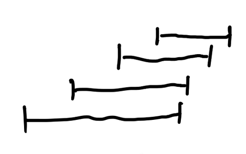
GREEDY ALGORITHMS

TYPICAL GREEDY ALGORITHMS PROOFS HYPOTHESIZE THE EXISTENCE OF AN OPTIMAL SOLUTION $\mathcal{O}$.

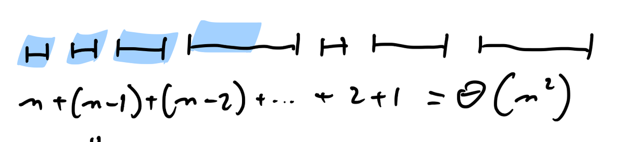
AS THE GREEDY ALGO PROGRESSES, ONE SHOWS THAT ITS SOLUTION (I) MATCHES THE OPTIMAL SOLUTION $\mathcal{O}$, OR (II) MATCHES ANOTHER OPTIMAL SOLUTION (WHICH IS USUALLY OBTAINED BY MODIFYING $\mathcal{O}$).

IN THE END, THIS PROVES THAT GREEDY RETURNS AN OPTIMAL SOLUTION.

THE GREEDY ALGORITHM FOR INTERVAL SCHEDULING, IN THE WORST CASE, TAKES TIME $\Theta(n^2)$.



$O(n)$



$n+(n-1)+(n-2)+\dots+2+1=\Theta(n^2)$

$\frac{n+1}{2}n=\frac{(n+1)n}{2}=\Theta(n^2)$

FAST ALG (I):

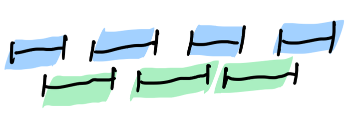
- SORT THE INTERVALS OF I INCREASINGLY BY FINISHING TIME $O(n \log n)$
- LET $I=\{I_1, I_2, \dots, I_n\}$ WITH $f(I_1) \le f(I_2) \le \dots \le f(I_n)$
- SET $T \leftarrow -\infty$, SET $S \leftarrow \emptyset$ $O(1)$
- FOR $i=1,\dots,n$ // n ITERATIONS
  - IF $s(I_i) > T$:
    $S = S \cup \{I_i\}$
    $T = f(I_i)$ $\Big\}\ O(1)$
- RETURN S.

$O(n \log n) + O(1) + n \cdot O(1) = O(n \log n)$

EX: PROVE THAT FAST ALG RETURNS AN OPTIMAL SOLUTION.

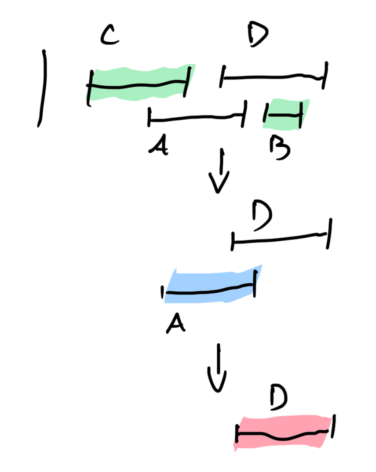
INTERVAL PARTITIONING

WE ARE GIVEN A SET OF INTERVALS I.
WE AIM TO SCHEDULE EACH INTERVAL ON THE MINIMUM POSSIBLE NUMBER OF RESOURCES.



TO SCHEDULE EACH INTERVAL I NEED 2 RESOURCES.

HERE, ONE RESOURCE IS SUFFICIENT



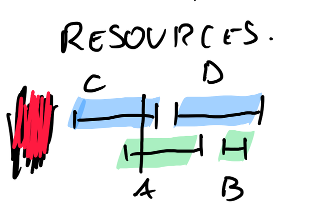
M^* = "EARLIEST TO FINISH"

SELECT$_{M^*}$

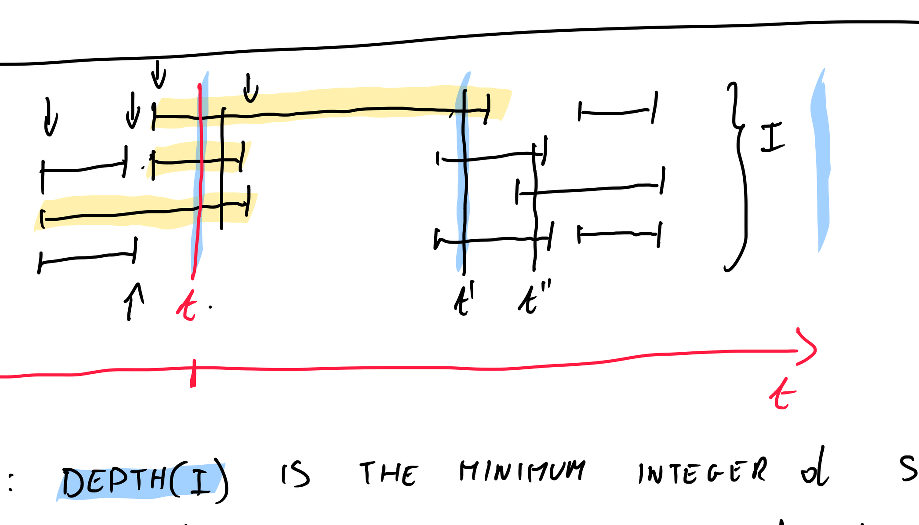
$\{C, B\}$

$\{A\}$ $\rightarrow$ SELECT$_{M^*}$ APPLIED GREEDILY USES 3 RESOURCES.

$\{D\}$

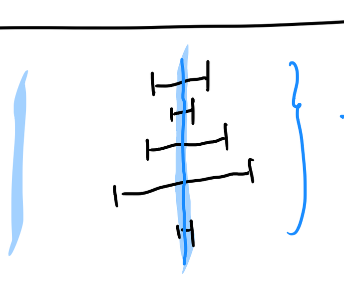


BUT 2 RESOURCES ARE OPTIMAL!



DEPTH $(I)=3$

DEF: DEPTH(I) IS THE MINIMUM INTEGER d S.T. $\forall t \in \mathbb{R}$
$|\{I_i \mid I_i \in I \wedge t \in I_i\}| \le d$.

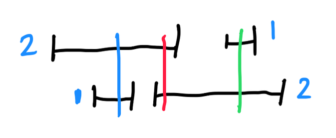


DEPTH $(I')=5$

DEF: LET OPT(I) BE THE MINIMUM NUMBER OF RESOURCES TO SCHEDULE EACH INTERVAL IN I.

L1: OPT$(I) \ge$ DEPTH(I).

P: THERE MUST EXIST A TIME t WHEN EXACTLY DEPTH(I) INTERVALS ARE RUNNING AT THE SAME TIME. AT TIME t, WE THEN NEED DEPTH(I) RESOURCES TO SCHEDULE ALL THE INTERVALS: OPT$(I) \ge$ DEPTH(I) $\square$



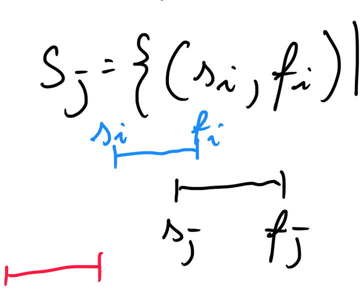
ALG(I):

- LET d BE $d=$ DEPTH(I)
- SORT THE INTERVALS BY THEIR STARTING TIME, INCREASINGLY
- LET $I=\{(s_1,f_1),(s_2,f_2),\dots,(s_n,f_n)\}$ WITH $s_1 \le s_2 \le s_3 \le \dots \le s_n$.
- FOR $j=1\dots n$
  - $L \leftarrow \{1,2,\dots,d\}$
  - FOR $i=1\dots j-1$
    IF (s_i,f_i) IS INCOMPATIBLE WITH (s_j,f_j):
    $L \leftarrow L-\{\ell(i)\}$ // $\ell(i)$
  - IF $|L| \ge 1$:
    LET $e \in L$
    SET $\ell(j)=e$
  - ELSE:
    FAIL
- RETURN THE LABELLING $\ell(1), \ell(2), \dots, \ell(n)$.

L2: THE ALGORITHM NEVER FAILS.

P: CONSIDER A GENERIC ITERATION j OF THE LOOP.
LET S_j BE THE SET OF INTERVALS THAT (I) THE ALGORITHM CONSIDERED BEFORE (s_j,f_j) AND THAT (II) END AFTER s_j.

$S_j=\{(s_i,f_i) \mid i \le j-1 \text{ AND } f_i \ge s_j\}$.



THE ALGORITHM WILL REMOVE FROM THE SET OF AVAILABLE LABELS FOR (s_j,f_j) ALL AND ONLY THE LABELS ASSIGNED TO THE INTERVALS IN S_j.

IN PARTICULAR, AFTER THE INNER LOOP ENDS,
$|L| \ge d-|S_j|$. (EACH INTERVAL IN S_j REMOVES AT MOST ONE LABEL).

WE WILL PROVE $|L| \ge 1$, BY PROVING $d-|S_j| \ge 1$. THE LATTER IS EQUIVALENT TO $|S_j| \le d-1$.

CLAIM: $|S_j| \le d-1$

P: EACH INTERVAL IN S_j PASSES THROUGH s_j, AND COMES BEFORE (s_j,f_j) IN THE ORDERING. THEN, $(s_j,f_j) \notin S_j$.

SUPPOSE, BY CONTRADICTION, THAT $|S_j| \ge d$.

THEN, THE SET $S_j \cup \{(s_j,f_j)\}$ HAS A CARDINALITY OF AT LEAST $d+1$.

NOW, EACH INTERVAL IN $S_j \cup \{(s_j,f_j)\}$ PASSES THROUGH TIME s_j.
BUT, THEN DEPTH$(I) \ge |S_j \cup \{(s_j,f_j)\}| \ge d+1$.
CONTRADICTION. $\square$

THUS, THE ALGORITHM NEVER FAILS. $\square$

L3: THE ALGORITHMS RETURNS A VALID LABELLING.

P: EXERCISE!

T: THE ALGORITHM RETURNS AN OPTIMAL SOLUTION TO INTERVAL PARTITIONING.

P: APPLY L1, L2, L3. $\square$

EX 2: PROVE THAT, IF INTERVALS ARE SORTED INCREASINGLY BY FINISHING TIME, TODAY'S ALGORITHM FAILS TO FIND AN OPTIMAL SOLUTION IN GENERAL.