Notation and definitions for question:

Any functions discussed throughout are polynomials.

If $p(x) = a_n x^n + \cdots + a_1 x + a_0$, then $\deg(p) = n$, where deg is the function telling you the degree of the polynomial.

We define the highest common factor of two polynomials f and g as the polynomial which divides both f and g and has a positive leading coefficient.

Example:

$$f(x) = -(x+1)(x-3)(x+7), \qquad g(x) = (x+1)(x-3)(x+10)$$

then

$$\text{hcf}(f, g) = (x+1)(x-3).$$

QUESTION

i) Let $f(x) = x^2 - 5x - 6$ and $g(x) = x^3 - 4x^2 + x + 6$.
Given $g(-1) = 0$, find $\text{hcf}(f, g)$. [4 marks]

ii) Let $\deg(f) = m$ and $\deg(g) = n$, where $m \geq n$.
Explain why $\deg(\text{hcf}(f, g)) \leq n$. [3 marks]

iii) Given $\text{hcf}(f, g) = k$, where k is a constant, how many roots do f and g share? [3 marks]

iv) Let $\deg(f) = \deg(g) = m$.
Given $\deg(\text{hcf}(f, g)) = m$, explain why $f(x) = Ag(x)$ for some constant A. [3 marks]

v) Let $\deg(f) = m$, where $m > 1$. Let $g = f(-x)$.

a) Describe the graph transformation that maps f to g. [1 mark]

b) Given $\deg(\text{hcf}(f, g)) = 1$, deduce that $f(0) = 0$. [3 marks]

c) Given that $\deg(\text{hcf}(f, g)) = m$ and $f(0) \neq 0$,
show that m is an even number. [3 marks]

vi) Let $\deg(f) = m$.
Given $\text{hcf}(f, f') = (x - r_1) \cdots (x - r_{m-1})$ and $f(x_0) = y_0 \neq 0$,
explain how we can find the function $f(x)$.
(DO NOT find the function; give a worded explanation only) [5 marks]

i) Let $f(x) = x^2 - 5x - 6$ and $g(x) = x^3 - 4x^2 + x + 6$. Given $g(-1) = 0$, find $\mathrm{hcf}(f, g)$.

$f(x) = x^2 - 5x - 6$

$(x-6)(x+1)$ $\quad g(x) = x^3 - 4x^2 + x + 6$

$(x-(-1)) = x+1$ $\quad (x+1)(x^2 - 5x + 6)$

$(x-2)(x-3)(x+1)$ $\quad hcf(f,g) = x+1$

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ii) Let $\deg(f) = m$ and $\deg(g) = n$, where $m \geq n$. Explain why $\deg(\mathrm{hcf}(f, g)) \leq n$.

$\deg(f) = m$ $\deg(g) = n$

$m \geq n$

Suppose tha $hcf(f,g) > n$. Then $g(x)$ ccu be written as a product of n_k terms p_k where $\deg(p_1 \cdots p_n) > n$ this implies that $\deg(g(x)) > n$ contr.

$\Rightarrow \deg(hcf(f,g)) \leq n$

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iii) Given $\mathrm{hcf}(f, g) = k$, where k is a constant, how many roots do f and g share?

Let's assue that f, g share at least a root.

Then we know that $\exists\, \lambda \in \mathbb{R}$ such that

$f(\lambda) = g(\lambda) = 0$. This mean that both f, g are divisible by $(x-\lambda)$. Now $\deg(hcf(f,g)) \geq 1$

this is a contr. becaus $\deg(hcf(f,g)) = 0$

$= \deg(\text{cost})$ $\Rightarrow$ share no root.

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iv) Let $\deg(f) = \deg(g) = m$.
Given $\deg(\text{hcf}(f,g)) = m$, explain why $f(x) = Ag(x)$ for some constant A.

Suppose they have no com factor

$hcf(f,g) = 0$ if $m=0$ f, g are cost.

there exist a const. $\frac{f}{g} = A \Rightarrow f = Ag$

$m \neq 0$ then suppose n com factors

$n < m$ $\deg(hcf(f,g)) = n < m$

so they share all comon factor

$$\Rightarrow \frac{f}{g} = \frac{c_1}{c_2} \frac{\cancel{p_1 \cdots p_m}}{\cancel{p_1 \cdots p_m}} \Rightarrow \frac{f}{g} = \frac{c_1}{c_2}$$

$f = \frac{c_1}{c_2} g$ where $\frac{c_1}{c_2} = A$ $\Rightarrow f = Ag$

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v) Let $\deg(f) = m$, where $m > 1$. Let $g = f(-x)$.

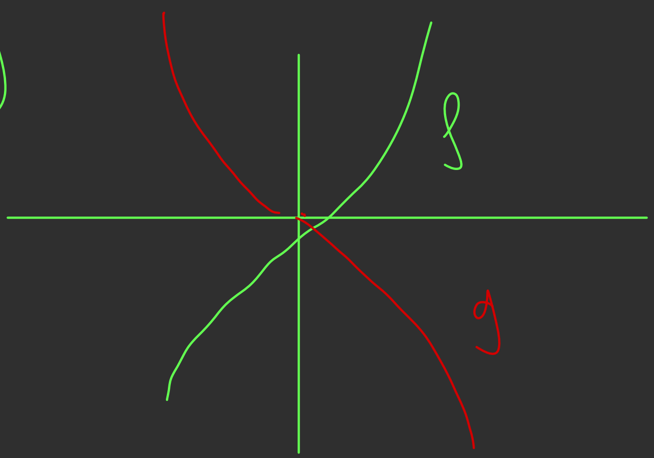
a) Describe the graph transformation that maps f to g.

b) Given $\deg(\text{hcf}(f, g)) = 1$, deduce that $f(0) = 0$.

c) Given that $\deg(\text{hcf}(f, g)) = m$ and $f(0) \neq 0$, show that m is an even number.

$\deg(f) = m \quad g(x) = f(-x)$

a)



$g(x)$ acts as a reflect. around the y axis.

b) $\deg(\text{hcf}(f, g)) = 1 \quad f(0) = 0$

f, g are divis. by $(x - \lambda)$ λ is a root of f, g

$\Rightarrow f(\lambda) = 0 = g(\lambda)$. $g(\lambda) = 0 \Rightarrow f(-\lambda) = 0$

$-\lambda$ is also a root of g, f

$\Rightarrow$ both. g, f are div. by $x + \lambda$

$\Rightarrow$ g, f are divis. by $(x+\lambda)(x-\lambda)$

if $\lambda \neq 0$ then $\text{hcf}(f, g) = (x+\lambda)(x-\lambda)$

$\deg(\text{hcf}) = 2 \neq 1 \Rightarrow \lambda = 0$

$\Rightarrow f(x) = g(x) = f(0) : g(0) = 0 \Rightarrow f(0) = 0$ ✓

c) Since $\gcd(f, g) = m$ and $g(x) = f(-x)$

then $\deg(g) = \deg(f) \Rightarrow \deg(\gcd(f, g)) = \deg(f) = \deg(g)$

$= m$

$\Rightarrow f(x) = k\,g(x)$. $f(x) = k\,f(-x)$

$f(-x) = k\,f(x)$ $\qquad f(-x) = k^2 g(x)$

$\Rightarrow f(-x) = k^2 f(-x)$ $\qquad f(x) \neq 0 \quad \forall x$.

$\Rightarrow k^2 = 1 \Rightarrow k = \pm 1$

- $k = -1$ $\quad f(-x) = -f(-x)$ $\qquad f(y) = -f(y)$

this means tha f is odd.

$\Rightarrow f(0) = 0$ for odd simmetry properties

contz. $f(0) \neq 0$

- $k = 1$ $\quad f(x) = g(x) \Rightarrow f(x) = f(-x)$

$\Rightarrow f$ is even !

The fact that for a polynomial to be even it needs to be the sum of even power of of x. $\Rightarrow$ $\deg(hcf(f,g)) = m$ is even for f to be even. ✓

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vi) Let $\deg(f) = m$.
Given $\text{hcf}(f, f') = (x - r_1)\cdots(x - r_{m-1})$ and $f(x_0) = y_0 \neq 0$, explain how we can find the function $f(x)$.
(DO NOT find the function; give a worded explanation only)

$\deg(f) = m$

$hcf(f, f') = (x - r_1)\cdots(x - r_{m-1})$

Derivat. oper. $D: \mathbb{R}[t]_m \to \mathbb{R}[t]_{m-1}$

$\Rightarrow hcf(f, f') = (x - r_1)\cdots(x - r_{m-1})$

$\deg(f') = m-1 \qquad \deg(f) = m$

$\deg(hcf) = \deg(f')$

$\Rightarrow f' = k(hcf(f, f'))$

$f = \int k(hcf(f, f'))\,dx$

$$f = k \int \mathrm{hcf}(f, f') \, dx$$

$$f = k \int (x - r_1) \cdots (x - r_{m-1}) \, dx$$

$$f = k(p_m) + c$$

$$\mathrm{hcf}(f, f') = k f'$$

any root of f' is also a root of f

$$\Rightarrow f(\lambda) = c \qquad \text{where } \lambda \text{ is a root of } f'$$

$$f = k(p_m) + f(\lambda)$$

$$\Rightarrow y_0 = k\, p_m(x_0) + f(\lambda)$$

$$k = \frac{y_0 - f(\lambda)}{p_m(x_0)}$$