) = \frac{P(FH)}{P(H)}$$

$$= \frac{P(H|F)\,P(F)}{P(H|F)\,P(F) + P(H|\overline{F})\,P(\overline{F})}$$

$$P(F|H) = \frac{\frac{3}{4}\binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)}{\frac{3}{4}\binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right) + \frac{1}{4}\binom{6}{5}(0.8)^5(0.2)}$$

$$= 0.417$$

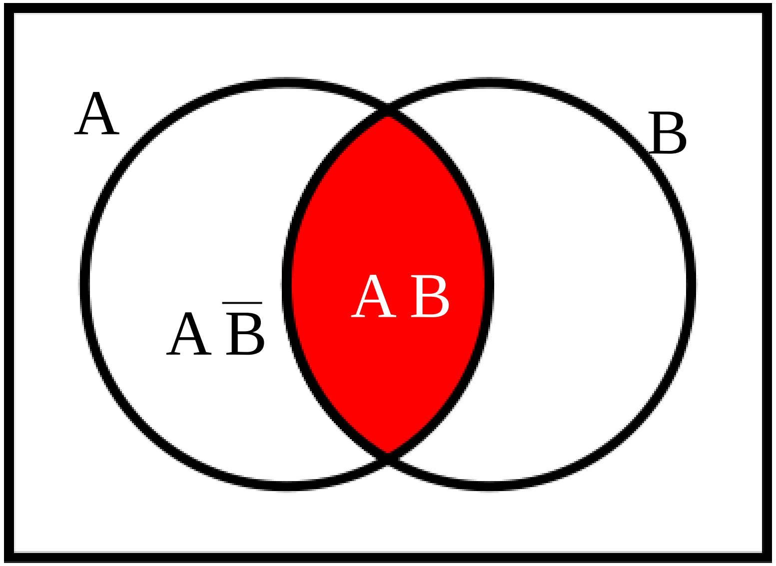
4.4

Let A and B be events defined on the same sample space, with $P(A) = 0.3$, $P(B) = 0.4$ and $P(A|B) = 0.5$.

(a) Given that event B does not occur, what is the probability of event A?

$P(A|\overline{B})$???

$$P(A \mid \overline{B}) = \frac{P(A\overline{B})}{P(\overline{B})}$$



A
B
$A\overline{B}$
A B

$$P(AB) = P(A|B)\,P(B)$$
$$= (0.5)(0.4)$$
$$= 0.20$$

$$P(A\,\overline{B}) = P(A) - P(AB)$$
$$= 0.30 - 0.20$$
$$= 0.1$$

$$P(\overline{B}) = 1 - P(B)$$
$$= 1 - 0.4 = 0.6$$

$$P(\,A|\overline{B}) = 0.1/0.6$$

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(b) What is $P(\bar{A} \mid B)$? $P(A) = 0.3$, $P(B) = 0.4$ and $P(A|B) = 0.5$.

$P(A \mid B) + P(\bar{A} \mid B) = 1$

$P(\bar{A} \mid B) = 1 - 0.5 = 0.5$

Review of Useful Series and Sums

1. <u>Geometric Series</u>:

$$\boldsymbol{a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r}}$$

$$= \frac{a(r^n - 1)}{r - 1} \text{ for } r \neq 1$$

- If |r|<1, then

$$a + ar + ar^2 + \ \ldots = \frac{a}{1 - r}$$

(infinite sum)

2. Binomial Theorem: Various forms exist, but we shall use

$$(\mathbf{1}+\boldsymbol{a})^{\boldsymbol{n}} = 1 + \binom{n}{1} a^1 + \binom{n}{2} a^2 + \cdots + \binom{n}{n} a^n$$

$$= \sum_{x=0}^{n} \binom{n}{x} a^x$$

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3. A more generalised version of the Binomial Theorem exists where we consider an infinite series:

$$\sum_{x=0}^{\infty} \binom{n}{x} a^x = (1 + a)^n \text{ if } |a| < 1$$

- The proof follows by using the Maclaurin series.

4. <u>Multinomial Theorem</u>: This is an extension of the Binomial Theorem

$$(\boldsymbol{a_1 + a_2 + \cdots + a_k})^{\boldsymbol{n}}$$

$$= \sum_{x_1, x_2, \ldots, x_k} \frac{\boldsymbol{n!}}{\boldsymbol{x_1! \ldots x_k!}} \boldsymbol{a_1^{x_1} \ldots a_k^{x_k}}$$

Where $\sum x_i = n$ and $\frac{n!}{x_1! \ldots x_k!} = \binom{n}{x_1, \ldots, x_k}$

5. Hypergeometric Identity:

$$\sum_{x=0}^{\infty} \binom{a}{x} \binom{b}{n-x} = \binom{a+b}{n}$$

6. Exponential Series:

Here we use the Maclaurin series expansion again. Let $f(x) = e^x$, then

- $f^{(r)}(0) = 1$, where $f^{(r)}(x)$ represents the r^{th} derivative, So

$$\boldsymbol{e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

- We can also use the limit definition: for all real x,

$$\boldsymbol{e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n}$$

7. Special Series involving integers:

- $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

- $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

- $1^3 + 2^3 + \cdots + n^3 = [\frac{n(n+1)}{2}]^2$

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