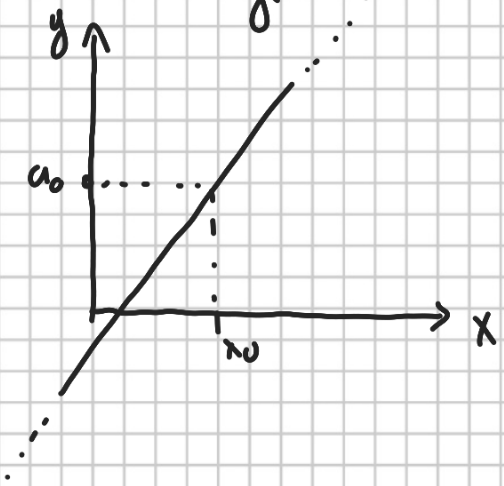
A way of approximating a function to a polynomial.

A polynomial can be defined as $\sum_{k=0}^{n} a_k (x-x_0)^k$.

An approximation of $n=1$ is the tangent line:

$a_0 + a_1(x-x_0)$



So, I fix a generic point of a $f(x)$ $(x_0, f(x_0))$, and I make $a_0 = f(x_0)$ and $a_1 = f'(x_0)$, I have the tangent in that point.

This is a decent approximation around a fixed point.
The x_0 in the polynomial's expression is, indeed, this point.

This approximation brings an error with itself:

$$f(x) - [f(x_0) + a_1(x-x_0)].$$

Why choosing $a_1 = f'(x_0)$?

So, in general:

$$\lim_{x \to x_0} f(x) - [f(x_0) + a_1(x-x_0)] = 0$$

But what if

$$\lim_{x \to x_0} \frac{f(x) - [f(x_0) + a_1(x-x_0)]}{x - x_0} = ? \qquad \left(\frac{0}{0}\right)$$

This is because we want to see how fast the numerator is going to 0.

Let's manipulate it:

$$\lim_{x\to x_0} \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\text{but this is } f'(x)!!} - a_1 = f'(x_0) - a_1$$

So, for $f'(x_0) - a_1 = 0$, we choose $a = f'(x_0)$.
But why do we choose this? Because it means that the numerator is going to 0 faster than the denominator! It's like $\lim_{x\to 0} \frac{x^2}{x}$.

So, the Taylor Polynomial at $n=1$ is

$$P_1(x, x_0) = f(x_0) + f'(x_0)(x-x_0)$$

(So, the equation for the tangent line is $P_1(x, x_0)$)
Its property, as seen before, is

$$\lim_{x\to x_0} \frac{f(x)-P_1(x,x_0)}{(x-x_0)^1} \qquad R_1 = f(x) - P_1(x, x_0)$$

And it's the only P doing this limit.

Now, we could keep on going with the same reasoning for $f''(x)$ (if it exists):

$$P_2(x, x_0) = \overset{a_0}{f(x_0)} + \overset{a_1}{f'(x_0)}(x-x_0) + a_2(x-x_0)^2$$

So, must $a_2 = f''(x_0)$?
Let's do the same (the error must go to 0 faster than $(x-x_0)^2$)

$$\lim_{x\to x_0} \frac{R_2(x-x_0)}{(x-x_0)^2} = ?$$

$$\lim_{x\to x_0} \frac{f(x)-P_2(x-x_0)}{(x-x_0)^2} =$$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)-a_2(x-x_0)^2}{(x-x_0)^2} =$$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} - a_2 =$$

Let's apply de L'Hôpital to the first term

$$\lim_{x \to x_0} \underbrace{\frac{f'(x) - f'(x_0)}{2(x - x_0)}}_{\frac{1}{2}f(x_0)'' \; !!!} - a_2 = \frac{1}{2}f''(x_0) - a_2$$

So, following the same reasoning:

$\frac{1}{2}f''(x_0) - a_2 = 0$

$a_2 = \frac{1}{2}f''(x_0)$

Thus,

$$P_2(x, x_0) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

Note that the coefficient of a_n is $\frac{1}{n!}$. This is coming from de l'Hôpital differentiation (the exponent going to the coefficient)

Let's keep doing this with $n=3$:

$P_3 = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + a_3(x - x_0)^3$

$$\lim_{x \to x_0} \frac{R_3(x - x_0)}{(x - x_0)^3} = ?$$

$$\lim_{x \to x_0} \frac{f(x) - P_3(x - x_0)}{(x - x_0)^3} =$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0) - \frac{1}{2}f''(x_0)(x - x_0)^2 - a_3(x - x_0)^3}{(x - x_0)^3}$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0) - \frac{1}{2}f''(x_0)(x - x_0)^2}{(x - x_0)^3} - a_3 =$$

Let's apply de L'Hôpital twice to the first term

$$\lim_{x \to x_0} \frac{f'(x) - f'(x_0) - f''(x_0)(x - x_0)}{3(x - x_0)^2} - a_3 =$$

$$\lim_{x \to x_0} \underbrace{\frac{f''(x) - f''(x_0)}{6(x - x_0)}}_{\frac{1}{6} f(x_0)''' \ !!!} - a_3 = \frac{1}{6} f(x_0)''' - a_3$$

So, following the same reasoning:

$\frac{1}{6} f'''(x_0) - a_3 = 0$

$a_3 = \frac{1}{6} f'''(x_0)$

So, Taylor expansion at n is:

$$P_n(x, x_0) = \sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!} (x - x_0)^k$$

And there's a theorem for the Taylor expansion at n so that:

Peano reminder: $\lim_{x \to x_0} \frac{R_n(x, x_0)}{(x - x_0)^n} = 0$

And the Taylor series is the sum of the two (T. poly + P. remindor).

$$f(x) = P_n(x, x_0) + R_n(x, x_0)$$