HOW TO GRAPH FUNCTIONS!!

9. (a) Briefly analyze the function

$$f(x) = \frac{(x-1)^2}{\sqrt{x}}$$

(domain, sign, limits at the boundaries of the domain, differentiability, and monotonicity) and provide a qualitative sketch of its graph. (b) Assert if the integral function

$$F(x) = \int_2^x f(t)dt$$

is well defined over $[0, +\infty[$. Briefly analyze the function (sign, limits at the boundaries of the domain, asymptotes, differentiability, monotonicity, extrema, and concavity) and sketch a qualitative graph.

$$f(x) = \frac{(x-1)^2}{\sqrt{x}}$$

$\sqrt{x} \neq 0 \wedge x \geq 0 \Rightarrow \boxed{x > 0}$

$\text{dom}(f) =]0, +\infty[$

$f(x) \geq 0 \quad \frac{(x-1)^2}{\sqrt{x}} \geq 0 \quad (x-1)^2 \geq 0$

$f(x) = 0 \Leftrightarrow x - 1 = 0 \Rightarrow x = 1$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{(x-1)^2}{\sqrt{x}} = \left[\frac{1}{0^+}\right] = +\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{(x-1)^2}{\sqrt{x}} \sim \frac{x^2}{\sqrt{x}} = x^{\frac{3}{2}} \to +\infty$$

$$f'(x) = \frac{2(x-1)(\sqrt{x}) - (x-1)^2 \frac{1}{2\sqrt{x}}}{x} = \frac{2(x-1)x - \frac{1}{2}(x-1)^2}{x\sqrt{x}}$$

$$= \frac{1}{2x\sqrt{x}} \cdot \left(4(x-1)x - (x-1)^2\right)$$

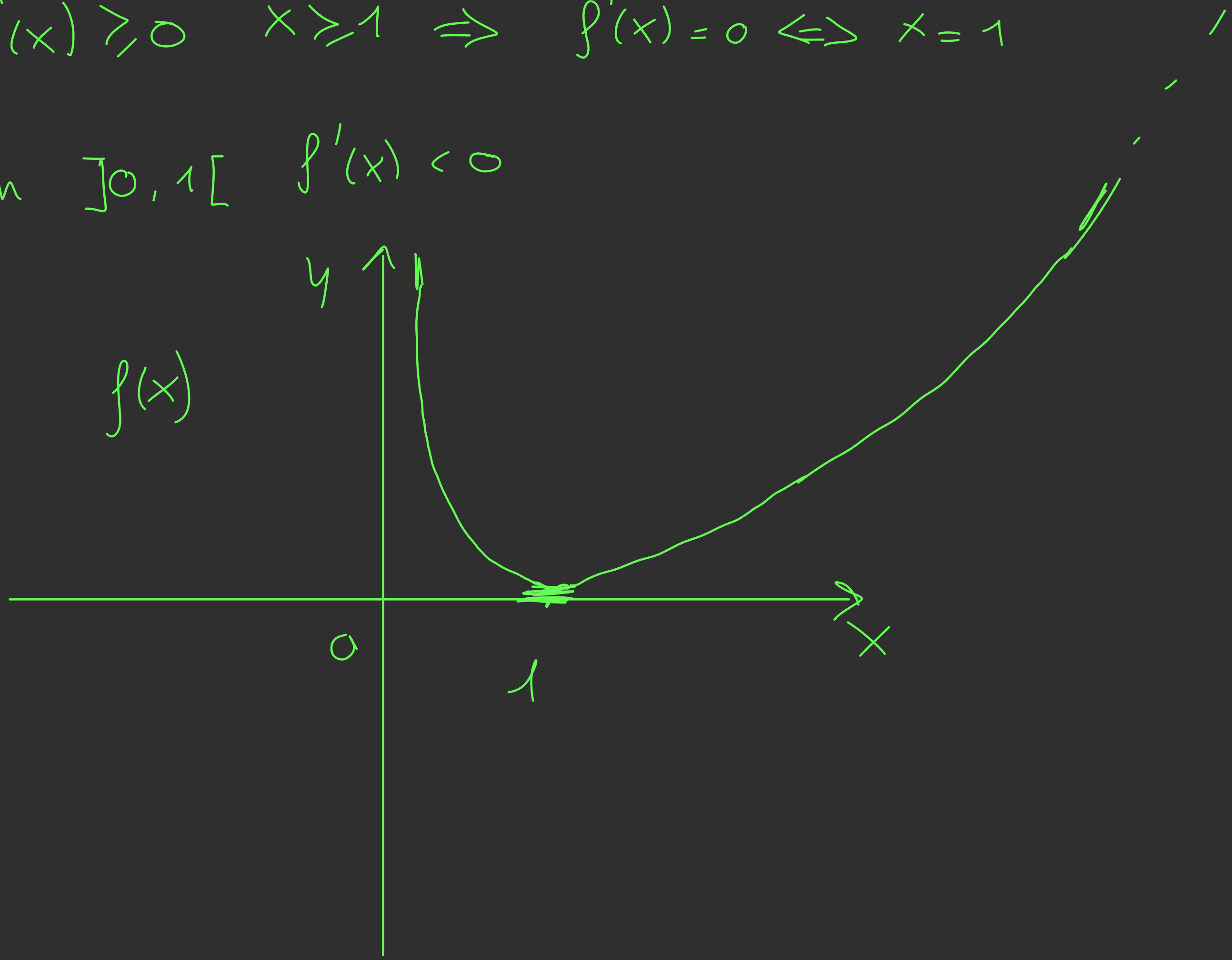
$$= \frac{1}{2x\sqrt{x}} \cdot (x-1)\left(4x-(x-1)\right) = \frac{(x-1)(3x+1)}{2x\sqrt{x}}$$

$$f'(x) = \frac{(x-1)(3x+1)}{2x\sqrt{x}} \geq 0 \qquad (x-1)(3x+1) \geq 0$$

$$x \geq 1 \qquad x \geq -\frac{1}{2}$$

$$f'(x) \geq 0 \quad x \geq 1 \Rightarrow f'(x) = 0 \Leftrightarrow x = 1$$

in $]0,1[$ $\quad f'(x) < 0$



$$F(x) = \int_2^x f(t)\,dt \qquad f(t) \geq 0 \quad \forall x \in]0,+\infty[$$

$F(x) \geq 0$ if $x \geq 2$

- $x < 2$ $\quad F(x) = \int_2^x f(t)\,dt = -\underbrace{\int_x^2 f(t)\,dt}_{>0}$

$F(x) = 0 \iff x = 2$

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} -\int_x^2 f(t)\,dt$$

$f(t) \underset{x \to 0^+}{\sim} \dfrac{1}{\sqrt{x}}$ $\quad f(t) \sim \dfrac{1}{x^\alpha}$ $\quad \alpha < 1$

$x^{\frac{1}{2}}$ $\quad \frac{1}{2} < 1 \Rightarrow \int_0^2 f(t)\,dt$ converges

$\Rightarrow$ the lim $\exists$ and it's finite

$$\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} \int_0^\infty f(t)\,dt$$

$f(t) \sim x^{\frac{3}{2}}$ $\quad f(x) \sim \dfrac{1}{x^\alpha}$ $\quad \alpha > 2$

$\Rightarrow \lim_{x \to +\infty} F(x) = +\infty$

$$F'(x) = f(x) = \frac{(x-1)^2}{\sqrt{x}}$$

$F'(x) = 0 \iff x = 1$

$F'(x) \geq 0$

$F''(x) = f'(x) = \dfrac{(x-1)(3x+1)}{2x\sqrt{x}}$

$x = 1$ $\quad F''(x) < 0 \quad 0 < x < 1 \quad F''(x) > 0 \quad x > 1$

$F'(x) \quad x \to 0^+$

$\lim_{x \to 0^+} F'(x) = \lim_{x \to 0^+} f(x) = +\infty$