Immersed Finite Element Methods for Interface Problems with Multi-Domains and Triple-Junction Points

Yuan Chen[1], Songming Hou[2] & Xu Zhang[3]

[1] The George Washington University, Washington, DC
[2] Louisiana Tech University, Ruston, LA
[3] Oklahoma State University, Stillwater, OK

Presented by:
Yuan Chen
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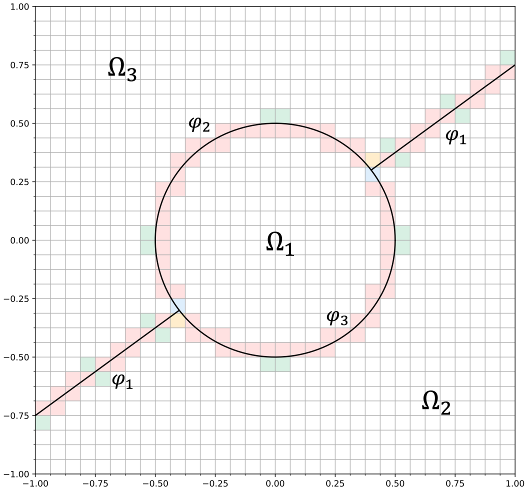
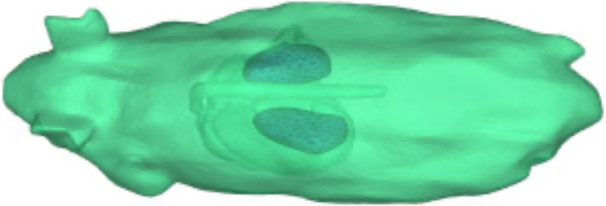
Presentation Outline

Abstract Form:

$$-\nabla\cdot(\beta\nabla u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x}\in\Omega$$
$$\beta = \beta_i, \quad \text{on } \Omega_i, i = 1,2,3$$
$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x}\in\partial\Omega$$
$$[u]_{\Gamma_i} = 0, \quad \forall i = 1,2,3.$$
$$[\beta\nabla u\cdot\mathbf{n}]_{\Gamma_i} = b_i(\mathbf{x}) \quad \text{on } \Gamma_i,$$



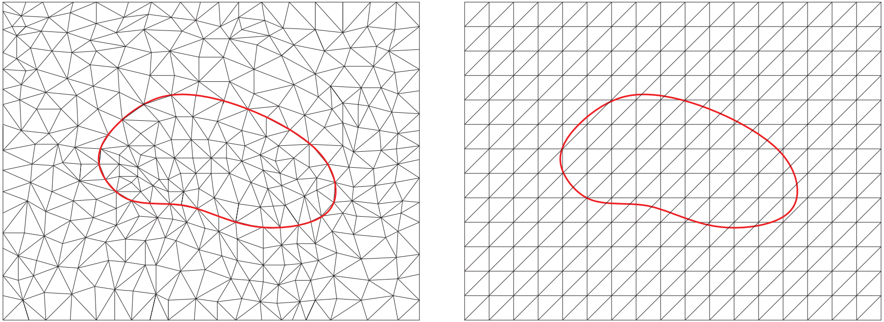
Example Application in biomedical imaging of mouse:
In the bio-imaging, Ω_1: Organ, Ω_2: muscle, Ω_3: other computing region [1].



[1]

[1] Source: Duan Chen, Guo-Wei Wei, Wen-Xiang Cong, and Ge Wang. Computational methods for optical molecular imaging. Communications in numerical methods in engineering, 25(12):1137–1161, 2009.

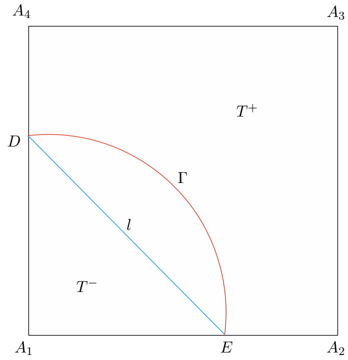
- Non-body fitting: efficiency

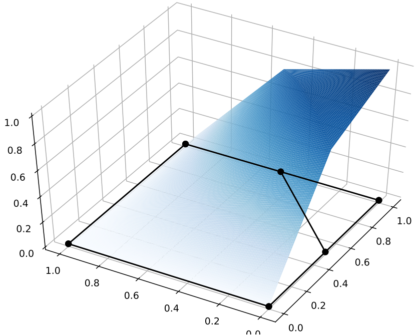


[2]

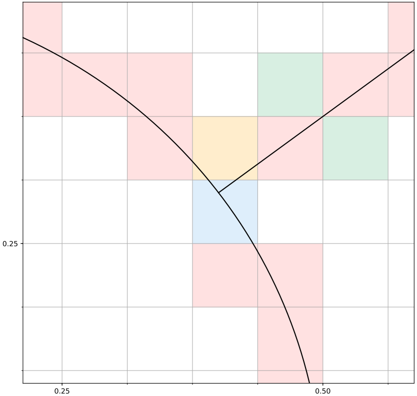
[2] Source: Xu Zhang. Nonconforming Immersed Finite Element Methods for Interface Problems. Thesis (Ph.D.). Virginia Polytechnic Institute and State University, 2013

- "Immersed" basis function

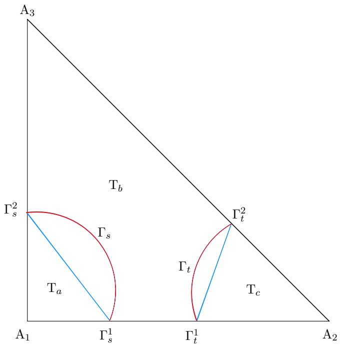


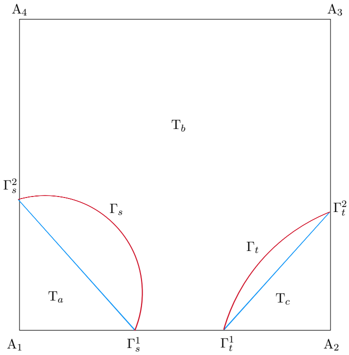


- "Complexity" and "Challenge" for multi-domain and triple points



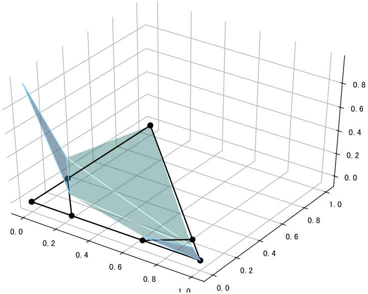
Case 1: $T \in \mathcal{T}_{h,2}^{i}$: all elements intersecting two interfaces

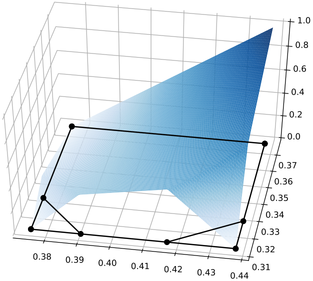




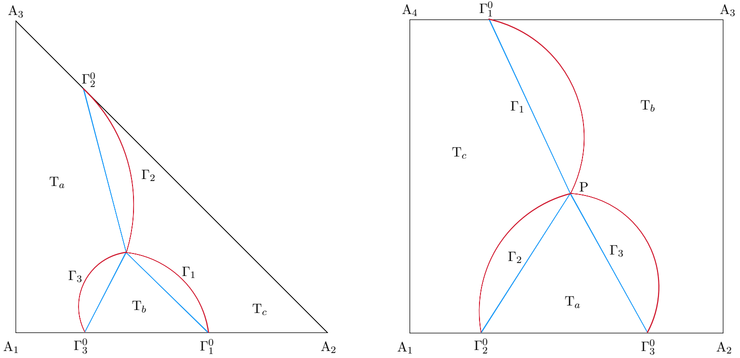
$\mathcal{T}_{h,2}^{i}$ Basis function construction:

- Nodal-value conditions: $\phi_{i,T}(A_j) = \delta_{ij}, \quad \forall i,j \in \mathcal{I}_h$
- Continuity conditions of the basis functions on two interfaces
- Two conditions of normal flux continuity



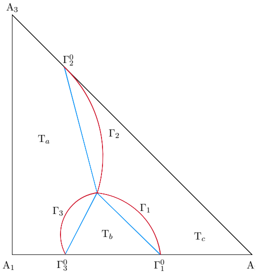


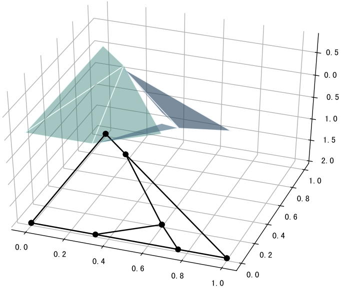
Case 2: $T \in \mathcal{T}_{h,3}^{i}$: all elements intersecting three interfaces with triple point



$\mathcal{T}^i_{h,3}$ Triangular Basis function construction:

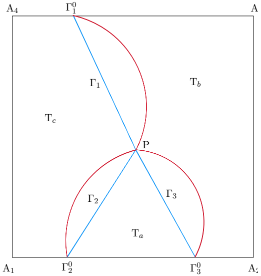
- Nodal-value conditions: $\phi_{i,T}(A_j) = \delta_{ij}, \quad \forall i, j \in \mathcal{I}_h$
- Continuity conditions of the basis functions on triple point and interface intersection points
- Three conditions of normal flux continuity
- Lack of freedom: **least square method**

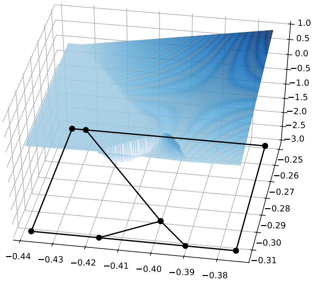




$\mathcal{T}^i_{h,3}$ Rectangular Basis function construction:

- Nodal-value conditions: $\phi_{i,T}(A_j) = \delta_{ij}, \quad \forall i,j \in \mathcal{I}_h$
- Continuity conditions of the basis functions on triple point and interface intersection points
- Three conditions of normal flux continuity
- Degree of freedom **enough**





Weak form

$$a_h(u_h, v_h) = (f, v_h) - a_h(u_h^J, v_h) + \sum_{i=1}^{3} \int_{\Gamma_i} Q_i v_h ds, \ \forall v_h \in S_h. \tag{1}$$

Galerkin

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K \beta \nabla u_h \cdot \nabla v_h dX \tag{2}$$

Partially Penalized Immersed Finite Element Method

$$\begin{aligned} a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K \beta \nabla u_h \cdot \nabla v_h dX - \sum_{B \in \mathring{\mathcal{E}}_h^i} \int_B \{\beta \nabla u_h \cdot \mathbf{n}_B\}[v_h] ds \\ + \epsilon \sum_{B \in \mathcal{E}_h^i} \int_B \{\beta \nabla v_h \cdot \mathbf{n}_B\}[u_h] ds + \sum_{B \in \mathcal{E}_h^i} \int_B \frac{\sigma_B^0}{|B|^\alpha}[v_h][u_h] ds \end{aligned} \tag{3}$$

Presentation Outline

Interpolation error of numerical example with circle interface, $\beta_1 = 10$, $\beta_2 = 1$, $\beta_3 = 100$

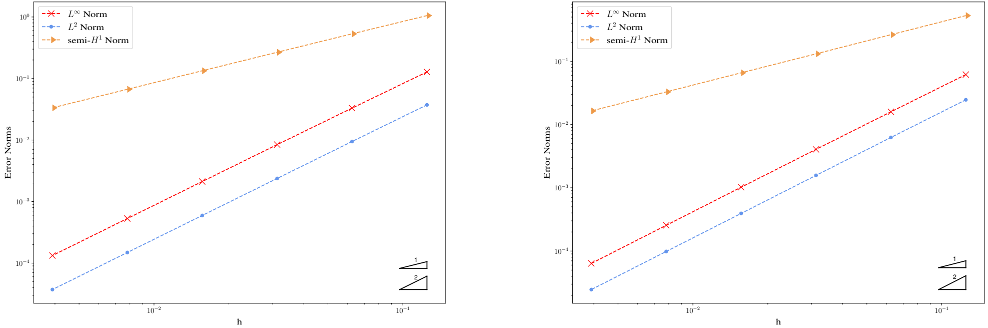


Figure 1: Interpolation error of triangular mesh(l) and Interpolation error of rectangular mesh(r)

Numerical solution with circle interface, $\beta_1 = 10$, $\beta_2 = 1$, $\beta_3 = 100$

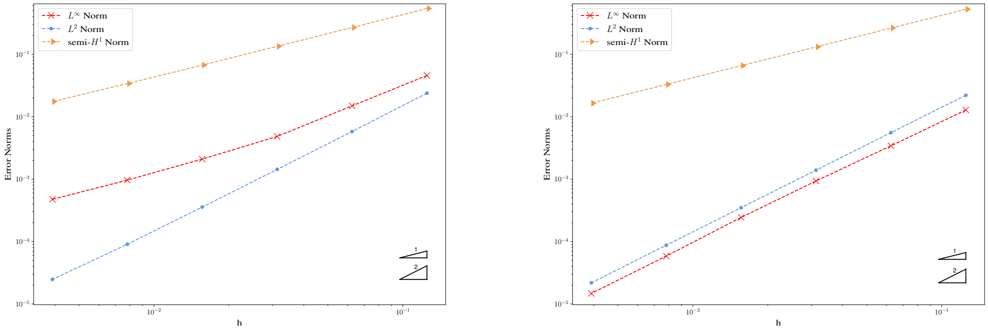


Figure 2: Error of $u - u_h$ using IFEM(l) and SPPIFEM(r)

Solution difference of IFEM and PPIFEM can be revealed in Error plot

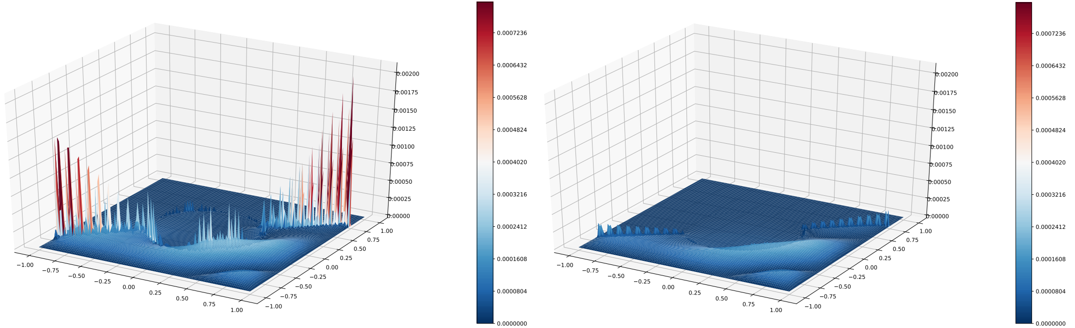


Figure 3: Error plot of $|u - u_h|$ using IFEM(l) and SPPIFEM(r)

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The above work cites [2, 3].

Duan Chen, Guo-Wei Wei, Wen-Xiang Cong, and Ge Wang.
Computational methods for optical molecular imaging.
Communications in numerical methods in engineering, 25(12):1137–1161, 2009.

Yuan Chen, Songming Hou, and Xu Zhang.
A bilinear partially penalized immersed finite element method for elliptic interface problems with multi-domains and triple junction points.
Results in Applied Mathematics, in press, [arXiv: 2003.01601].

Yuan Chen, Songming Hou, and Xu Zhang.
An immersed finite element method for elliptic interface problems with multi-domain and triple junction points.
Advances in Applied Mathematics and Mechanics, 11(5):1005–1021, 2019.

Thank You!

Questions?