STAT230

PROBABILITY

Chapter 9 (a)

UNIVERSITY OF WATERLOO

Multivariate Distributions
Chapter 9 : Chapter objectives

- Understand the probability models for the joint behavior of several random variables.

- Independent random variables.

- Conditional distributions: discrete case.

- Functions of Random Variables.

Basic Terminology and Techniques

- Many problems involve more than a single random variable.

- When there are multiple random variables associated with an experiment or process we usually denote them as X, Y, . . . or as X_1, X_2, ...

Example

Your final mark in a course might involve

X_1=Your assignment mark.

X_2=Your midterm test mark.

X_3 =Your final exam mark.

UNIVERSITY OF WATERLOO

Multivariate Problems

- Discrete multivariate problems.

Example

Suppose a fair coin is tossed 3 times. Define the r.v.'s

X = number of Heads and Y = 1(0) if H(T) occurs on the first toss.

- Continuous multivariate problems.

Example

Consider a person's height X and weight Y.

Joint Probability Functions:

- Joint probability function of two discrete random variables.

Let X and Y be two discrete r.v.'s defined on the sample space **S** of an experiment. The *joint probability function* $f(x, y)$ is

$$f(x, y) = P(X = x \text{ and } Y = y)$$

$$= P(X = x, Y = y)$$

UNIVERSITY OF WATERLOO

The function $f\,(x, y)$ is the joint probability function of the discrete random variables X and Y if and only if its values satisfy the conditions

- $f\,(x, y) \geq 0$, for all (x, y)
- $\sum\sum_{\text{all }(x,\ y)} f\,(x\,,y\,) = 1$,

In general, if there are n random variables $X_1, X_2, \ldots, X_n$

$$f\,(x_1, x_2, x_3, \ldots, x_n) = P(X_1{=}x_1, X_2{=}x_2, \ldots X_n{=}x_n)$$

UNIVERSITY OF WATERLOO

Example

Let the joint probability function of X and Y is given in the following table

		y		
	$f(x, y)$	0	1	2
x	0	1/16	1/8	1/8
	1	1/16	1/8	1/16
	2	1/8	1/4	1/16

Find

1) $f(0, 0)$

2) $P(y \geq 1)$

3) $P(1 \leq X < 3, Y > 1)$

1) $f(0, 0) = P(X=0, Y=0) = 1/16$

2) $P(y \geq 1) =$

{ Is computed by summing probabilities of all (x, y) pairs for which $y \geq 1$ }

$= P(X=0, Y=1) + P(X=0, Y=2) + P(X=1, Y=1) + P(X=1, Y=2)$
$+ P(X=2, Y=1) + P(X=2, Y=2)$
$= 1/8+1/8+1/8+1/16+1/4+1/16$

3) $P(1 \leq X < 3, Y > 1)$

$= P(X=1, Y=2) + P(X=2, Y=2) = 1/16 + 1/16$

Marginal Distributions:

❑ Marginal Probability Functions

- The *marginal probability functions* of X and Y, denoted $f_X(x)$ and $f_Y(y)$ are given by

- $f_X(x) = f_1(x) = \sum_{y} f(x, y)$ and

- $f_Y(y) = f_2(y) = \sum_{x} f(x, y)$

- We can extend the idea of the marginal distribution to any number of variables (beyond two variables).

$$f_1(x_1) = \sum_{all\,(x_2,x_3)} f(x_1, x_2, x_3) = P(\ X_{1=}x_1\ )$$

$$f_{1,3}(x_1, x_3) = \sum_{all\,(x_2)} f(x_1, x_2, x_3) = P(\ X_{1=}x_1, X_{3=}x_3\ )$$

Where $f_{1,3}(x_1,x_3)$ is the marginal joint probability function of (X_1, X_3).

UNIVERSITY OF WATERLOO

- The joint cumulative probability distribution function of X and Y such that $X \le x$ and $Y \le y$,given by

$$F(x, y) = p(X \le x, Y \le y) = \sum_{x_i \le x} \sum_{y_i \le y} f(x_i, y_i)$$

$$\text{for } -\infty < x < \infty \quad , \quad -\infty < y < \infty$$

Example

Let the joint probability function of X and Y is given in the following table

	x				
y	f(x, y)	0	1	2	f(y)
	0	1/16	1/8	1/8	5/16
	1	1/16	1/8	1/16	4/16
	2	1/8	1/4	1/16	7/16
	f(x)	4/16	8/16	4/16	1

Find

1) F(1, 1).

2) Find the marginal distribution of X.

3) Find the marginal distribution of Y.

(1) $F(1, 1) = P(X \leq 1, Y \leq 1)$

$$= f(1, 0) + f(1, 1) + f(0,0) + f(0,1)$$

$$= 1/16 + 1/8 + 1/16 + 1/8$$

2) Marginal of Y

y	0	1	2	Sum
$f_2(y)$	5/16	4/16	7/16	1

3) Marginal of X

x	0	1	2	Sum
$f_1(x)$	4/16	8/16	4/16	1

UNIVERSITY OF WATERLOO

Example

Suppose that 3 balls are randomly selected from an urn(or box) containing 3 red, 4 green, and 5 blue balls. If we let X and Y denote, respectively, the number of red and green balls chosen, then the joint probability function of X and Y,

$f(i, j) = P(X = i, Y = j)$, is given by

$$f(0,0) = \binom{5}{3} / \binom{12}{3} = 10 \ / \ 220$$

$$f(0,1) = \binom{4}{1}\binom{5}{2} / \binom{12}{3} = 40 \ /220$$

$$f(0,2) = \binom{4}{2}\binom{5}{1} / \binom{12}{3} = 30 \ /220$$

....

....

$$f(3,0) = \binom{3}{3} / \binom{12}{3} = 1 \ / \ 220$$

3 red

4 green

5 blue balls

3 balls

	y					
x	f(x=i,y=j)	**0**	**1**	2	**3**	f(x=i)
	0	10/220	40/220	30/220	4/220	84/220
	1	30/220	60/220	18/220	0	108/220
	2	15/220	12/220	0	0	27/220
	3	1/220	0	0	0	1/220
	f(y =j)	56/220	112/220	48/220	4/220	1

Independent Random Variables

X and Y are independent random variables if

$$f\ (x, y) = f_1(x)\, f_2(y) \quad \text{for all values of } (x, y)$$

The random variables $X_1, X_2,\ldots,X_n$ are **independent if for every subset** of the variables, the joint p.f of the subset is equal to the product of the marginal p.f's.

$$f(x_1, x_2,\ldots, x_n) = f_1(x_1)\, f_2(x_2) \ldots\ldots f_n(x_n) \text{ for all } x_1, x_n,\ldots, x_n$$

Example

The joint probability distribution of X and Y as in the table.

Are X and Y independent?

		x				
y	$f(x,y)$	0	1	2	3	$f(y)$
	-3	1/8	0	0	0	1/8
	-1	0	3/8	0	0	3/8
	1	0	0	3/8	0	3/8
	3	0	0	0	1/8	1/8
	$f(x)$	1/8	3/8	3/8	1/8	1

UNIVERSITY OF WATERLOO

No because $P(X = 0, Y = -3) = 1/8$,

however $P(X=0) = 1/8$

$P(Y = -3) = 1/8$

so $(1/8) * (1/8)$ not equal $1/8$

Conditional Probability Functions

- For events A and B, recall that

$$P(A \mid B) = \frac{P(AB)}{P(B)}, \text{ where } P(B) > 0$$

- The conditional probability function of X given Y = y

$$f(x \mid y) = \frac{f(x, y)}{f_Y(y)}, \text{ where } f_Y(y) > 0$$

Similarly,

$$f(y \mid x) = \frac{f(x, y)}{f_X(x)}$$

where $f_X(x) > 0$

UNIVERSITY OF WATERLOO

Example

Suppose that $f(x, y)$, the joint probability function of X and Y, is given by

$f(0, 0) = 0.4$ $f(0, 1) = 0.2$ $f(1, 0) = 0.1$ $f(1, 1) = 0.3$

Calculate the conditional probability function of X given that Y = 1.

UNIVERSITY OF WATERLOO

$$P(X = x \mid Y=1) = \frac{P(X = x, Y=1)}{P(Y=1)}$$

$$f_Y(y=1) = \sum_x f(x, 1) = f(0, 1) + f(1, 1) = 0.5$$

$$f_{X|Y}(x=0|y=1) = \frac{f(x=0, y=1)}{f_Y(y=1)} = 0.2 / 0.5 = 2/5$$

$$f_{X|Y}(x=1|y=1) = \frac{f(x=1, y=1)}{f_Y(1)} = 0.3 / 0.5 = 3/5$$

UNIVERSITY OF WATERLOO

Example

Suppose a fair coin is tossed 3 times. Define the r.v.'s X = number of Heads and Y = 1(0) if H(T) occurs on the first toss. Find $f(x|Y=1)$.

$f(x, y)$		x: 0 (no H)	1 (one H)	2 (two H)	3 (three H)	$f_Y(y)$
y	0 First toss T	1/8	2/8	1/8	0	4/8
	1 First toss H	0	1/8	2/8	1/8	4/8
$f_X(x)$		1/8	3 /8	3/8	1/8	1

UNIVERSITY OF WATERLOO

$$f(x \mid Y = 1) = \frac{f(x, 1)}{f_Y(1)}$$

x	0	1	2	3
$f(x, Y=1)$	0	1/8	2/8	1/8

$$f_Y(1) = 4/8$$

x	0	1	2	3
$f(x \mid Y=1)$	0	1/4	2/4	1/4

Note: marginal and conditional probability functions are probability functions in that they are always ≥ 0 and their sum is 1.

Functions of Random Variables

Example

Suppose X and Y have joint probability function as in the table below.

Find the probability function of $U = 2(Y - X)$.

$f(x, y)$	x = 0	x = 1	x = 2	$f_Y(y)$
y = 1	0.1	0.2	0.3	0.6
y = 2	0.2	0.1	0.1	0.4
$f_X(x)$	0.3	0.3	0.4	1

Let $U = 2(Y - X)$, the possible values of U are seen by looking at the value of $u = 2(y - x)$ for each (x, y) in the range of (X, Y)

	u	x: 0	1	2
y	1	2	0	-2
	2	4	2	0

$u=2(y-x)$

- When y=1 and x=0, $u=2(1-0)=2$
- When y=1 and x=1 $u=2(1-1)=0$

....

	$f(x, y)$	x: 0	1	2	$f_Y(y)$
y	1	0.1	0.2	0.3	0.6
	2	0.2	0.1	0.1	0.4
	$f_X(x)$	0.3	0.3	0.4	1

Then

$P(U = -2) = P(X = 2 \text{ and } Y = 1) = f(2, 1) = 0.3$

$P(U = 0) = P(X = 1 \text{ and } Y = 1, \text{ or } X = 2 \text{ and } Y = 2)$
$= f(1, 1) + f(2, 2) = 0.3$

$P(U = 2) = f(0, 1) + f(1, 2) = 0.2$

$P(U = 4) = f(0, 2) = 0.2$

u	-2	0	2	4
$f(u)$	0.3	0.3	0.2	0.2

Poisson and Binomial Random Variables

- If X and Y are independent Poisson random variables with respective parameters μ_1 and μ_2, then X + Y has a Poisson distribution with parameter $\mu_1 + \mu_2$.

- Let X and Y be independent Binomial random variables with respective parameters (n, p) and (m, p), then the distribution of X + Y is a Binomial random variable with parameters (n + m, p)

Example

Let X~ Poisson(μ_1) and independently Y ~ Poisson(μ_2). Find the probability function of T= X+Y.

Solution

The joint probability function of X and Y is given by

$$f(x,y) = \frac{e^{-\mu 1}\mu_1^x}{x!} \frac{e^{-\mu 2}\mu_2^y}{y!} \quad \text{for } x=0,1,2,\ldots \text{ and } y=0,1,2,\ldots$$

$$P(T = t) = P(X + Y = t) = \sum_{\text{all } x} P(X = x, Y = t - x)$$

$$= \sum_{\text{all } x} P(X = x)\, P(Y = t - x)$$

$$= \sum_{x=0}^{t} \frac{e^{-\mu 1}\, \mu_1^{\,x}}{x!} \frac{e^{-\mu 2}\, \mu_2^{\,t-x}}{(t-x)!}$$

$$= e^{-(\mu 1+\mu 2)} \sum_{x=0}^{t} \frac{\mu_1^{\,x}\, \boxed{\mu_2^{\,t-x}}}{x!\,(t-x)!}$$

$$\boxed{\mu_2^{\,t}\, \mu_2^{\,-x}}$$

using the Binomial Theorem

$$= \frac{e^{-(\mu 1+\mu 2)}\, \mu_2^{\,t}}{t!} \boxed{\sum_{x=0}^{t} \frac{t!}{x!\,(t-x)!} \left(\frac{\mu_1}{\mu_2}\right)^{x}}$$

UNIVERSITY OF WATERLOO

$$P(T=t) = \frac{e^{-(\mu 1+\mu 2)}\ \mu_2^{\,t}}{t!}\left(1+\frac{\mu_1}{\mu_2}\right)^t$$

$$= \frac{e^{-(\mu 1+\mu 2)}\ \cancel{\mu_2^{\,t}}}{t!}\ \ \frac{\boxed{\mu_2 + \mu_1}^{\,t}}{\cancel{\mu_2^{\,t}}}$$

$$= \frac{e^{-(\mu 1+\mu 2)}\ (\mu_1+\mu_2)^t}{t!} \qquad \text{for } t=0,1,2,....$$

Thus, $X_1 + X_2$ has a Poisson distribution with parameter $\mu_1 + \mu_2$.

The Multinomial Distribution

- Suppose an experiment is repeated independently n times with k distinct types of outcome each time.

- Suppose that each experiment can result in any one of k possible outcomes, with respective probabilities $p_1, p_2, . . . , p_k$, each time.

- If we let X_i denote the number of times the i^{th} type occurs, then

$(X_1,X_2, \quad ,X_k)$ has a Multinomial distribution.

X_1 the number of times the 1^{st} type occurs.

X_2 the number of times the 2^{nd} type occurs.

....

X_k the number of times the k^{th} type occurs.

UNIVERSITY OF WATERLOO

Notes

1) $p_1 + p_2 + . . . + p_k = \sum_{i=1}^{k} p_i = 1.$

2) $X_1 + X_2 + ... + X_k = n$

3) You can drop one of the variables (say the last), and just note that X_k equals

$$n - X_1 - X_2 - ... - X_{k-1}.$$

The Multinomial Distribution

The Joint probability function of $X_1, X_2,\dots, X_k$ is given by

$$P(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k) = f(x_1, x_2,\dots, x_k) =$$

$$\frac{n!}{x_1!\, x_2! \cdots x_k!}\, p_1^{x1}\, p_2^{x2} \cdots p_k^{xk}$$

The restriction on the x_i's are

$$x_i = 0, 1, 2,\dots, n$$

$$\sum_{i=1}^{k} x_i = n$$

UNIVERSITY OF WATERLOO

Notes

(1) If $k = 2$ and there are two possible outcomes (Success and Failure) then we simply have a Binomial distribution.

(2) If $k = 3$ and there are three possible outcomes this distribution is also called the Trinomial distribution.

Example

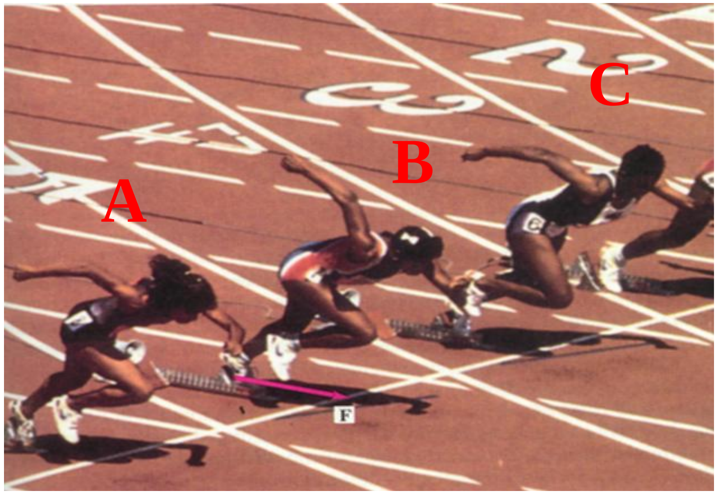
Suppose that a fair die is rolled 12 times. The probability that 1 appears 4 times, 2 and 3 three each, 4 and 5 once each, and 6 not at all is

$$f(x_1, x_2, \ldots, x_6) = \frac{12!}{4!3!3!1!1!0!} (1/6)^4 (1/6)^3 (1/6)^3 (1/6)^1 (1/6)^1 (1/6)^0 =$$

$$\frac{12!}{4!3!3!} (1/6)^{12}$$

Example

- Three sprinters A, B and C, compete against each other in 10 independent 100 m. races.
- The probabilities of winning any single race are 0.5 for A, 0.4 for B, and 0.1 for C. Let X_1, X_2 and X_3 be the number of races A, B and C win respectively.



(a) Find the joint probability function, $f(x_1, x_2, x_3)$.

$x_1 + x_2 + x_3 = 10$ since there are 10 races in all.

$$f(x_1, x_2, x_3) = \begin{cases} \dfrac{10!}{x_1!x_2!x_3!} (0.5)^{x1}(0.4)^{x2}(0.1)^{x3} & \begin{array}{l} x_1 = 0,1,\ldots 10 \\ x_2 = 0,1,\ldots 10 \\ x_3 = 0,1,\ldots 10 \\ x_1 + x_2 + x_3 = 10 \end{array} \\ 0 & \text{o/w} \end{cases}$$

UNIVERSITY OF WATERLOO

(b) Find the marginal probability function, $f(x_1)$.

Note

Each race is either won by A ("success") or it is not won by A ("failure").

Since the races are independent and X_1 is now just the number of "success" outcomes, X_1 must have a Binomial distribution, with n = 10 and p = 0.5.

$$f(x_1) = \begin{cases} \binom{10}{x_1} (0.5)^{x_1}(0.5)^{10-x_1} & \text{for } x_1 = 0, 1, \ldots, 10 \\ 0 & \text{o/w} \end{cases}$$

(c) Find the conditional probability function, $f\,(x_2 \mid x_1)$.

$$f\,(x_2 \mid x_1) = \frac{f(x_1, x_2)}{f\,(x_1)}$$

$X_3 = 10 - X_1 - X_2$

$$f(x_1, x_2) = \frac{10!}{x_1!x_2!\,(10 - x_1 - x_2)!} \quad (0.5)^{x1}\,(0.4)^{x2}\,(0.1)^{10-x1-x2}$$

$x_1 = 0, 1,\ldots,10;$

$x_2 = 0, 1,\ldots,10$ and

$x_1 + x_2 \leq 10.$

$x_2 \leq 10 - x_1$

UNIVERSITY OF WATERLOO

$$\frac{f(x_1, x_2)}{f(x_1)} = \frac{\dfrac{\cancel{10!}}{\cancel{x_1!}x_2!\,(10 - x_1 - x_2)!}\cancel{(0.5)^{x1}}\,(0.4)^{x2}(0.1)^{10-x1-x2}}{\binom{10}{x_1}\cancel{(0.5)^{x1}}\,(0.5)^{10-x1}}$$

$$\boxed{\frac{\cancel{10!}}{\cancel{x_1}!\,(10 - x_1)!}}$$

$$\boxed{\begin{aligned}(0.5)^{10-x1-x2+x2} &= \\ (0.5)^{x2}\,(0.5)^{10-x1-x2}&\end{aligned}}$$

$$= \binom{10-x_1}{x_2}\left(\frac{4}{5}\right)^{x_2}\left(\frac{1}{5}\right)^{(10 - x_1 - x_2)} \qquad \text{for } x_2 = 0, 1, \ldots, (10 - x_1)$$

(d) Are X_1 and X_2 independent? Why?

X_1 and X_2 are clearly not independent random variables since the more races A wins, the fewer races there are for B to win.

$$f(x_1)\ f(x_2) \neq f(x_1,x_2)$$

$$\binom{10}{x_1}(0.5)^{x1}(0.5)^{10-x1}\ \binom{10}{x_2}(0.4)^{x2}(0.6)^{10-x2} \neq$$

$$\frac{10!}{x_1!x_2!\,(10-x_1-x_2)!}\ (0.5)^{x1}(0.4)^{x2}(0.1)^{10-x1-x2}$$

UNIVERSITY OF WATERLOO

Note

In general, if the range of X_1 depends on the value of X_2, then X_1 and X_2 can not be independent random variables.

(e) Let $T = X_1 + X_2$. Find its probability function, $f_T(t) = P(T = t)$.

If $T = X_1 + X_2 \Longrightarrow X_2 = T - X_1$

$$f_T(t) = P(T = t) = \sum_{x_1=0}^{t} f(x_1, t - x_1)$$

$$\boxed{f(x_1, x_2) = \frac{10!}{x_1!x_2!\,(10 - x_1 - x_2)!}\,(0.5)^{x1}\,(0.4)^{x2}\,(0.1)^{10-x1-x2}}$$

$$f_T(t) = \sum_{x_1=0}^{t} \frac{10!}{x_1!(t - x_1)!\,\underbrace{(10 - x_1 - (t - x_1))!}_{(10-t)!}}\,(0.5)^{x1}(0.4)^{t-x1}\,(0.1)^{10-x1-(t-x1)}$$

$$f_T(t) = \frac{10!}{(10-t)!}(0.4)^t (0.1)^{10-t} \sum_{x_1=0}^{t} \frac{1}{x_1!(t - x_1)!} \left(\frac{0.5}{0.4}\right)^{x1}$$

Multiply by t on the top and bottom

$$f_T(t) = \frac{10!}{t!\,(10-t)!}(0.4)^t (0.1)^{10-t} \sum_{x_1=0}^{t} \frac{t!}{x_1!(t - x_1)!} \left(\frac{0.5}{0.4}\right)^{x1}$$

$$f_T(t) = \binom{10}{t}(0.4)^t (0.1)^{10-t} \left(1 + \frac{0.5}{0.4}\right)^t$$

using the Binomial Theorem

UNIVERSITY OF WATERLOO

$$f_T(t) = \binom{10}{t} \cancel{(0.4)^t} \quad (0.1)^{10-t} \quad \frac{\left(0.4 + 0.5\right)^t}{\cancel{\left(0.4\right)}^t}$$

$$f_T(t) = \binom{10}{t} \quad (0.9)^t \quad (0.1)^{10-t} \qquad \text{for} \quad t = 0, 1, \ldots, 10$$

Expectation for Multivariate Distributions

If X is a discrete r.v. with probability function $f(x) = P(X=x)$ then

- The expected value of X is

$$E(X) = \mu_x = \sum_{\text{all } x} x \; f(x)$$

$$E[g(X)] = \sum_{\text{all } x} g(x) \, f(x)$$

UNIVERSITY OF WATERLOO

- Suppose that X and Y are two discrete random variables and g is a function of the two variables.

If X and Y have a joint probability function $f(x,y)$, then

$$E[g(X,Y)] = \sum_{\text{all}(x,\, y)} g(x, y)\, f(x, y)$$

$$E[g(X_1,X_2,\ldots,X_n)] = \sum_{\text{all } (x_1,x_2,\ldots,x_n)} g(x_1,x_2,\ldots,x_n)\, f(x_1,x_2,\ldots,x_n)$$

UNIVERSITY OF WATERLOO

Example

Let the joint probability function, $f(x, y)$ be given by

$f(x, y)$		x = 0	1	2	$f_2(y)$
y	1	0.1	0.2	0.3	0.6
	2	0.2	0.1	0.1	0.4
$f_1(x)$		0.3	0.3	0.4	1

Find E(XY) and E(X).

$$E(XY) = \sum_{\text{all } (x,y)} xy\, f(x, y)$$

$$= (0 * 1)(0.1) + (1*1)(0.2) + (2*1)(0.3) + (0*2)(0.2) + (1*2)(0.1) + (2*2)(0.1)$$

$$= 1.4$$

$$E(X) = \sum_{\text{all } (x,y)} x\, f(x, y)$$

$$= (0*0.1) + (1*0.2) + (2*0.3) + (0*0.2) + (1*0.1) + (2* 0.1)$$

$$= 1.1$$

$$E(X) = \sum_{x=0}^{2} x\, f(x) = (0*0.3) + (1*0.3) + (2*0.4) = 1.1$$

UNIVERSITY OF WATERLOO

Example

Suppose that the joint probability of X_1 and X_2 is given below

$$f(x_1, x_2) = \frac{10!}{x_1!x_2!\,(10 - x_1 - x_2)!}(0.5)^{x1}(0.4)^{x2}(0.1)^{10-x1-x2}$$

$x_1 = 0, 1,\ldots,10;$

$x_2 = 0, 1,\ldots,10$ and

$x_1 + x_2 \leq 10.$

Find $E(X_1X_2)$.

$$E(X_1X_2) = \sum x_1x_2 \ f(x_1, x_2)$$

$$= \sum_{\substack{x_1 \neq 0 \\ x_2 \neq 0}} x_1x_2 \frac{10!}{x_1!x_2!\,(10 - x_1 - x_2)!} (0.5)^{x1}(0.4)^{x2}(0.1)^{10-x1-x2}$$

$$= \sum_{\substack{x_1 \neq 0 \\ x_2 \neq 0}} \cancel{x_1x_2} \frac{10 * 9 * 8!}{\cancel{x_1}(x_1 - 1)!\ \cancel{x_2}\,(x_2 - 1)!\ (10 - x_1 - x_2)!} *$$

$$(0.5)\ (0.5)^{x1-1}(0.4)\ (0.4)^{x2-1}(0.1)^{(10-2)-(x1-1)-(x2-1)}$$

$E(X_1X_2) =$

$$10 * 9 * 0.5 * 0.4 \sum_{\substack{x_1 \neq 0 \\ x_2 \neq 0}} \frac{8!}{(x_1 -1)!\,(x_2 -1)!\,(8 - (x_1 -1) - (x_2 -1))!} \;*$$

$$(0.5)^{x1 -1}\,(0.4)^{x2-1}(0.1)^{(8-(x1-1)-(x2-1))}$$

Let

$y_1 = x_1 - 1$ and $y_2 = x_2 - 1$

$$E(X_1X_2) = 18 \sum_{\substack{y_1 \neq 0 \\ y_2 \neq 0}} \frac{8!}{y_1!\, y_2!\, (8 - y_1 - y_2)!} (0.5)^{y1} (0.4)^{y2} (0.1)^{(8-y1-y2)}$$

$$= 18(\ 0.5+0.4+0.1)^8 \quad = 18$$

By the Multinomial theorem.

$$\boxed{(\ a_1 + a_2 + \ldots + a_k)^n = \sum_{\substack{\text{All} \\ x1,\, x2,..,\, xk}} \frac{n!}{x_1!\, x_2!\, \ldots,\, x_k!} a_1^{X1}\, a_2^{X2} \ldots\, a_k^{Xk}}$$

Exercise

Four electronic ovens that were dropped during shipment are inspected and classified as containing either a major, a minor, or no defect. In the past, 60% of dropped ovens had a major defect, 30% had a minor defect, and 10% had no defect. Assume that the defects on the four ovens occur independently.

(a) Is the probability distribution of the count of ovens in each category multinomial? Why or why not?

The probability distribution is multinomial because the result of each trial (a dropped oven) results in either a major, minor or no defect with probability 0.6, 0.3 and 0.1 respectively. Also, the trials are independent

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \frac{n!}{x_1!x_2!\ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

for $x_1 + x_2 + \ldots + x_k = n$ and $p_1 + p_2 + \ldots + p_k = 1.$

(b) What is the probability that, of the four dropped ovens, two have a major defect and two have a minor defect?

Let X, Y, and Z denote the number of ovens in the sample of four with major, minor, and no defects, respectively

$$P(X = 2, Y = 2, Z = 0) = \frac{4!}{2!2!0!} 0.6^2 0.3^2 0.1^0 = 0.1944$$

UNIVERSITY OF WATERLOO

c) What is the probability that no oven has a defect?

$$P(X=0,Y=0,Z=4)=\frac{4!}{0!0!4!}0.6^{0}0.3^{0}0.1^{4}=0.0001$$

(d) The joint probability function of the number of ovens with a major defect and the number with a minor defect.

$$f_{XY}(x,y) = \sum_{R} f_{XYZ}(x,y,z)$$

where R is the set of values for z such that x + y + z = 4.

That is, R consists of the single value z = 4 – x - y and

$$f_{XY}(x,y) = \frac{4!}{x!\,y!(4-x-y)!}0.6^{x}0.3^{y}0.1^{4-x-y} \quad for\;\; x+y \le 4.$$

(e) The expected number of ovens with a major defect.

$$E(X) = np_1 = 4(0.6) = 2.4$$

(f) The expected number of ovens with a minor defect.

$$E(Y) = np_2 = 4(0.3) = 1.2$$

(g) The conditional probability distribution of the number of ovens with major defects given that two ovens have minor defects

$$P(X=2 \mid Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{0.1944}{0.2646} = 0.7347$$

$$P(Y=2) = \binom{4}{2} 0.3^2 0.7^4 = 0.2646$$

(h) The conditional probability that three ovens have major defects given that two ovens have minor defects.

Not possible, $x + y + z = 4$, the probability is zero.

Properties

- $E(X + Y) = E(X) + E(Y)$

If $E(X_i)$ is finite for all $i = 1, \ldots, n$, then

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)$$

- Suppose that, for random variables X and Y, $X \geq Y$ then $E(X) \geq E(Y)$

- If X and Y are independent, then, for any functions h and g,

$$E[\, g(X)\, h(Y)\,] = E[g(X)]\ E[h(Y)]$$

- $E\,[a\, g_1(X, Y\,) + b g_2(X, Y\,)] = a\, E[g_1(X, Y\,)] + b\, E[g_2(X, Y\,)]$