STAT230

PROBABILITY

Chapter 8(a)

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Continuous Probability Distributions
Chapter 8: Chapter objectives

- Know the definition and properties of a continuous random variables
- Characterize a continuous r.v. by its p.d.f and c.d.f.
- Compute the expected value and the variance of any function of a continuous random variable.
- Understand the assumptions for some common continuous probability distributions.
- Calculate probabilities, determine means and variances for some common continuous probability distributions.

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Please Do Chapter 8 Problems (8.1- 8.18)

Continuous Random Variables and Probability Density Functions

A random variable that can assume any value in an entire interval of real numbers is said to be continuous, that is for some $a < b$, any number x between a and b is possible, and $P(X = x) = 0$ for each x.

Examples

- The time that a train arrives at a specified stop.
- The life time of transistor.

- The amount of time in hours that a computer functions before braking down.

- A probability density function(p.d.f) of a continuous random variable X is a function $f(x)$, such that for any two numbers $a \le b$

$$P[a \le X \le b] = \int_a^b f(x)\,dx.$$

Note: for any number c, P(X =c)=0

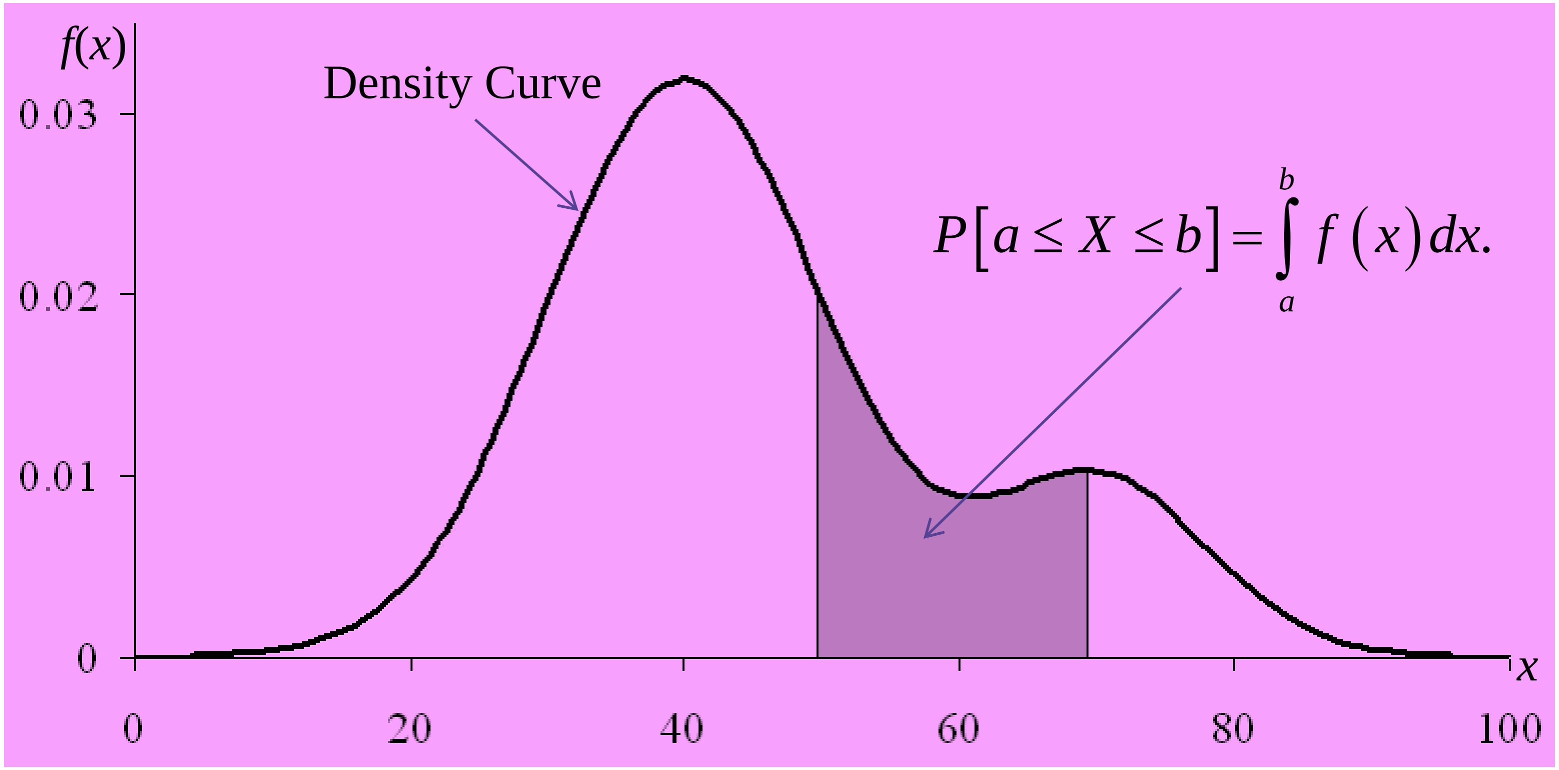
which has the following properties :

1. It is non-negative i.e. $f(x) \ge 0$.

2. $$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$ Total area =1

= area under the entire graph of $f(x)$

Probability density function, *f*(*x*)



Area under the entire graph of *f*(x) = ???

Note:

If X is a continuous random variable, for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2)$$

$$= P(x_1 \le X < x_2)$$

$$= P(x_1 < X < x_2)$$

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Example

Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of C?

(b) Find P(X > 1).

Since f is a probability density function, we must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1,$$

$$C \int_{0}^{2} (4x - 2x^2)\,dx = 1$$

$$C \left[2x^2 - 2x^3/3 \right]_0^2 = 1 \Rightarrow C = 3/8$$

$$f(x) = \begin{cases} (3/8)(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) $P(X > 1) = \int_{1}^{\infty} f(x)\,dx = 3/8 \int_{1}^{2} (4x - 2x^2)\,dx$

$= 1/2$

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The Cumulative Distribution Function(c.d.f)

The cumulative distribution function, $F(x)$ for a continuous r.v. X is defined for every number $x \in R$ by

$$F(x) = P\left(X \leq x\right) = \int_{-\infty}^{x} f(y)dy$$

For each x, $F(x)$ is the area under the density curve to the left of x.

Properties

- The function *F* is non-negative: $F(x) \geq 0$. Probabilities are never negative.

- The function *F* is non-decreasing . If $b \geq a$, then $F(b) \geq F(a)$.

- The function *F* is continuous for all $x \in R$.

$$\boxed{\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to +\infty} F(x) = 1.}$$

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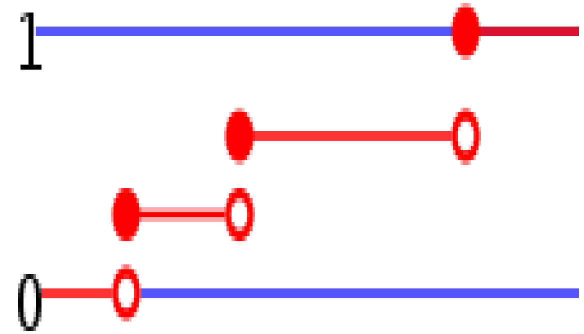
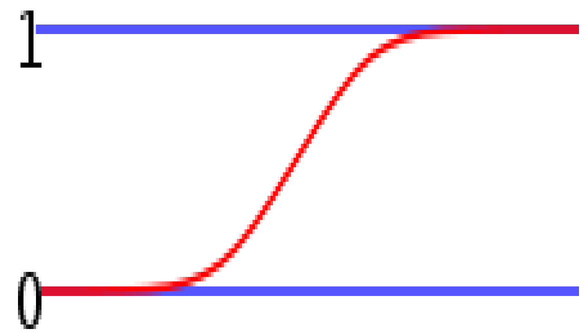
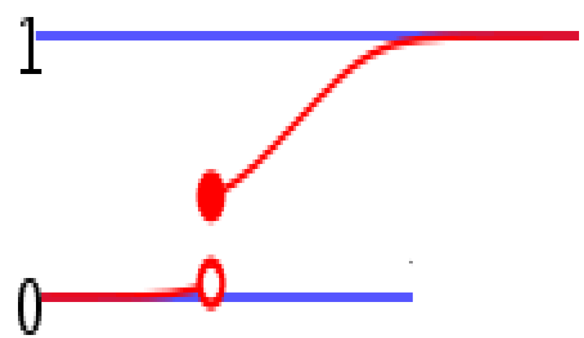
- If X is a purely discrete random variable, then it attains values x_1, x_2, ... with probability $p_i = P(x_i)$, and the CDF of X will be discontinuous at the points x_i and constant in between.



- The cumulative distribution function of a continuous probability distribution.



- The cumulative distribution function which has both a continuous part and a discrete part.



Notes

1. $P(a < X \leq b) = F(b) - F(a)$.

2. $P(X = a) = P(a \leq X \leq a) = F(a) - F(a) = 0$.

When we say that the probability is zero that a continuous random variable assumes a specific value, we do not necessarily mean that a particular value cannot occur. We in fact, mean that the point (event) is one of an infinite number of possible outcomes.

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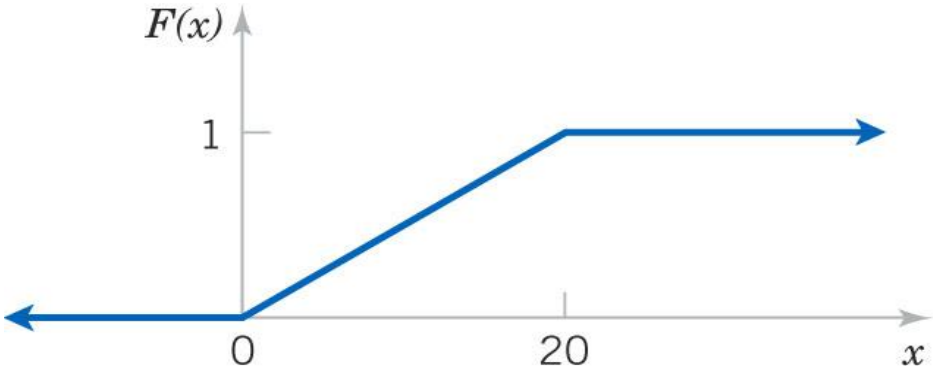
Example

The cumulative distribution function of the random variable X consists of three expressions.



$$F(x) = \begin{cases} 0 & x < 0 \\ 0.05\,x, & 0 \le x < 20 \\ 1 & 20 \le x \end{cases}$$

Using F(x) to Compute Probabilities

- Let X be a continuous r.v. with p.d.f. $f(x)$ and c.d.f. $F(x)$. Then for any number a

- $P(X > a) = 1 - P(X \leq a) = 1 - F(a)$

- $P(a \leq x \leq b) = F(b) - F(a)$

Example

Suppose the p.d.f of the X is given by

$$f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}x, & 0 \le x < 2 \\ 0 & o/w \end{cases}$$

Find

a) F(x)

b) $P(1 \le X \le 1.5)$

c) $P(X > 1)$

$$F(x) = \int_{-\infty}^{x} f(y)\, d\, y$$

$$F(x) = \int_0^x \left(\frac{1}{8} + \frac{3}{8}\, y\right) dy = \frac{y}{8} + \frac{3y^2}{8(2)} \Big|_0^x$$

$$F(x) = \frac{x}{8} + \frac{3x^2}{16}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{8} + \dfrac{3x^2}{16}, & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

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$P(1 \leq X \leq 1.5) = F(1.5) - F(1)$

$$= \left(\frac{1.5}{8} + \frac{3(1.5)^2}{16}\right) - \left(\frac{1}{8} + \frac{3(1)^2}{16}\right) = 0.297$$

$P(x > 1) = 1 - P(X \leq 1) = 1 - F(1)$

$$= 1 - \left(\frac{1}{8} + \frac{3(1)^2}{16}\right) = 0.688$$

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Example

The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda\, e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) Find λ?

(b) What is the probability that a computer will function between 50 and 150 hours before breaking down?

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(a) Since

$$1 = \int_0^\infty f(x)\,dx = \int_0^\infty \lambda\, e^{-x/100}\,dx$$

we obtain

$$1 = -\lambda\,(100)e^{-x/100}\Big|_0^\infty$$

$$1 = 100\lambda \quad \text{or} \quad \lambda = 1/100$$

(b)

$$P(50 < X < 150) = \int_{50}^{150} \frac{1}{100} e^{\frac{-x}{100}} dx = 0.383$$

Obtaining $f(x)$ from $F(x)$

- Suppose X is a continuous random variable with cumulative distribution function F(x).
- The probability density function (p.d.f.) of X is defined as

$$F'(x) = f(x).$$

$$f(x) = \frac{d}{dx} F(x)$$

where the derivative exists.

Example

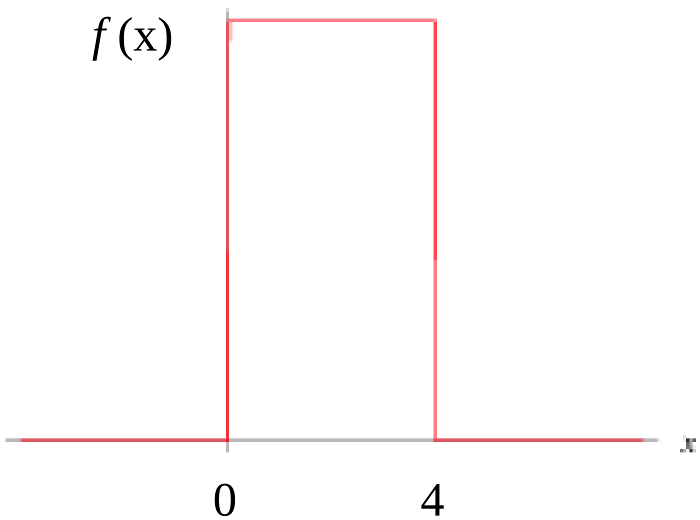
$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x/4 & \text{for } 0 < x \leq 4 \\ 1 & \text{for } x > 4 \end{cases}$$

Thus, the p.d.f. is

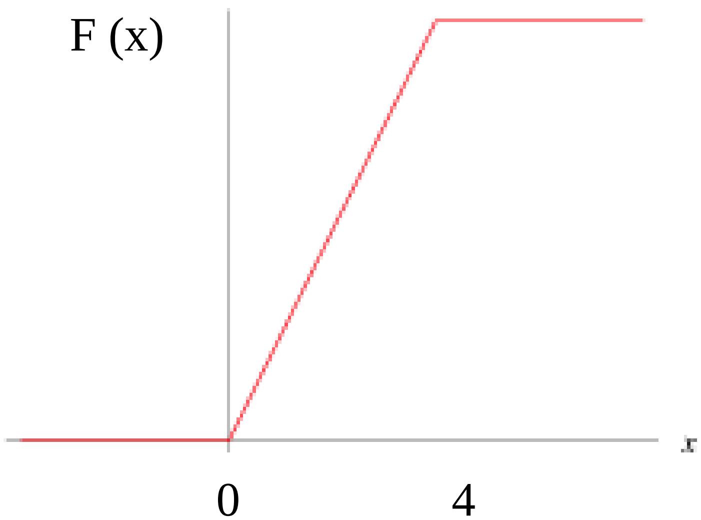
$$F'(x) = f(x).$$

$$f(x) = \begin{cases} 1/4 & \text{for } 0 < x < 4 \\ 0 & \text{o/w} \end{cases}$$

This is called a “uniform” distribution.



f (x)
x
0
4



F (x)
x
0
4

Example

Let

$$f(x) = \begin{cases} kx^2 & 0 < x \leq 1 \\ k(2-x) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

be a p.d.f.

Find

a) k

b) F (x)

c) $P\left(\frac{1}{2} < X < 1\frac{1}{2}\right)$

a) $$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\int_{-\infty}^{\infty} f(\mathrm{x})\mathrm{dx} = \int_0^1 \mathrm{k\ x^2dx} + \int_1^2 \mathrm{k(2-x)\ dx}$$

$$1 = \mathrm{k}\int_0^1 \mathrm{x^2dx} + \mathrm{k}\int_1^2 \mathrm{(2-x)dx}$$

$$1 = k\,\frac{x^3}{3}\,\Big|_1^0 + k\left(2x - \frac{x^2}{2}\right)\Big|_1^2$$

1 =5k/6

k=6/5

b) $F(x) = 0 \quad \text{if} \quad x \leq 0$

$F(x) = 1 \quad \text{if} \quad x \geq 2$

<u>For $0 < x < 1$</u>

$$P(X \leq x) = \int_0^x \frac{6}{5} z^2 \, dz = \frac{6}{5} \frac{z^3}{3} \Big|_0^{x} = \frac{2}{5} x^3$$

<u>For $1 < x < 2$</u>

$$P(X \leq x) = \int_0^1 \frac{6}{5} z^2 \, dz + \int_1^x \frac{6}{5}(2 - z) dz$$

$$= \frac{6}{5} \frac{z^3}{3} \Big|_0^1 + \frac{6}{5}\left(2z - \frac{z^2}{2}\right)\Big|_1^x$$

$$= \frac{12x - 3x^2 - 7}{5}$$

$$F(x) = \begin{cases} 0 & x \le 0 \\ 2x^3/5 & 0 < x \le 1 \\ \dfrac{12x - 3x^2 - 7}{5} & 1 < x < 2 \\ 1 & x \ge 2 \end{cases}$$

$$c)\ P\left(\frac{1}{2} < X < 1\,\frac{1}{2}\right) = F(1.5) - F(0.5)$$

$$= \frac{18 - 6.75 - 7}{5} - \frac{2(0.5)^3}{5} = 0.8$$

Defined Variables or Change of Variable

When we know the p.d.f. or c.d.f. for a continuous random variable X we sometimes want to find the p.d.f. or c.d.f. for some other random variable Y which is a function of X

1) Write the c.d.f. of Y as a function of X.

2) Use $F_X(x)$ to find $F_Y(y)$. Then if you want the p.d.f. $f_Y(y)$, you can differentiate the expression for $F_Y(y)$.

3) Find the range of values of y.

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Example

$$f(x) = \begin{cases} 1/4 & 0 < x \le 4 \\ 0 & 0/w \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

Let Y = 1/X. Find $f(y)$.

Solution:

Step(1)

$F_Y(y) = G(y) = P(Y \leq y) = P(1/X \leq y) = P(X \geq 1/y)$

$= 1 - P(X < 1/y)$

$= 1 - F_x(1/y)$

Step(2) substituting 1/y for x in $F_X(x)$

$$F_Y(y) = 1 - \frac{\left(\frac{1}{y}\right)}{4} = 1 - \frac{1}{4y}$$

Therefore

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \begin{cases} \dfrac{1}{4y^2} & \text{for } y \geq 1/4 \\ 0 & \text{o/w} \end{cases}$$

(As x goes from 0 to 4, y = 1/x goes between ∞ and 1/4 .)

Expected value for the Continuous Random Variables

- The expected or mean of continuous r.v. X with p.d.f $f(x)$ is

$$\mu_x = E(X)$$

$$E(X) = \int_{-\infty}^{+\infty} x\, f(x)\, dx$$

Note:

- If X is a continuous r.v. with pdf. $f(x)$ and g(x) is any function of X, then

$$E(g(x)) = \mu_{g(x)} = \int_{-\infty}^{+\infty} g(x)\, f(x)\, dx$$

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Example

Find E(X) when the density function of X is

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{+\infty} x \; f(x)\, dx$$

$$= \int_{0}^{1} 2x^2\, dx \quad = \quad 2x^3/3 \;\Big|_{0}^{1}$$

$$= 2 / 3$$

Example

Find $E(X^k)$ for k=1, 2, 3,...when the density function of X is

$$f(x) = \begin{cases} \lambda x & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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$$E(X^k) = \int_{-\infty}^{+\infty} x^k \; f(x) \, dx$$

$$= \int_0^1 x^k \; \lambda x \, dx = \int_0^1 \lambda \, x^{k+1} \, dx = \left. \frac{\lambda \, x^{k+2}}{k+2} \right|_0^1$$

$$= \lambda \, / \, (k+2)$$

Example

Suppose

$$f(\mathrm{x}) = \begin{cases} 1 & 0 \leq \mathrm{x} \leq 1 \\ 0 & \mathrm{o/w} \end{cases} \qquad h(\mathrm{x}) = \begin{cases} 1-\mathrm{x} & 0 \leq \mathrm{x} < 1/2 \\ \mathrm{x} & 1/2 \leq \mathrm{x} \leq 1 \end{cases}$$

Find E(h(x))

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$$E(h(x)) = \int_{0}^{1/2} (1-x)\ .1\ \ dx + \int_{1/2}^{1} x.1\ dx$$

$$= (x - x^2/2)\ \Big|_{0}^{1/2} + x^2/2\ \Big|_{1/2}^{1}$$

Example

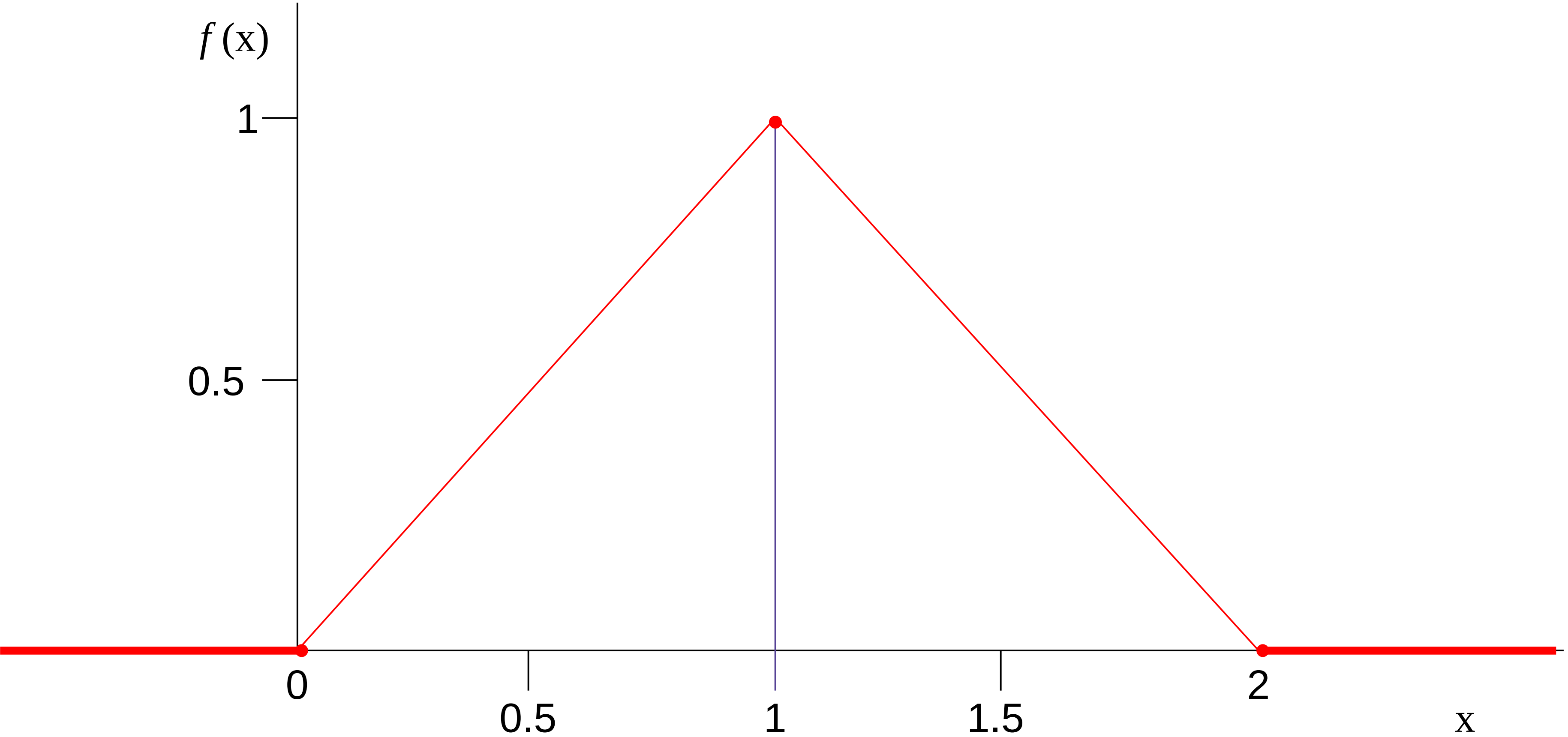
Let

$$f(x) = \begin{cases} x & 0 < x \leq 1 \\ 2(1 - x/2) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

be a p.d.f.

a) Sketch $f(x)$.

b) Find E(2X+3)



(a) Symmetric around 1.

By symmetry of the probability density function $\mu = E(X) = 1$

(b) $E(2X+3) = 2E(X) + 3 = 2(1) + 3 = 5$

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The Variance of a Continuous Random Variable

- If the random variable *X* is continuous with probability density function *f*(*x*), then the variance is given by

$$\mathrm{Var}(X) = E((X - \mu)^2)$$

$$\mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)\, dx = \int_{-\infty}^{+\infty} x^2 f(x)\, dx - \mu^2$$

where μ is the expected value,

The standard deviation σ is the square root of the variance.

Example

Let X have the probability density function

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find Var(X) where E(X)=2/3.

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Solution.

We first compute $E[X^2]$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^1 x^2 (2x)\, dx \quad = \; 2\, x^4/4 \Big|_0^1 = \; 1/2$$

Since, $E(X) = 2/3$

$Var(X) = (1/2) - (2/3)^2 = 1/18$

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Example

Let X have the probability density function

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $E(X) = 2/3$ and $Var(X) = 1/18$. Find $E(X^2)$ WITHOUT using integration.

$V(X) = E(X^2) - [E(X)]^2$

$V(X) + [E(X)]^2 = E(X^2)$

$E(X^2) = (1/18) + (2/3)^2$

$= 1/2$

Example

The probability density function of X is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = e^X$

Find

1) $F_Y(y)$
2) $f_Y(y)$
3) $E(e^X)$

1) We start by determining F_Y, the c.d.*f* of Y

<u>Step 1</u>

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln(y))$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

<u>Step 2</u>

$$F_Y(y) = \begin{cases} 0 & y < 1 \\ \ln(y) & 1 \leq y < e \\ 1 & y \geq e \end{cases}$$

If $x=0 \Rightarrow y=1$ and if $x=1 \Rightarrow y = e$

2) By differentiating $F_Y(y)$, we can conclude that the probability density function of Y is given by

$$f_Y(y) = \begin{cases} 1/y & 1 \leq y \leq e \\ 0 & \text{o/w} \end{cases}$$

3)

$$E(e^X) = \int_{-\infty}^{\infty} e^x \, f_X(x)\, dx$$

$$= \int_0^1 e^x \, dx = e - 1$$

Discrete versus Continuous

	Discrete	Continuous
c.d.f.	$F(x)=P(X\le x)=\sum_{t\le x}P(X=t)$ *F* is a right continuous step function for all real *x*	$F(x)=P(X\le x)=\int_{-\infty}^{x}f(t)dt$ *F* is a continuous function for all real *x*
p.f./p.d.f.	$f(x)=P(X=x)$	$f(x)=\frac{d}{dx}F(x)$ $f(x)\neq P(X=x)$
Probability of an event	$P(X\in A)=\sum_{x\in A}P(X=x)=\sum_{x\in A}f(x)$	$P(a<X\le b)=F(b)-F(a)=\int_{a}^{b}f(x)dx$
Total probability	$\sum_{\text{all } x}P(X=x)=\sum_{\text{all } x}f(x)=1$	$\int_{-\infty}^{\infty}f(x)dx=1$
Expectation	$E[g(X)]=\sum_{\text{all } x}g(x)f(x)=\sum_{\text{all } x}g(x)P(X=x)$	$E[g(X)]=\int_{-\infty}^{\infty}g(x)f(x)dx$