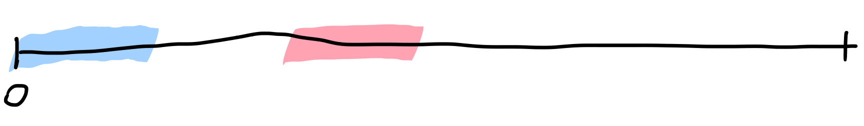
GREEDY ALGORITHMS

INTERVAL SCHEDULING

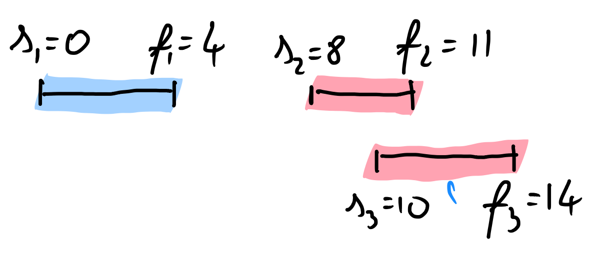
1 RESOURCE (CLASSROOM, COMPUTER, CAR)



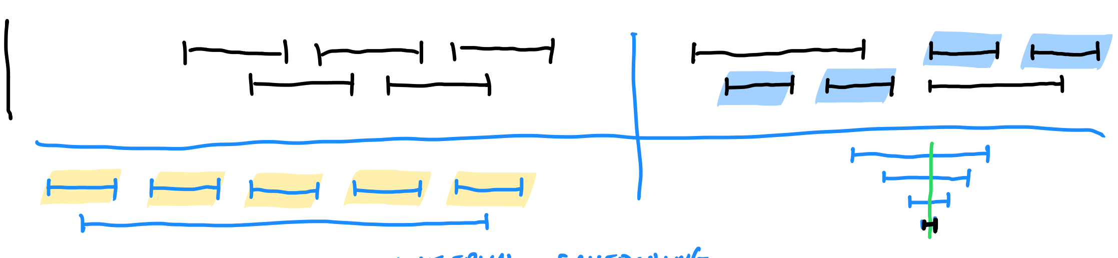
n JOBS, EACH CHARACTERIZED BY A STARTING TIME AND AN ENDING TIME.

$$I = \{(s_1, f_1), (s_2, f_2), \dots, (s_n, f_n)\}$$

(s_i: STARTING TIME OF JOB i,
f_i: ENDING " " " ")



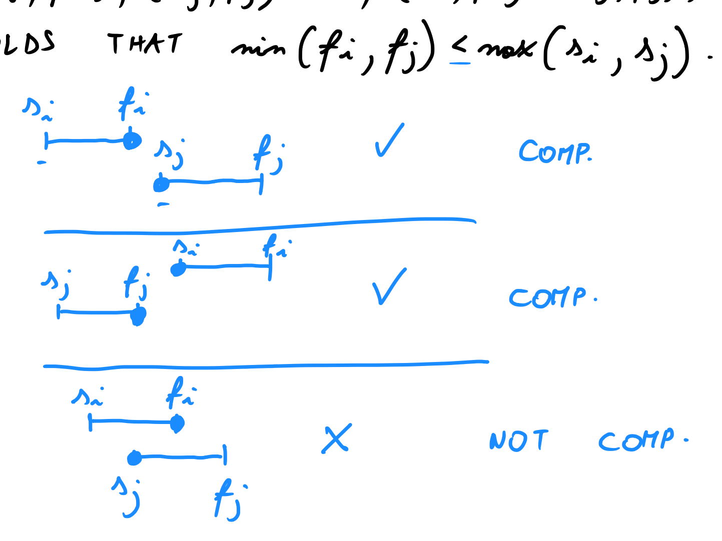
(JOBS 2 AND 3 ARE INCOMPATIBLE — THEY NEED THE RESOURCE AT THE SAME TIME)



INTERVAL SCHEDULING

Q: FIND A SUBSET OF THE JOBS THAT IS LARGEST AMONG THE SUBSETS OF COMPATIBLE JOBS.

DEF.: A SET $S \subseteq I$ OF JOBS IS COMPATIBLE IF $\forall (s_i, f_i), (s_j, f_j) \in S$, $(s_i, f_i) \neq (s_j, f_j)$, IT HOLDS THAT $\min(f_i, f_j) \leq \max(s_i, s_j)$.



GIVEN AN INPUT/INSTANCE I FIND $S \subseteq I$ SUCH THAT S IS COMPATIBLE, AND $|S|$ IS LARGEST.

A GREEDY APPROACH

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SELECT_M (I):
    S = []
    WHILE |I| ≥ 1:
        PICK (s_j, f_j) ∈ I ACCORDING TO RULE M
        N = {(s_i, f_i) | (s_i, f_i) ∈ I AND
                ((s_i, f_i) AND (s_j, f_j) ARE INCOMP.)}
        I -= N      ( I = I - N )
        S.APPEND((s_j, f_j))
    RETURN S
```

A=SET([2,4,5])
B=SET([2,6,5])

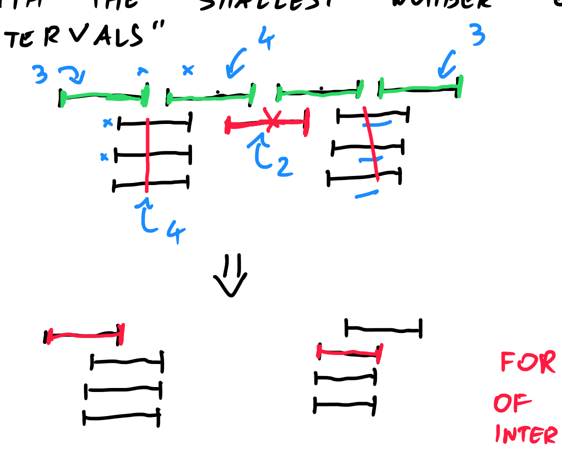
A-B IS THE SET: SET([4]).

L1: $\forall$ RULE M, SELECT$_M$ RETURNS A SET OF COMPATIBLE JOBS/INTERVALS, THAT IS, IT RETURN A FEASIBLE SOLUTION.

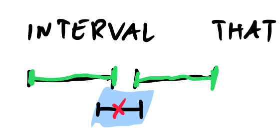
HOW CAN WE CHOOSE M?

LET US TRY WITH THE FOLLOWING M:

M= " PICK AN INTERVAL THAT IS INCOMPATIBLE WITH THE SMALLEST NUMBER OF REMAINING INTERVALS" NO



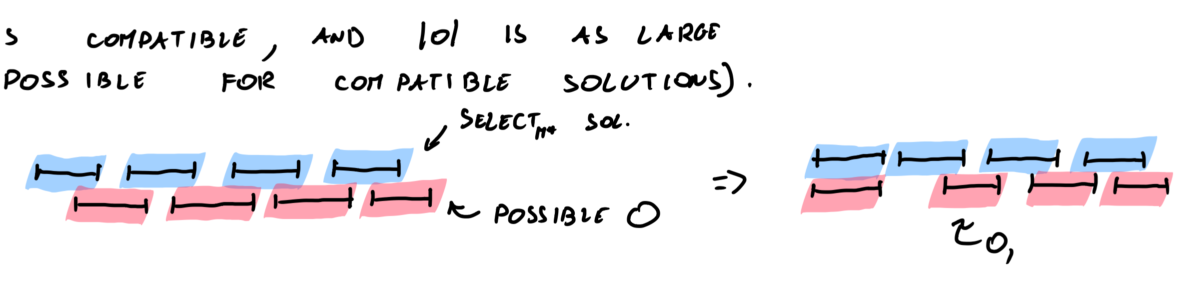
M= " PICK AN INTERVAL THAT IS SHORTEST " NO



M^*= " PICK AN INTERVAL THAT ENDS SOONEST "

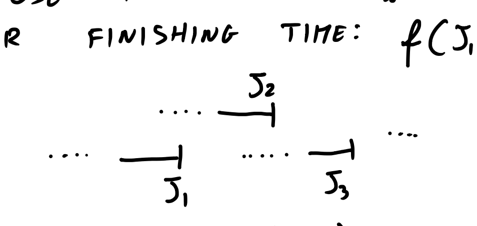
L2: SELECT$_{M^*}$ RETURNS AN OPTIMAL SOLUTION.

P: LET $O \subseteq I$ BE AN OPTIMAL SOLUTION (O IS COMPATIBLE, AND $|O|$ IS AS LARGE AS POSSIBLE FOR COMPATIBLE SOLUTIONS).



W.L.O.G., $O = \{J_1, J_2, \dots, J_m\}$, WHERE $s(A)$ IS THE STARTING TIME OF INTERVAL A, AND $f(A)$ IS ITS FINISHING TIME.

SUPPOSE THAT THE J_i'S ARE SORTED BY THEIR FINISHING TIME: $f(J_1) < f(J_2) < \dots < f(J_m)$.



CLEARLY, $s(J_1) < f(J_1)$; MOREOVER $f(J_1) < s(J_2)$ (OTHERWISE J_1 AND J_2 WOULD BE INCOMPATIBLE). MORE GENERALLY, THEN,

$$s(J_1) < f(J_1) < s(J_2) < f(J_2) < s(J_3) < \dots < f(J_m).$$

LET S BE THE SOLUTION RETURNED BY SELECT$_{M^*}$.

BY L1, S IS A SET OF COMPATIBLE JOBS.

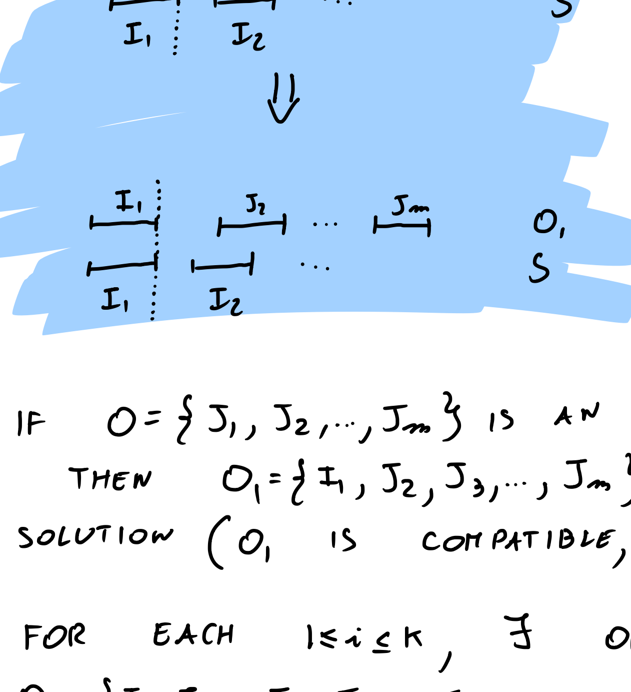
THEN, WE CAN LABEL THE INTERVALS IN $S = \{I_1, I_2, \dots, I_k\}$ SO THAT

$$s(I_1) < f(I_1) < s(I_2) < f(I_2) < s(I_3) < \dots < f(I_k).$$

WE WOULD LIKE TO PROVE THAT $k \geq m$, THAT IS, THAT $|S| \geq |O|$.

CLAIM: $f(I_1) \leq f(J_1)$

P: BY OUR RULE M^*, I_1 IS GOING TO BE A JOB THAT FINISHES EARLIEST. THUS, IT CANNOT BE THAT J_1 FINISHES EARLIER THAN I_1. □



THUS, IF $O = \{J_1, J_2, \dots, J_m\}$ IS AN OPTIMAL SOLUTION THEN $O_1 = \{I_1, J_2, J_3, \dots, J_m\}$ IS AN OPTIMAL SOLUTION (O_1 IS COMPATIBLE, $|O| = |O_1|$).

CLAIM: FOR EACH $1 \leq i \leq k$, $\exists$ OPTIMAL SOLUTION $O_i = \{I_1, I_2, \dots, I_i, J_{i+1}, J_{i+2}, \dots, J_m\}$, AND SUCH THAT $f(I_i) \leq f(J_i)$.

P: WE ALREADY PROVED THE $i=1$ CASE.

LET US ASSUME, BY INDUCTION, THAT THE CLAIM HOLDS FOR i. WE PROVE IT FOR $i+1$.

BY INDUCTION, $O_i = \{I_1, I_2, \dots, I_i, J_{i+1}, \dots, J_m\}$ IS AN OPTIMAL SOLUTION WITH $f(I_i) \leq f(J_i)$.

THEN, LET US DEFINE $O_{i+1} = \{I_1, I_2, \dots, I_i, I_{i+1}, J_{i+2}, \dots, J_m\}$, (THAT IS, LET ME SUBSTITUTE J_{i+1} WITH I_{i+1}).

THEN, $|O_{i+1}| = |O_i|$.

SELECT$_{M^*}$ CHOOSES, AS ITS $(i+1)$TH INTERVAL, THE INTERVAL THAT FINISHES EARLIEST AMONG THOSE COMPATIBLE WITH THE INTERVALS $\{I_1, I_2 \dots I_i\}$.

THUS, $f(I_{i+1}) \leq f(J_{i+1})$ (J_{i+1} IS COMPATIBLE WITH $I_1 \dots I_i$).

THEN, O_{i+1} IS COMPATIBLE. □

THUS, $O_k = \{I_1, I_2, \dots, I_k, J_{k+1}, \dots, J_m\}$. BUT, GIVEN THAT SELECT$_{M^*}$ STOPS WHEN THERE ARE NO MORE INTERVALS COMPATIBLE WITH THE ONES PICKED SO FAR, $J_{k+1} \dots J_m$ CANNOT EXIST.

THUS $m = k$, AND $S = \{I_1, \dots, I_k\}$ IS AN OPTIMAL SOLUTION. □