$a \in A$ "MALE" SIDE $\qquad$ $b \in B$ "FEMALE" SIDE

LEMMA 1: EACH $b \in B$ REMAINS MATCHED FROM THE FIRST TIME SHE GETS A PROPOSAL, UNTIL THE END OF THE EXECUTION OF THE ALGORITHM. $\qquad$ T TIME

ALSO, THE PARTNERS OF b "GET BETTER" (FROM HER PERSPECTIVE) OVER TIME.

P: WHEN b GETS HER FIRST PROPOSAL SHE ACCEPTS IT (AND BECOMES MATCHED/ENGAGED).

LATER, SHE COULD GET OTHER PROPOSALS.

- IF SHE ACCEPTS ONE SUCH (SUBSEQUENT) PROPOSAL, SHE'LL REMAIN MATCHED (EVEN THOUGH WITH SOMEONE ELSE, THAT SHE LIKES BETTER THAN HER PREVIOUS PARTNER).
- IF SHE REJECTS A PROPOSAL, SHE'LL JUST KEEP HER PARTNER. □

L2: FOR EACH $a \in A$, THE SEQUENCE OF PROPOSALS OF a GET WORSE (FROM a'S PERSPECTIVE) OVER TIME.

P: TRIVIAL (BY ALG.'S DEFINITION).

THM 1: THE ALGORITHM ENDS AFTER AT MOST n^2 ITERATIONS OF ITS WHILE LOOP.

P: EACH $a_i \in A$ CAN PROPOSE TO AT MOST $|B| = n$ PEOPLE FROM B.

IN EACH ITERATION OF THE LOOP, SOME a_i PROPOSES TO SOME b_j THAT HE HADN'T PROPOSED TO EARLIER.

THEREFORE, THERE CAN BE AT MOST $|A| \cdot |B| = n^2$ ITERATIONS. □

SO, WE NOW KNOW THAT THE ALGORITHM TERMINATES AFTER $\leq n^2$ ITERATIONS.

IT REMAINS TO BE PROVED THAT THE OUTPUT OF THE ALGORITHM:

1. IS A PERFECT MATCHING, AND THAT IT
2. " " STABLE "

L3: IF $a \in A$ IS FREE AT SOME POINT IN THE EXECUTION OF THE ALGORITHM, THEN THERE EXISTS SOME $b \in B$ TO WHICH a HAS YET TO PROPOSE TO.

P: BY CONTRADICTION, SUPPOSE THAT — AT SOME POINT — $a^* \in A$ IS FREE AND THAT HAS PROPOSED TO EACH $b \in B$.

BY LEMMA 1, EACH $b \in B$ IS ENGAGED FROM THE FIRST PROPOSAL SHE GETS UNTIL THE END.

THEN, FOR a^* TO BE FREE AFTER $|B| = n$ PROPOSALS, IT MUST BE THAT — AT HIS nTH (LAST) PROPOSAL — EACH $b \in B$ WAS ENGAGED.

BUT, $|A| = |B| = n$, IF EACH $b \in B$ IS ENGAGED, IT MUST HOLD THAT EACH $a \in A$ IS ENGAGED — THUS, a^* CANNOT BE FREE. CONTRADICTION. □

L4: THE ALGORITHM OUTPUTS A PERFECT MATCHING

P: THE SET OF ENGAGED PAIRS ALWAYS FORMS A MATCHING (IF a MAKES A PROPOSAL, THEN HE IS FREE; MOREOVER WHEN b ACCEPTS A PROPOSAL SHE IS EITHER FREE, OR SHE BREAKS DOWN HER CURRENT ENGAGEMENT).

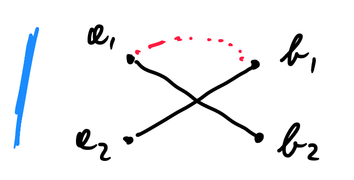
SUPPOSE THAT, IN THE END, $a \in A$ IS FREE. THEN a HAS PROPOSED TO EACH $b \in B$. BUT, THIS CONTRADICTS L3.

THUS, NO $a \in A$ CAN END UP FREE.

GIVEN THAT $|A| = |B| = n$, NO $b \in B$ CAN END UP FREE, EITHER. THUS THE ALGORITHM RETURNS A PERFECT MATCHING. □

$a_1 : b_1 > b_2$ $\qquad$ $b_1 : a_1 > a_2$

$a_2 : b_1 > b_2$ $\qquad$ $b_2 : a_1 > a_2$



$\Rightarrow \{a_1, b_2\}, \{a_2, b_1\}$ FORM AN INSTABILITY

T2: THE ALGORITHM OUTPUTS A STABLE MATCHING

P: BY L4, THE OUTPUT MATCHING M IS PERFECT.

BY CONTRADICTION, SUPPOSE THAT $\{a_i, b_j\}, \{a_k, b_\ell\} \in M$ AND THAT $\{a_i, b_j\}, \{a_k, b_\ell\}$ IS UNSTABLE.

THEN,

1. a_i PREFERS b_ℓ TO b_j, AND
2. b_ℓ PREFERS a_i TO a_k.

BY THE ALGORITHM, a_i'S LAST PROPOSAL WAS TO b_j.

WE CONSIDER TWO CASES:

- a_i PROPOSED TO b_ℓ BEFORE PROPOSING TO b_j. THEN, SINCE b_ℓ ENDED UP WITH a_k, L1 ENTAILS THAT b_ℓ PREFERS a_k TO a_i. CONTRADICTION.
- a_i DID NOT PROPOSE TO b_ℓ BEFORE b_j. THEN, a_i DID NOT PROPOSE TO b_ℓ (b_j IS a_i'S LAST PROPOSAL). BUT, THEN, a_i PREFERS b_j TO b_ℓ. CONTRADICTION.

THUS, M IS A STABLE MATCHING. □