STAT 230

PROBABILITY

(Chapter 3)

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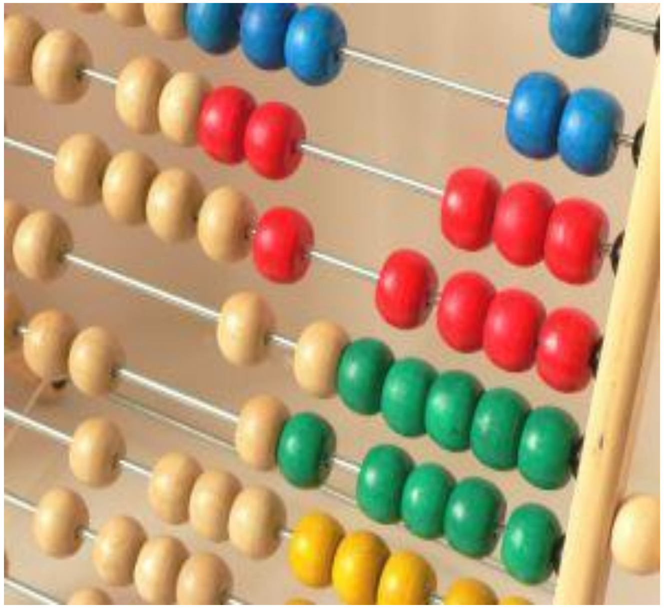
Probability-Counting Techniques
Chapter three objectives

- The Basic Principle of Counting.
- Permutations.
- Combinations.



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Events in a Uniform Probability Model

- Uniform model, in which all of the outcomes have the same probability.

- The probability of any event A is

$$P(A) = \frac{\text{Number of outcomes in A}}{\text{Total Number of outcomes in S}}$$

where A is a compound event.

The Basic Principle of Counting

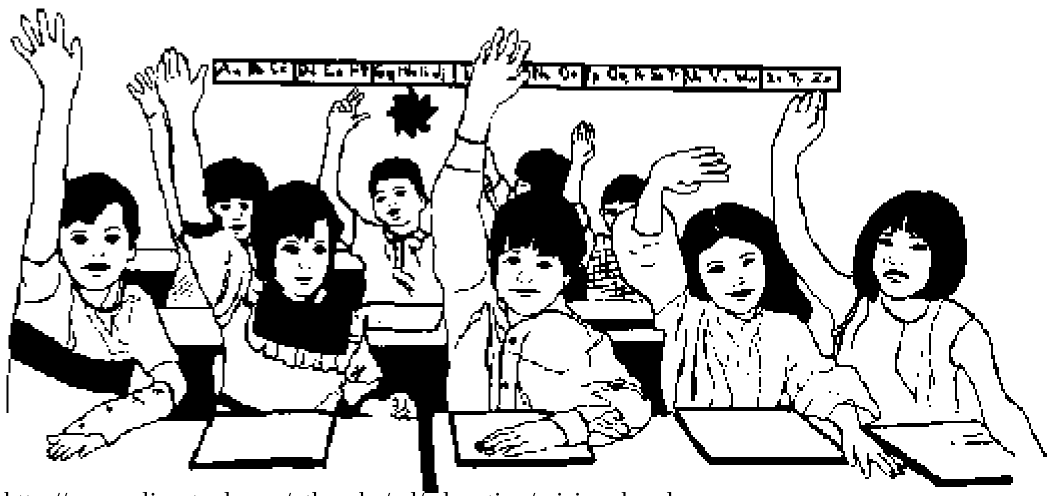
- The Addition Rule

Suppose that two experiments (jobs) are to be performed. Then if experiment 1 can result in any one of p possible outcomes and there are q possible outcomes of experiment 2.

Then we can do either experiment 1 OR experiment 2 (but not both), in $p + q$ ways.

Example

Suppose a class has 30 men and 25 women. How many ways the instructor can pick one student to answer a question ?



http://www.clipartpal.com/_thumbs/pd/education/raising_hands.png

25 + 30=55 ways.

Example

If there are 5 odd numbers and 9 even and you must pick one number. How many ways you can do that?

http://images.sodahead.com/polls/004503031/3749288762_glossary_even_numbers_answer_2_xlarge_answer_2_xlarge.gif

5 + 9 = 14 ways.

- The Multiplication (Product) Rule

Suppose that two experiments (jobs) are to be performed. Then if experiment 1 can result in any one of p possible outcomes and if, for each outcome of experiment 1, there are q possible outcomes of experiment 2, then together (AND) there are p q possible outcomes of two experiments.

$$S = \left\{ \begin{array}{llll} (1,1) & (1,2), & \ldots\ldots. & (1,q), \\ (2,1) & (2,2), & \ldots\ldots. & (2,q), \\ . & . & \ldots\ldots\ldots\ldots & \\ . & . & \ldots\ldots\ldots\ldots. & \\ (p,1) & (p,2), & \ldots\ldots. & (p,q) \end{array} \right\}$$

The Generalized Basic Principle of Counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if . . . , then there is a total of $n_1\, n_2 \ldots n_r$ possible outcomes of the r experiments.

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Example

A small community consists of 10 women, each whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

$$S = \left\{ \begin{array}{l} (W1,C1),\ (W1,C2),\ (W1,C3), \\ (W2,C1),\ (W2,C2),\ (W2,C3), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (W10,C1),\ (W10,C2),\ (W10,C3) \end{array} \right\}$$

10 x 3= 30

Example

1, 2, 3, 4, 5

(a) How many ways can we choose 2 numbers at random (equally likely) from the above 5, with replacement ?

Note:

"With replacement" means that after the first number is picked it is "replaced" in the set of numbers, so it could be picked again as the second number.

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1, 2, 3, 4, 5

5	5

Since the first number can be chosen in 5 ways AND the second in 5 ways, S contains 5 x 5 = 25 points.

(b) Find the probability that one number is even?

- "The first number is even AND the second is odd, OR, the first is odd AND the second is even."
- We can then use the addition and multiplication rules to calculate that.
- There are

(2 x 3) + (3 x 2) = 12 ways for this event to occur

- P (one number is even) = 12 /25

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Example

How many different 7-place license plates are possible

(a) If the first 3 places are to be occupied by letters and the final 4 by numbers? (with replacement)

26 x 26 x 26 x 10 x 10 x 10 x 10 = 175,760,000

(b) If repetition among letters or numbers were prohibited?

(without replacement)

26 x 25 x 24 x 10 x 9 x 8 x 7 = 78624000

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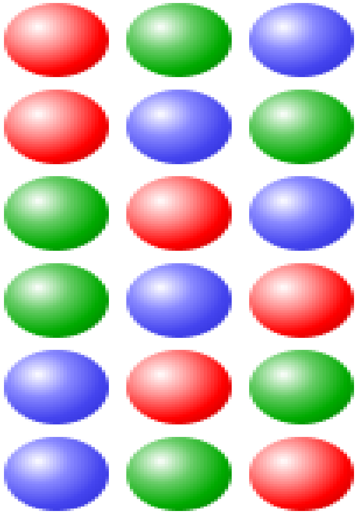
Notes

- When objects are selected and replaced after each draw, the addition and multiplication rules are generally sufficient to find probabilities.
- When objects are drawn **without being replaced, some special** rules may simplify the solution.

Permutations

A permutation, also called an "arrangement number" or "order," are arrangement of the elements of an ordered list.



http://en.wikipedia.org/wiki/Permutation#mediaviewer/File:Permutations_RGB.svg

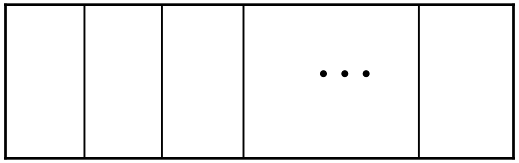
Each of the six rows is a different permutation of three distinct balls.

(1) Suppose that we have n distinct objects, then the number of ways to arrange these n objects in a row (using each symbol once and only once) is

$$n! = n\,(n-1)\,(n-2)\ldots1$$

Explanation:

We can fill the first position in n ways. Since this object can’t be used again, there are only $(n-1)$ ways to fill the second position. So we keep having 1 fewer object available after each position is filled.



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Factorial Notation:

For any positive integer m, m! “m factorial” is

$$m! = m\,(m-1)(m-2)\ \ldots\ldots 2 \times 1$$

Examples

1!=1; 2!=2 x 1=2 ; 3!= 3 x 2 x 1=6; 4!=24; 5!=120;

- Special definition: 0!=1

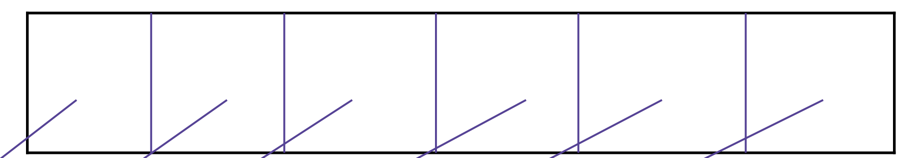
Example

Suppose the letters (a, b, c, d, e, f) are arranged at random to form a six-letter word (an arrangement) – we must use each letter once only (.?...)

(without replacement)

(a) What is the sample space?

S = { abcdef, abcdfe,…, fedcba }



6!=720

Or using the basic principle of counting

6 x 5 x 4 x 3 x 2 x 1 = 720

(b) What is the sample space if the second letter should be e or f ?

S = {afbcde, aebcdf,…, efdcba}

Start with the second box.

	2				

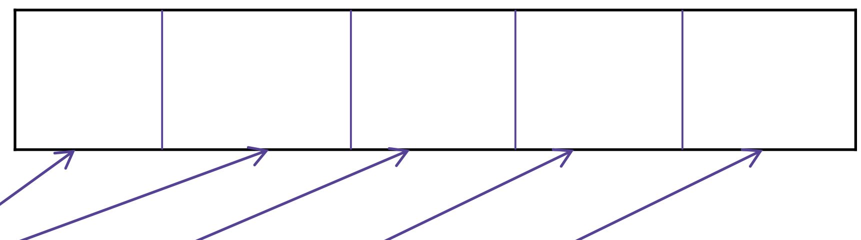
5 x   x 4 x 3 x 2 x 1 = 240

(c) What is the probability that the second letter should be e or f ?

240 / 720

Example

If a test has a 5-questions, how many different versions of the test are available if all possible arrangements of the questions are included?



There are 5 possible choices for the first question,

4 remaining choices for the second question,

3 choices for the third question,

2 choices for the fourth question,

and 1 choice for the fifth question.

- The number of possible arrangements is therefore

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$n! = 5! = 120$$

Example

Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

$\boxed{4!}\ 4!\ \ 3!\ \ 2!\ \ 1! = 6912$

Possible orderings of the subjects.

(2) The number of ways to arrange k objects selected from n distinct objects is called a permutations of size k of the objects denoted $n^{(k)}$ (read n to the k factors), (nPk, P_k^n).

$$n^{(k)} = \frac{n!}{(n-k)!}$$

$$n^{(k)} = n\,(n-1)\,(n-2)\ldots\ldots\,(n-k+1)$$

$$n^{(n)} = \frac{n!}{(n-n)!} = n!$$

Example

1, 2, 3, 4, 5

How many ways can we choose 2 numbers from the above 5, without replacement, when the order in which we choose the numbers is important?

$$5^{(2)} = \frac{5!}{(5-2)!} = 20$$

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Example

A pin number of length 4 is formed by randomly selecting and arranging 4 digits from the set {0,1, 2, 3,. . . 9} with replacement. Find the probability of the event

A: The pin number is even.

B: The pin number has only even digits.

C: All of the digits are unique.

D: The pin number contains at least one 1.

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$$P(A) = \frac{5 \text{ x } 10^3}{10^4}$$

$$P(B) = \frac{5^4}{10^4}$$

$$P(C) = \frac{10 \text{ x } 9 \text{ x } 8 \text{ x } 7}{10^4} = \frac{10^{(4)}}{10^4}$$

$$P(D) = \frac{10^4 - 9^4}{10^4}$$

Example

A pin number of length 4 is formed by randomly selecting and arranging 4 digits from the set { 0,1, 2, 3,. . . 9} without replacement. Find the probability of the event

A: The pin number is even.

B: The pin number has only even digits.

C: The pin number begins or ends with a 1.

D: The pin number contains 1(“Exactly 1” because without replacement).

$$P(A) = \frac{5 \text{ x } 9 \text{ x } 8 \text{ x } 7}{10^{(4)}} = 1/2$$

$$P(B) = \frac{5^{(4)}}{10^{(4)}}$$

$$P(C) = \frac{2 \text{ x } 9^{(3)}}{10^{(4)}}$$

$$P(D) = 1 - \frac{9^{(4)}}{10^{(4)}} = \frac{10^{(4)} - 9^{(4)}}{10^{(4)}}$$

Combinations

Any Unordered (order does not matter) sequence of k objects taken from a set of n distinct objects is called a combinations of size k of the objects denoted

$$C^n_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Read (n choose k)

For n and k both non-negative integers with $n \geq k$.

Example

Suppose there are 8 students in a group and that 5 of them must be selected to form a basketball team.

(a) How many *different* teams could be formed?

(b) What is the probability of ending up with one specific team?

(a)

Use the combination rule with n = 8 and k = 5 as shown below:

$$\binom{8}{5} = \frac{8!}{5! \cdot (3!)}$$

$$= 56$$

(b) 1/56

Example

1, 2, 3, 4, 5

How many ways can we choose 2 numbers from the above 5, without replacement, when the order in which we choose the numbers is not important?

$$\binom{5}{2} = \frac{5!}{2!\,3!} = 10$$

Example

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

$$\binom{20}{3} = \frac{20!}{3!\,17!}$$

Number of Arrangements When Some Symbols are Alike

To determine the number of arrangements of a set of n objects when certain of the objects are indistinguishable (are alike) from each other

There are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_k}{n_k}=\frac{n!}{n_1!\;n_2!\;\ldots.\;n_k!}$$

Multinomial Coefficient (CN).

Example

A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the US, 2 are from Canada, and one from Brazil. If the tournament results lists just **the nationalities** of the players in the order in which they placed, how many outcomes are possible?

$$\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1} = \frac{10!}{4!\,3!\,2!\,1!}$$

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Example

How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same **color are identical**?

$$\binom{9}{4}\binom{5}{3}\binom{2}{2} = \frac{9!}{4!\,3!\,2!}$$

Example

5 men and 3 women sit together in a row. Find the probability that

(a) The same gender is at each end?

(b) The women all sit together.

(c) What are you assuming in your solution? Is it likely to be valid in real life?

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Note: Treat each gender as alike (one type)

- If we treat the people as being 8 objects – 5M and 3W , our sample space will have

$$\binom{8}{5}\binom{3}{3} = \frac{8!}{5!\,3!}$$

$$= 56 \text{ points.}$$

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(a) To get the same gender at each end we need either

$$M------M \text{ OR } W------W$$

The number of distinct arrangements with a man at each end is

$$\frac{6!}{3!\,3!} = 20,$$

since we are arranging 3M's and 3W 's in the middle 6 positions.

The number with a woman at each end is

$$\frac{6!}{5!\,1!} = 6$$

$$P(\text{same gender at each end}) = \frac{20 + 6}{56}$$

Assuming each arrangement is equally likely.

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(b) Treating WWW as a single unit, we are arranging 6 objects – 5M's and 1 WWW. There are

$$\frac{6!}{5!\,1!} = 6 \text{ arrangements.}$$

Thus, $P(\text{women sit together}) = \dfrac{6}{56}$

- Our solution is based on the assumption that all points in S are equally probable.
- This would mean the people sit in a purely random order.
- In real life this isn't likely, for example, since friends are more likely to sit together.

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Example

The letters of the word STATISTICS are arranged in a random order.

Find the probability of the event G that the arrangement begins and ends with S?

S: 3 letters

T: 3 letters

A: 1 letter

I: 2 letters

C: 1 letter

Total: 10 letters

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- The number of outcomes in S is

$$\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \frac{10!}{3!\,3!\,2!\,1!\,1!} = 50400$$

S									S

- The number of outcomes in G (we must have S in the fist and last position) is

$$\binom{8}{3}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{8!}{3!\,2!\,\,1!\,1!\,1!} = 3360$$

$$P(G) = \frac{3360}{50400} = \frac{1}{15}$$

Examples on Chapter 3

Example ($1_{\text{page 28}}$)

In the Lotto 6/49 lottery, six numbers are drawn at random, without replacement, from the numbers 1 to 49. Find the probability that

http://upload.wikimedia.org/wikipedia/en/thumb/f/f1/Lotto_649_logo.svg/1280px-Lotto_649_logo.svg.png

(a) The numbers drawn are 1, 2, 3, 4, 5, 6 (in any order).

(b) No even number is drawn.

(a) Let the sample space S consist of all combinations of 6 numbers from 1, ..., 49; there are $\binom{49}{6} = C_6^{49}$

Since 1, 2, 3, 4, 5, 6 consist of one of these 6-tuples,

P ({1, 2, 3, 4, 5, 6}) = 1 / C_6^{49} = 1 / 13.9million.

(b) There are 25 odd and 24 even numbers, so there are

C_6^{25} choices in which all the numbers are odd.

P (no even number) = P (all odd numbers)

$= C_6^{25} \ / \ C_6^{49}$

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Example (2 page 25)

Suppose we make a random arrangement of length 3 using letters from the set {a, b, c, d, e, f, g, h, i, j}. What is the probability of the event B the letters are in alphabetic order if

(a) Letters are selected without replacement?

(b) Letters are selected with replacement?

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(a) The S consist of $10^{(3)}$ equally probable outcomes.

The number of outcomes in B are

$$C_3^{10} = \binom{10}{3}$$

Note : First select the 3 (different) letters to form the arrangement and then there is only 1 way to make arrangement in alphabetic order

$$P(B) = \frac{1\binom{10}{3}}{10^{(3)}} = 1/6$$

(b) The S consist of 10^3 outcomes.

Case 1 :

All three letters are the same (there are 10 such arrangements)

{ aaa, bbb,..., jjj}

Case 2:

There are 2 different letters, for each of these we can then make 2 arrangements with the letters are in alphabetic order e.g.

{aab, abb}

There are 2 $\binom{10}{2}$

Case 3:

All 3 letters are different. We can select the three letters in $\binom{10}{3}$ ways and then make 1 arrangement that is in alphabetic order

Combining the three cases, we have

$$P(B) = \frac{10 + 2\binom{10}{2} + 1\binom{10}{3}}{10^3}$$

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Example ($3_{\text{page 25}}$)

We form a 4 digit number by randomly selecting and arranging 4 digits from 1, 2,..., 7 without replacement.

Find the probability the number formed is

(a) Even

(b) Over 3000

(c) An even number over 3000.

a) $$\frac{3 \times 6^{(3)}}{7^{(4)}}$$

(b) $$\frac{5 \times 6^{(3)}}{7^{(4)}}$$

(c) $$\frac{2 \times 2 \times 5^{(2)} + 3 \times 3 \times 5^{(2)}}{7^{(4)}}$$

Over 3000 [3, 4, 5, 6, 7(for the first digit)]

From 1, 2,..., 7 the total even numbers (2,4,6)

Case I:

2 numbers

Even: 4 or 6

Case II :

3 numbers

Odd: 3,5,7

Exercise ($4_{\text{page 23}}$)

A binary sequence is an arrangement of zeros and ones. Suppose we have a uniform probability model on the sample space of all binary sequences of length 10.

What is the probability that the sequence has exactly 5 zeros?

- The sample space is

$$S = \{0000000000,\ 0000000001, \ldots, 1111111111\}$$

- S has 2^{10} outcomes each with probability $1 / 2^{10}$
- The event E with exactly 5 zeros and 5 ones is

$$E = \{0000011111, 1000001111, \ldots, 1111100000\}$$

$$P(E) = \frac{\binom{10}{5}}{2^{10}}$$

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Example

Suppose a box contains 10 balls of which 3 are red, 4 are black and 3 are green. A sample of 4 balls is selected at random without replacement. Find the probability of the events:

E: the sample contains 2 red balls.

F: the sample contains 2 red, 1 **black** and 1 green ball.

G: the sample contains 2 or more red balls.

E: the sample contains 2 red balls.

$$P(E) = \frac{\binom{3}{2}\binom{7}{2}}{\binom{10}{4}}$$

To satisfy the condition, the 2 red balls can be chosen from the 3 of which are red and the remaining **2** balls can be chosen from the 7 of which are **black** and green balls.

F: the sample contains 2 red, 1 **black** and 1 green ball.

$$P(F) = \frac{\binom{3}{2}\binom{4}{1}\binom{3}{1}}{\binom{10}{4}}$$

To satisfy the condition:

2 red balls can be chosen from the 3 of which are red, 1 **black** ball can be chosen from the **4 black** balls and the remaining 1 ball can be chosen from the 3 green balls.

G: the sample contains 2 or more red balls.

$$P(G) = \frac{\binom{3}{2}\binom{7}{2} + \binom{3}{3}\binom{7}{1}}{\binom{10}{4}}$$

To satisfy the condition:

2 red balls can be chosen from the 3 of which are red, and the other 2 balls from the remaining 7 **black** and green balls.
OR
3 red balls can be chosen from the 3 of which are red, and the other 1 balls from the remaining 7 **black** and green balls.