2022 :

ALGEBRE :

1 _ $P_{M_\alpha}(\lambda) = \det \begin{pmatrix} 1-\lambda & -1 & 2-\alpha \\ 0 & 2-\lambda & \alpha-2 \\ 1 & 1 & \alpha-\lambda \end{pmatrix}$

$= (1-\lambda)(2-\lambda)(\alpha-\lambda)$

a _ si $(\alpha \neq 1)$ et $(\alpha \neq 2)$: $M\alpha$ st diagonalisable

b _ $(\alpha = 1)$ $P_{M_1}(\lambda) = (1-\lambda)^2 \cdot (2-\lambda)$

$E_1 = \mathrm{Ker}(M_1 - 1 I_n)$: solution du système $\begin{cases} 0x - y + z = 0 \\ 0x + y - z = 0 \\ x + y + z = 0 \end{cases}$

$\Rightarrow \begin{cases} x = -2y \\ z = y \end{cases} \Rightarrow E_1 = \{ y(-2, 1, 1) \,/\, y \in \mathbb{R} \}$

multiplicité de $1 \neq$ card E_1 $\Rightarrow$ si $(\alpha = 1)$ M_α n'st pas diagonalis.

c - $(\alpha = 2)$. $P_{M_2}(\lambda) = (2-\lambda)^2 (1-\lambda)$.

$E_2 = \mathrm{Ker}(M_2 - 2 I_n)$: $\begin{cases} -x - y + 0z = 0 \\ 0x + 0y + 0z = 0 \\ x + y + 0z = 0 \end{cases}$

$E_2 = \{ x(1, -1, 0) + z(0, 0, 1) \,/\, x, y \in \mathbb{R} \}$

$\begin{cases} \text{cardinale de } E_2 = 2 = \text{multiplicité de } 2 \\ \text{et } 1 \text{ valeur propre simple} \end{cases} \Rightarrow$ M_2 diagonalisable

Conclusion :

$M\alpha$ st diagonalisable ssi $\alpha \neq 1$.

2_a) $P_{M_0}(\lambda) = -\lambda(1-\lambda)(2-\lambda) \Rightarrow 0$ valeur propre de M_0

$\Rightarrow \det(M_0) = 0 \Rightarrow M_0$ n'est pas inversible

b) Cherchons les vecteurs propres: $M_0 = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 1 & 1 & 0 \end{pmatrix}$

$E_0 = \ker(f) \quad \begin{cases} x - y + 2z = 0 \\ 2y - 2z = 0 \\ x + y = 0 \end{cases} \quad \begin{cases} y = -x \\ z = -x \end{cases}$

$E_0 = \{ x(1,-1,-1) \mid x \in \mathbb{R} \} \quad v_0 = (1,-1,-1)$

$E_1 = \ker(f - I_n) \quad \begin{cases} -y + 2z = 0 \\ y - 2z = 0 \\ x + y - z = 0 \end{cases} \quad \begin{cases} y = 2z \\ x = -z \end{cases}$

$E_1 = \{ z(-1,2,1) \mid z \in \mathbb{R} \} \quad v_1 = (-1,2,1).$

$E_2 = \ker(f - 2I_n) \quad \begin{cases} -x - y + 2z = 0 \\ 0x + 0y - 2z = 0 \\ x + y - 2z = 0 \end{cases} \Rightarrow \begin{cases} y = -x \\ z = 0 \end{cases}$

$E_2 = \{ x(1,-1,0) \mid x \in \mathbb{R} \} \quad v_2 = (1,-1,0).$

$P = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix} ; \quad P^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\begin{cases} v_0 = e_1 - e_2 - e_3 \\ v_1 = -e_1 + 2e_2 + e_3 \\ v_2 = e_1 - e_2 \end{cases} \Rightarrow \begin{cases} e_1 = v_0 + v_1 + v_2 \\ e_2 = v_0 + v_1 \\ e_3 = -v_0 + v_2 \end{cases}$

$$D = P^{-1} M_0 P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

c- $M_0 = P D P^{-1} \Rightarrow M_0^n = P^{-1} D^n P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} P$

$$M_0^n = \begin{pmatrix} 0 & 1 & -2^n \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$M_0^n = \begin{pmatrix} -1 + 2^n & -1 - 2^{n+1} & 1 + 2^n \\ -1 & 2 & -1 \\ -2^n & 2^n & 0 \end{pmatrix}$$

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