# *Lecture Notes Calculus Unit 2, ACSAI*

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# Contents

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# Chapter 1

# Riemann Integral

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## 1.1 Riemann Integral

We assume $f : [a, b] \mapsto \mathbb{R}$ bounded function defined on a closed bounded interval and bounded $[a, b]$; hence, there exists $m, M \in \mathbb{R}$ such that

$$m \leq f(x) \leq M \qquad \forall x \in [a, b]\,.$$

A **p**artition $P$ of $[a, b]$ is a finite sets of points $x_i$ for $i = 0, \cdots, n$ such that

$$a = x_0 \leq x_1 \leq \cdots \leq x_i \leq \cdots \leq x_n = b\,.$$

where we label the endpoints of the subintervals by $x_0, x_1$, ..., so that the leftmost point is $a = x_0$ and the rightmost point is $b = x_n$. We denote by $I_i = [x_{i-1}, x_i]$, $i = 1, \cdots, n$. Assume for simplicity that $f$ is also a continuous function then by Weierstrass theorem there exist

$$\begin{aligned} m_i &:= \min\{f(x) : \ x \in I_i\} = \min_{x \in I_i} f(x)\,, \\ M_i &:= \max\{f(x) : \ x \in I_i\} = \max_{x \in I_i} f(x)\,. \end{aligned}$$

We define then

$$\begin{aligned} s(f, P) &= \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \\ S(f, P) &= \sum_{i=1}^{n} M_i (x_i - x_{i-1})\,, \end{aligned}$$

where $s(f, P)$ and $S(f, P)$ denote the **l**ower sum and the **upper sum**, respectively, of $f$ with respect to the partition $P$. Note that, by definition

$$s(f, P) \leq S(f, P)$$

for every $P$ partition of $[a, b]$. Moreover, we define

$$s(f) := \sup_P s(f, P)\,, S(f) := \inf_P S(f, P)\,.$$

Note that $s(f) \le S(f)$. The following example clarify the role of lower and upper sum in the Definition 1.1.5 of the Riemann integral.

**Example 1.1.1 (The area underneath the graph of a function).** Let us consider an increasing positive monotone function $f(x) = x^2$ in $[0, 1]$. We investigate the area of the region that is under the graph of $f(x)$ and above the interval $[a, b]$ on the $x$-axis. The region under the graph of $f(x)$ has such a strange shape, calculating its area is too difficult. But calculating the area of rectangles is simple. Let's simplify our life by pretending the region is composed of a bunch of rectangles. We divide $[0, 1]$ into $n$ subintervals of length $x_i - x_{i-1} = (1/n)$ and we construct rectangles with base given by $(x_i - x_{i-1})$ and height equal to $M_i$ or $m_i$. Hence, by adding up the areas of all the rectangles we get

$$\begin{aligned} s(f, P_n) &= \sum_{i=1}^{n} \frac{m_i}{n} \\ S(f, P_n) &= \sum_{i=1}^{n} \frac{M_i}{n}\,, \end{aligned}$$

where $s(f, P_n)$ and $S(f, P_n)$ denote the **lower sum** and the **upper sum**, respectively, of $f$ with respect to the partition $P_n$. The lower and upper sum, $s(f, P_n)$ and $S(f, P_n)$, represent only an approximation to the actual area underneath the graph of $f$. To make the approximation better, we have to increase the number of subintervals $n$, which makes the subinterval width $1/n$ decrease; that is,

$$\sum_{i=1}^{n} \frac{m_i}{n} \le Area(\mathcal{G}_f) \le \sum_{i=1}^{n} \frac{M_i}{n}$$

where $\mathcal{G}_f = \{(x, y) \in \mathbb{R} \times \mathbb{R} :\ x \in [a, b], 0 \le y \le x^2\}$. Since $f$ is an increasing function we have that $m_i = f(x_{i-1})$ and $M_i = f(x_i)$; hence,

$$s(f, P_n) = \frac{1}{n} \sum_{i=1}^{n} (\frac{(i-1)}{n})^2 = (\frac{1}{n})^3 \sum_{k=0}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6}\,,$$

similarly,

$$S(f, P_n) = \frac{1}{n} \sum_{i=1}^{n} (\frac{i}{n})^2 = (\frac{1}{n})^3 \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

Moreover, $s(f, P_n) \le s(f, P_{n+1}) \le S(f, P_{n+1}) \le S(f, P_n)$. Therefore,

$$s(f, P_n) = \frac{(n-1)n(2n-1)}{6n^3} \le Area(\mathcal{G}_f) \le \frac{n(n+1)(2n+1)}{6n^3} = S(f, P_n)\,,$$

and

$$\begin{aligned} \sup_n \Big(\frac{(n-1)n(2n-1)}{6n^3}\Big) &= \lim_{n\to+\infty} \frac{(n-1)n(2n-1)}{6n^3} \\ &= \frac{1}{3} \\ &= \lim_{n\to+\infty} \frac{n(n+1)(2n+1)}{6n^3} = \inf_n \Big(\frac{n(n+1)(2n+1)}{6n^3}\Big)\,. \end{aligned}$$

Hence, by the comparison theorem for sequences, we have that

$$\begin{aligned} Area(\mathcal{G}_f) &= \sup_n s(f, P_n) \\ &= \inf_n S(f, P_n) \\ &= \frac{1}{3}\,. \end{aligned}$$

The example shows that when the number of subintervals increase the lower and the upper sum approach the area $Area(\mathcal{G}_f)$ from above and from below therefore is quite intuitive to reach $Area(\mathcal{G}_f)$ as the infimum of $S(f, P)$ among all possible partition $P$ as well as the supremum of $s(f, P)$.

**Example 1.1.2.** Let $f(x) = c$, with $c \in \mathbb{R}$, then $f$ si integrable and the integral is

$$\int_a^b c\,dx = c(b-a)\,.$$

Indeed,

$$s(f,P) = \sum_{i=1}^{n} c\,(x_i - x_{i-1}) = S(f,P) = c\,(b-a)\,;$$

hence,

$$s(f) = S(f) = c\,(b-a)\,.$$

If $f$ is bounded but not necessarily continuous in $[a, b]$ we can generalize the definition of $s(f, P)$ and $S(f, P)$ by using inf and sup instead of min and max. More precisely, we denote by

$$m_i := \inf_{x\in[x_{i-1},x_i]} f(x)\,, \qquad M_i := \sup_{x\in[x_{i-1},x_i]} f(x)\,.$$

**Example 1.1.3.** Let $f : [0, 2] \mapsto \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & x \in [0,1) \\ 1/2 & x = 1 \\ 0 & x \in (1,2]\,. \end{cases} \tag{1.1}$$

Let $0 < \varepsilon < 1$ and consider $P_\varepsilon = \{0 < x_1 := 1-\varepsilon < x_2 := 1+\varepsilon < 2\}$ then we have

$$\begin{aligned} m_1 &= \inf_{x\in[0,1-\varepsilon]} f(x) = 1 & M_1 &= \sup_{x\in[0,1-\varepsilon]} f(x) = 1 \\ m_2 &= \inf_{x\in[1-\varepsilon,1+\varepsilon]} f(x) = 0 & M_2 &= \sup_{x\in[1-\varepsilon,1+\varepsilon]} f(x) = 1 \\ m_3 &= \inf_{x\in[1+\varepsilon,2]} f(x) = 0 & M_3 &= \sup_{x\in[1+\varepsilon,2]} f(x) = 0\,. \end{aligned}$$

Hence,

$$\begin{aligned} s(f,P_\varepsilon) &= m_1(1-\varepsilon) + m_2 2\varepsilon + m_3(1-\varepsilon) = 1-\varepsilon \\ S(f,P_\varepsilon) &= M_1(1-\varepsilon) + M_2 2\varepsilon + M_3(1-\varepsilon) = 1-\varepsilon+2\varepsilon = 1+\varepsilon\,. \end{aligned}$$

Therefore,

$$\inf_P S(f,P) - \sup_P s(f,P) < S(f,P_\varepsilon) - s(f,P_\varepsilon) = 2\varepsilon\,.$$

By the arbritrariness of $\varepsilon$ we have that

$$\inf_P S(f,P) = \sup_P s(f,P) = \int_0^2 f(x)\,dx = 1\,.$$

Nevertheless, if $f$ is bounded but not continuous it may also happen that...

**Example 1.1.4** (Dirichlet function)**.** Consider the Dirichlet's function over $[0,1]$

$$f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \cap \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

for any partition $P$ we have that $m_i = 0$ and $M_i = 1$ for all $i = 1, \cdots, n$. Therefore, $s(f,P) = 0$ and $S(f,P) = 1$ and

$$0 = \sup_P s(f,P) < \inf_P S(f,P) = 1\,.$$

We are now ready to define the Riemann integral.

**Definition 1.1.5.** *Let $f : [a,b] \mapsto \mathbb{R}$ be a bounded function with $-\infty < a < b < +\infty$. We define lower Riemann integral and upper Riemann integral as*

$$s(f) := \sup_P s(f,P)\,, S(f) := \inf_P S(f,P)\,;$$

*respectively, where*

$$\begin{aligned} s(f,P) &= \sum_{i=1}^n m_i(x_i - x_{i-1}) \\ S(f,P) &= \sum_{i=1}^n M_i(x_i - x_{i-1})\,, \end{aligned}$$

*with*

$$m_i := \inf_{x\in[x_{i-1},x_i]} f(x)\,, \qquad M_i := \sup_{x\in[x_{i-1},x_i]} f(x)\,.$$

*We say that $f$ is* **Riemann integrable** *if*

$$s(f) = S(f)\,.$$

*We call the (Riemann)* **Integral** *of $f$ over $[a,b]$ the real number*

$$\int_a^b f(x)\,dx := s(f) = S(f)\,.$$

**Math vocabulary**: $a$ and $b$ are also called extremes of integration, $f$ is the integrand, $dx$ states that "it's summing" all increments of the function $f$. Since the extremes of integration are fixed we call $\int_a^b f(x)\,dx$ also definite integral.

**Proposition 1.1.6** (Integrability Criterion)**.** *Let $f : [a,b] \mapsto \mathbb{R}$ be a bounded function, we say that $f$ is Riemann integrable over $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon$ of $[a,b]$ such that*

$$S(f, P_\varepsilon) - s(f, P_\varepsilon) < \varepsilon\,.$$

**Theorem 1.1.7.** *Let $f : [a,b] \mapsto \mathbb{R}$ be a bounded function.*

*a If $f$ is continuous on $[a,b]$ then $f$ is Riemann integrable.*

*b If $f$ is monotone on $[a,b]$ then $f$ is Riemann integrable.*

*c If $f$ has a finitely many discontinuities (i.e. $f$ is continuous on $[a,b]$ except on a finite number of points), then $f$ is Riemann integrable.*

*Proof.* (b). Assume $f$ is an increasing function (for decreasing function the argument is the same). We consider the partition $P_n$ which divides the interval $[a,b]$ into $n$ sub-intervals of equal length (and the length is $(b-a)/n$) meaning $x_i = a + i(b-a)/n$, $i = 0 \cdots, n$. Then, since $f$ is increasing, we have $m_i = \inf_{[x_{i-1},x_i]} f(x) = f(x_{i-1})$ and $M_i = \sup_{[x_{i-1},x_i]} f(x) = f(x_i)$; the lower and upper sums are

$$\begin{aligned} S(f,P_n) - s(f,P_n) &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ &= (f(x_i) - f(x_{i-1}))\Big(\frac{b-a}{n}\Big) \\ &= \Big(\frac{b-a}{n}\Big)(f(b) - f(a))\,. \end{aligned}$$

Hence, for every $\varepsilon > 0$ there exists $P_n$ with $n = \lceil 1/\varepsilon \rceil$ (that is the ceiling function, $1/\varepsilon \le n$), such that $S(f,P_n) - s(f,P_n) < \varepsilon$.

(c) We may divide the interval $[a,b]$ into finitely many subintervals $[a_{i-1}, a_i]$ where $f$ is continuous on the interior: $[a,b] = [a_1,a_2] \cup [a_2,a_3] \cup \cdots \cup [a_{n-1},a_n]$. By using the additivity result with respect to the domain of integration we have that

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{a_i}^{b_i} f(x)\,dx\,.$$

□

**Example 1.1.8.**

$$f(x) = \begin{cases} 1 + \sin\frac{1}{x} & x \in (0,1] \\ 0 & x = 0\,. \end{cases}$$

## 1.2 Properties of Riemann Integral

From now on we will refer to Riemann integrable function or Riemann integral as integrable function or integral, respectively.

Let $f$ and $g$ be integrable functions on $[a,b]$ and let $\lambda \in \mathbb{R}$. Then:

1) (**Additivity**) the integral of the sum or difference of two functions can be computed by integrated each term separately

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx\,; \tag{1.2}$$

2) (**Linearity**) a constant factor $\lambda \in \mathbb{R}$ can be moved outside the integral sign

$$\int_a^b \lambda\, f(x)\,dx = \lambda \int_a^b f(x)\,dx\,; \tag{1.3}$$

3) (**Additivity with respect to the interval of integration**): for every $c \in (a, b)$ we have that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,; \tag{1.4}$$

4) (**Positivity**) if $f \geq 0$ (respectively, $f(x) \leq 0$) then

$$\int_a^b f(x)\,dx \geq 0 \quad (resp. \leq 0)\,. \tag{1.5}$$

As a consequence of (1)-(2)-(4) we have the following comparison properties between integrals:

5) (**Monotonicity**) If $f \leq g$ on $[a, b]$ then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx\,; \tag{1.6}$$

6) (**Modulus**)

$$\Big|\int_a^b f(x)\,dx\Big| \leq \int_a^b |f(x)|\,dx\,. \tag{1.7}$$

Finally, note that so far we always assume that $a < b$ for the intervals of integration. In general, we set

- $$\int_a^a f(x)\,dx = 0\,;$$

- $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx\,.$$

**Remark 1.2.1. Geometric Interpretation.**
Example 1.1.1 shows that if $f$ is a positive function the area of the region that is under the graph of $f(x)$ and above the interval $[a, b]$ on the $x$-axis can be compute by the Riemann

integral. If $f$ has a changing sign, for example, $f$ is positive on $[a, c]$ and negative on $[c, b]$, then the integral

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= Area(\{(x, y) \in \mathbb{R} \times \mathbb{R} :\ x \in [a, c], 0 \leq y \leq f(x)\}) \\ &\quad - Area(\{(x, y) \in \mathbb{R} \times \mathbb{R} :\ x \in [c, b], 0 \leq y \leq -f(x)\})\,;\end{aligned}$$

that is, the integral is the sum of the areas with sign! In particular, we may have that

$$\int_a^b f(x)\,dx = 0\,.$$

**Theorem 1.2.2** (Mean value Theorem for Integrals)**.** *Let $f$ be a continuous function on $[a, b]$ then there exists $x_0 \in [a, b]$ such that*

$$\frac{1}{(b-a)} \int_a^b f(x)\,dx = f(x_0)\,.$$

*Proof.* By the monotonicity of the integral, Example **??**(1) and Weierstrass's theorem for continuous functions on a bounded and closed interval we have that

$$(b-a) \min_{x\in[a,b]} f(x) \leq \int_a^b f(x)\,dx \leq (b-a) \max_{x\in[a,b]} f(x)$$

which implies

$$\min_{x\in[a,b]} f(x) \leq \frac{1}{(b-a)} \int_a^b f(x)\,dx \leq \max_{x\in[a,b]} f(x)\,.$$

Since the continuous functions takes any given values between their minimum value and their maximum value (this is statement of the Intermediate Value theorem), we can conclude that there exists $x_0 \in [a, b]$ such that

$$f(x_0) = \frac{1}{(b-a)} \int_a^b f(x)\,dx\,.$$

□

**Definition 1.2.3.** *A function $F : [a, b] \mapsto \mathbb{R}$ is called* **primitive** *or* **antiderivative** *of a function $f : [a, b] \mapsto \mathbb{R}$ if $F$ is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.*

The Theorem 1.2.2 allows us to prove a very important theorem: the **Fundamental Theorem of Integral Calculus**. Such theorem is extremely powerful since it establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums. The theorem is comprised of two parts, the first of which, establishes the relationship between differentiation and integration but also it guarantees that any continuous function has a primitive function that is its integral function. The second part of the theorem guarantees that given a primitive function we can evaluate the definite integral by evaluating the primitive $F$ at the extremes of integration and subtracting.

**Theorem 1.2.4** (Fundamental Theorem of Integral Calculus)**.** *Let $f : [a,b] \mapsto \mathbb{R}$ be a continuous function. We define the bf integral function $F : [a,b] \mapsto \mathbb{R}$ as follows*

$$F(x) := \int_a^x f(t)\,dt\,. \tag{1.8}$$

*Then $F$ is differentiable on $[a,b]$ and*

$$F'(x) = f(x)\,, \qquad \forall x \in [a,b]\,. \tag{1.9}$$

*Moreover, if $G$ is a differentiable function such that $G'(x) = f(x)$; i.e., $G$ is any primitive of $f$, then*

$$\int_a^x f(t)\,dt = G(x) - G(a)\,.$$

*Proof.* Applying the definition of the derivative, we have

$$\begin{aligned}
\frac{F(x)-F(x_0)}{x-x_0} &= \frac{1}{(x-x_0)}\Big(\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt\Big) \\
&= \frac{1}{(x-x_0)}\Big(\int_{x_0}^x f(t)\,dt\Big)\,.
\end{aligned}$$

By the Mean value Theorem for Integrals 1.2.2 we have that there exists $\xi$ between $x$ and $x_0$ such that

$$\frac{F(x)-F(x_0)}{x-x_0} = f(\xi)\,;$$

hence, by continuity of $f$ we have that there exists the limit as $x$ tends to $x_0$

$$\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$$

for any $x_0 \in [a,b]$. It follows that $F$ is differentiable and $F'(x) = f(x)$.

Let $G$ be such that $G'(x) = f(x)$, then by definition of $G$ and the first part of the theorem we have that

$$F'(x) - G'(x) = 0 \to F(x) - G(x) = c = F(a) - G(a) = -G(a)$$

with $c$ an arbitrary constant. Hence, $F(x) = G(x) - G(a)$; *i.e.*,

$$\int_a^x f(t)\,dt = G(x) - G(a)\,, \quad \forall x \in [a,b]\,.$$

□

**Remark 1.2.5.** Note that if $F$ and $G$ are both primitive of $f$ then $F$ and $G$ differ by an additive constant; *i.e.*,

$$G(x) = F(x) + c\,,$$

$c \in \mathbb{R}$. Moreover, if we change the extreme of integration $a$ with any other $x_1 \in [a,b]$ then

$$\int_a^x f(t)\,dt = \int_a^{x_1} f(t)\,dt + \int_{x_1}^x f(t)\,dt = C + \int_{x_1}^x f(t)\,dt\,;$$

hence, without loss of generality we can always consider $a$ has endpoints in the definition of integral function $F$.

By Theorem 1.2.4 we have then any definite integral

$$\int_a^b f(t)\,dt = G(b) - G(a)\,;$$

equivalently,

$$\int_a^b f'(t)\,dt = f(x)\Big|_a^b = f(b) - f(a)\,.$$

**Definition 1.2.6** (**Indefinite Integral).** *The indefinite integral of $f$ is the set of all possible primitives or antiderivatives of $f$; i.e.,*

$$\int f(x)\,dx = \{F(x) + c\,, \quad \forall c \in \mathbb{R}\}\,.$$

| **Function** | **Primitive** | **Indefinite Integrals** |
|---|---|---|
| $\sin x$ | $-\cos x$ | $\int \sin x\, dx = -\cos x + c$ |
| $\cos x$ | $\sin x$ | $\int \cos x\, dx = \sin x + c$ |
| $x^\alpha$, $\alpha \neq -1$ | $\dfrac{x^{\alpha+1}}{\alpha+1}$ | $\int x^\alpha\, dx = \dfrac{x^{\alpha+1}}{\alpha+1} + c$ |
| $\dfrac{1}{x}$, $x \neq 0$ | $\ln\|x\|$ | $\int \dfrac{1}{x}\, dx = \ln\|x\| + c$ |
| $e^x$ | $e^x$ | $\int e^x\, dx = e^x + c$ |
| $a^x$, $(a > 0,\, a \neq 1)$ | $\dfrac{a^x}{\ln a}$ | $\int a^x\, dx = \dfrac{a^x}{\ln a} + c$ |

Let us study the case of $x^\alpha$, $\alpha \neq -1$. We expect that the primitive of $x^\alpha$ is still a power function, $x^\beta$, since its derivative should gives back a power function too. Hence,

$$\frac{d}{dx}x^\beta = \beta x^{\beta-1}$$

which implies that

$$x^\alpha = \frac{d}{dx}(\frac{x^\beta}{\beta}) = x^{\beta-1}$$

and therefore, $\beta = \alpha + 1$.

Let us now consider $\alpha = -1$ ; *i.e.*, $1/x$. If $x > 0$ this the derivative of $\ln x$ and therefore, the last one is its primitive. If $x < 0$, then we can rewrite

$$\int \frac{1}{x}\, dx = -\int \frac{1}{(-x)}\, dx = \ln(-x) + c\,,$$

which implies that the primitive of $1/x$ for every $x \neq 0$ is $\ln|x|$.

## 1.3 Methods of Integration: substitution and integration by parts

In this section we discuss the two main strategies for calculating an integral: integration by parts and change of variable.

### 1.3.1 Method of substitution: change of variables

Regarding the substitution technique, we need to first establish some reasonable hypothesis: if we set $s = g(t)$, where the variable $t \in [a, b]$ and the new (dependent) variable $s$ varies in a new interval, say $[\alpha, \beta]$, we need the mapping $t \to g(t) = s$ to be a bijection between $[a, b]$ and $[\alpha, \beta]$; moreover, we need to require that if such map is regular on $[a, b]$ (say, it is a continuous function with continuous derivative), also its inverse has the same regularity. Such properties are certainly satisfied if we assume the following hypothesis in the next proposition.

**Proposition 1.3.1** (Change of variables). *Let $f : [\alpha, \beta] \mapsto \mathbb{R}$ be a continuous function and let $g : [a, b] \mapsto \mathbb{R}$ be a continuous function with continuous derivative $g'(t) \neq 0$ for all $t \in [a, b]$. Then*

$$\int_a^b f(g(t))g'(t)\,dt = \int_{g(a)}^{g(b)} f(s)\,ds\,. \tag{1.10}$$

### 1.3.2 Examples

(1) Find

$$\int t^2(t^3+2)^2\,dt\,.$$

Let us choose $g(t) = t^3 + 2$ and $f(x) = x^2$. Note that, $g'(t) = 3t^2$; hence, $dx = g'(t)\,dt$ and

$$\begin{aligned} \int t^2(t^3+2)^2\,dt &= \frac{1}{3}\int f(g(t))\,g'(t)\,dt \\ &= \frac{1}{3}\int 3t^2(t^3+2)^2\,dt \\ &= \frac{1}{3}\int x^2\,dx = \frac{x^3}{9} + c \\ &= \frac{(t^3+2)^3}{9} + c \end{aligned}$$

The exercise suggests a general rule: if $\alpha \neq -1$ then

$$\int [g(t)]^\alpha\, g'(t)\,dt = \frac{[g(t)]^{\alpha+1}}{(\alpha+1)} + c\,, \quad \forall \alpha \neq -1\,;$$

if $\alpha = -1$ then

$$\int \frac{g'(t)}{g(t)}\,dt = \log|g(t)| + c\,.$$

For example, let us consider

$$\begin{aligned} \int \tan t\,dt &= \int \frac{\sin t}{\cos t}dt \\ &= \int -\frac{g'(t)}{g(t)}\,dt \\ &= -\log|\cos t| + c\,. \end{aligned}$$

(2) Find

$$\begin{aligned} \int e^{3x} - (x-3)^4\,dx &= \int e^{3x}\,dx - \int (x-3)^4\,dx \\ &= \frac{e^{3x}}{3} - \frac{(x-3)^5}{5} + c\,; \end{aligned}$$

(3) Find

$$\begin{aligned}\int \sqrt{x+3} - \frac{1}{x}\,dx &= \frac{(x+3)^{1/2+1}}{(\frac{1}{2}+1)} - \ln|x| + c \\ &= \frac{2}{3}(x+3)^{3/2} - \ln|x| + c\,.\end{aligned}$$

(4) Find

$$\int \frac{3}{1+2x^2}\,dx$$

we make the following substitution $t = \sqrt{2}x,\ dt = \sqrt{2}\,dx,\ t^2 = 2x^2$, then

$$\begin{aligned}\int \frac{3}{1+2x^2}\,dx &= \frac{3}{\sqrt{2}}\int \frac{1}{1+t^2}\,dt \\ &= \frac{3}{\sqrt{2}}\arctan t + c \\ &= \frac{3}{\sqrt{2}}\arctan\sqrt{2}x + c\,.\end{aligned}$$

(5) Find

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx\,.$$

We make the following substitution $t = \sqrt{x},\ dt = \frac{1}{2\sqrt{x}}\,dx,\ 2t\,dt = dx$, then

$$\begin{aligned}\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx &= \int 2t\frac{\sin t}{t}\,dt \\ &= \int 2\sin t\,dt = -2\cos t + c \\ &= -2\cos\sqrt{x} + c\,.\end{aligned}$$

(6) Find

$$\int_e^{e^2} \frac{dx}{x\ln x} = \int_e^{e^2} \frac{g'(x)}{g(x)}\,dx$$

with $g(x) = \ln x$; hence, we solve it

$$\begin{aligned}\int_e^{e^2} \frac{dx}{x\ln x} &= \ln(\ln x)\Big|_e^{e^2} \\ &= \ln(\ln e^2) - \ln(\ln e) = \ln\Big(\frac{\ln e^2}{\ln e}\Big) \\ &= \ln\Big(2\frac{\ln e}{\ln e}\Big) = \ln 2\,.\end{aligned}$$

Note that, in case of logarithm with respect to a base $a \neq e$, we recall that $(\log_a x)' = 1/x(\ln a)$; hence, if we take $g(x) = \log_a x$ we have that

$$\begin{aligned}\int_e^{e^2} \frac{dx}{x \log_a x} &= \frac{1}{\ln a} \int_e^{e^2} \frac{g'(x)}{g(x)}\, dx \\ &= \frac{1}{\ln a} \ln(\log_a x)\Big|_e^{e^2} = \frac{1}{\ln a} \ln\Big(2 \frac{\log_a e}{\log_a e}\Big) \\ &= \frac{\ln 2}{\ln a}\,.\end{aligned}$$

(7) Find

$$\int \frac{\cos x}{4 + \sin x}\, dx = \int \frac{g'(x)}{g(x)}\, dx = \ln|4 + \sin x| + c\,,$$

with $g(x) = 4 + \sin x$.

(8) Find

$$\int_0^1 \frac{1}{e^x + e^{-x}}\, dx$$

by substitution $t = e^x$, $dt = e^x\, dx$ we have that the indefinite integral

$$\begin{aligned}\int \frac{1}{e^x + e^{-x}}\, dx &= \int \frac{1}{t + \frac{1}{t}} \frac{1}{t}\, dt \\ &= \int \frac{1}{1 + t^2}\, dt = \arctan t + c\,.\end{aligned}$$

To compute the definite integral we can proceed in two different way:

(a) since $t = e^x$ for every $0 \leq x \leq 1$ we have that $1 \leq t \leq e$; hence,

$$\int_0^1 \frac{1}{e^x + e^{-x}}\, dx = \int_1^e \frac{1}{1 + t^2}\, dt = \arctan t\Big|_1^e = \arctan e - \arctan 1\,;$$

(b)

$$\int \frac{1}{e^x + e^{-x}}\, dx = \arctan t + c = \arctan e^x + c\,;$$

hence,

$$\int_0^1 \frac{1}{e^x + e^{-x}}\, dx = \arctan e^x\Big|_0^1 = \arctan e - \arctan 1\,.$$

### 1.3.3 Integration by parts

The integration by parts is a straightforward consequence of the Leibniz rule for the derivative of the product:

$$(f(t)g(t))' = f'(t)g(t) + f(t)g'(t)\,;$$

hence,

$$\int_a^b f(t)g'(t)\, dt = f(t)g(t)\Big|_a^b - \int_a^b f'(t)g(t)\, dt \tag{1.11}$$

### 1.3.4 Examples

(1)

$$\begin{aligned}\int x\sin x\,dx &= -x\cos x - \int 1\cdot(-\cos x)\,dx \\ &= -x\cos x + \sin x + c\,.\end{aligned}$$

Note that, in this case it's not convenient to choose $g'(x) = x$ since we are increasing the power of $x$; *i.e.*,

$$\int x\sin x\,dx = \frac{x^2}{2}\sin x - \int \frac{x^2}{2}(\cos x)\,dx\,.$$

In general, if we have an integrand function of the type $x^n\,h(x)$ it is always more convenient to reduce the power of $x^n$ and identify $h(x) = g'(x)$ such that

$$\int x^n\,h(x)\,dx = x^n\,g(x) + \int nx^{n-1}\,g(x)\,dx\,.$$

For example,

$$\begin{aligned}\int x^2\sin x\,dx &= -x^2\cos x - \int 2x(-\cos x)\,dx \\ &= -x^2\cos x + 2x\sin x - \int 2x\sin x\,dx \\ &= -x^2\cos x + 2x\sin x + 2\cos x + c\,.\end{aligned}$$

(2)

$$\begin{aligned}\int e^x\,\sin x\,dx &= e^x\,\sin x - \int e^x\,\cos x\,dx \\ &= e^x\,\sin x - \Big(e^x\,\cos x + \int e^x\,\sin x\,dx\Big)\,;\end{aligned}$$

hence,

$$2\int e^x\,\sin x\,dx = e^x\,\sin x - e^x\,\cos x + c\,;$$

which implies

$$\int e^x\,\sin x\,dx = \frac{e^x(\sin x - \cos x)}{2} + c\,.$$

(3)

$$\begin{aligned}\int \arctan x\,dx &= \int 1\cdot\arctan x\,dx \\ &= x\arctan x - \int \frac{x}{1+x^2}\,dx \\ &= x\arctan x - \frac{1}{2}\ln(1+x^2) + c = x\arctan x - \ln\sqrt{1+x^2} + c \\ &= x\arctan x + \ln\Big(\frac{1}{\sqrt{1+x^2}}\Big) + c\end{aligned}$$

(4)

$$\int_1^2 x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \int x\ln x\,dx\,.$$

We proceed again as above

$$\begin{aligned}\int x\ln x\,dx &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,dx\\ &= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c\,;\end{aligned}$$

hence,

$$\begin{aligned}\int_1^2 x(\ln x)^2\,dx &= \left[\frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4}\right]\Big|_1^2\\ &= 2(\ln 2)^2 - 2\ln 2 + \frac{3}{4}\,.\end{aligned}$$

(5)

$$\int_0^{2\pi} e^{-x}|\sin x|\,dx = \int_0^{\pi} e^{-x}\sin x\,dx - \int_{\pi}^{2\pi} e^{-x}\sin x\,dx$$

Since $\sin x$ is an odd function we can make the change of variable $t=-x$, $\sin(-t) = -\sin t$, $dt=-dx$; hence, by Exercise (2) we have that

$$\begin{aligned}\int e^{-x}\sin x\,dx = \int e^{t}\sin t\,dt &= \frac{e^t(\sin t - \cos t)}{2} + c\\ &= \frac{e^{-x}(-\sin x - \cos x)}{2} + c\,,\end{aligned}$$

and,

$$\begin{aligned}\int_0^{2\pi} e^{-x}|\sin x|\,dx &= -\frac{e^{-x}(\sin x + \cos x)}{2}\Big|_0^{\pi} + \frac{e^{-x}(\sin x+\cos x)}{2}\Big|_{\pi}^{2\pi}\\ &= -\frac{e^{-x}(\cos x)}{2}\Big|_0^{\pi} + \frac{e^{-x}(\cos x)}{2}\Big|_{\pi}^{2\pi}\\ &= \frac{e^{-\pi}+1}{2} + \frac{e^{-2\pi}+e^{-\pi}}{2} = \frac{1+e^{-2\pi}}{2} + e^{-\pi}\,.\end{aligned}$$

(6)

$$\begin{aligned}\int_0^4 \frac{x^3}{\sqrt{1+x^2}}\,dx &= \int_0^4 \frac{x^2}{2}\frac{2x}{\sqrt{1+x^2}}\,dx\\ &= x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2}\,dx \qquad \Big(\text{where } (\sqrt{1+x^2})' = \frac{x}{\sqrt{1+x^2}})\Big)\\ &= \left[x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2}\right]\Big|_0^4\\ &= 16\sqrt{17} - \frac{2}{3}(17)^{3/2} + \frac{2}{3} = -\frac{530}{3}\sqrt{17} + \frac{2}{3}\,.\end{aligned}$$

### 1.3.5 Substitution for $\sqrt{a^2 - x^2}$, $\sin x$ and $\cos x$

Let start with the irrational function

$$f(x) = \sqrt{a^2 - x^2}\,,$$

by substitution $x = a\sin t$, $dx = a\cos t\, dt$ we get

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin t^2)} = \sqrt{(a\cos t)^2} = |a\ \cos t|\,.$$

**Example 1.3.2.** (1)

$$\int_0^1 \sqrt{1 - x^2}\, dx = \int_0^{\pi/2} |\cos t|\, dt = \sin t\Big|_0^{\pi/2} = 1\,.$$

(2)

$$\begin{aligned}
\int_0^1 x^2\sqrt{1 - x^2}\, dx &= \int_0^{\pi/2} (\sin t)^2\sqrt{(\cos t)^2}\cos t\, dt \\
&= \int_0^{\pi/2} (\sin x)^2\,(\cos t)^2\, dt \\
&= \int_0^{\pi/2} (\sin x)^2 - (\sin x)^4\, dt\,.
\end{aligned}$$

How to solve these integrals is explained below.

In case of rational functions depending on $\cos x$ and/ or $\sin x$ we may proceed with the following substitution

$$t = \tan\frac{x}{2}\,, \quad x = 2\ \arctan t\,, \quad dx = \frac{2}{1+t^2}\, dt$$

which implies

$$\cos x = \frac{1-t^2}{1+t^2}\,, \quad \sin t = \frac{2t}{1+t^2}\,.$$

For example,

$$\begin{aligned}
\int (\cos x)^2\, dx &= \int \Big(\frac{1-t^2}{1+t^2}\Big)^2 \frac{2}{1+t^2}\, dt \\
&= 2\int \frac{(1-t^2)^2}{(1+t^2)^3}\, dt\,.
\end{aligned}$$

Note that, we can also integrate by parts

$$\begin{aligned}
\int (\cos x)^2\, dx &= \int (\cos x)\cdot(\cos x)\, dx \\
&= \int (\sin x)'\cdot(\cos x)\, dx \\
&= \sin x\ \cos x + \int (\sin x)^2\, dx \\
&= \sin x\ \cos x + \int 1 - (\cos x)^2\, dx\,,
\end{aligned}$$

then

$$\int (\cos x)^2 \, dx = \frac{\sin x \, \cos x + x}{2} + c\,.$$

Similarly,

$$\int (\sin x)^2 \, dx = \frac{-\sin x \, \cos x + x}{2} + c\,.$$

## 1.4 Integration of rational functions

A rational function is a function that can be written as the quotient of two polynomials

$$f(x) = \frac{P(x)}{Q(x)}\,,$$

with $Q$ not zero. In this section we learn how to integrate rational function where the degree of $P$ is strictly less then the degree of $Q$, in particular, we consider $P(x) = p_0 + p_1 x$ and $Q(x) = q_0 + q_1 x + q_2 x^2$ polynomials of degree 1 and 2, respectively.

If the degree of the $P$ is greater than $Q$ we first perform a long division of $P$ into $Q$ as shown in the following example: let $x^3 + x$ be the dividend and $x^2 + x + 1$ the divisor

$$\frac{x^3 + x}{x^2 + x + 1}\,.$$

We divide the first term of the dividend, $x^3$, by the highest term of the divisor, $x^2$, which gives $x$ (the first term of the quotient). Hence

$$x(x^2 + x + 1) = x^3 + x^2 + x$$

if we make the difference $(x^3 + x) - (x^3 + x^2 + x) = -x^2$ the remainder that has the same degree of the divisor; hence, we have to repeat the previous step. We divide $-x^2$ by the highest term of the divisor, $x^2$, and we get $-1$ (this will be the second term of the quotient). It remains to multiply then $-1$ times $x^2 + x + 1$ and makes the difference with $-x^2$; *i.e.*, $-x^2 - [-(x^2 + x + 1)] = x + 1$. Now we stop the algorithm since the remainder $r(x) = x + 1$ has degree less then the divisor. Then we can conclude that

$$\frac{x^3 + x}{x^2 + x + 1} = (x - 1) + \frac{x + 1}{x^2 + x + 1}\,.$$

In this way, we have reduced the computation of the integral into a sum of two integrals

$$\int \frac{x^3 + x}{x^2 + x + 1} \, dx = \int (x - 1) \, dx + \int \frac{x + 1}{x^2 + x + 1} \, dx\,,$$

the first one is the integral of the quotient, the second one is still an integral of a rational function where the degree of the polynomial at numerator is not greater than the degree of the polynomial at denominator.

If $Q(x)$ is a polynomial of degree 2 we may distinguish between three different cases:

(1) there exist **two real roots** $x_1 \neq x_2$ such that $Q(x_1) = Q(x_2) = 0$, then

$$Q(x) = (a_1 x + b_1) \times (a_2 x + b_2)$$

and we have to find $A \neq B$ such that

$$\frac{P(x)}{Q(x)} = \frac{A}{(a_1 x + b_1)} + \frac{B}{(a_2 x + b_2)}\,.$$

For example, let us consider the following rational function

$$\int \frac{x+2}{x^2+x-6}\,dx\,.$$

The solutions to $x^2 + x - 6 = 0$ are $2, -3$; hence,

$$\begin{aligned}\frac{x+2}{x^2+x-6} &= \frac{A}{x-2} + \frac{B}{x+3}\\ &= \frac{(A+B)x + 3A - 2B}{(x-2)(x+3)}\end{aligned}$$

if and only if $A + B = 1$ and $3A - 2B = 2$; *i.e.*, $A = 4/5$ and $B = 1/5$. Therefore,

$$\begin{aligned}\int \frac{x+2}{x^2+x-6}\,dx &= \frac{4}{5}\int \frac{1}{x-2}\,dx + \frac{1}{5}\int \frac{1}{x+3}\,dx\\ &= \frac{4}{5}\ln|x-2| + \frac{1}{5}\ln|x+3| + c\,.\end{aligned}$$

(2) there exists **one real root** $x_1$ such that $Q(x_1) = 0$, then

$$Q(x) = (ax+b)^2\,.$$

In this case we decompose in sum of terms of the form

$$\frac{P(x)}{Q(x)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}\,.$$

For example, let us consider the following rational function

$$\int \frac{(x+1)}{(3x+2)^2}\,dx\,;$$

hence,

$$\begin{aligned}\frac{(x+1)}{(3x+2)^2} &= \frac{A}{(3x+2)} + \frac{B}{(3x+2)^2}\\ &= \frac{3Ax + (2A+B)}{(3x+2)^2}\,.\end{aligned}$$

We get then

$$\begin{aligned}\int \frac{(x+1)}{(3x+2)^2}\,dx &= \frac{1}{3}\int \frac{1}{(3x+2)}\,dx + \frac{1}{3}\int \frac{1}{(3x+2)^2}\,dx\\ &= \frac{1}{9}\ln|3x+2| - \frac{1}{9}\frac{1}{(3x+2)} + c\,.\end{aligned}$$

Similarly, we may proceed by substitution $t = 3x+2$, $dt = 3\,dx$, then

$$\begin{aligned}\int \frac{(x+1)}{(3x+2)^2}\,dx &= \int \frac{(\frac{t-2}{3})+1}{t^2}\frac{dt}{3}\\ &= \frac{1}{9}\int \frac{t+1}{t^2}\,dt = \frac{1}{9}\Big(\int \frac{1}{t}\,dt + \int \frac{1}{t^2}\,dt\Big)\\ &= \frac{1}{9}\ln|3x+2| - \frac{1}{9}\frac{1}{(3x+2)} + c\,.\end{aligned}$$

(3) there are **not real roots**; *i.e.*, $Q(x) \neq 0$ for very $x \in \mathbb{R}$. In this case we decompose the rational function in sum of terms of the form

$$\frac{P(x)}{Q(x)} = A\frac{Q'(x)}{Q(x)} + \frac{B}{Q(x)}\,,$$

where $Q'(x)$ denotes the first derivative of the polynomial $Q(x)$. Hence

$$\begin{aligned}\int \frac{P(x)}{Q(x)}\,dx &= A\int \frac{Q'(x)}{Q(x)}\,dx + B\int \frac{1}{Q(x)}\,dx\\ &= A\ln|Q(x)| + B\int \frac{1}{Q(x)}\,dx + c\,.\end{aligned}$$

For example, let us consider the following rational function

$$\begin{aligned}\frac{x}{x^2+2x+4} &= A\frac{2x+2}{x^2+2x+4} + \frac{B}{x^2+2x+4}\\ &= \frac{2Ax + (2A+B)}{x^2+2x+4}\,.\end{aligned}$$

Therefore we get $A = 1/2$, $B = -1$ and

$$\begin{aligned}\int \frac{x}{x^2+2x+4}\,dx &= \frac{1}{2}\int \frac{2x+2}{x^2+2x+4}\,dx - \int \frac{1}{x^2+2x+4}\,dx\\ &= \frac{1}{2}\ln|x^2+2x+4| - \int \frac{1}{x^2+2x+4}\,dx\,.\end{aligned}$$

To solve the last integral we rewrite the polynomial $x^2+2x+4$ as the sum of two squared terms such that the primitive is the arctan function. More precisely,

$$x^2+2x+4 = (x+1)^2+3\,;$$

hence, by a substitution $t = (x+1)/\sqrt{3}$, we get

$$\begin{aligned}\int \frac{1}{x^2+2x+4}\,dx &= \int \frac{1}{(x+1)^2+3}\,dx\\ &= \frac{\sqrt{3}}{3}\int \frac{1}{t^2+1}\,dt = \frac{\sqrt{3}}{3}\arctan t + c = \frac{\sqrt{3}}{3}\arctan\Big(\frac{x+1}{\sqrt{3}}\Big) + c\,.\end{aligned}$$

**General rule**:

$$ax^2+bx+c = a\Big[\Big(x+\frac{b}{2a}\Big)^2 + \frac{4ac-b^2}{4a^2}\Big]$$

where $4ac - b^2 = -\Delta$ that is the discriminant changed of signed.

## 1.5 Exercises

(1) Compute the area of the following region $A \subset \mathbb{R}^2$:

$$A = \{(x, y) \in \mathbb{R}^2 \, : \, 0 \le x \le 1, \, x^2 \le y \le \sqrt{x}\}.$$

(2) Find the following integrals:

$$\int_0^\pi \sin^3(x)\, dx \qquad \int_{-1}^1 \frac{x}{(4x^2+1)^5}\, dx \qquad \int_0^{\frac{\pi}{2}} \cos^3(x)\, dx$$

$$\int_0^1 \left(x^4 + 5x^3 + 1\right) dx, \qquad \int_0^{\frac{\pi}{4}} \cos(x)\, dx, \qquad \int_{-2}^{-1} \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx,$$

$$\int_0^2 e^{-2x}\, dx \qquad \int_0^1 \frac{1}{x^2+4}\, dx \qquad \int_{-2}^{-1} \frac{2x+1}{(x-1)^2}\, dx$$

(3) Find the following integrals:

$$\int (3x+1)^3\, dx \qquad \int_0^1 x^2(3x^3+1)^2\, dx \qquad \int x(3x^2+1)^3\, dx$$

$$\int_0^1 \left(x^{5/6} + 2x^{-2} - 3x^{-1} + 2\right) dx \qquad \int_0^{\frac{\pi}{2}} e^x\, (\sin x)^2\, dx, \qquad \int_0^{\pi/4} \cos 2x \sin x\, dx,$$

$$\int \left(\sqrt{x} + \sqrt[3]{x+2} + \frac{1}{x^2}\right) dx \qquad \int e^{-x^2} x^3\, dx \qquad \int \frac{3x}{x^2 - x - 2}$$

$$\int_{-1}^1 \frac{x}{1+x^2}\, dx \qquad \int \left(\frac{e^{2x} - e^{-x}}{3}\right) dx \qquad \int \frac{(x-2)}{(2x-1)^2}$$

$$\int_{-1}^1 e^{-|x|}\, dx \qquad \int \frac{(x+3)}{(x^2 - x + 4)} \qquad \int (x^2 + 5x + 4) e^x\, dx$$

$$\int_0^1 x\, e^{3x}\, dx \qquad \int_1^3 x^2 \ln x\, dx \qquad \int_0^\pi x \cos x\, dx$$

## 1.6 Solving Exercises Chapter 1

(1) The following integral can be solve by substitution $t = 3x + 1$; *i.e.*,

$$\int (3x+1)^3\, dx = \int \frac{t^3}{3}\, dt = \frac{t^4}{12} + c\,.$$

If we consider

$$\int x(3x^2+1)^3\, dx$$

it can be still solved by substitution by identifying

$$\int x(3x^2+1)^3 \ dx = \int g'(x) f(g(x))\, dx\,;$$

hence,

$$\int x(3x^2+1)^3\,dx = \int \frac{y^3}{6}\,dy = \frac{(3x^2+1)^4}{6\cdot 4} + c\,.$$

Note that, $x$ multiplying $(3x^2+1)^3$ makes easier the computation of the integral because, up to a constant, it coincides with the derivative of $3x^2+1$ and it allows us to apply the method by substitution. Let us consider another case slightly different

$$\int x\,(3x+1)^3\,dx\,.$$

Here, $x$ does not have "the same role"as in the previous case, we do not have to interprete $x$ as derivative of ... better make it disappear... How? By parts! Let's do it

$$\begin{aligned}\int x\,(3x+1)^3\,dx &= x\,\frac{(3x+1)^4}{4\cdot 3} - \int \frac{(3x+1)^4}{12}\,dx\\ &= x\,\frac{(3x+1)^4}{12} - \frac{(3x+1)^5}{12\cdot 5} + c\,.\end{aligned}$$

(2) Find

$$\int_0^{\frac{3\pi}{2}} e^x\,(\sin x)^2\,dx\,.$$

We use the method of integration by parts

$$\int_0^{\frac{3\pi}{2}} e^x\,(\sin x)^2\,dx = e^x\,(\sin x)^2 - \int e^x\,2\sin x\,\cos x\,dx\,.$$

Now, we may proceed in two different way: 1) use the trigonometric formula $\sin(2x) = 2\sin x\,\cos x$; 2) we use integration by parts again and the trigonometric identity $1 = (\sin x)^2 + (\cos x)^2$.

In the first case

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^x\,(\sin x)^2\,dx &= e^x\,(\sin x)^2 - \int_0^{\frac{\pi}{2}} e^x\,2\sin x\,\cos x\,dx\\ &= e^x\,(\sin x)^2\Big|_0^{\pi/2} - \int_0^{\frac{\pi}{2}} e^x\,(\sin 2x)\,dx\\ &= e^{\pi/2} + \text{And now what?}\end{aligned}$$

The second strategy is longer than the first one but probably most useful....

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^x\,(\sin x)^2\,dx &= e^x\,(\sin x)^2 - \int e^x\,2\sin x\,\cos x\,dx\\ &= e^x\,(\sin x)^2\Big|_0^{\pi/2} - 2\,e^x\,\sin x\,\cos x\Big|_0^{\pi/2} + 2\int_0^{\pi/2} e^x\,[-(\sin x)^2 + (\cos x)^2]\,dx\\ &= e^{\pi/2} + 2\int_0^{\pi/2} e^x\,[1-2(\sin x)^2]\,dx\\ &= e^{\pi/2} + 2(e^{\pi/2}-1) - 4\int_0^{\pi/2} e^x\,(\sin x)^2\,dx\,.\end{aligned}$$

Hence,

$$5\int_0^{\frac{\pi}{2}} e^x\,(\sin x)^2\,dx \quad = \quad e^{\pi/2} + 2(e^{\pi/2} - 1)\,,$$

and

$$\int_0^{\frac{\pi}{2}} e^x\,(\sin x)^2\,dx = \frac{3e^{\pi/2} - 2}{5}\,.$$

# Chapter 2

# Improper Integral

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The Riemann integral has been defined for functions $f : [a, b] \mapsto \mathbb{R}$ satisfying the following assumptions:

(1) $[a, b]$ is a bounded interval;

(b) $f$ is a bounded function.

Now the question is:

*What does it happen when we remove one or both assumptions?*

From the geometric interpretation of the integral we may immediately observe that we pass from a bounded region (underneath the graph of $f$) to an unbounded region! Still, is it possibile to have (at least in some cases, we don' t expect always....) a finite area?

The answer is:

*We may still have, in some cases, that the "Integral' is finite!*

Let us consider the following function

$$f(x) = \frac{1}{x^\alpha}\,, \qquad x > 0\,.$$

The function $f$ is not bounded in $(0, 1]$ (or in any interval $(0, a]$) since $\lim_{x\to 0^+} 1/x^\alpha = +\infty$; hence, we can not perform the Riemann integral on $(0, 1]$ but on $[a, 1]$ for every $a > 0$. More precisely,

$$\int_a^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{1-\alpha}(1 - a^{1-\alpha}) & \text{if } \alpha \neq 1 \\ \\ -\ln a & \text{if } \alpha = 1\,, \end{cases}$$

passing to the limit on $a$ we get

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{1-\alpha} & \text{if } \alpha < 1 \\ \\ +\infty & \text{if } \alpha \geq 1\,. \end{cases} \tag{2.1}$$

Let us consider the function $f$ on $[1,+\infty)$. Again we can not apply the theory of Riemann integral since $[1,+\infty)$ is an unbounded interval. We may reason as above; hence,

$$\int_1^L \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{1-\alpha}(L^{1-\alpha}-1) & \text{if } \alpha \neq 1 \\ \ln L & \text{if } \alpha = 1\,, \end{cases}$$

passing to the limit on $L$ we get

$$\lim_{L\to+\infty}\int_1^L \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{\alpha-1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \leq 1\,. \end{cases} \tag{2.2}$$

This example suggests that we may introduce a notion of integrability for unbounded functions and unbounded intervals such that

$$\frac{1}{x^\alpha} \text{ is " integrable"on } (0,1] \text{ for every } \alpha < 1 \text{ and on } [1,+\infty) \text{ for every } \alpha > 1\,.$$

## 2.1 Definitions and Examples

**Definition 2.1.1** (Improper Integral-def1)**.** *Let $f:(a,b]\mapsto\mathbb{R}$ be a Riemann integrable function in $[c,b]$ for every $c\in(a,b)$. Then the improper integral of $f$ in $(a,b]$ is defined as*

$$\int_a^b f(x)\,dx := \lim_{c\to a^+}\int_c^b f(x)\,dx \in \mathbb{R}\,.$$

*Similarly, let $f:[a,+\infty)\mapsto\mathbb{R}$ be a Riemann integrable function on $[a,L]$ for every $L>a$. Then the improper integral of $f$ in $[a,+\infty)$ is defined as*

$$\int_a^{+\infty} f(x)\,dx := \lim_{L\to+\infty}\int_a^L f(x)\,dx \in \mathbb{R}\,.$$

**Remark 2.1.2.** Similarly, if $f:[a,b)\mapsto\mathbb{R}$ is Riemann integrable in $[a,c]$ for every $c\in(a,b)$, then the improper integral of $f$ in $[a,b)$ is given by

$$\int_a^b f(x)\,dx := \lim_{c\to b^-}\int_a^c f(x)\,dx \in \mathbb{R}\,;$$

if $f:(-\infty,b]\mapsto\mathbb{R}$ is Riemann integrable in $[L,b]$ for every $L<b$, then the improper integral of $f$ in $(-\infty,b]$ is given by

$$\int_{-\infty}^b f(x)\,dx := \lim_{L\to-\infty}\int_L^b f(x)\,dx \in \mathbb{R}\,.$$

**Example 2.1.3.** Let us consider $f(x)=e^{-x}$. It is continuous on $\mathbb{R}$; hence, it is Riemann integrable in $[a,b]$ for every $a,b\in\mathbb{R}$. There exists the improper integral of $f$ on $(-\infty,a]$ and/or on $[a,+\infty)$?

$$\int_a^b e^{-x}\,dx = -e^{-x}\Big|_a^b = -e^{-b}+e^{-a}\,;$$

hence,

$$\lim_{b\to+\infty} -e^{-b}+e^{-a} = e^{-a}\,,$$
$$\lim_{a\to-\infty} -e^{-b}+e^{-a} = +\infty\,.$$

Then the answer is: for every $a, b \in \mathbb{R}$ there exists the improper integral of $f$ on $[a, +\infty)$ but not in $(-\infty, b]$ !

In general, we integrate also a function $f$ where all cases, listed above, are combined as in case of function $f$ that is not defined in the two extremes of the interval or in an internal point. In addition, we may require to integrate an unbounded function in an unbounded interval as $f(x) = 1/x^\alpha$, $x \in (0, +\infty$. Then we may integrate with respect to Riemann in a bounded interval where the function is bounded as follows

$$\int_a^L \frac{1}{x^\alpha}\,dx = \frac{1}{\alpha-1}(L^{1-\alpha} - a^{1-\alpha})$$

and, finally, we pass to the limit as $a \to 0^+$ and $L \to +\infty$.

Note that, there exists the improper integral if the result does not depend on the order of the limit as $a \to 0^+$ and $L \to +\infty$, since by the additivity of the integral we may always separate the two limits as follows

$$\begin{aligned}\int_0^{+\infty} \frac{1}{x^\alpha}\,dx &:= \lim_{a\to0^+}\lim_{L\to+\infty}\Big(\int_a^L \frac{1}{x^\alpha}\,dx\Big)\\ &= \lim_{L\to+\infty}\lim_{a\to0^+}\Big(\int_a^L \frac{1}{x^\alpha}\,dx\Big)\\ &= \lim_{a\to0^+}\int_a^{x_0} \frac{1}{x^\alpha}\,dx + \lim_{L\to+\infty}\int_{x_0}^L \frac{1}{x^\alpha}\,dx\,.\end{aligned}$$

**Definition 2.1.4** (Improper Integral-def2)**.** *Let $-\infty \le a \le b \le +\infty$ and let $f : (a, b) \mapsto \mathbb{R}$ be a Riemann integrable function in $[c, d]$ for every $[c, d] \subset (a, b)$. Then there exists the improper integral of $f$ in $(a, b)$ if there exist the two following limit*

$$l_- := \lim_{c\to a^+}\int_c^{x_0} f(x)\,dx \in \mathbb{R}\,, \qquad l_+ := \lim_{d\to b^-}\int_{x_0}^d f(x)\,dx \in \mathbb{R}\,.$$

*and*

$$\int_a^b f(x)\,dx := l_- + l_+\,,$$

*for any arbitrary fixed $x_0 \in \mathbb{R}$. In particular,*

$$\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx := \lim_{c\to-\infty}\int_c^{x_0} f(x)\,dx + \lim_{d\to+\infty}\int_{x_0}^d f(x)\,dx\,.$$

**Example 2.1.5.** Find the improper integral of the following function on $(-1, 1)$

$$f(x) = \frac{1}{\sqrt{1-x^2}}\,.$$

Note that $f$ is not defined in $\pm 1$ and it is unbounded on $(-1,1)$. We study separately the integrability on $(-1,x_0]$ and $[x_0,1)$ for an arbitrary $x_0 \in (-1,1)$, for example we may choose $x_0 = 0$. Then,

$$\begin{aligned}
\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{c\to(-1)^+} \int_c^0 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{c\to(-1)^+} \arcsin x\Big|_c^0 = -\arcsin(-1) = \frac{\pi}{2} \\
\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{c\to1^-} \int_0^c \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{c\to1^-} \arcsin x\Big|_0^c = \arcsin(1) = \frac{\pi}{2}\,.
\end{aligned}$$

Hence,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi\,.$$

**Example 2.1.6.** Find the following improper integral

$$\int_0^1 \frac{1}{x\sqrt{1-x}}\,dx\,.$$

The integrand function is not define in $x = 0,1$ and is unbounded in the interval $(0,1)$. Hence, we first find the Riemann integral in $[c,x_0] \subset (0,1)$. More precisely, we make the following change of variable $\sqrt{1-x} = t$

$$\int \frac{1}{x\sqrt{1-x}}\,dx = \int \frac{-2}{1-t^2}\,dt$$

By Definition 2.1.4 we have that

$$\begin{aligned}
\int_0^1 \frac{1}{x\sqrt{1-x}}\,dx &= \lim_{c\to0^+} \int_c^{x_0} \frac{1}{x\sqrt{1-x}}\,dx + \lim_{d\to1^-} \int_{x_0}^{d} \frac{1}{x\sqrt{1-x}}\,dx \\
&= \lim_{c\to0^+} \int_{\sqrt{1-x_0}}^{\sqrt{1-c}} \frac{2}{1-t^2}\,dt + \lim_{d\to1^-} \int_{\sqrt{1-d}}^{\sqrt{1-x_0}} \frac{2}{1-t^2}\,dt\,.
\end{aligned}$$

By Section 1.4 we have that

$$\int \frac{2}{1-t^2}\,dt = \int \Big(\frac{1}{1-t} + \frac{1}{1+t}\Big)\,dt = \ln\Big(\frac{1+t}{1-t}\Big) + c\,.$$

Hence,

$$\begin{aligned}
\lim_{c\to0^+} \int_c^{x_0} \frac{1}{x\sqrt{1-x}}\,dx &= \lim_{c\to0^+} \ln\Big(\frac{1+\sqrt{1-c}}{1-\sqrt{1-c}}\Big) - \ln\Big(\frac{1+\sqrt{1-x_0}}{1-\sqrt{1-x_0}}\Big) = +\infty\,; \\
\lim_{d\to1^-} \int_{x_0}^{d} \frac{1}{x\sqrt{1-x}}\,dx &= \lim_{d\to1^-} \ln\Big(\frac{1+\sqrt{1-x_0}}{1-\sqrt{1-x_0}}\Big) - \ln\Big(\frac{1+\sqrt{1-d}}{1-\sqrt{1-d}}\Big) = \ln\Big(\frac{1+\sqrt{1-x_0}}{1-\sqrt{1-x_0}}\Big)\,.
\end{aligned}$$

Since, one of the two limit is infinite we conclude that does not exist the improper integral on $(0,1)$. Indeed, it exists on every interval $[a,1)$ with $a > 0$.

**Example 2.1.7.** Let us consider

$$f(x) = \frac{1}{x^2+1}\,, \qquad x \in \mathbb{R}\,.$$

We may fix $x_0 = 0$; hence,

$$\lim_{c\to-\infty}\int_c^0 \frac{1}{x^2+1}\,dx = \lim_{c\to-\infty} -\arctan c = \frac{\pi}{2}$$

and

$$\lim_{d\to+\infty}\int_0^d \frac{1}{x^2+1}\,dx = \lim_{d\to+\infty} \arctan d = \frac{\pi}{2}\,.$$

Therefore, there exists the improper integral on $\mathbb{R}$

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = \pi\,.$$

**Example 2.1.8.** Let us consider

$$f(x) = \frac{1}{2\sqrt{|x|}}\,, \qquad x \in [-1,1]\setminus\{0\}\,.$$

The function $f$ is not definite in 0 then we may integrate in $[-1,0)$ and $(0,1]$. More precisely, since $f$ is even and by definition of improper integral, we have that

$$\begin{aligned}\int_0^1 \frac{1}{2\sqrt{|x|}}\,dx &= \int_{-1}^0 \frac{1}{2\sqrt{|x|}}\,dx \\ &= \lim_{\varepsilon\to0^+}\int_\varepsilon^1 \frac{1}{2\sqrt{|x|}}\,dx = \lim_{\varepsilon\to0^+} 1-\sqrt{\varepsilon} = 1\,;\end{aligned}$$

hence, there exists the improper integral in $[-1,1]\setminus\{0\}$ and it is given by

$$\int_{-1}^1 \frac{1}{2\sqrt{|x|}}\,dx = 2\,.$$

**Example 2.1.9.** Let us consider for $\alpha > 0$ the following function

$$f(x) = \frac{1}{x|\ln x|^\alpha}\,, \qquad x \in (0,+\infty)\setminus\{1\}$$

Determine the value of $\alpha$ and the intervals where there exists the improper integral.

Note that the function $f$ is unbounded and not definite in 0 and 1. Moreover, we want to integrate in an unbounded interval. Hence, according to Definition 2.1.4, we have to study the existence of the improper integral in $(0,1)$ and $(1,+\infty)$, separately. More precisely, fixing $x_0 = 1/2$, we have that

$$\int_0^1 \frac{1}{x|\ln x|^\alpha}\,dx := \lim_{\varepsilon\to0^+}\int_\varepsilon^{1/2} \frac{1}{x|\ln x|^\alpha}\,dx + \lim_{\varepsilon\to0^+}\int_{1/2}^{1-\varepsilon} \frac{1}{x|\ln x|^\alpha}\,dx\,;$$

and, fixing $x_0 = 2$, we have that

$$\int_1^{+\infty} \frac{1}{x|\ln x|^\alpha}\,dx := \lim_{\varepsilon\to0^+}\int_{1+\varepsilon}^{2} \frac{1}{x|\ln x|^\alpha}\,dx + \lim_{L\to+\infty}\int_{2}^{L} \frac{1}{x|\ln x|^\alpha}\,dx\,.$$

In $(0,1)$ we have that $|\ln x| = -\ln x$; hence, we make the following change of variable $t = -\ln x$. Since $-\ln\varepsilon \to +\infty$ as $\varepsilon \to 0^+$, by (2.2), we get that there exists the improper integral on $(0,1/2)$, for any $\alpha > 1$; *i.e.*,

$$\int_0^{1/2} \frac{1}{x|\ln x|^\alpha}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^{1/2} \frac{1}{x|\ln x|^\alpha}\,dx = \lim_{\varepsilon\to 0^+} \int_{\ln 2}^{-\ln\varepsilon} \frac{1}{t^\alpha}\,dt = \frac{(\ln 2)^{1-\alpha}}{\alpha - 1}\,, \quad \alpha > 1\,.$$

Since $-\ln(1-\varepsilon) \to 0$ as $\varepsilon \to 0^+$, by (2.1), we have that there exists the improper integral on $(1/2,1)$, for any $\alpha < 1$; *i.e.*,

$$\lim_{\varepsilon\to 0^+} \int_{1/2}^{1-\varepsilon} \frac{1}{x|\ln x|^\alpha}\,dx = \lim_{\varepsilon\to 0^+} \int_{-\ln(1-\varepsilon)}^{\ln 2} \frac{1}{t^\alpha}\,dt = \frac{(\ln 2)^{1-\alpha}}{1-\alpha}\,, \quad \alpha < 1\,.$$

Let us study the improper integral in $(1,+\infty)$. More precisely, we study the improper integral in $(1,2)$ and $(2,+\infty)$. By applying the same change of variable as above, we get that

$$\lim_{\varepsilon\to 0^+} \int_{1+\varepsilon}^{2} \frac{1}{x|\ln x|^\alpha}\,dx = \lim_{\varepsilon\to 0^+} \int_{\ln(1+\varepsilon)}^{\ln 2} \frac{1}{t^\alpha}\,dt = \frac{(\ln 2)^{1-\alpha}}{1-\alpha}\,, \quad \alpha < 1\,,$$

that is, since, $\lim_{\varepsilon\to 0^+} \ln(1+\varepsilon) = 0$, we can conclude that there exists the improper integral if and only if $\alpha < 1$. Finally,

$$\lim_{L\to+\infty} \int_2^L \frac{1}{x|\ln x|^\alpha}\,dx = \lim_{L\to+\infty} \int_{\ln 2}^{\ln L} \frac{1}{t^\alpha}\,dt = \frac{(\ln 2)^{1-\alpha}}{\alpha-1}\,, \quad \alpha > 1\,.$$

Summing up, there exists the improper integral on $[a,b]$ for every $a < 1 < b$ and $0 < \alpha < 1$; on $(0,a]$ and $[b,+\infty)$ for every $a < 1 < b$ and $\alpha > 1$.

**Remark 2.1.10.** In case the improper integral has to be worked out by dividing the interval in subintervals, as in the previous example where the function was not defined in 0 and 1, then also the corresponding limits have to be performed separately, otherwise we could have a contradiction. Let us consider $1/x$ in $(-1,1)$, the improper integrals has to be worked out in $(0,1)$ and $(-1,0)$. Hence,

$$\begin{aligned}\int_{-1}^1 \frac{1}{x}\,dx &= \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x}\,dx + \lim_{\varepsilon\to 0^+} \int_{-1}^{-\varepsilon} \frac{1}{x}\,dx \\ &= \lim_{\varepsilon\to 0^+} -\ln\varepsilon + \lim_{\varepsilon\to 0^+} \ln\varepsilon\,;\end{aligned}$$

while, if we do not separate the computation of the limit we would have

$$\int_{-1}^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+} -\ln\varepsilon + \ln\varepsilon = 0\,,$$

that is absolutely FALSE! Actually, it might be confusing the choice of the same parameter $\varepsilon$ in both limits... Indeed, since the two limits have to be worked out independently, it would be better to use two different parameters $\varepsilon$ and $\delta$

$$\begin{aligned}\int_{-1}^1 \frac{1}{x}\,dx &= \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x}\,dx + \lim_{\delta\to 0^+} \int_{-1}^{-\delta} \frac{1}{x}\,dx \\ &= \lim_{\varepsilon\to 0^+} -\ln\varepsilon + \lim_{\delta\to 0^+} \ln\delta\,.\end{aligned}$$

## 2.2 Comparison Theorems

Let $f : [a, +\infty) \mapsto [0, +\infty)$ be a non negative function, Riemann integrable on very closed and bounded subintervals $[a, b]$. The integral function

$$F(L) := \int_a^L f(x)\, dx$$

it is a non-decreasing function and the limit $\lim_{L\to+\infty} F(L)$ always exists. Hence, there exists the improper integral of $f$ on $[a, +\infty)$ if and only if such limit is finite and it does not exists if $\lim_{L\to+\infty} F(L) = +\infty$. Note that in this case, as for the Riemann integral, there is a natural geometric interpretation of the improper integral as the 'measure' of an unbounded area underneath the graph of the function $f$ that is finite if the improper integral of $f$ is finite.

Similarly, if we consider the improper integral on bounded interval $[a, b)$, $(a, b]$, $(a, b)$.

The geometric interpretation suggests also some other considerations: if a graph of a positive function $g$ is above the graph of another positive function $f$ then by the monotonicity of the Riemann integral we have that also the improper integrability are related to each other. More precisely, we have the following comparison theorem

**Theorem 2.2.1** (Comparison Test)**.** *Let $f, g : [a, b) \mapsto \mathbb{R}$ be two Riemann integrable function on every $[a, c]$ foe any $c \in (a, b)$ and $b \le +\infty$. Let us assume that*

$$0 \le f(x) \le g(x)\,, \quad \forall x \in [a, b)\,.$$

*Then,*

$$0 \le \int_a^b f(x)\, dx \le \int_a^b g(x)\, dx\,,$$

*and*

$$\begin{aligned} &\text{if} \quad \int_a^b f(x)\, dx = +\infty \quad \Longrightarrow \quad \int_a^b g(x)\, dx = +\infty\,; \\ &\text{if} \quad \int_a^b g(x)\, dx < +\infty \quad \Longrightarrow \quad \int_a^b f(x)\, dx < +\infty\,. \end{aligned}$$

**Example 2.2.2.** prove that the following improper integral exists and is finite

$$\int_1^{+\infty} \frac{|\cos x|}{x^2}\, dx\,.$$

It is not easy to find the primitive of $|\cos x|/x^2$ but we can establish by comparison if the integral is finite or not. Indeed,

$$\frac{|\cos x|}{x^2} \le \frac{1}{x^2}\,, \forall x \ge 1\,,$$

moreover, by (2.2), we have that

$$\int_1^{+\infty} \frac{1}{x^2}\, dx = 1\,;$$

hence, by Theorem 2.2.1, we have that

$$\int_1^{+\infty} \frac{|\cos x|}{x^2}\, dx \leq 1\,,$$

which implies not only the existence of the improper integral but also an estimate for above.

**Exercise 2.2.3.** *Say if the following improper integral is finite or infinite*

$$\int_0^1 \frac{1+e^x}{x}\, dx\,.$$

**Theorem 2.2.4** (Asymptotic Comparison Test)**.** *Let $f, g : [a, b) \mapsto \mathbb{R}$ be two Riemann integrable function on every $[a, c]$ foe any $c \in (a, b)$ and $b \leq +\infty$. Let us assume that $f(x) \geq 0$ and $g(x) > 0$ for every $x \in [a, b)$.*

- *Assume that*

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = l > 0\,,$$

*then,*

$$\begin{aligned} &\text{if} \quad \int_a^b f(x)\, dx = +\infty \quad \Longleftrightarrow \quad \int_a^b g(x)\, dx = +\infty\,; \\ &\text{if} \quad \int_a^b g(x)\, dx < +\infty \quad \Longleftrightarrow \quad \int_a^b f(x)\, dx < +\infty\,. \end{aligned}$$

- *Assume that*

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = 0\,,$$

*then,*

$$\begin{aligned} &\text{if} \quad \int_a^b f(x)\, dx = +\infty \quad \Longrightarrow \quad \int_a^b g(x)\, dx = +\infty\,; \\ &\text{if} \quad \int_a^b g(x)\, dx < +\infty \quad \Longrightarrow \quad \int_a^b f(x)\, dx < +\infty\,. \end{aligned}$$

- *Assume that*

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = +\infty\,,$$

*then,*

$$\begin{aligned} &\text{if} \quad \int_a^b g(x)\, dx = +\infty \quad \Longrightarrow \quad \int_a^b f(x)\, dx = +\infty\,; \\ &\text{if} \quad \int_a^b f(x)\, dx < +\infty \quad \Longrightarrow \quad \int_a^b g(x)\, dx < +\infty\,. \end{aligned}$$

**Example 2.2.5.** Say if the following improper integral is finite or infinite

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx\,.$$

The function $e^{-x^2}$ is continuous, bounded and even on $\mathbb{R}$, hence the improper integral on $\mathbb{R}$ is finite if

$$\lim_{L\to+\infty}\int_0^L e^{-x^2}\,dx < +\infty\,.$$

Note that

$$\lim_{x\to\pm\infty}\frac{e^{-x^2}}{1/(x^2+1)} = 0\,.$$

Hence, by the Asymptotic comparison test Theorem 2.2.4, we have that the improper integral on $\mathbb{R}$ is finite since

$$\int_0^{+\infty}\frac{1}{x^2+1}\,dx = \int_{-\infty}^{0}\frac{1}{x^2+1}\,dx = \frac{\pi}{2}\,,$$

(see Example 2.1.7).

## 2.3 Exercises

(1) Say if the following improper integrals are finite or infinite

$$\int_0^{+\infty}\frac{1+2\sin(\arctan x)-e^{-x}}{(1+x^2)}\,dx\,; \qquad \int_1^{\infty}\frac{\log(x^2+4)}{x^2}\,dx;$$

$$\int_0^1\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx\,,\ \text{with } k^2<1\,.$$

(2) Find the following improper integrals with $p,q\in\mathbb{R}$

$$\int_{-1}^{3}\frac{x}{x^2-9}\,dx \qquad \int_0^1\frac{\sin x}{x^p}\,dx, \qquad \int_0^1\frac{(\sin x)^q}{x}\,dx$$

$$\int_1^2\frac{1}{(x-1)\,(x^2+1)}\,dx, \qquad \int_0^1\frac{1}{x\,\log x}\,dx, \qquad \int_1^2\frac{x^2}{x^3-8}\,dx,$$

## 2.4 Solving Exercises Chapter 2

(1) Say if the following improper integral is finite or infinite.

$$\int_0^1\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx \qquad k^2<1\,. \tag{2.3}$$

First of all we note that since $k^2<1$ then the function $1/\sqrt{(1-k^2x^2)}$ is bounded while $1/\sqrt{(1-x^2)}$ is unbounded for every $x\in[0,1]$; more precisely,

$$1\le\frac{1}{\sqrt{(1-k^2x^2)}}\le\frac{1}{\sqrt{(1-k^2)}}\,, \qquad \lim_{x\to1^-}\frac{1}{\sqrt{(1-x^2)}} = +\infty\,.$$

Hence, the contribution which makes unbounded the function $1/\sqrt{(1-x^2)}$ is just:

$$\frac{1}{\sqrt{1-x}}\,. \tag{2.4}$$

Thus we actually need to compare the whole function to that function:

$$\lim_{x\to 1^-}\frac{\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}}{\frac{1}{\sqrt{1-x}}}=\frac{1}{\sqrt{2(1-k^2)}<\infty}\,. \tag{2.5}$$

By Theorem 2.2.4 we have that the improper integral is finite by comparison with

$$\int_0^1\frac{1}{\sqrt{1-x}}\,dx=\int_0^1\frac{1}{\sqrt{t}}\,dt=\lim_{\varepsilon\to 0}2\sqrt{t}\Big|_{\varepsilon}^{1}=2\,.$$

Note that, in Example 2.1.5 we solve the improper integral of $1/\sqrt{(1-x^2)}$ on $[0,1]$ by recognizing the function as a primitive of the arcsin function.

# Chapter 3

# Numerical Sequences and Series

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## 3.1 Sequences of real numbers

A sequence is a function $f : \mathbb{N} \mapsto \mathbb{R}$ that takes values $f(1), f(2), \cdots, f(n), \cdots$. An intuitive understanding of the definition of a sequence is that a sequence it is an itemized collection of elements, it is a chain of ordered term. Then, the best notation for sequences is rather

$$a_1, a_2, \cdots, a_n, \cdots$$

where $n$ is the index of the sequence and $(a_n) = \{a_1, a_2, \cdots, a_n, \cdots\} = f(\mathbb{N})$ is the set of all values assumed by the function $f$ and denoted by $a_n := f(n)$.

Before giving the formal definitions of important notions for sequences, we take a look at the behaviour of the sequence in the example below.

**Example 3.1.1.** *The sequence of the inverse of the integers:*

$$a_n = \frac{1}{n}, \quad (a_n) = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}.$$

*Note that, the sequence $a_n$ is bounded from below and from above; i.e., $0 < a_n \leq 1$. Moreover, $a_{n+1} \leq a_n$ for every $n \in \mathbb{N}$; i.e., in this sequence the values are decreasing as $n$ increases and $(1/n)$ seems to "approach" a single point as $n$ increases that is $0$.*

*Indeed, for every arbitrarily fixed $\varepsilon > 0$ we have that*

$$0 < \frac{1}{n} < \varepsilon \Longleftrightarrow n > \frac{1}{\varepsilon},$$

*which implies that the sequence $(a_n) \in (0, \varepsilon)$ definitively; i.e., for every $n > 1/\varepsilon$ and for every arbitrarily small $\varepsilon > 0$.*

This very simple example suggests the introduction of very important notions already familiar in the context of sets and functions as: boundedness, monotonicity and limits.

First of all we can apply the notion of bounded sets to the sequences; *i.e.*, to the set $(a_n)$. More precisely, there alway exist

$$\sup_n a_n := \sup\{a_n,\ \forall n \in \mathbb{N}\} \leq +\infty\,, \qquad \inf_n a_n := \inf\{a_n,\ \forall n \in \mathbb{N}\} \geq -\infty\,,$$

hence,

- we say that the sequence $(a_n)$ is bounded above if $\sup_n a_n \in \mathbb{R}$ or equivalently, there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for every $n \in \mathbb{N}$;
- we say that the sequence $(a_n)$ is bounded below if $\inf_n a_n \in \mathbb{R}$ or equivalently, there exists $m \in \mathbb{R}$ such that $a_n \geq m$ for every $n \in \mathbb{N}$;
- we say that the sequence $(a_n)$ is bounded if $\inf_n a_n\,, \sup_n a_n \in \mathbb{R}$ or equivalently, there exists $m, M \in \mathbb{R}$ such that $m \leq a_n \leq M$ for every $n \in \mathbb{N}$.

**Definition 3.1.2** (Monotone sequences)**.** *We say that a sequence $(a_n)$ is increasing if $a_n \leq a_{n+1}$ for very $n \in \mathbb{N}$ (or strictly increasing if $a_n < a_{n+1}$). We say that a sequence $(a_n)$ is decreasing if $a_n \geq a_{n+1}$ for very $n \in \mathbb{N}$ (or strictly decreasing if $a_n > a_{n+1}$). Sequences which are either increasing or decreasing sequences are called monotone (or strictly monotone).*

Let us now state the formal definition of convergence. We distinguish sequences whose elements approach a single point as $n$ increases from those sequences whose elements do not.

**Definition 3.1.3** (Convergence of a sequence)**.** *We say that a sequence $(a_n)$ converges if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a natural number $n_\varepsilon$ (depending on $\varepsilon > 0$) such that $a_n \in (L - \varepsilon, L + \varepsilon)$ or, equivalently, $|a_n - L| < \varepsilon$, for all $n \geq n_\varepsilon$.*

*It can be easily verified that if such a number $L$ exists then it is unique. In this case, we say that the sequence $(a_n)$ converges to $L$ and we call $L$ the limit of the sequence $(a_n)$ and we denotes it*

$$L := \lim_{n\to+\infty} a_n\,.$$

By Example 3.1.1 we have then the sequence $1/n$ converges to 0 since we may apply the Definition 3.1.3 by taking any natural number $n_\varepsilon > 1/\varepsilon$; hence,

$$\lim_{n\to+\infty} \frac{1}{n} = 0\,.$$

**Definition 3.1.4** (Divergence of a sequence)**.** *We say that a sequence $(a_n)$ diverges to $+\infty$ if for every $M > 0$, there exists a natural number $k$ such that $a_n > M$ for all $n \geq k$. Similarly, we say that a sequence $(a_n)$ diverges to $-\infty$ if for every $m < 0$, there exists a natural number $k$ such that $a_n < m$ for all $n \geq k$. In these cases, we write*

$$\lim_{n\to+\infty} a_n = +\infty\,, \qquad \lim_{n\to+\infty} a_n = -\infty\,,$$

*respectively.*

**Example 3.1.5.** The oscillating sequence $a_n = (-1)^n$ is bounded since $(a_n) = \{1, -1\}$ for every $n \in \mathbb{N}$, it is not monotone and it does not converges since it does not approach a single point as $n$ increase. Indeed, if there exists a limit $L = \lim_{n\to+\infty}(-1)^n$ then for a fixed $\varepsilon$ we should have by definition

$$L - \varepsilon \leq (-1)^n \leq L + \varepsilon\,, \quad \forall n \geq n_\varepsilon$$

and

$$a_{n+1} - a_n \leq \Big(L + \frac{1}{2}\Big) - \Big(L - \frac{1}{2}\Big) = 1\,, \quad \forall n \geq n_\varepsilon\,.$$

This is in general not true when $n$ is odd.

Finally, we note that

$$a_n = \begin{cases} 1 & n = 2k\,,\ k \in \mathbb{N} \\ -1 & n = 2k+1\,,\ k \in \mathbb{N}\,. \end{cases}$$

which means that from $a_n$ we can extract two sequences $b_k = a_{2k} = 1$ and $c_k = a_{2k+1} = -1$ and we call $b_k$ and $c_k$ subsequences of $a_n$. More in general, a *subsequences* of a sequence $a_n$ is a sequence defined as $b_k = a_{n_k}$ where $n_1 < n_2 < \cdots < n_k < \cdots$ with $k \in \mathbb{N}$.

The two subsequences $b_k$ and $c_k$ trivially converge to 1 and $-1$, respectively. Hence, $a_n$ is a bounded sequence that does not converge but we can extract two subsequences converging. This is true not only for this particular example, but in general!

**Theorem 3.1.6** (Bolzano-Weierstrass)**.** *Any bounded sequence $(a_n) \subset \mathbb{R}$ has a converging subsequence.*

The following theorem gives a necessary condition for the convergence of a sequence.

**Proposition 3.1.7.** *Every convergent sequence is a bounded sequence.*

*Proof.* Let us assume $\lim_{n\to+\infty} a_n = L$. Then, by Definition 3.1.3, we have that $|a_n| < L + \varepsilon$ for every $n \geq n_\varepsilon$; hence,

$$|a_n| < L + \varepsilon + M\,, \quad \forall n \in \mathbb{N}$$

with $M = \max\{|a_1|, \cdots, |a_{n_\varepsilon - 1}|\}$; *i.e.*, $(a_n)$ is bounded. □

**Definition 3.1.8** (Cauchy Sequences)**.** *We say that $(a_n)$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists natural number $n_\varepsilon$ such that*

$$|a_m - a_n| \leq \varepsilon\,, \qquad \forall m, n \geq n_\varepsilon\,.$$

**Theorem 3.1.9** (Cauchy Criterion)**.** *A sequence converges if and only if it is a Cauchy sequence.*

We will see soon an important application of the Criterion to the study of converging series.

### 3.1.1 Limits Theorem

**Theorem 3.1.10** (Operations of Limits- I)**.** *Suppose that $a_n \to a$ and $b_n \to b$ as $n \to +\infty$, then*

- $(a_n \pm b_n) \to a \pm b\,,$
- $a_n \cdot b_n \to a \cdot b\,,$
- *if* $b_n, b \neq 0$ *then* $a_n/b_n \to a/b.$

**Remark 3.1.11** (Indeterminate Forms)**.** If the limits are infinite or 0 they may give rise to indeterminate forms as:

$$\infty - \infty\,, \quad \frac{\infty}{\infty}\,, \quad \frac{0}{0}\,, \quad \infty \cdot 0$$

This does not means that the limits do not necessarily exist but their computation need a further investigate. In case of indeterminate forms of limits of function the De l'Hopital's theorem gives an instrument to compute the limits $0/0$ and $\infty/\infty$ and as a consequence of these two cases also $\infty \cdot 0$ and $\infty - \infty$. Thanks to the next Theorem 3.1.13 we may take advantage of that to study the indeterminate forms for sequences.

**Remark 3.1.12** (Operations of Limits- II)**.** Here we collect other cases that can not be included in the previous theorem but for which we still can predict the limit:

- if $a_n \to +\infty$ and $(b_n)$ is bounded below then $(a_n + b_n) \to +\infty\,,$
- if $a_n \to -\infty$ and $(b_n)$ is bounded above then $(a_n + b_n) \to -\infty\,,$
- if $a_n \to \pm\infty$ and $b_n \to L \neq 0$ then $(a_n \cdot b_n) \to \pm\infty$ according to the sign of infinity and of $L\,,$
- if $a_n \to \pm\infty$ then $(1/a_n) \to 0\,,$
- if $a_n \to \pm\infty$ and $b_n \to L$ then $(b_n/a_n) \to 0\,,$
- if $a_n \to 0$ then $1/|a_n| \to +\infty$. In particular, if $a_n > 0$ or $< 0$ then $1/a_n \to +\infty$ or $-\infty\,.$

**Theorem 3.1.13** (A bridge between limit of functions and of sequences)**.** *Let $L \in \mathbb{R} \cap \{+\infty, -\infty\}$ and $x_0 \in \mathbb{R} \cap \{+\infty, -\infty\}$ then*

$$\lim_{x \to x_0} f(x) = L$$

*if and only if for every $x_n \to x_0$ as $n \to +\infty$, $x_n \neq x_0$, we have that*

$$\lim_{n \to +\infty} f(x_n) = L\,.$$

**Example 3.1.14.** Theorem 3.1.13 is very useful if we have to compute limit of sequences $f(x_n)$ by knowing the limit of the function $f(x)$ as shown the following examples.

- The limit of
$$\lim_{n \to +\infty} e^{\sqrt{n}} = +\infty\,,$$
since $\lim_{n \to +\infty} \sqrt{n} = +\infty$ and $\lim_{x \to +\infty} e^x = +\infty$.

- The limit of
$$\lim_{n \to +\infty} n \sin\frac{1}{n} = 1\,,$$
since, $\lim_{n \to +\infty} 1/n = 0$ and $\lim_{x \to 0} \sin x/x = 1$. Note that, if we compute the limit as the product of the limits we would have an indeterminate form $\infty \cdot 0$ or, equivalently, $\infty/\infty$ by writing $n\,\sin(1/n) = \sin(1/n)/(1/n)$.

Less known application of the theorem is to prove the no existence of a limit of a function. Let us prove, for example, that

$$\nexists \lim_{x\to+\infty} \sin x\,.$$

We recall that Theorem 3.1.13 states that the $\lim_{x\to x_0} f(x) = L$ if and only $\lim_{n\to+\infty} f(x_n) = L$ for EVERY sequences $x_n \to x_0$. This implies that if we find two sequences $b_n$ and $c_n$ both converging to $x_0$ such that $\lim_{n\to+\infty} f(b_n) \neq \lim_{n\to+\infty} f(c_n)$ then we can conclude that the limit $\lim_{x\to x_0} f(x)$ does not exists!

Indeed, if we consider $b_n = 2\pi n$ and $c_n = \pi/2 + 2\pi n$, both they are diverging to $+\infty$ and

$$\lim_{n\to+\infty} \sin(2\pi n) = 0 \neq \lim_{n\to+\infty} \sin(\frac{\pi}{2} + 2\pi n) = 1\,.$$

**Theorem 3.1.15** (Sandwiches Theorem)**.** *Suppose that $a_n \le b_n \le c_n$ for every $n \in \mathbb{N}$ (or there exists $n_0$ such that the inequality is satisfied for every $n \ge n_0$). Then, if there exists $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = L \in \mathbb{R}$ then there exists also $\lim_{n\to+\infty} b_n = L$.*

**Theorem 3.1.16** (Comparison Test for sequences)**.** *Suppose that $a_n \le b_n$ for every $n \in \mathbb{N}$ (or there exists $n_0$ such that the inequality is satisfied for every $n \ge n_0$). Then,*

- *if $\lim_{n\to+\infty} a_n = +\infty$ then $\lim_{n\to+\infty} b_n = +\infty$ ;*
- *if $\lim_{n\to+\infty} b_n = -\infty$ then $\lim_{n\to+\infty} a_n = -\infty$ ;*
- *if $|a_n| \le b_n$ and $\lim_{n\to+\infty} b_n = 0$ then $\lim_{n\to+\infty} a_n = 0$.*

**Theorem 3.1.17** (Ratio test for sequences)**.** *Let $(a_n)$ be a sequence of real numbers such that $a_n > 0$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l\,.$$

*Then*

1. *if $l < 1$ then $\lim_{n\to+\infty} a_n = 0$ ,*
2. *if $l > 1$ then $\lim_{n\to+\infty} a_n = +\infty$ .*

*Proof.* If $l < 1$ then we can fix $\varepsilon$ such that $l + \varepsilon < 1$ and, for every $n \ge n_\varepsilon$, we have that

$$\frac{a_{n+1}}{a_n} \le (l+\varepsilon) < 1$$

which implies that

$$\begin{array}{rcl} a_{n_\varepsilon+1} & < & (l+\varepsilon)a_{n_\varepsilon} \\ a_{n_\varepsilon+2} & < & (l+\varepsilon)a_{n_\varepsilon+1} < (l+\varepsilon)^2 a_{n_\varepsilon} \\ & & \dots \\ 0 < a_{n_\varepsilon+k} & < & (l+\varepsilon)^k a_{n_\varepsilon}\,. \end{array}$$

Since $l + \varepsilon < 1$ then $\lim_{n\to+\infty}(l+\varepsilon)^k = 0$. By Theorem 3.1.16 we get that $\lim_{n\to+\infty} a_n = 0$.

If $l > 1$ then we can fix $\varepsilon$ such that $l - \varepsilon > 1$ and, for every $n \ge n_\varepsilon$, we have that

$$\frac{a_{n+1}}{a_n} \ge (l-\varepsilon) > 1\,.$$

Reasoning as above, we have that

$$a_{n_\varepsilon+k} > (l-\varepsilon)^k a_{n_\varepsilon}$$

for every $k \in \mathbb{N}$. By Theorem 3.1.16 we get that $\lim_{n\to+\infty} a_n = +\infty$. □

**Remark 3.1.18.** If $l = 1$ we can not predict the limit of $(a_n)$. Indeed, if we consider $a_n = n$ then $a_{n+1}/a_n \to 1$, as $n \to +\infty$, and $\lim_{n\to+\infty} a_n = +\infty$, while if we consider $a_n = 3 + 1/n$ then $a_{n+1}/a_n \to 1$ and $\lim_{n\to+\infty} a_n = 3$.

In the following proposition we study how fast some sequences approach infinity.

**Proposition 3.1.19** (Hierarchy of sequences)**.** *If $\alpha > 0$ and $a > 1$ then we have*

$$\lim_{n\to+\infty} \frac{\log_a n}{n^\alpha} = 0\,, \qquad \lim_{n\to+\infty} \frac{n^\alpha}{a^n} = 0\,.$$

*If $a > 0$ then we have*

$$\lim_{n\to+\infty} \frac{a^n}{n!} = 0\,, \qquad \lim_{n\to+\infty} \frac{n!}{n^n} = 0\,. \tag{3.1}$$

*Proof.* The proof of (3.1) is an application of the ratio test Theorem 3.1.17. □

**Example 3.1.20** (Indeterminate forms: $\infty^0$, $0^0$, $1^\infty$)**.** Another limit of sequence that can give rise to an indeterminate form is

$$\lim_{n\to+\infty} \sqrt[n]{n}\,.$$

This limit can be rewritten as $\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} f(n)^{g(n)}$ where $f(x) = x$ and $g(x) = 1/x$. We recall that if there exist finite $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ then $\lim_{x\to x_0} f(x)^{g(x)} = (\lim_{x\to x_0} f(x))^{(\lim_{x\to x_0} g(x))}$. If we apply directly Theorem 3.1.13 we get then an indeterminate form $\infty^0$.

Still we can use the same approach by rewriting

$$\sqrt[n]{n} = e^{\ln \sqrt[n]{n}} = e^{\frac{1}{n}\ln n}\,;$$

hence, by Theorem 3.1.13 and Proposition 3.1.19

$$\lim_{n\to+\infty} \sqrt[n]{n} = e^{\lim_{n\to+\infty} \frac{1}{n}\ln n} = e^0 = 1\,. \tag{3.2}$$

As a consequence of the previous result we can prove that

$$\lim_{n\to+\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{n}} = 1\,.$$

Note that in this case, reasoning as above, we could have an indeterminate form by applying directly Theorem 3.1.13 as following $\lim_{n\to+\infty} f(n)^{g(n)} = 0^0$ with $f(x) = 1/x$ and $g(x) = 1/x$.

Finally, we recall that the Euler number $e$ can be defined as limit of the sequence

$$e := \lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n. \tag{3.3}$$

The proof, that such sequence converges, is an interesting application of the convergence of monotone sequence (see Theorem 3.1.21 below) to the sequence $(a_{n-1}/a_n)$ after proving that $(a_n)$ is bounded; *i.e.*, $2 \leq a_n < 4$. Note that, reasoning as above, by a direct application of Theorem 3.1.13 the limit in (3.3) gives rise to the indeterminate form $1^\infty$.

More in general, if $\lim_{n\to+\infty} a_n = \pm\infty$ still we may prove that

$$e = \lim_{n\to+\infty} \Big(1 + \frac{1}{a_n}\Big)^{a_n}.$$

**Theorem 3.1.21** (Limits of monotone sequences)**.** *Suppose $(a_n)$ is a bounded and increasing sequence. Then there exists the limit of $(a_n)$ and*

$$\lim_{n\to+\infty} a_n = \sup_{n\in\mathbb{N}} a_n \in \mathbb{R}.$$

*If $(a_n)$ is not bounded above then $\lim_{n\to+\infty} a_n = +\infty$.*

*Suppose $(a_n)$ is a bounded and decreasing sequence. Then there exists the limit of $(a_n)$ and*

$$\lim_{n\to+\infty} a_n = \inf_{n\in\mathbb{N}} a_n \in \mathbb{R}.$$

*If $(a_n)$ is not bounded below then $\lim_{n\to+\infty} a_n = -\infty$.*

*Proof.* Suppose $\sup_{n\in\mathbb{N}} a_n = M$. Then for given $\varepsilon > 0$, there exists $n_\varepsilon$ such that $M - \varepsilon \leq a_{n_\varepsilon}$. Since $(a_n)$ is increasing, we have $a_{\leq} a_{n_\varepsilon}$ for all $n \geq n_\varepsilon$. This implies that $M - \varepsilon \leq a_n \leq M \leq M + \varepsilon$ for all $n \geq n_\varepsilon$. That is $\lim_{n\to+\infty} a_n = M$.

For decreasing sequences the proof is similar. □

**Example 3.1.22.** (1) Let $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$ for $n > 1$. Then use induction to see that $0 \leq a_n \leq 2$ and $(a_n)$ is increasing. Therefore, by previous result $(a_n)$ converges. Indeed, $a_2 = \sqrt{2 + \sqrt{2}}$ and $(a_2)^2 = 2 + \sqrt{2} < 4$; hence $a_2 < 2$. similarly, if $a_{n-1} < 2$ then $(a_n)^2 = 2 + a_{n-1} < 4$ which implies $a_n < 2$. Suppose $\lim_{n\to+\infty} a_n = L$. Then $\lim_{n\to+\infty} a_n = L = \sqrt{2 + (\lim_{n\to+\infty} a_{n-1})} = \sqrt{2 + L}$. This implies that $L = 2$.

(2) Prove that the

$$\lim_{n\to+\infty} \sqrt{n} - \sqrt{n+1} = 0.$$

The limit can not be computed by applying Theorem 3.1.10, it gives rise to the indeterminate form $\infty - \infty$. But, we can rationalised

$$\begin{aligned} \sqrt{n} - \sqrt{n+1} &= \frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n} + \sqrt{n+1}} \Big(\sqrt{n} + \sqrt{n+1}\Big) \\ &= \frac{n - (n+1)}{\sqrt{n} + \sqrt{n+1}} = \frac{1}{\sqrt{n} + \sqrt{n+1}} \to 0, \end{aligned}$$

as $n \to +\infty$.

(3) Prove that

$$\lim_{n\to+\infty} \frac{3^n + n}{n^3 + 1} = +\infty. \tag{3.4}$$

It is convenient to rewrite the sequence as follows

$$\begin{aligned}\frac{3^n+n}{n^3+1} &= \frac{3^n}{n^3+1}+\frac{n}{n^3+1}\\ &= \left(\frac{3^n}{n^3}\right)\left(\frac{n^3}{n^3+1}\right)+\frac{1}{\frac{n^3}{n}+\frac{1}{n}}\,.\end{aligned}$$

By Proposition 3.1.19 we have that $(3^n/n^3)\to+\infty$, moreover by Remark 3.1.12 we have

$$\frac{n^3}{n^3+1}=\frac{1}{1+\frac{1}{n^3}}\to 1\,,\qquad \frac{1}{\frac{n^3}{n}+\frac{1}{n}}=\frac{1}{n^2+\frac{1}{n}}\to 0\,,$$

which gives (3.4).

(4) Find the following limit of sequences

$$\lim_{n\to+\infty}\frac{n+2}{n^2+3n}\,.$$

It is convenient to rewrite the sequence as follows

$$\begin{aligned}0\le\frac{n+2}{n^2+3n} &= \frac{n}{n^2+3n}+\frac{2}{n^2+3n}\\ &= \left(\frac{1}{n+3}\right)+\left(\frac{2}{n^2+3n}\right);\end{aligned}$$

by Remark 3.1.12 we get that

$$\lim_{n\to+\infty}\frac{n+2}{n^2+3n}=0\,.$$

(5) Find the following limit of sequences

$$\lim_{n\to+\infty}\frac{3n^3+2n+1}{4n^4+3n^3+2}\,.$$

We divide the numerator and the denominator by $n^3$ then

$$\frac{3n^3+2n+1}{4n^4+3n^3+2} = \frac{3+\frac{2}{n^2}+\frac{1}{n^3}}{4n+3+\frac{2}{n^3}}\,.$$

By Remark 3.1.12 we get that

$$\lim_{n\to+\infty}3+\frac{2}{n^2}+\frac{1}{n^3}=3\,,\qquad \lim_{n\to+\infty}4n+3+\frac{2}{n^3}=+\infty\,,$$

and,

$$\lim_{n\to+\infty}\frac{3n^3+2n+1}{4n^4+3n^3+2}=0\,.$$

(6) Find the following limit of sequences

$$\lim_{n\to+\infty}\frac{3^n}{2^n+4^n}\,.$$

We divide the numerator and the denominator by $3^n$ then

$$\frac{3^n}{2^n+4^n}=\frac{1}{(\frac{2}{3})^n+(\frac{4}{3})^n}\,.$$

Since $2/3<1$ and $4/3>1$ we have that

$$\lim_{n\to+\infty}(\frac{2}{3})^n=0\,,\qquad \lim_{n\to+\infty}(\frac{4}{3})^n=+\infty\,.$$

Hence,

$$\lim_{n\to+\infty}(\frac{2}{3})^n+(\frac{4}{3})^n=+\infty\,,$$

and, by Remark 3.1.12 we have that

$$\lim_{n\to+\infty}\frac{1}{(\frac{2}{3})^n+(\frac{4}{3})^n}=0\,.$$

(7) Find the limit by varying the parameter $\alpha\in\mathbb{R}$ of the sequence

$$\lim_{n\to+\infty}\frac{n-1}{n^\alpha+2}\,.$$

We divide the numerator and denominator by $n$; hence,

$$\lim_{n\to+\infty}\frac{1-\frac{1}{n}}{n^{\alpha-1}+\frac{2}{n}}=\begin{cases}+\infty & \text{if } \alpha<1\\ 1 & \text{if } \alpha=1\\ 0 & \text{if } \alpha>1\,.\end{cases}\tag{3.5}$$

Indeed,

$$\lim_{n\to+\infty}\frac{2}{n}=\lim_{n\to+\infty}\frac{1}{n}=0\,,$$

hence, the limit in (3.5) depends on the sign of $(\alpha-1)$ since $\lim_{n\to+\infty}n^{\alpha-1}=+\infty,1,0$ if $(\alpha-1)>0,0,<0$, respectively.

## 3.2 Numerical series

Given a sequence of real numbers $(a_n)=\{a_0,a_1,a_2,\cdots,a_n,\cdots\}$ we associate to $(a_n)$ a recursive sequence $(s_n)$ defined as

$$\begin{aligned} s_0 &= a_0\\ s_{n+1} &= s_n+a_{n+1}\,,\quad n\in\mathbb{N}\,;\end{aligned}$$

that is,

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ s_2 &= a_0 + a_1 + a_2 \\ &\cdots \\ s_n &= a_0 + \cdots + a_n \,. \end{aligned}$$

The sequence $(s_n)$ is called *series* and it is denoted by

$$\sum_{n=0}^{\infty} a_n \,,$$

the single element of the sequence $s_n = \sum_{k=0}^{k=n} a_k$ is called *partial sums.* Then, $(s_n)$ is also called the sequence of partial sums.

**Definition 3.2.1.** *We say that a series is convergent if there exists finite the limit* $\lim_{n\to+\infty} s_n = s \in \mathbb{R}$*, it is called the sum of the series and it is denoted by*

$$s = \sum_{n=0}^{\infty} a_n \,.$$

*The series is divergent if* $\lim_{n\to+\infty} s_n = \pm\infty$*. The series does not converge if does not exists* $\lim_{n\to+\infty} s_n$*. The remainder associated to the series is defined by*

$$R_n := s - s_n = \sum_{k=n+1}^{+\infty} a_k \,.$$

Note that the series converges if and only if $\lim_{n\to+\infty} R_n = 0$.

**Example 3.2.2** (Geometric series)**.** The geometric series is defined as

$$\sum_{k=0}^{\infty} x^k \,,$$

for every $x \in \mathbb{R}$. Note that $x = x^{k+1}/x^k$; *i.e.*, $x$ represents a common ratio between all terms of the series that is constant. It is easy to prove that the partial sum is given by

$$s_n = \begin{cases} \dfrac{1 - x^{n+1}}{1 - x} & \text{if } x \neq 1 \\[2ex] n & \text{if } x = 1 \,. \end{cases}$$

We get then

$$\lim_{n\to+\infty} s_n = \begin{cases} \dfrac{1}{1-x} & \text{if } |x| < 1 \\[2ex] +\infty & \text{if } x \geq 1 \\[2ex] \nexists & \text{if } x \leq -1 \,; \end{cases} \tag{3.6}$$

that is,

- the geometric series converges for every $|x| < 1$ and $\sum_{n=0}^{\infty} x^n = 1/(x-1)$ ,
- the geometric series diverges for every $x \geq 1$ and $\sum_{n=0}^{\infty} x^n = +\infty$ .

**Theorem 3.2.3** (Cauchy Criterion)**.** *The series $\sum_{n=0}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$ and $p \in \mathbb{N}$ we have that*

$$\Big|\sum_{k=n}^{n+p} a_k\Big| \leq \varepsilon \,.$$

*Proof.* By Theorem 3.1.9 we have that the sequence $(s_n)$ converges if and only if it is a Cauchy sequence; *i.e.*, for every $\varepsilon > 0$ there exists natural number $n_\varepsilon$ such that

$$|s_m - s_l| \leq \varepsilon \,, \qquad \forall l, m \geq n_\varepsilon \,,$$

(see Definition 3.1.8 of Cauchy sequence). Without loss of generality we may always assume that $l = n - 1$ and $m = n + p$; hence, we get the thesis since $|s_m - s_l| = \Big|\sum_{k=n}^{n+p} a_k\Big|$. □

A very important consequence of the theorem is the following Corollary that provides a necessary condition for the convergence of a series.

**Corollary 3.2.4.** *If the series $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to+\infty} a_n = 0$.*

*Proof.* It is a straightforward consequence of Theorem 3.2.3 with $p = 0$. □

**Example 3.2.5.** The $\lim_{n\to+\infty} a_n = 0$ is not sufficient to guarantee that the associated series $\sum_{n=0}^{\infty} a_n$ converges. Indeed, let us consider

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

it is called harmonic series. Then if we choose $m = 2n$ we have that

$$s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq n\frac{1}{2n} = \frac{1}{2}$$

since we are summing up $n$-terms with the denominators $(n+1), (n+2), \cdots, (2n-1), 2n \leq 2n$. This proves that $(s_n)$ is not a Cauchy sequence and therefore it can not be convergent.

### 3.2.1 Series with non-negative terms

A series $\sum_{n=0}^{\infty} a_n$ with non-negative terms $a_n \geq 0$, $n \in \mathbb{N}$, it is always convergent or divergent since the sequence of partial sums $(s_n)$ is monotone increasing; *i.e.*,

$$s_{n+1} = s_n + a_{n+1} \geq s_n \,, \qquad \forall n \in \mathbb{N} \,;$$

hence, as already observed in Theorem 3.1.21, there always exists

$$\lim_{n\to+\infty} s_n = \sup_{n\in\mathbb{N}} s_n \leq +\infty \,.$$

In particular, a series with non-negative terms converges if and only if the sequence of partial sums $(s_n)$ is bounded.

Here below, we collect some methods that allow to study the divergence or convergence of a series with non negative terms.

**Theorem 3.2.6** (Integral test)**.** *Let $f : [1, +\infty) \mapsto \mathbb{R}$ be a positive, monotone decreasing function. Then*

$$\sum_{n=1}^{\infty} f(n)\,, \qquad \text{and} \qquad \int_1^{\infty} f(x)\,dx$$

*converge or diverge simultaneously.*

**Example 3.2.7** ($\alpha$-series)**.** We consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}\,, \quad \alpha > 0\,.$$

The function $1/x^{\alpha}$, $\alpha > 0$, satisfies the assumption of Theorem 3.2.6. Moreover, by (2.2), we have that

$$\int_1^{\infty} \frac{1}{x^{\alpha}}\,dx = \begin{cases} \dfrac{1}{\alpha - 1} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \le 1\,; \end{cases}$$

hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} = \begin{cases} \text{converges} & \text{if } \alpha > 1 \\ +\infty & \text{if } \alpha \le 1\,. \end{cases} \tag{3.7}$$

**Theorem 3.2.8** (Comparison test)**.** *Let $(a_n)$ and $(b_n)$ be nonnegative sequences and assume that there exists $N \in \mathbb{N}$ such that*

$$a_n \le b_n\,, \qquad n \ge N\,.$$

*Then, we have*

- *if $\sum_{n=0}^{\infty} a_n$ diverges then $\sum_{n=0}^{\infty} b_n$ diverges,*
- *if $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges.*

**Theorem 3.2.9** (Limit Comparison Test)**.** *Let $(a_n)$ be a nonnegative sequence, let $(b_n)$ be a positive sequence and assume that*

$$\lim_{n \to +\infty} \frac{a_n}{b_n} = L\,.$$

*Then, we have*

- *if $L \in (0, +\infty)$ then the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge or diverge simultaneously,*
- *if $L = 0$ then*

  $$\sum_{n=0}^{\infty} b_n < +\infty \Longrightarrow \sum_{n=0}^{\infty} a_n < +\infty$$

  *or*

  $$\sum_{n=0}^{\infty} a_n = +\infty \Longrightarrow \sum_{n=0}^{\infty} b_n = +\infty\,,$$

- *if* $L = +\infty$ *then*
$$\sum_{n=0}^{\infty} a_n < +\infty \Longrightarrow \sum_{n=0}^{\infty} b_n < +\infty$$
*or*
$$\sum_{n=0}^{\infty} b_n = +\infty \Longrightarrow \sum_{n=0}^{\infty} a_n = +\infty\,.$$

*Proof.* If $L \in (0, +\infty)$, then we may choose $\varepsilon > 0$ such that $L - \varepsilon > 0$; hence,

$$-\varepsilon < \left|\frac{a_n}{b_n} - L\right| < \varepsilon \quad \Longrightarrow \quad (L-\varepsilon)b_n < a_n < (L+\varepsilon)b_n\,, \qquad \forall n \geq n_\varepsilon\,.$$

If $L = 0$ then, by definition of limit, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$0 \leq \frac{a_n}{b_n} < \varepsilon \quad \Longrightarrow \quad 0 \leq a_n < \varepsilon b_n\,, \qquad \forall n \geq n_\varepsilon\,.$$

Finally, if $L = +\infty$ then, for every $M > 0$ there exists $k \in \mathbb{N}$

$$\frac{a_n}{b_n} > M \quad \Longrightarrow \quad a_n > M b_n\,, \qquad \forall n \geq k\,.$$

By Comparison Test 3.2.8, we get the thesis. □

**Example 3.2.10** (Bertrand series)**.** Let us consider

$$\sum_{n=2}^{\infty} \frac{1}{n^\alpha (\ln n)^\beta}\,, \quad \alpha, \beta \in \mathbb{R}\,. \tag{3.8}$$

- If $\alpha < 1$, we can write $\alpha = \gamma - \varepsilon$ with $\alpha < \gamma < 1$ and $\varepsilon > 0$; hence,
$$\frac{1}{n^\alpha (\ln n)^\beta} = \frac{n^\varepsilon}{n^\gamma (\ln n)^\beta}\,.$$
By Proposition 3.1.19, for every $\varepsilon > 0$ and $\beta \in \mathbb{R}$, we have that
$$\lim_{n\to+\infty} \frac{n^\varepsilon}{(\ln n)^\beta} = +\infty \implies \frac{n^\varepsilon}{(\ln n)^\beta} > 1\,, \quad \forall n \geq N\,,$$
which implies that
$$\frac{n^\varepsilon}{n^\gamma (\ln n)^\beta} \geq \frac{1}{n^\gamma}\,.$$
Since $0 < \gamma < 1$, by Comparison Test 3.2.8 and (3.7), we have that the series in (3.8) diverges for every $\alpha < 1$ and $\beta \in \mathbb{R}$.

- If $\alpha > 1$, we can write $\alpha = \gamma + \varepsilon$ with $1 < \gamma < \alpha$ and $\varepsilon > 0$; hence,
$$\frac{1}{n^\alpha (\ln n)^\beta} = \frac{1}{n^\gamma} \frac{1}{n^\varepsilon (\ln n)^\beta}\,.$$
Since
$$\lim_{n\to+\infty} \frac{1}{n^\varepsilon (\ln n)^\beta} = 0 \Longrightarrow \frac{1}{n^\varepsilon (\ln n)^\beta} < 1\,, \quad \forall n \geq N\,,$$

which implies that
$$\frac{1}{n^\gamma}\frac{1}{n^\varepsilon(\ln n)^\beta} \le \frac{1}{n^\gamma}\,.$$
Since $\gamma > 1$ by Comparison Test 3.2.8 and (3.7), we have that the series in (3.8) converges for every $\alpha > 1$ and $\beta \in \mathbb{R}$.

- If $\alpha = 1$ and $\beta \le 0$ then
$$\frac{(\ln n)^{-\beta}}{n} \ge \frac{(\ln 2)^{-\beta}}{n}$$
By Comparison Test 3.2.8 and (3.7), we have that the series in (3.8) diverges.

- If $\alpha = 1$ and $\beta > 0$ then we can not use comparison argument with $\alpha$-series. We apply then the Integral test since we can associate to the series a function $f : [2, +\infty) \mapsto \mathbb{R}^+$ defined as below
$$f(x) = \frac{1}{x(\ln x)^\beta}$$
that is continuous and decreasing . Now, by the change of variable $y = \ln x$ we have for every $L > 2$
$$\int_2^L \frac{1}{x(\ln x)^\beta}\,dx = \int_{\ln 2}^{\ln L} \frac{1}{y^\beta}\,dy = \begin{cases} \left.\dfrac{y^{-\beta+1}}{(-\beta+1)}\right|_{\ln 2}^{\ln L} & \text{if } \beta \neq 1 \\ \left.\ln y\right|_{\ln 2}^{\ln L} & \text{if } \beta = 1\,. \end{cases}$$
If $\beta > 1$ we have that the series in (3.8) converges since there exists the improper integral; *i.e.*,
$$\begin{aligned} \int_2^{+\infty} \frac{1}{x(\ln x)^\beta}\,dx &= \lim_{L\to+\infty} \frac{(\ln L)^{-\beta+1} - (\ln 2)^{-\beta+1}}{(-\beta+1)} \\ &= \frac{(\ln 2)^{1-\beta}}{(\beta-1)}\,. \end{aligned}$$
Since the series is with positive terms and it converges if and only if $\beta > 1$ we can conclude that it diverges for $0 < \beta \le 1$

**Example 3.2.11.** Determine whether the following series converges or diverges.

- $$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right).$$
Since
$$\lim_{n\to+\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1\,,$$
by the Limit Comparison Test 3.2.9, we have that the two series
$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right), \quad \sum_{n=1}^{\infty} \frac{1}{n}$$
do the same; *i.e.*, both they diverge.

- $$\sum_{n=1}^{\infty} \frac{1}{n(1+\sin n)}\,.$$

  Note that, this is a series with positive terms since $0 < 1 + \sin n < 2$, in particular we have that
  $$\frac{1}{n(1+\sin n)} \geq \frac{1}{2n}\,.$$
  By Comparison Test 3.2.8 and (3.7) we have that the series diverges.

- $$\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^4+4}}\,.$$

  We compare $(2n+1)/\sqrt{n^4+4}$ with $1/n$. Indeed,
  $$\begin{aligned} \lim_{n\to+\infty} \frac{\frac{2n+1}{\sqrt{n^4+4}}}{\frac{1}{n}} &= \lim_{n\to+\infty} \frac{2n^2+n}{\sqrt{n^4+4}} \\ &= \lim_{n\to+\infty} \frac{2+\frac{1}{n}}{\sqrt{1+\frac{4}{n^4}}} \\ &= 2\,. \end{aligned}$$

  By the Limit Comparison Test 3.2.9 we have that the series diverges.

**Theorem 3.2.12** (Root test)**.** *Let $(a_n)$ be a nonnegative sequence and assume that there exists $L \in [0,1)$ and $N \in \mathbb{N}$ such that*
$$\sqrt[n]{a_n} \leq L\,, \quad n \geq N\,,$$
*then, the series converges. If there exists $L > 1$ and $N \in \mathbb{N}$ such that*
$$\sqrt[n]{a_n} \geq L\,, \quad n \geq N\,,$$
*then, the series diverges to $+\infty$.*

As an immediate consequence we have that

**Corollary 3.2.13** (Limit Root test )**.** *Let $(a_n)$ be a nonnegative sequence and assume that*
$$\lim_{n\to+\infty} \sqrt[n]{a_n} = L\,.$$
*Then, we have*

- *if $L < 1$ then the series converges,*
- *if $L > 1$ then the series diverges.*

**Example 3.2.14.**

$$\sum_{n=0}^{\infty} \frac{3^n+5^n}{2^n+6^n}\,.$$

Note that

$$\sqrt[n]{\frac{3^n+5^n}{2^n+6^n}} = \left(\frac{5}{6}\right) \frac{\sqrt[n]{1+\left(\frac{3}{5}\right)^n}}{\sqrt[n]{1+\left(\frac{2}{6}\right)^n}}\,;$$

hence, if we pass to the limit

$$\lim_{n\to+\infty} \sqrt[n]{\frac{3^n+5^n}{2^n+6^n}} = \lim_{n\to+\infty} \left(\frac{5}{6}\right) \frac{\sqrt[n]{1+\left(\frac{3}{5}\right)^n}}{\sqrt[n]{1+\left(\frac{2}{6}\right)^n}} = \left(\frac{5}{6}\right).$$

By the Limit Root Test, since $(5/6) < 1$ we have that the series converges.

We can study the convergence of the series also by comparing it with suitable geometric series. Indeed,

$$\begin{aligned} \frac{3^n+5^n}{2^n+6^n} &= \frac{3^n}{2^n+6^n} + \frac{5^n}{2^n+6^n} \\ &\le \frac{3^n}{6^n} + \frac{5^n}{6^n} = \left(\frac{1}{2}\right)^n + \left(\frac{5}{6}\right)^n. \end{aligned}$$

Hence, by (3.6), we have that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{3^n+5^n}{2^n+6^n} &\le \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n + \left(\frac{5}{6}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n \\ &= 2+6\,. \end{aligned}$$

Note that in the previous formula we use the following general fact: if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge then also $\sum_{n=0}^{\infty}(a_n+b_n)$ converges and

$$\sum_{n=0}^{\infty}(a_n+b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n\,.$$

Could you prove it?

**Theorem 3.2.15** (Ratio test)**.** *Let $(a_n)$ be a positive sequence and assume that there exists $L \in (0,1)$ and $N \in \mathbb{N}$ such that*

$$\frac{a_{n+1}}{a_n} \le L\,, \quad n \ge N\,,$$

*then, the series converges. If*

$$a_{n+1} \ge a_n\,, \quad n \ge N\,,$$

*then, the series diverges to $+\infty$.*

As an immediate consequence we have that

**Corollary 3.2.16** (Limit Ratio test ). *Let $(a_n)$ be a positive sequence and assume that*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = L\,.$$

*Then, we have*

- *if $L < 1$ then the series converges,*
- *if $L > 1$ then the series diverges.*

**Example 3.2.17** (Case $L = 1$ in Limit Root and Ratio test). As observed, if $L = 1$ the series may be either divergent or convergent as the following examples show.

Let us consider

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

We already know that the series diverges for $\alpha \le 1$ and it converges for $\alpha > 1$ (see Example 3.7). Nevertheless, if we apply the limit root test, by (3.2), we have that

$$\lim_{n\to+\infty} \frac{1}{\sqrt[n]{n^{\alpha}}} = \lim_{n\to+\infty} \frac{1}{(\sqrt[n]{n})^{\alpha}} = 1\,, \qquad \forall \alpha\,.$$

**Example 3.2.18.** Let us consider

$$\sum_{n=1}^{\infty} \frac{1}{n!}\,.$$

By Limit Ratio test we have that

$$\lim_{n\to+\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to+\infty} \frac{n!}{(n+1)!} = \lim_{n\to+\infty} \frac{1}{(n+1)} = 0\,.$$

### 3.2.2 Conditional and absolute convergence

If a series $\sum_{n=0}^{\infty} a_n$ has both positive and negative terms $a_n$ then we can consider a new series by taking the absolute value of each terms; *i.e.*, $\sum_{n=0}^{\infty} |a_n|$ and we may ask if such series converges.

**Definition 3.2.19.** *If $\sum_{n=0}^{\infty} |a_n|$ converges then we say that $\sum_{n=0}^{\infty} a_n$ converges absolutely. If $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ does not, we say that $\sum_{n=0}^{\infty} a_n$ converges conditionally.*

**Theorem 3.2.20.** *If $\sum_{n=0}^{\infty} |a_n|$ converges then $\sum_{n=0}^{\infty} a_n$ is convergent; i.e., a series absolutely convergent is always conditionally convergent.*

*Proof.* For every $n, p \in \mathbb{N}$ we have that

$$|s_{n+p} - s_{n-1}| = \Big|\sum_{k=n}^{n+p} a_k\Big| \le \sum_{k=n}^{n+p} |a_k|\,;$$

hence, $\sum_{n=0}^{\infty} |a_n|$ converges, by the Cauchy Criterion 3.2.3, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$ and $p \in \mathbb{N}$ we have

$$|s_{n+p} - s_{n-1}| = \Big|\sum_{k=n}^{n+p} a_k\Big| \leq \varepsilon\,,$$

which implies that $\sum_{n=0}^{\infty} a_n$ converges. □

The Vice versa, it is not true as show the following example.

**Example 3.2.21.** Let us consider

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}\,.$$

Such series does not converge absolutely since $|(-1)^n/n| = 1/n$. Nevertheless we can prove that such series converge conditionally thanks to the following theorem.

**Theorem 3.2.22** (Leibniz Criterion)**.** *Suppose that $a_n > 0$ is a decreasing sequence which tends to zero as $n \to +\infty$. Then,*

- *the series $\sum_{n=1}^{\infty} (-1)^n\, a_n$ converges,*
- $|R_n| = |s - s_n| \leq a_{n+1}\,, \qquad \forall n \in \mathbb{N}\,.$

**Remark 3.2.23.** It is possible to prove that in particular

$$s_{2n+1} \leq s \leq s_{2n}\,, \quad \forall n \in \mathbb{N}\,,$$

this implies that we have more information on the approximate the sum of the series $s$ from below with $s_{2n+1}$ and from above with $s_{2n}$.

By Theorem 3.2.22 we can prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges conditionally. The Leibniz criterion is the only method that we are able to provide for series $\sum_{n=0}^{\infty} a_n$ that has both positive and negative terms $a_n$.

**Remark 3.2.24.** The geometric series $\sum_{n=0}^{\infty} x^n$ with $x \in \mathbb{R}$ is absolutely convergent for every $|x| < 1$. We can apply either the Ratio or the Root Test to prove it. Indeed,

$$\frac{|x|^{n+1}}{|x|^n} = |x|\,, \qquad \sqrt[n]{|x|^n} = |x|\,;$$

hence, by Theorems 3.2.15 and 3.2.12 we get that the series is absolutely convergent for every $|x| < 1$ and it is divergent for every for $x \geq 1$. Finally, for $x \leq -1$ the series $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (-1)^n |x|^n$ but the Liebniz Criterion does not apply and we can prove that the series neither converge nor diverge as already observed in (3.6).

**Example 3.2.25.** Say if the following series converges.

- $$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}\,.$$

  We apply the Comparison Test 3.2.8 to the series of the absolute value

  $$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty\,. \tag{3.9}$$

- $$\sum_{n=0}^{\infty} \frac{3+(-1)^n 2^n}{6^n}\,.$$

  As already observed in Example 3.2.14 if we prove that the two following series

  $$\sum_{n=0}^{\infty} \frac{3}{6^n}\,, \quad \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{6^n}$$

  are both converging then we may conclude that also (3.9) converges. Indeed,

  $$\sum_{n=0}^{\infty} \frac{3}{6^n} = 3\sum_{n=0}^{\infty} \Big(\frac{1}{6}\Big)^n$$

  that is a geometric series with ratio $1/6 < 1$, then it converges. By Leibniz Criterion also the second series is convergent

  $$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{6^n} = \sum_{n=0}^{\infty} (-1)^n \Big(\frac{1}{3}\Big)^n\,.$$

  since $(1/3)^n$ is a decreasing sequence, positive and converging to 0.

- $$\sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n}\,.$$

  Note that

  $$\sin(n\frac{\pi}{2}) = \begin{cases} 0 & \text{if} n = 2k \\ -1 & \text{if} n = 2k+1\,, \end{cases}$$

  for $k \in \mathbb{N}$. Hence, the series can be rewritten as

  $$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}\,,$$

  and it converges by the Leibniz criterion.

**Exercise 3.2.26.** Does the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^2+2n+5}\,, \qquad \sum_{n=1}^{\infty} (-1)^n \,\frac{n^2+2n+5}{2^n}$$

converges absolutely, converges conditionally or diverges?

**Solutions:**

- $$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n+3}{n^2+2n+5}\right| = \sum_{n=1}^{\infty}\frac{n+3}{n^2+2n+5}$$

  using the limit comparison test, comparing with $1/n$ we get the series diverge. Then the series does not converge absolutely.

  Using the alternating series test, we have to check that the sequence

  $$\frac{n+3}{n^2+2n+5}$$

  is decreasing. Then

  $$\frac{(n+1)+3}{(n+1)^2+2(n+1)+5} \leq \frac{n+3}{n^2+2n+5}$$

  if and only if

  $$\frac{(n+1)+3}{n+3} = 1+\frac{1}{n+3} \leq \frac{(n+1)^2+2(n+1)+5}{n^2+2n+5} = 1+\frac{2n+3}{n^2+2n+5}\,.$$

  Hence

  $$\frac{1}{n+3} \leq \frac{2n+3}{n^2+2n+5}$$

  if and only if $n^2+7n+4 \geq 0$, that is true for very $n \geq 1$. Moreover, it is easy to check that

  $$\lim_{n\to+\infty}\frac{n+3}{n^2+2n+5} = 0\,;$$

  hence, by Theorem 3.2.22 we have that the series converges conditionally.

- $$\sum_{n=1}^{\infty}\left|(-1)^n\ \frac{n^2+2n+5}{2^n}\right| = \sum_{n=1}^{\infty}\frac{n^2+2n+5}{2^n}\,.$$

  Using the ratio test we have

  $$\begin{aligned}\frac{\frac{(n+1)^2+2(n+1)+5}{2^{n+1}}}{\frac{n^2+2n+5}{2^n}} &= \left(\frac{2^n}{2^{n+1}}\right)\frac{(n+1)^2+2(n+1)+5}{n^2+2n+5} \\ &= \frac{1}{2}\left(1+\frac{2n+3}{n^2+2n+5}\right).\end{aligned}$$

  Then

  $$\lim_{n\to+\infty}\frac{1}{2}\left(1+\frac{2n+3}{n^2+2n+5}\right) = \frac{1}{2}\,,$$

  which gives the absolutely convergence of the series.

# Chapter 4

# Power series in $\mathbb{R}$

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Given a sequence of real numbers $(a_n)$ and a fixed $x_0 \in \mathbb{R}$ we defined a power series centered at $x_0$ with coefficients $(a_n)$ as

$$\sum_{n=0}^{+\infty} a_n(x-x_0)^n$$

with $x \in \mathbb{R}$. Note that, if the series converges then the sum of the series is given by a function

$$f(x) := \sum_{n=0}^{+\infty} a_n(x-x_0)^n$$

defined in the set $A = \{x \in \mathbb{R} \ : \sum_{n=0}^{+\infty} a_n(x-x_0)^n < +\infty\}$.

*Where else do we see such series?*

Example 3.2.2: the Geometric series ! Indeed, if we consider $a_n \equiv 1$ and $x_0 = 0$, the power series takes the form

$$\sum_{n=0}^{+\infty} x^n \, .$$

Of course, we may fix $a_n \equiv c$ for any fixed constant $c \in \mathbb{R}$ and $x_0 \neq 0$ and still we have a geometric series $\sum_{n=0}^{+\infty} c(x-x_0)^n = c\sum_{n=0}^{+\infty}(x-x_0)^n$ that can be solved as in Example 3.2.2 by replacing $x$ with $(x-x_0)$.

## 4.1 Convergence of a power series: the radius of convergence

We now want to study the convergence of the power series, we expect that it will depend on $x$.

First of all we observe that such series converges at least at $x = x_0$ since in this case $\sum_{n=0}^{+\infty} a_n(x_0-x_0)^n = a_0$. The following Theorem provides a characterisation of the set of convergence of the series.

**Theorem 4.1.1.** *Given $\sum_{n=0}^{+\infty} a_n(x-x_0)^n$. Then, one of the following statements holds true:*

*(1) the series converges only at $x = x_0$,*

*(2) the series converges absolutely for every $x \in \mathbb{R}$,*

*(3) there exists $R > 0$ such that the series converges absolutely for every $|x-x_0| < R$ and it does not converge for every $|x-x_0| > R$. For $|x-x_0| = R$ there is no general statement on the convergence of the series.*

*Proof.* Without loss of generality we may assume that $x_0 = 0$.This is always possible by a change of variable $y = x - x_0$. Suppose that there exists $x_1 \neq 0$ such that $\sum_{n=0}^{+\infty} a_n\, x_1^n$ converges. We now prove that the series converges absolutely for every $|x| < |x_1|$. Indeed, since $\sum_{n=0}^{+\infty} a_n\, x_1^n$ converges we have that $\lim_{n\to+\infty} a_n\, x_1^n = 0$ then it is bounded; *i.e.*, there exists $M > 0$ such that

$$|a_n\, x_1^n| \leq M\,, \quad \forall n \in \mathbb{N}\,.$$

Therefore, for every $n \in \mathbb{N}$

$$|a_n\, x^n| = |a_n x_1^n| \Big|\frac{x}{x_1}\Big|^n \leq M \left|\frac{x}{x_1}\right|^n,$$

and, by the comparison test, we have that

$$\Big|\frac{x}{x_1}\Big| < 1 \Longrightarrow \sum_{n=0}^{+\infty} \Big|\frac{x}{x_1}\Big|^n < +\infty\,, \Longrightarrow \sum_{n=0}^{+\infty} |a_n\, x^n| < +\infty\,.$$

We now set

$$R := \sup\Big\{|x| \in \mathbb{R} \ :\ \sum_{n=0}^{+\infty} a_n\, x^n < +\infty\Big\}\,.$$

Note that, the set is not empty since at least $x = 0$ belongs to it; hence, the definition of $R$ is well posed.

If $R = 0, +\infty$ then (1) and (2) holds true. If $R \in (0, +\infty)$ then the series converges absolutely for every $|x| < R$ and it does not converge not only absolutely but also conditionally for $|x| > R$, otherwise we will have a contraction by the definition of $R$ as supremum. Then also (3) holds true.

In the general case $x_0 \neq 0$, we define

$$R := \sup\Big\{|x-x_0| \in \mathbb{R} \ :\ \sum_{n=0}^{+\infty} a_n\, (x-x_0)^n < +\infty\Big\}\,,$$

and we reason as above by making a change of variable $y = x - x_0$. □

The radius of convergence can be computed by applying the Root and Ratio test for series.

**Proposition 4.1.2.** *Given $\sum_{n=0}^{+\infty} a_n(x-x_0)^n$ assume that there exists*

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} =: L \in [0, +\infty]\,.$$

*Then, the radius of convergence of the series is given by*

$$R = \begin{cases} \dfrac{1}{L} & \text{if } L \in (0, +\infty) \\ +\infty & \text{if } L = 0 \\ 0 & \text{if } L = +\infty\,. \end{cases} \tag{4.1}$$

*Proof.* For $x \neq x_0$ we have that

$$\lim_{n\to+\infty} \sqrt[n]{|a_n(x-x_0)^n|} = |x-x_0| \lim_{n\to+\infty} \sqrt[n]{|a_n|} = L\,|x-x_0|\,.$$

By the Root Test, the series converges absolutely if $L\,|x-x_0| < 1$; *i.e.*, for every $|x-x_0| < 1/L$ and it does not converge if $|x-x_0| > R = 1/L$. Then $R = 1/L$.

If $L = +\infty$, the series converges only for $x = x_0$ then $R = 0$. Finally, if $L = 0$ then the series converges for every $x \in \mathbb{R}$. $\square$

Similarly,

**Proposition 4.1.3.** *Given $\sum_{n=0}^{+\infty} a_n(x-x_0)^n$ assume that there exists*

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| =: L \in [0,+\infty]\,.$$

*Then, the radius of convergence of the series is given by*

$$R = \begin{cases} \dfrac{1}{L} & \textit{if } L \in (0,+\infty) \\ +\infty & \textit{if } L = 0 \\ 0 & \textit{if } L = +\infty\,. \end{cases} \tag{4.2}$$

**Example 4.1.4.** Find the interval of convergence of the following power series.

- $$\sum_{n=1}^{+\infty} \frac{(x+3)^{3n}}{8^n n^2}\,.$$

  The series is centered at $x = -3$. We study the absolutely convergence by applying the ratio test:

  $$\begin{aligned} \frac{\frac{|x+3|^{3(n+1)}}{8^{n+1}(n+1)^2}}{\frac{|x+3|^{3n}}{8^n n^2}} &= \frac{|x+3|^{3(n+1)}}{8^{n+1}(n+1)^2}\cdot\frac{8^n n^2}{|x+3|^{3n}} \\ &= \frac{|x+3|^3}{8}\cdot\frac{n^2}{(n+1)^2} \end{aligned}$$

  then

  $$\lim_{n\to+\infty}\frac{\frac{|x+3|^{3(n+1)}}{8^{n+1}(n+1)^2}}{\frac{|x+3|^{3n}}{8^n n^2}} = \frac{|x+3|^3}{8}\,.$$

  If $|x+3| < 2$ the series converges absolutely, then $R = 2$. Let us study what happens on the extremes of the interval; *i.e.*, $|x+3| = 2$. If we evaluate the serie $x = -5$ then

  $$\sum_{n=1}^{+\infty}(-1)^n\,\frac{2^{3n}}{8^n n^2} = \sum_{n=1}^{+\infty}(-1)^n\,\frac{8^n}{8^n n^2} = \sum_{n=1}^{+\infty}(-1)^n\,\frac{1}{n^2}$$

  that is a series absolutely convergent. If we evaluate the series at $x = -1$ then

  $$\sum_{n=1}^{+\infty}\frac{1}{n^2}$$

  which is again convergent. Hence, the power series converges absolutely in $[-5,-1]$.

- $$\sum_{n=1}^{+\infty} \frac{x^n}{n!}\,.$$

  By Ratio test we have that

  $$\lim_{n\to+\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to+\infty} \frac{1}{(n+1)} = 0\,.$$

  Hence, by (4.2), we have that $R = +\infty$; *i.e.*, the series converges absolutely for every $x \in \mathbb{R}$.

- $$\sum_{n=1}^{+\infty} \frac{x^n}{n}\,.$$

  By Root test we have that

  $$\lim_{n\to+\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{n}} = 1\,.$$

  Hence, by (4.1), we have that $R = 1$; *i.e.*, the series converges absolutely for every $|x| < 1$. If $|x| = 1$ then we have

  $$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} < +\infty\,, \qquad \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty\,,$$

  which implies that the power series converges in the interval $[-1, 1)$.

- For $\alpha \in \mathbb{R}$

  $$\sum_{n=1}^{+\infty} n^\alpha\, x^n\,.$$

  We recall that

  $$\lim_{n\to+\infty} \frac{n^\alpha}{a^n} = 0\,, \quad \forall \alpha \in \mathbb{R}\,, a > 1\,;$$

  hence, the necessary condition for the convergence of the series is satisfied only if $|x| < 1$ for every $\alpha \in \mathbb{R}$ or if $|x| = 1$ and $\alpha < 0$. Moreover,

  $$\lim_{n\to+\infty} \sqrt[n]{n^\alpha} = 1\,, \quad \forall \alpha \in \mathbb{R}\,;$$

  hence, by Proposition 4.1.2 we have that $R = 1$ and the series converges absolutely for $|x| < 1$ and every $\alpha \in \mathbb{R}$. If $x = 1$ the series converges for $\alpha < -1$; if $x = -1$, by Leibniz Criterion, the series converges for every $\alpha < 0$.

  Sumarising, we have that the series converges in $(-1, 1)$ for every $\alpha \in \mathbb{R}$, in $[-1, 1]$ for every $\alpha < -1$, in $[-1, 1)$ for every $\alpha \in [-1, 0)$.

## 4.2 Taylor series

The Taylor series of a function $f$ that is infinitely differentiable at a point $x_0$ is a power series centered at $x_0$ with coefficients $f^{(n)}(x_0)/n!$; *i.e.*,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)}{2}(x-x_0)^2 + \frac{f^3(x_0)}{3!}(x-x_0)^3 + \frac{f^4(x_0)}{4!}(x-x_0)^4 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + ....$$

where $f^{(n)}(x_0)$ denotes the $n$-derivative of the function $f$ at $x_0$ and $n! = 1\cdot 2\cdot 3\cdot 4\cdots(n-1)\cdot n$ $(0! = 1)$. Many questions arise naturally!

(1) What about the convergence of the series?

(2) The series converges to the function $f$?

(3) What about functions that are not infinitely differentiable but only $n$-times differentiable?

We will start by answering to *Question 3* !

### 4.2.1 Taylor polynomials

First of all we prove that if a function is $n$-differentiable at a point $x_0$ it can be approximated by (Taylor) polynomials! How? How do we determine the accuracy when we use a (Taylor) polynomial to approximate a function?

Let's start by the case $n = 1$!

Let $f : I \mapsto \mathbb{R}$ be a function differentiable in $x_0 \in I$ then we can state the equation of the tangent line at the point $(x_0, f(x_0))$ as $y = f(x_0) + f'(x_0)(x-x_0)$ where $f'(x_0)$ is the angular coefficient. The tangent line is in particular a polynomial of degree 1 that we can denote by

$$P_1(x; x_0) := f(x_0) + f'(x_0)(x-x_0)\,,$$

which satisfies the following property

$$\lim_{x\to x_0} \frac{f(x) - P_1(x;x_0)}{(x-x_0)} = 0\,.$$

Indeed, if we denote by $R_1(x;x_0) := f(x) - P_1(x;x_0)$; *i.e.*, the remainder between the function and the tangent line at $(x_0, f(x_0))$, then by definition of derivative we have that

$$\lim_{x\to x_0} \frac{R_1(x;x_0)}{(x-x_0)} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{(x-x_0)} - f'(x_0) = 0\,. \tag{4.3}$$

Formula (4.3) says that *the remainder* $R_1(x;x_0)$ *goes to zero, as* $x \to x_0$, *faster than* $(x-x_0)$ which gives us an important information on the capacity to approximate the graph of the function $f$ with the tangent line at the graph around $(x_0, f(x_0))$; *i.e.*, with the polynomial $P_1(x;x_0)$.

We need a language to describe the behaviour of a function that is vanishing as $x \to x_0$.

**Definition 4.2.1.** *Given two functions $f$ and $g$ defined in an interval $I$ with $x_0 \in I$ such that $f(x_0) = g(x_0) = 0$, we say that $f$ is a "small o" of $g$ and we write*

$$f(x) = o(g(x)) \iff \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0\,;$$

*that is, the function $f$ approaches $0$, as $x \to x_0$, faster than $g$.*

Hence, by Definition 4.2.1 we have that $R_1(x;x_0) = o(x - x_0)$ and

$$\begin{aligned} f(x) &= P_1(x;x_0) + o(x-x_0) \\ &= f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)\,. \end{aligned}$$

**Theorem 4.2.2** (Taylor Formula)**.** *If $f : I \mapsto \mathbb{R}$ has $n$-derivatives on an open interval $I$ that contains $x_0$, we define, for every $x \in I$, the Taylor polynomial $P_n(x;x_0)$ and the remainder $R_n(x;x_0)$ as*

$$P_n(x;x_0) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\,, \quad R_n(x;x_0) := f(x) - P_n(x;x_0)\,.$$

*Then,*

$$\lim_{x\to x_0} \frac{R_n(x;x_0)}{(x-x_0)^n} = 0\,,$$

*or, equivalently,*

$$R_n(x;x_0) = o((x-x_0)^n)\,.$$

As a consequence of the theorem above we have that

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f^2(x_0)}{2}(x-x_0)^2+\frac{f^3(x_0)}{3!}(x-x_0)^3+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o((x-x_0)^n)$$

and the error we incur by approximating $f$ with a polynomial $P_n(x;x_0)$ near $x_0$ is given by $R_n(x;x_0)$ that goes to zero faster than $(x-x_0)^n$, as $x \to x_0$.

**Corollary 4.2.3** (Lagrange Formula for the remainder)**.** *If $f$ has $(n+1)$-derivatives on an open interval $I$ that contains $x_0$ then the remainder in Taylor's theorem can be written as*

$$R_n(x;x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

*with $\xi \in \mathbb{R}$ between $x$ and $x_0$.*

**Remark 4.2.4** (Maclaurin Formula)**.** If $x_0 = 0$ the Taylor formula is also called Maclaurin Formula.

**Example 4.2.5.**
- There functions that we can not keep taking derivatives as $f(x) = |x|^3$ that is differentiable only two times at $x_0 = 0$:

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{|x|^3}{x} = 0 \\ f^2(0) &= \lim_{x\to 0} 3|x| = 0 \\ &\nexists f^3(0)\,. \end{aligned}$$

- The Taylor Polynomial for the exponential function $f(x) = e^x$ with $x_0 = 0$ is given by

$$P_n(x:0) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \tag{4.4}$$

Since $e^x$ is infinitely differentiable we can use the Lagrange formula for the remainder

$$R_n(x;0) = e^{\xi}\frac{x^{n+1}}{(n+1)!} \qquad \Longrightarrow \qquad \Big|R_n(x;0)\Big| = e^{\xi}\frac{|x|^{n+1}}{(n+1)!} \leq M\,\frac{|x|^{n+1}}{(n+1)!}$$

where $M = e^x$ for every fixed $x > 0$ ( since $0 < \xi < x$) or $M = 1$ for every fixed $x < 0$. It follows then that $\lim_{n\to+\infty} R_n(x;0) = 0$ which implies the convergences of the series

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

for every fixed $x \in \mathbb{R}$ as already showed in Examples 4.1.4 (see also Definition 3.2.1). Therefore we can approximate $e^x$ as closely we wish by Taylor polynomials, in particular,

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}\,.$$

### 4.2.2 From the Taylor polynomials to the Taylor series

In Section 4.2.1 we have computed the Taylor polynomials for a differentiable function and we have used it to approximate that function. We have also observed that if the function is infinitely differentiable we can compute a Taylor series and since it is in particular a power series, we know how to study its convergence (see Section 4.1). By the general theory developed in Section 3.2 for numerical series we recognise the Taylor polynomials as the partial sums of the Taylor series. This answers to *Question 1* !

Now, we can address also *Question 2*:

*When does a function equal its Taylor series?*

The function $f(x)$ will equal the Taylor series in $x$ provided the remainder $R_n(x;x_0)$ goes to zero as $n \to +\infty$; *i.e.*, for every fixed $x$ "near $x_0$" ( and of course in the domain of $f$)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n \qquad \Longleftrightarrow \qquad \lim_{n\to+\infty}\Big(f(x) - P_n(x;x_0)\Big) = 0\,.$$

**Example 4.2.6** (VIE: very important example)**.** Let us consider the following function defined on $\mathbb{R}$

$$f(x) = \begin{cases} e^{-1/x^2}\,, & x \neq 0 \\ 0\,, & x = 0\,. \end{cases}$$

The function $f$ is infinitely differentiable in $x = 0$. Indeed, by applying L' Hopital rules and the hierarchy between power and exponential, we have that

$$f'(0) = \lim_{x\to 0}\frac{e^{-1/x^2}}{x} = \lim_{x\to 0}\frac{2}{x^3}e^{-1/x^2} = 0\,.$$

If we continue to derive we have that $f^n$ is a sum of terms of the type $e^{-1/x^2}/x^h$ (up to some constants), which implies that passing into the limit as $x \to 0$ we always get $f^n(0) = 0$! Therefore, the Taylor series of $f$ at $x_0 = 0$ is identically zero and then, trivially, it converges for every $x$ "near to 0" but the function $f$ equals the Taylor series only at $x = 0$!

We conclude this section by saying:

*What can go wrong?*

(1) For some functions the derivatives exist at $x = x_0$ but are too big (they grow too quickly)...Imagine if $f^n(0) = (n!)^2$ then $f^n(0)/n!n!$ and the Taylor series around 0 is
$$\sum_{n=0}^{\infty} n!x^n$$
which has a radius of convergence $R = 0$. There exists such a function but the formula is quite complicated....

(2) Example 4.2.6 shows that for some functions Taylor series converges but not to the function itself! In this case the polynomials are not the right object to describe around 0 an exponential like $e^{-1/x^2}$ that probably decreases too fast near to zero ...

**Example 4.2.7.** • Let's study the Taylor series of the sine function $f(x) = \sin x$. We recall that

$$f(x) = \sin x\,, \quad f'(x) = \cos x\,, \quad f^2(x) = -\sin x\,, \quad f^3(x) = -\cos x\,,$$
$$f^4(x) = \sin x\,, \quad \cdots$$

From the derivative of order 4 the pattern repeats itself then if we consider $x_0 = 0$ we have that the even derivative are always zero at 0 and the odd derivatives are alternatively 1 and $-1$
$$f(0) = 0\,, \quad f^2(0) = 0\,, \quad f^4(0) = 0\,,$$
$$f'(0) = 1\,, \quad f^3(0) = -1\,, \quad f^5(0) = 1\,, \cdots.$$

Therefore in the Taylor polynomials only odd powers appear and this is not surprising since the sine function is and odd function and it is infinitely differentiable; hence, for every $n \in \mathbb{N}$ we have that
$$\sin x = \sum_{h=0}^{n} (-1)^h \frac{x^{2h+1}}{(2h+1)!} + (-1)^{(n+1)} \frac{x^{2n+3}}{(2n+3)!} \cos\xi\,.$$

Since
$$\lim_{n\to+\infty} |R_{2n+1}(x;0)| \le \lim_{n\to+\infty} \frac{|x|^{2n+3}}{(2n+3)!} = 0\,,$$
we have that the Taylor series converges to the sine function
$$\sin x = \sum_{h=0}^{+\infty} (-1)^h \frac{x^{2h+1}}{(2h+1)!}\,.$$

Similarly, we can prove that

$$\cos x = \sum_{h=0}^{n} (-1)^h \frac{x^{2h}}{(2h)!} + (-1)^{(n+1)} \frac{x^{2n+2}}{(2n+2)!} \cos \xi \,;$$

hence,

$$\lim_{n\to+\infty} |R_{2n}(x;0)| \le \lim_{n\to+\infty} \frac{|x|^{2n+2}}{(2n+2)!} = 0\,,$$

and

$$\cos x = \sum_{h=0}^{+\infty} (-1)^h \frac{x^{2h}}{(2h)!}\,.$$

- Let us consider

$$f(x) = \frac{1}{1-x}\,,$$

$$\begin{aligned} f(x) &= \frac{1}{1-x}\,, \quad f'(x) = \frac{1}{(1-x)^2}\,, \quad f^2(x) = \frac{2}{(1-x)^3}\,, \quad f^3(x) = \frac{3!}{(1-x)^4}\,, \\ & \cdots f^n(x) = \frac{n!}{(1-x)^n}\,; \end{aligned}$$

hence, there exists $\xi$ between 0 and $x$ such that

$$\frac{1}{1-x} = \sum_{h=0}^{n} x^h + \frac{x^{n+1}}{(1-\xi)^{n+1}}\,.$$

Note that, the remainder can be also rewritten in the following form

$$\begin{aligned} R_n(x;0) &= \frac{1}{1-x} - \sum_{h=0}^{n} x^h = \frac{1 - \sum_{h=0}^{n} x^h (1-x)}{1-x} \\ &= \frac{1 - \sum_{h=0}^{n} (x^h - x^{h+1})}{1-x} \\ &= \frac{1 - (1 - x + x - x^2 + x^2 - \cdots + x^n - x^{n+1})}{1-x} \\ &= \frac{x^{n+1}}{1-x}\,. \end{aligned}$$

Hence, if $|x| < 1$ we have that $\lim_{n\to+\infty} R_n(x;0) = 0$ and we have the convergence of the series

$$\frac{1}{1-x} = \sum_{h=0}^{+\infty} x^n\,, \qquad \forall |x| < 1\,,$$

which agrees with the analysis that we performed for the geometric series in Example 3.2.2.

- We come now to the logarithm function

$$f(x) = \ln(1+x)$$

which is defined in $(-1, +\infty)$ and on that interval has derivatives of all orders:

$$\begin{aligned} f'(x) &= \frac{1}{1+x}\,, \quad f^2(x) = -\frac{1}{(1+x)^2}\,, \quad f^3(x) = \frac{2}{(1+x)^3}\,, \quad f^4(x) = -\frac{3!}{(1+x)^4}\,, \\ f^5(x) &= \frac{4!}{(1+x)^5}\,, \cdots \end{aligned}$$

The pattern is clear

$$f^h(x) = (-1)^{h-1}\frac{(h-1)!}{(1+x)^h}\,, \quad \forall k \geq 1\,, \Longrightarrow \quad f^h(0) = (-1)^{h-1}(h-1)!\,.$$

Therefore,

$$\ln(1+x) = \sum_{h=1}^{n} \frac{(-1)^{h-1}}{h}\,x^h + \frac{(-1)^n}{(n+1)}\frac{x^{n+1}}{(1+\xi)^{n+1}}\,.$$

If $0 < \xi < x$ then $1/(1+\xi) < 1$ and

$$\lim_{n\to+\infty} |R_n(x;0)| \leq \lim_{n\to+\infty} \frac{|x|^{n+1}}{(n+1)} = 0\,, \quad \forall\; 0 < x < 1\,.$$

Actually, we can prove that the remainder by Lagrange formula goes to zero, as $n \to +\infty$, for every $-1/2 < x < 1$. To recover $(-1, 1)$ we have to apply another formula that allows us to represent the remainder term in integral form. Indeed, by the general theory of power series, we can prove that the radius of convergence of the Taylor series is $R = 1$ and it is possible to prove (by using the integral formula of the remainder) that the sum of series egual the function; *i.e.*,

$$\ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}\,x^n\,, \quad \forall |x| < 1\,.$$

# Chapter 5

# Ordinary differential equations

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An ordinary differential equation is an equation that relates an unknown function, $y$, to one or more of its derivatives $y', y'', \cdots, y^n$; *i.e.*,

$$F(t, y, y', y'', \cdots, y^n) = 0\,; \tag{5.1}$$

where $F : A \mapsto \mathbb{R}$ is a given function of $(n+2)$ variables $(t, y, z_1, \cdots, z_n) \in A \subset \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$, $y : I \mapsto \mathbb{R}$ and $t \in I$. Solving the equation (5.1) means to find an interval $I \subset \mathbb{R}$ and a $n$-times derivable function $y : I \mapsto \mathbb{R}$ such that $(t, y(t), y'(t), y''(t), \cdots, y^n(t)) \in A$ for every $t \in I$ and $F(t, y(t), y'(t), y''(t), \cdots, y^n(t)) = 0$. The function $y$ is said a solution to the ordinary differential equation. An ordinary differential equation may have infinitely many solutions, or finitely many solutions or even no solutions at all. We define the general integral of the equation as the following set (which may be infinite, finite, a singleton or empty)

$$\mathcal{I} = \{y : I \mapsto \mathbb{R} \mid I \subset \mathbb{R}\,, \quad y \ \text{ is a solution to o.d.e.}\}$$

In other words the general integral is the set of all solutions of the equation, everyone defined on its interval of definition. The equation (5.1) is in the so-called non-normal form, that is the highest order derivative, $y^n$, is not "isolated" with respect to the other derivatives.

In this chapter we will focus on first and second-order partial differential equations ($n = 1, 2$) in normal form.

## 5.1 First-order ODE

**Definition 5.1.1.** *A first-order ODE, in normal form, on the unknown $y$ and $f : A \mapsto \mathbb{R}$ given, is*

$$y'(t) = f(t, y)\,. \tag{5.2}$$

*Solving the equation (5.2) means to find an interval $I \subset \mathbb{R}$ and a derivable function $y : I \mapsto \mathbb{R}$ such that $(t, y(t), y'(t)) \in A$ and $y'(t) = f(t, y(t))$ for every $t \in I$. The function $y$ is said a solution to the ordinary differential equation.*

*We define the general integral of the equation as the following set (which may be infinite, finite, a singleton or empty)*

$$\mathcal{I} = \{y : I \mapsto \mathbb{R} \mid I \subset \mathbb{R}\,, \quad y \quad \textit{is a solution to } (5.2)\}\,.$$

### 5.1.1 ODE by separation of variables

A very important class of ODE is the one with the $f$ given by

$$f(t, y) = a(t)\, g(y)\,,$$

where $a : I \mapsto \mathbb{R}$ and $g : J \mapsto \mathbb{R}$ are continuous functions in $I$ and $J$ (intervals of $\mathbb{R}$), respectively. Given $(t_0, y_0) \in I \times J$ we can also defined the initial value problem, the so-called Cauchy Problem

$$\begin{cases} y'(t) = a(t)\, g(y)\,, \\ y(t_0) = y_0\,, \end{cases}$$

where $y(t_0) = y_0$ represents the initial condition.

If $g(y_0) = 0$ then $y(t) \equiv y_0$ is a solution for every $t \in I$ to the Cauchy Problem. If $g(y_0) \neq 0$ then we can solve the differential equation by separation of variables; *i.e.*, we first rewrite the equation as

$$\frac{y'(t)}{g(y(t))} = a(t)\,,$$

then we integrate with respect to the variable $t$ on both sides of the equation and we get an integral equation

$$\int \frac{y'(t)}{g(y(t))}\, dt = \int a(t)\, dt\,. \tag{5.3}$$

In particular,

$$\int \frac{y'(t)}{g(y(t))}\, dt = \int \frac{1}{g(s)}\, ds\,,$$

hence,

$$\int \frac{1}{g(s)}\, ds = \int a(t)\, dt\,.$$

To find the general integral of the equation we have to solve the two indefinite integrals. To find the solution of the Cauchy Problem we can evaluate the general integral in $t_0$ and fix the constant (of the indefinite integrals) by imposing that $y(t_0) = y_0$. In alternative, we can use directly the initial condition in the integral equation (5.3) by fixing one of the extremes of integration; *i.e.*,

$$\int_{t_0}^{t} \frac{y'(\tau)}{g(y(\tau))}\, d\tau = \int_{t_0}^{t} a(\tau)\, d\tau \quad \Longrightarrow \quad \int_{y_0}^{y(t)} \frac{1}{g(s)}\, ds = \int_{t_0}^{t} a(\tau)\, d\tau\,.$$

If $a(t) \equiv 1$ then we have an autonomous differential equation; *i.e.*,

$$y'(t) = g(y)$$

and the rate of change of $y(t)$, the derivative $y'(t)$, does not depends explicitly on $t$. This implies that given a solution $y$ to the Cauchy Problem

$$\begin{cases} y'(t) = g(y)\,, \\ y(0) = y_0\,, \end{cases}$$

then, any other solution to

$$\begin{cases} \bar{y}'(t) = g(\bar{y})\,, \\ \bar{y}(t_0) = y_0\,, \end{cases}$$

it can be obtained by translating $y$ in the variable $t$ as follows $\bar{y}(t) := y(t - t_0)$. Indeed, $\bar{y}'(t) = y'(t - t_0) = g(y(t - t_0)) = g(\bar{y}(t))$ and satisfies the initial condition $\bar{y}(t_0) = y(0) = y_0$ as show the following example.

**Example 5.1.2.** Solve

$$\begin{cases} y'(t) = y^2\,, \\ y(0) = 1\,, \end{cases}$$

By separation of variables we have from one side that

$$\int_0^t \frac{y'(t)}{y^2(t)}\,dt = \int_1^{y(t)} \frac{1}{s^2}\,ds = 1 - \frac{1}{y(t)}\,.$$

On the other side, we have

$$\int_0^t dt = t \quad \Longrightarrow \quad 1 - \frac{1}{y(t)} = t \quad \Longrightarrow \quad y(t) = \frac{1}{1-t} \quad \forall t \in [0,1).$$

Note that in this case the solution exists and is unique on the bounded interval $[0, 1)$ even if the function $a(t) \equiv 1$ and $g(y) = y^2$ are both defined on the whole real line $I = \mathbb{R}$ and $J = \mathbb{R}$. This example suggests that, in general, the interval of existence of a solution is contained in $I$ and $(t, y(t))$ may take values in a subset strictly contained in $I \times J$.

Moreover, this equation ia an example of autonomous differenatial equation; hence, we expect the phenomena described above; *i.e.*, if we consider $t_0 < 1$ with $\bar{y}(t_0) = 1$ we get that the solution $\bar{y}$ is defined on $[t_0, 1)$ and is given by

$$\bar{y}(t) = \frac{1}{1-(t-t_0)} \quad \forall t \in [t_0, 1)\,. \tag{5.4}$$

Indeed,

$$\int_{t_0}^t \frac{\bar{y}'(t)}{\bar{y}^2(t)}\,dt = \int_1^{\bar{y}(t)} \frac{1}{s^2}\,ds = 1 - \frac{1}{\bar{y}(t)} = (t - t_0)\,.$$

Hence, we get (5.4).

Finally, if $y_0 = 0$ then $y \equiv 0$ is a solution on $\mathbb{R}$!

**Theorem 5.1.3** (Existence and uniqueness of solutions to equation with separation of variables)**.** *Let $a : I \mapsto \mathbb{R}$ and $g : J \mapsto \mathbb{R}$ be two continuous functions in the intervals $I$ and $J$ with $J$ open. Then, for every $(t_0, y_0) \in I \times J$ the Cauchy Problem*

$$\begin{cases} y'(t) = a(t)\,g(y)\,, \\ y(t_0) = y_0\,, \end{cases}$$

*admits at least a solution. If $g$ is locally Lipschitz (for example $g \in C^1$) then the solution is also unique in the maximal interval of existence.*

Let's see some other examples.

**Example 5.1.4.**

- Solve
$$y'(t) = \Big(\frac{2}{t}\Big) y(t)\,.$$
By separation of variables we have that
$$\int \frac{y'(t)}{y(t)}\,dt = \int \frac{2}{t}\,dt\,, \quad \Longrightarrow \ln|y(t)| = \ln t^2 + c_0$$
where the constant $c_0 \in \mathbb{R}$ is due to the computation of two indefinite integrals. Then we get the general integral of the equation
$$y(t) = \pm e^{c_0}\,t^2 \quad \Longrightarrow y(t) = c\,t^2\,, \quad c \in \mathbb{R}\,.$$
If we have also an initial condition then we can fix the constant $c$ by evaluating the general integral at $t_0$. For example, if fix the initial condition $y(-1) = 2$ then $c = 2$ and the unique solution to the Cauchy Problem is $y(t) = 2\,t^2$.

- Solve
$$y'(t) = \sqrt{y(t)}\,.$$
By separation of variables we have that
$$\int \frac{y'(t)}{\sqrt{y(t)}}\,dt = \int dt\,, \quad \Longrightarrow \quad 2\sqrt{y(t)} = t + c_0\,,$$
that is,
$$y(t) = \Big(\frac{t + c_0}{2}\Big)^2\,.$$
If we consider the Cauchy problem
$$\begin{cases} y'(t) = \sqrt{y(t)}\,, \\ y(0) = 0\,, \end{cases}$$
then $y \equiv 0$ is solution but also $y(t) = (t/2)^2$ with $c_0 = 0$.... In this case we DO NOT HAVE UNIQUENESS of the solution! The reason is that the function $g(y) = \sqrt{y}$ is not Lipschitz at $y = 0$; *i.e.*, the derivative $g'$ is not bounded "around" $y = 0$.

  In this course we will not further investigate the uniqueness of the solutions. By the way, it is important to be aware that the uniqueness of the solution of a Cauchy problem depends on the regularity of the function $y \mapsto f(t, y)$.

  If we change initial condition, for example,
$$\begin{cases} y'(t) = \sqrt{y(t)}\,, \\ y(0) = 1\,, \end{cases}$$

then, by evaluating $2\sqrt{y(t)} = t + c_0$ at $t_0 = 0$ we have $2\sqrt{y(0)} = c_0$, which implies $2 = c_0$. Therefore, the solution to the cauchy problem in this case is unique and it is given by

$$y(t) = \Big(\frac{t+2}{2}\Big)^2 .$$

Finally, note that we can not choose a negative initial condition, $y(0) < 0$, since $g(y) = \sqrt{y(t)}$ is defined only on $J = [0, +\infty)$ and the initial datum $(t_0, y_0) \in I \times J$.

- Solve

$$\begin{cases} y'(t) = 2t\sqrt{1-y^2} \\ y(0) = 0 . \end{cases}$$

The function $g(y) = \sqrt{1-y^2}$ is defined on $J = [-1, 1]$ and $g(\pm 1) = 0$; hence, $y(t) = \pm 1$ are constant solutions to the equation with initial condition $y_0 = \pm 1$, respectively.

The other solutions can be obtained by separation of variable

$$\int \frac{y'(t)}{\sqrt{1-y^2}}\, dt = \int 2t\, dt = t^2 + c_0 .$$

Hence, since $y(0) = 0$ we have that $c_0 = 0$ and

$$\arcsin y(t) = t^2 \in [0, \frac{\pi}{2}) \quad \Longrightarrow \quad y(t) = \sin(t^2)\, , \forall t \in (-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}) .$$

- Solve

$$\begin{cases} y'(t) = 1 - y^2 \\ y(0) = 0 . \end{cases}$$

- Solve

$$y'(t) = 2t\, y^2 .$$

By separation of variables we have from one side that

$$\int \frac{y'(t)}{y^2(t)}\, dt = \int \frac{1}{s^2}\, ds = -\frac{1}{y(t)} .$$

On the other side, we have

$$\int 2t\, dt = t^2 + c_0 \quad \Longrightarrow \quad -\frac{1}{y(t)} = t^2 + c_0 \quad \Longrightarrow \quad y(t) = -\frac{1}{t^2 + c_0} .$$

If $c_0 > 0$ then the solution exists for every $t \in \mathbb{R}$; if $c_0 < 0$ then $t^2 + c_0 = 0$ if and only if $t = \pm\sqrt{-c_0}$. It implies that we may have three different type of solutions according to the initial condition $y(t_0) = y_0$; *i.e.*,

$$\begin{aligned} y(t) &= -\frac{1}{t^2 + c_0}, \qquad t \in (-\infty, -\sqrt{-c_0})\, , \\ y(t) &= -\frac{1}{t^2 + c_0}, \qquad t \in (-\sqrt{-c_0}, \sqrt{-c_0})\, , \\ y(t) &= -\frac{1}{t^2 + c_0}, \qquad t \in (-\sqrt{-c_0}, +\infty)\, . \end{aligned} \tag{5.5}$$

Finally, if $c_0 = 0$ we have two solutions defined in two different intervals

$$y(t) = -\frac{1}{t^2}\,, \quad t \in (0,+\infty)\,, \qquad y(t) = -\frac{1}{t^2}\,, \quad t \in (-\infty, 0)\,.$$

Let 's make an example by fixing the initial condition $y(0) = 1$. By replacing in

$$y(t) = -\frac{1}{t^2 + c_0} \quad \Longrightarrow 1 = -\frac{1}{c_0} \quad \Longrightarrow c_0 = -1\,;$$

hence, we are in one of the three cases in (5.5). More precisely, since $t_0 = 0$ the interval of existence of the solution should contain 0; *i.e.*,

$$y(t) = -\frac{1}{t^2}\,, \qquad \forall t \in (-1, 1)\,.$$

- Solve

$$\begin{cases} y'(t) = \dfrac{t^2+1}{2t(t-1)y^2} \\ \\ y(2) = -\dfrac{\ln 2}{6} \end{cases}$$

By separation of variables we have

$$y^2\, y'(t) = \frac{t^2+1}{2t(t^2-1)} \quad \Longrightarrow \quad \int y\, y'(t)\, dt = \int \frac{t^2+1}{2(t^2-1)t}\, dt\,.$$

Hence,

$$\frac{y^3(t)}{3} = \frac{1}{2}\int \frac{t^2+1}{t(t^2-1)}\, dt$$

in particular, we can split in tow parts and solve the first integral by integration by parts the second integral by the method of integration of rational functions (Section 1.4); *i.e.*,

$$\begin{aligned} \int \frac{t^2+1}{2t(t^2-1)}\, dt &= \int \frac{t}{t-1}\, dt + \int \frac{1}{t(t-1)}\, dt \\ &= (t-1)\ln(t-1) - 3t + \int \frac{-1}{t} + \frac{1}{t-1}\, dt \\ &= (t-1)\ln(t-1) - 3t - \ln t + \ln(t-1) + c_0 \\ &= t\ln(t-1) - 3t - \ln t + c_0\,. \end{aligned}$$

Therefore

$$y^3(t) = \frac{1}{6}(t\ln(t-1) - 3t - \ln t + c_0) \quad \Longrightarrow \quad y^3(2) = -1 - \frac{\ln 2}{6} + c_0 = -\frac{\ln 2}{6} \Longrightarrow c_0 = 1\,,$$

and

$$y(t) = \sqrt[3]{\frac{1}{6}(t\ln(t-1) - 3t - \ln t + 1}\,.$$

### 5.1.2 First-order linear differential equations

A first order linear differential equation can be written in the form

$$y'(t) = a(t)y(t) + b(t) \tag{5.6}$$

with $a, b : I \mapsto \mathbb{R}$ continuous functions.

### 5.1.3 Homogeneous linear first-order ode

If $b(t) \equiv 0$ then the equation is called *homogeneous*

$$y'(t) = a(t)y(t) \tag{5.7}$$

and we can solved it by separation of variables. In particular,

$$\int \frac{y'(t)}{y(t)}\, dt = \int a(t)\, dt \quad \Longrightarrow \quad \int \frac{1}{s}\, ds = \int a(t)\, dt\,,$$

and, the general integral of the homogeneous equation (5.7) is given by

$$\ln|y(t)| = A(t) + c_0 \quad \Longrightarrow \quad y_h(t) = c\, e^{A(t)}\,, \quad \forall t \in I\,,\ c \in \mathbb{R}$$

where

$$A(t) = \int a(t)\, dt\,.$$

The solution to the homogeneous ode is called *homogeneous solution.* If we fix the initial condition $y(t_0) = y_0$ then

$$y_h(t) = y_0\, e^{A(t)-A(t_0)}\,, \quad \forall t \in I \tag{5.8}$$

where

$$A(t) - A(t_0) = \int_{t_0}^{t} a(\tau)\, d\tau\,.$$

In particular, if $y_0 = 0$ then $y \equiv 0$.

### 5.1.4 Inhomogeneous linear first-order ode

The general integral for the equation (5.6) is the sum of the homogeneous solution (the solution to (5.7)) and te so-called *particular solution* that we denote by $y_p$ and that we get by the method of variation of constants. More precisely, we replace the constant $c$ with $C(t)$; *i.e.*,

$$y_p(t) = C(t)\, e^{A(t)}\,,$$

then we differentiate

$$y_p'(t) = C'(t)\, e^{A(t)} + C(t)A'(t)\, e^{A(t)}\,,$$

and we replace in (5.6); hence,

$$C'(t)\, e^{A(t)} + C(t)A'(t)\, e^{A(t)} = a(t)\, C(t)\, e^{A(t)} + b(t)\,.$$

since $A'(t) = a(t)$, we have that

$$C'(t)\, e^{A(t)} = b(t) \quad \Longrightarrow \quad C'(t) = e^{-A(t)}\, b(t)\,.$$

Therefore,

$$C(t) = \int e^{-A(\tau)}\, b(\tau)\, d\tau \qquad \Longrightarrow \qquad y_p(t) = e^{A(t)}\Big(\int e^{-A(\tau)}\, b(\tau)\, d\tau\Big)\,,$$

and the general integral is given by

$$y(t) = y_h + y_p = e^{A(t)}\Big(c + \int e^{-A(\tau)}\, b(\tau)\, d\tau\Big)\,. \tag{5.9}$$

In particular, the solution to the Cauchy Problem

$$\begin{cases} y'(t) = a(t)\, y(t) + b(t)\,, \\ y(t_0) = y_0\,, \end{cases}$$

is given, for every $t \in I$, by

$$y(t) = e^{\int_{t_0}^{t} a(\tau)\, d\tau}\Big(y_0 + \int_{t_0}^{t} e^{-\int_{t_0}^{s} a(\tau)\, d\tau}\, b(s)\, ds\Big)\,. \tag{5.10}$$

**Example 5.1.5.** • Find the general integral of

$$y'(t) = (\tan t) y(t) + \sin t\,.$$

We first note that we have to study the equation in an interval of continuity for $\tan t$ ( the function $\sin t$ is continuous in $\mathbb{R}$). We consider $I = (-\pi/2, \pi/2)$ and we compute the primitive of $a(t) = \tan t$; *i.e.*,

$$A(t) = \int \tan t\, dt = -\ln|\cos t| = -\ln(\cos t)\,, \quad \forall t(-\frac{\pi}{2}, \frac{\pi}{2})\,.$$

Therefore,

$$\begin{aligned} y(t) &= e^{-\ln(\cos t)}\Big(c + \int e^{\ln(\cos t)} \sin t\, dt\Big) \\ &= \frac{1}{\cos t}\Big(c + \int \cos t\, \sin t\, dt\Big) \\ &= \frac{1}{\cos t}(c - \frac{1}{2}(\cos t)^2)\,. \end{aligned}$$

• Find the solution to the ODE

$$t\, y'(t) = y(t) + t^2 \cos t\,.$$

First of all we note that we can always rewrite the equation in normal form

$$y'(t) = \frac{1}{t}\, y(t) + t\, \cos t$$

with $t \neq 0$. According to the initial condition $y(t_0) = y_0$, the solution then will exist in an interval contained in $(0, +\infty)$ or $(-\infty, 0)$ if $t_0 \in (0, +\infty)$ or $t_0 \in (-\infty, 0)$, respectively.

General solution

$$y(t) = (c + \sin t) t\,.$$

- Find the solution of the following ODE:

  a) $y'(t) = (y(t) - 3)\cos t$
  b) $y'(t) = e^{t+y}$
  c) $2t\,y'(t) + \frac{y^2}{t} = 0$
  d) $y'(t) + y + e^t = 0$
  e) $y'(t) + \frac{y}{t} = \sin t$
  f) $y'(t) + t^2 y = 2t^2$
  g) $y'(t) = -t\,y^2$ with $y(0) = 2$
  h) $y'(t) + t^2 y = 0$ with $y(0) = 3$
  i) $t\,y'(t) - y(t) = t^2 \cos t$ with $y(\frac{\pi}{2}) = \pi$.

## 5.2 Linear Second-order O.D.E. with constant coefficients

### 5.2.1 The Homogeneous case

The form for the linear second order ode with constant coefficients $a_0, a_1, a_2 \in \mathbb{R}$ is

$$a_2\, y''(t) + a_1\, y'(t) + a_0\, y(t) = 0\,, \tag{5.11}$$

with $a_2 \neq 0$ otherwise the equation would be of first order.

The solution is determined by assuming that there is a solution of the form $e^{\lambda t}$; hence,

$$a_2 \lambda^2\, e^{\lambda t} + a_1 \lambda\, e^{\lambda t} + a_0\, e^{\lambda t} = 0 \implies a_2\lambda^2 + a_1\lambda + a_0 = 0\,.$$

The polynomial $a_2\lambda^2 + a_1\lambda + a_0$ is called the characteristic polynomial of the ode and we have to solve the characteristic equation

$$a_2\lambda^2 + a_1\lambda + a_0 = 0\,. \tag{5.12}$$

**Theorem 5.2.1** (Solutions to Homogeneous equation with constant coefficients)**.** *Let $c_1, c_2 \in \mathbb{R}$ be two arbitrary constant and let $\lambda_1, \lambda_2$ be the roots to the characteristic equations* (5.12)*. Then any solutions to eqrefH-ode2 fulfilled only one of the following cases:*

- *if there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq \lambda_2$ then*

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\,;$$

- *if there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \equiv \lambda_2$ then*

$$y(t) = (c_1 + t c_2) e^{\lambda_1 t}\,;$$

- *if there exists $\lambda_1, \lambda_2 \in \mathbb{C}$ then $\lambda_1, \lambda_2 = \alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}$ and*

$$y(t) = c_1\, e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)\,.$$

*Moreover, for every fixed $t_0, y_0, y_1 \in \mathbb{R}$ there exists a unique solution to the initial value problem*

$$\begin{cases} a_2\, y''(t) + a_1\, y'(t) + a_0\, y(t) = 0\,, \\ y(t_0) = y_0\,, \\ y'(t_0) = y_1\,. \end{cases} \tag{5.13}$$

**Example 5.2.2.**

- Find the general solution to the following homogeneous equation

$$2y''(t) - 3\,y'(t) + y(t) = 0\,.$$

We associate to the ode the characteristic equation

$$2\lambda^2 - 3\lambda + 1 = 0\,;$$

hence, there exists two real and distinct roots $\lambda_1 = 1$, $\lambda_2 = 1/2$ and the general integral is

$$y(t) = c_1 e^t + c_2 e^{t/2}\,,$$

for any arbitrary constants $c_1, c_2$.

- Find the solution to the initial value problem

$$y''(t) + 2\,y'(t) + 3y = 0\,, \quad y(0) = 1\,, \quad y'(0) = 2\,.$$

We associate to the ode the characteristic equation

$$\lambda^2 + 2\lambda + 3 = 0\,;$$

hence, there exists $\lambda_1, \lambda_2 = -1 \pm i\sqrt{2}$ and the general integral is

$$y(t) = e^{-t}\Big(c_1\,\cos(\sqrt{2}\,t) + c_2\,\sin(\sqrt{2}\,t)\Big)\,.$$

We now find the constants $c_1$ and $c_2$ that satisfy the initial conditions; hence, $y(0) = c_1 = 1$ and

$$y'(t) = -e^{-t}\Big(c_1\,\cos(\sqrt{2}\,t) + c_2\,\sin(\sqrt{2}\,t)\Big) + e^{-t}\Big(-\sqrt{2}\,c_1\,\sin(\sqrt{2}\,t) + \sqrt{2}\,c_2\,\cos(\sqrt{2}\,t)\Big)$$

and

$$y'(0) = -c_1 + \sqrt{2}\,c_2 = 2 \quad\Longrightarrow\quad -1 + \sqrt{2}\,c_2 = 2 \Longrightarrow c_2 = \frac{3}{\sqrt{2}}\,.$$

Therefore, the unique solution to the initial value problem is

$$y(t) = e^{-t}\Big(\cos(\sqrt{2}\,t) + \frac{3}{\sqrt{2}}\,\sin(\sqrt{2}\,t)\Big)\,.$$

- Find the solution to the initial value problem

$$y''(t) + 4\,y'(t) + 4y = 0\,, \quad y(0) = 3\,, \quad y'(0) = 2\,.$$

We associate to the ode the characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0 \quad\Longrightarrow\quad (\lambda + 2)^2 = 0 \Longrightarrow \lambda_1 \equiv \lambda_2 = -2\,.$$

Therefore, the general integral is

$$y(t) = (c_1 + tc_2)e^{-2t}\,.$$

Let us now fix the constants $c_1, c_2$. In particular, $y(0) = c_1 = 3$ and $y'(t) = -2e^{-2t}(c_1 + tc_2) + c_2 e^{-2t}$, which implies $y'(0) = -6 + c_2 = 2$; *i.e.*, $c_2 = 8$. Therefore, the unique solution to the initial value problem is

$$y(t) = (3 + 8t)\, e^{-2t}\,.$$

**Exercise 5.2.3.** Find the solution to the initial value problem

$$y''(t) + 5\, y'(t) + 6y = 0\,, \quad y(0) = 1\,, \quad y'(0) = -1\,.$$

### 5.2.2 Inhomogeneous equation: method of undetermined coefficients

We now consider the case of linear second-order o.d.e. with constant coefficients not homogeneous; *i.e.*,

$$a_2\, y''(t) + a_1\, y'(t) + a_0\, y(t) = f(t)\,. \tag{5.14}$$

Similarly to the first order inhomogeneous case, the general integral is given by the sum of the solution to the homogeneous equation (that is the one with $f \equiv 0$) with a so-called *particular solution* $y_p$. More precisely,

$$y(t) = c_1 y_1 + c_2 y_2 + y_p$$

where $c_1, c_2 \in \mathbb{R}$ are two arbitrary constants and $y_1, y_2$ are as in Theorem 5.2.1.

There are cases where the particular solution $y_p$ belongs to the same class of functions of $f$. Here below the different cases.

(1) If $f(t) = P_n(t)$ is a polynomial of order $n$ then:

   (a) if $a_0 \neq 0$ then $y_p = Q_n(t)\,,$

   (b) if $a_0 = 0$ and $a_1 \neq 0$ then $y_p = t\, Q_n(t)\,,$

   (c) if $a_0 = 0$ and $a_1 = 0$ then $y_p = t^2\, Q_n(t)\,.$

(2) If $f(t) = A\cos at$ or $B\sin at$ or $f(t) = A\cos at + B\sin at$ then $y_p(t) = C\cos at + D\sin at$.

(3) If $f(t) = Ce^{Bt}$ where $B$ is not a root of the characteristic equation then $y_p(t) = Ae^{Bt}$. Note that the exponential is the same, we may find another constant.

**Example 5.2.4** ($f$ is a polynomial)**.** (1) Find the general integral of equation

$$y''(t) - 2y(t) = 1 + t^2\,.$$

Let's start by the particular solution that will be a polynomial of degree 2 since $f(t) = 1 + t^2$. Note that $a_0 \neq 0$ then $y_p = p_2 t^2 + p_1 t + p_0$. To fined the coefficients $p_0, p_1, p_2$ we have to compute

$$\begin{aligned} y_p' &= 2p_2 t + p_1 \\ & \qquad\qquad\qquad \Longrightarrow 2p_2 - 2(p_2 t^2 + p_1 t + p_0) = 1 + t^2 \\ y_p''(t) &= 2p_2 \qquad\qquad -2p_2 t^2 - 2p_1 t + 2p_2 - 2p_0 = t^2 + 1\,. \end{aligned}$$

Therefore,

$$-2p_2 = 1\,, \qquad -2p_1 = 0\,, \quad 2p_2 - 2p_0 = 1 \Longrightarrow p_0 = -1\,, p_1 = 0\,, p_2 = -1/2\,;$$

and the particular solution is

$$y_p = -\frac{1}{2}t^2 - 1\,.$$

The homogeneous solution is obtained by solving $\lambda^2 - 2 = 0$; *i.e.*, $\lambda = \pm\sqrt{2}$; hence,

$$y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} - \frac{t^2}{2} - 1\,.$$

(2) Find the general integral of equation

$$y''(t) + 2y'(t) = t\,.$$

Note that $a_0 = 0$; hence, we expect a particular solution of the type $y_p = t(p_0 + p_1 t)$. More precisely,

$$\begin{array}{rcl} y_p' & = & 2p_1 t + p_0 \\ & & \qquad\qquad \Longrightarrow 2p_1 + 4p_1 t + 2p_0 = t \\ y_p''(t) & = & 2p_1 \qquad\qquad\quad 4p_1 = 1\,, 2p_0 + 2p_1 = 0\,, \Longrightarrow p_1 = \dfrac{1}{4}\,, p_0 = -\dfrac{1}{4}\,. \end{array}$$

Therefore,

$$y_p = \frac{t}{4}(t-1)\,.$$

The homogeneous solution is obtained by solving $\lambda^2 + 2\lambda = 0$; *i.e.*, $\lambda = 0\,, -2$; hence,

$$y(t) = c_1 + c_2 e^{-2t} + \frac{t}{4}(t-1)\,.$$

It is important to underlying that, at difference of the previous case, now the particular solution is NOT given by a polynomial of the same degree of $f$.

(3) Find the general integral of equation

$$y''(t) = t\,.$$

Note that $a_0 = a_1 = 0$ then we expect a particular solution of the type $y_p = t^2(p_0 + p_1 t) = p_0 t^2 + p_1 t^3$. More precisely,

$$\begin{array}{rcl} y_p' & = & 3p_1 t^2 + 2p_0 t \\ & & \qquad\qquad \Longrightarrow 6p_1 t + 2p_0 = t \\ y_p''(t) & = & 6p_1 t + 2p_0 \qquad\qquad 6p_1 = 1\,, p_0 = 0\,, \Longrightarrow p_1 = \dfrac{1}{6}\,, p_0 = 0\,. \end{array}$$

Therefore,

$$y_p = t^2 \frac{t}{6} = \frac{t^3}{6}\,.$$

The homogeneous solution is obtained by solving $\lambda^2 = 0$; *i.e.*, $\lambda = 0$; hence,

$$y(t) = (c_1 + c_2 t) + \frac{t^3}{6}\,.$$

**Example 5.2.5** ($f$ is a combination of cos and sin)**.** (1) Find the general integral of equation

$$y''(t) + y'(t) + 2y = 2\cos t\,.$$

Let's start by the particular solution that will be of the form $y_p = A\cos t + B\sin t$. More precisely,

$$\begin{aligned} y_p' &= -A\sin t + B\cos t \\ y_p''(t) &= -A\cos t - B\sin t\,, \end{aligned}$$

which implies

$$\begin{aligned} &-A\cos t - B\sin t + -A\sin t + B\cos t + 2(A\cos t + B\sin t) \\ =\;& (A+B)\cos t + (B-A)\sin t \\ =\;& 2\cos t\,. \end{aligned}$$

Therefore,

$$\begin{cases} A + B = 2 \\ \\ B - A = 0\,, \end{cases} \Longrightarrow A = B = 1$$

and

$$y_p = \cos t + \sin t\,.$$

The homogeneous solution is obtained by solving $\lambda^2 + \lambda + 2 = 0$; *i.e.*, $\lambda = (-1 \pm i\sqrt{7})/2$ and

$$y(t) = e^{-t/2}(c_1\cos(\frac{\sqrt{7})}{2})t + c_2\sin(\frac{\sqrt{7})}{2})t) + \cos t + \sin t\,.$$

(2) Solve the initial value problem

$$y''(t) + 2y'(t) + 3y = 6\sin 3t\,, \quad y(0) = \frac{1}{2}\,, \quad y'(0) = -\frac{5}{2} - \sqrt{2}\,.$$

The corresponding homogeneous o.d.e. has a characteristic equation $\lambda^2 + 2\lambda + 3 = 0$ with roots $\lambda = -1 \pm i\sqrt{2}$ so the homogeneous solution is

$$y_h = e^{-t}\Big(c_1\ \cos(\sqrt{2}\,t) + c_2\ \sin(\sqrt{2}\,t)\Big)\,.$$

The method of undetermined coefficients says that we should try for a particular solution of the form

$$y_p = C\cos 3t + D\cos 3t\,.$$

Then,

$$y'(t) = -3C\sin 3t + 3D\cos 3t\,, \qquad y''(t) = -9C\cos 3t - 9D\sin 3t\,.$$

Substituting into the O.D.E. gives

$$\begin{aligned} &-9C\cos 3t - 9D\sin 3t + 2(-3C\sin 3t + 3D\cos 3t) + 3(C\cos 3t + D\sin 3t) \\ =\;& (6D - 6C)\cos 3t + (-6D - 6C)\sin 3t \\ =\;& 6\sin 3t\,. \end{aligned}$$

We can now compare the coefficients of $\cos 3t$ and $\sin 3t$ to get

$$(6D - 6C) = 0\,, \quad (-6D - 6C) = 6 \Longrightarrow D = C = -\frac{1}{2}\,.$$

Therefore, the general solution is

$$y(t) = e^{-t}\Big(c_1 \cos(\sqrt{2}\,t) + c_2 \sin(\sqrt{2}\,t)\Big) - \frac{1}{2}\cos 3t - \frac{1}{2}\sin 3t\,.$$

We now consider the initial conditions

$$y(0) = c_1 - \frac{1}{2} = \frac{1}{2} \quad \Longrightarrow c_1 = 1\,.$$

To consider the initial condition $y'(0)$ we have first to derivate the general integral

$$\begin{aligned} y'(t) &= -e^{-t}\Big(c_1 \cos(\sqrt{2}\,t) + c_2 \sin(\sqrt{2}\,t)\Big) + e^{-t}\Big(-\sqrt{2}c_1 \sin(\sqrt{2}\,t) + c_2\sqrt{2} \cos(\sqrt{2}\,t)\Big) \\ &\quad + \frac{3}{2}\sin 3t - \frac{3}{2}\cos 3t \\ y'(0) &= -c_1 + c_2\sqrt{2} - \frac{3}{2} = -\frac{5}{2} + c_2\sqrt{2} \\ &= -\frac{5}{2} - \sqrt{2}\,, \end{aligned}$$

so we get $c_2 = -1$. The unique solution to the initial value problem is then

$$y(t) = e^{-t}\Big(\cos(\sqrt{2}\,t) - \sin(\sqrt{2}\,t)\Big) - \frac{1}{2}\cos 3t - \frac{1}{2}\sin 3t\,.$$

**Example 5.2.6** ($f$ is the exponential function)**.** (1) Solve

$$3y''(t) - 2y'(t) - y = e^{2t}\,,$$

given that

$$y(0) = 0\,, y'(0) = 1\,.$$

The corresponding homogeneous o.d.e. has a characteristic equation $3\lambda^2 - 2\lambda - 1 = 0$ with roots $\lambda = 1, -1/3$. Hence the homogeneous solution is $y_h(t) = c_1 e^t + c_2 e^{-t/3}$. We have to look for a particular solution of the form $y_p = C\,e^{2t}$; hence,

$$y_p'(t) = 2C\,e^{2t}\,, \quad y_p''(t) = 4C\,e^{2t}\,.$$

Substituting into the O.D.E. gives

$$12C\,e^{2t} - 4C\,e^{2t} - C\,e^{2t} = 7C\,e^{2t}\,.$$

We now compare $7C\,e^{2t}$ with $f(t) = e^{2t}$ and we get $C = 1/7$ and the general integral is given by

$$y(t) = c_1 e^t + c_2 e^{-t/3} + \frac{1}{7}e^{2t}\,.$$

Now we use the initial conditions to find $c_1$ and $c_2$. Firstly,

$$y(0) = c_1 + c_2 + \frac{1}{7} = 0 \Longrightarrow c_1 + c_2 = -\frac{1}{7}\,.$$

Also,

$$y'(t) = c_1 e^t - \frac{1}{3} c_2 e^{-t/3} + \frac{2}{7} e^{2t} \Longrightarrow y'(0) = c_1 - \frac{1}{3} c_2 + \frac{2}{7} = 1$$

which gives $c_1 - c_2/3 = 5/7$. To find $c_1$ and $c_2$ we have to solve the following linear system

$$\begin{cases} c_1 + c_2 = -\frac{1}{7} \\ \\ c_1 - \frac{c_2}{3} = \frac{5}{7}\,. \end{cases} \Longrightarrow c_1 = \tfrac{1}{2}\,, c_2 = -\tfrac{9}{14}$$

Therefore, the unique solution to the initial value problem is

$$y(t) = \frac{1}{2} e^t - \frac{9}{14} e^{-t/3} + \frac{1}{7} e^{2t}\,.$$