ALGEBRE 2013 :

1. Le polynome caractéristique :

$$P_A(X) = \det(A - X I_4) = \det\begin{pmatrix} 3a-1-X & -3a+1 & -3a+b+1 & 2a-1 \\ 2a-1 & -2a+1-X & -2a & 2a-1 \\ 0 & 0 & 1-X & 0 \\ 0 & 0 & -b & a-X \end{pmatrix}$$

$$P_A(X) = (3a-1-X)\begin{vmatrix} -2a+1-X & -2a & 2a-1 \\ 0 & 1-X & 0 \\ 0 & -b & a-X \end{vmatrix} - (2a-1)\begin{vmatrix} -3a+1 & -3a+b+1 & 2a-1 \\ 0 & 1-X & 0 \\ 0 & -b & a-X \end{vmatrix}$$

$$P_A(X) = \overbrace{(3a-1-X)(-2a+1-X)}^{-6a^2+5a-1-aX+X^2}(1-X)(a-X) - \overbrace{(2a-1)(-3a+1)}^{-6a^2+5a-1}(1-X)(a-X)$$

$$= (1-X)(a-X)(X^2-aX) = X(X-1)(a-X)^2$$

2. Les valeurs propres de la matrice selon les paramètres :

si $(a \neq 0)$ et $(a \neq 1)$: 0, 1 ~~et a trois~~ valeurs p simples et a vp double

Si $(a = 0)$: 0 vp triple et 1 vp simple

si $(a = 1)$: 1 vp triple et 0 vp simple

3. A st diagonalisable ssi $(\dim(\ker(A - aI_4)) = \text{mult de } a)$

pour calculer $(E_a = \ker(A - aI_4))$ il faut resoudre le systeme :

$$\begin{cases} (2a-1)x + (-3a+1)y + (-3a+b+1)z + (2a-1)t = 0 \\ (2a-1)x + (-3a+1)y - 2a\,z + (2a-1)t = 0 \\ (1-a)z = 0 \\ -bz = 0 \end{cases}$$

Cas 1 : $(a \neq 1) \vee (b \neq 0)$

$$\begin{cases} \Rightarrow z = 0 \Rightarrow t = x + \dfrac{-3a+1}{2a-1} y \quad \text{avec } a \neq \dfrac{1}{2} \\ E_a = \left\{ x(1,0,0,1) + y\left(0,1,0,\dfrac{-3a+1}{2a-1}\right) / x, y \in \mathbb{R} \right\} \\ \Rightarrow A \text{ st diagonalisable si } (a \neq 0) \text{ et } (a \neq 1). \end{cases}$$

$$\begin{cases} \text{si } \left(a = \dfrac{1}{2}\right),\ E_a = \left\{ \underset{1}{x}(1,0,0,0) + \underset{2}{t}(0,0,0,1) / x, t \in \mathbb{R} \right\} \\ \Rightarrow A \text{ st diagonalisable.} \end{cases}$$

4- **Cas 2 :** $(a = 1)$ et $(b = 0)$ / $t = -x + 2y + 2z$

E_a / E_1 $= \{ x(1,0,0,-1) + y(0,1,0,2) + z(0,0,1,2) / x, y, z \in \mathbb{R} \}$

$\dim E_a = 3$ et $a = 1$ vp triple

alors A st diagonalisable.

$E_0 = \operatorname{Ker}(A)$

$$\begin{cases} 2x - 2y - 2z + t = 0 \\ x - y - 2z + t \\ z = 0 \\ t = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ z = 0 \\ t = 0 \end{cases}$$

$E_0 = \{ x(1,1,0,0) / x \in \mathbb{R} \}$