Exploring generalized logarithmic integrals using Lewin's polylogarithm inversion formula

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Abstract

Closed-form expressions for the five generalized logarithmic integrals

$$\int_0^1 \frac{x\ln(1+x)\ln^{2a}(x)}{1+x^4}\mathrm{d}x,\ \int_0^1 \frac{x\ln(1-x)\ln^{2a}(x)}{1+x^4}\mathrm{d}x,\ \int_0^1 \frac{\arctan(x)\ln^{2a-1}(x)}{1+x^2}\mathrm{d}x,$$

$$\int_0^1 \frac{\arctan(x)\ln^{a}(x)}{1+x}\mathrm{d}x,\ \text{and}\ \int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x}\mathrm{d}x. \tag{1}$$

New proofs for the four generalized logarithmic integrals

$$\int_0^1 \frac{\ln(1-x)\ln^{2a-1}(x)}{1+x}\mathrm{d}x,\ \int_0^1 \frac{x\ln(1+x)\ln^{2a-1}(x)}{1+x^2}\mathrm{d}x,\ \int_0^1 \frac{x\ln(1-x)\ln^{2a-1}(x)}{1+x^2}\mathrm{d}x,$$

$$\text{and}\ \int_0^1 \frac{\operatorname{arctanh}(x)\ln^{2a-1}(x)}{1-x^2}\mathrm{d}x, \tag{2}$$

where a is a positive integer greater than or equal to zero. Closed-form expressions for closely related generalized Euler-like sums are also presented.

1. Introduction

In [7, pp. 182–191], we derived the closed-form expressions for the five logarithmic integrals

$$\int_0^1 \frac{\ln(1+x)\ln^{2a-1}(x)}{1-x}\mathrm{d}x,\ \int_0^1 \frac{\ln(1+x)\ln^{2a}(x)}{1+x^2}\mathrm{d}x,\ \int_0^1 \frac{\ln(1-x)\ln^{2a}(x)}{1+x^2}\mathrm{d}x,$$

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln^{2a}(x)}{1+x^2}\mathrm{d}x,\ \text{and}\ \int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x^2}\mathrm{d}x$$

by utilizing parameterized logarithmic integrals given in Lemmas 6 and 7. We intend in this paper to present further generalized logarithmic integrals by applying the aforementioned lemmas, as well as by employing the Leibniz integral rule.

For the first integral in (2) we give a new proof that leads to the same form of an existing result already known in the literature [11, p. 37] while for the second and third integrals we give new proofs that lead to alternative forms of existing results already known in the literature [11, p. 64-65]. The result of the last integral in (2) will be proved by using only the result of the classical Euler's sum [3, p. 20, Thm 2.2]. As for the integrals in (1), their closed-form expressions are believed to be completely new.

The results of the generalized Euler-like sums

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2a+1)}}{2n+1}, \text{ and } \sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{(2n)^{2a}}, \tag{3}$$

where $H_n^{(a)}=\sum_{k=1}^{n}\frac{1}{k^a}$ is the nth generalized harmonic number of order a, are also presented in this paper. It's worth noting that by using the parity, the two sums in (3) can be rewritten as

$$\mathfrak{I}\sum_{n=1}^{\infty}\frac{i^n H_n^{(2a+1)}}{n}, \text{ and } \mathfrak{R}\sum_{n=1}^{\infty}\frac{i^n H_n}{n^{2a}}.$$

Also note that by using the result of the second sum in (3) along with the results of the classical Euler's sum and the alternating Euler's sum (see [2, II.1, pp. 4–7] and [11, p. 421]), we can find the closed-form expression of

$$\sum_{n=1}^{\infty}\frac{H_{4n}}{n^{2a}}. \tag{4}$$

The results of the three generalized sums in (3) and (4) are believed to be new since they are closely related to the third and fifth integrals in (1).

2. Special functions

In this section we present a number of special functions that are going to be needed for proving the main results in this paper. We start with the digamma function, defined by

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)},$$

this is entry 1.3.1 in [9], and the polygamma function, defined by

$$\psi^{(n)}(x)=\frac{d^n}{dx^n}\psi(x)=\frac{d^{n+1}}{dx^{n+1}}\ln\Gamma(x), \quad n\in\mathbb{Z}_{\geq 0}, \tag{5}$$

this is entry 6.4.1 in [1], where $\Gamma(x)$ is the gamma function defined by

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}\mathrm{d}t.$$

The Riemann zeta function is defined by

$$\zeta(s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} & \Re(s) > 1 \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \Re(s) > 0;\ s \neq 0 \end{cases}$$

This is entry 2.3.1 in [9]. The Dirichlet eta function is defined by

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \Re(s) > 0.$$

By comparing the definitions of zeta and eta functions, we see that

$$\eta(s) = (1-2^{1-s})\zeta(s), \quad \Re(s) > 0;\ s \neq 0. \tag{6}$$

The Dirichlet beta function is defined by

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad \Re(s) > 0.$$

This is entry 3.6.4 in [8]. The polylogarithm function is defined by

$$\mathrm{Li}_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a}, \quad |z| \leq 1.$$

This is entry 7.1.1 in [6].

3. The lemmas and their proofs

Lemma 1. *If $n \in \mathbb{Z}_{\geq 1}$ then*

$$\int_0^1 \frac{\ln^n(x)}{1-x}\, dx = (-1)^n n! \zeta(n+1).$$

Proof. Expand $\frac{1}{1-x}$ in series, interchange the integration and summation signs, integrate by parts n times, and then use the definition of Riemann zeta function. □

Also note that by integration by parts, we have

$$\int_0^1 \frac{\ln(1-x)\ln^n(x)}{x} \mathrm{d}x = \frac{1}{n+1} \int_0^1 \frac{\ln^{n+1}(x)}{1-x} \mathrm{d}x = (-1)^{n+1} n! \zeta(n+2). \tag{7}$$

Lemma 2. *If $n \in \mathbb{Z}_{\geq 0}$ then*

$$\int_0^1 \frac{\ln^n(x)}{1+x}\, dx = (-1)^n n! \eta(n+1).$$

Proof. Expand $\frac{1}{1+x}$ in series and use the definition of the Dirichlet eta function. $\square$

We also have, by integration by parts, that

$$\int_0^1 \frac{\ln(1+x)\ln^n(x)}{x}\mathrm{d}x = -\frac{1}{n+1}\int_0^1 \frac{\ln^{n+1}(x)}{1+x}\mathrm{d}x = (-1)^n n!\eta(n+2). \tag{8}$$

Lemma 3. *If $n \in \mathbb{Z}_{\geq 0}$ then*

$$\int_0^1 \frac{\ln^n(x)}{1+x^2}dx = (-1)^n n!\beta(n+1).$$

Proof. Expand $\frac{1}{1+x^2}$ in series and use the definition of the Dirichlet beta function. $\square$

Lemma 4. *If $n \in \mathbb{Z}_{\geq 1}$ and $x \leq 1$ then*

$$\int_0^1 \frac{x\ln^n(y)}{1-xy}dy = (-1)^n n!\operatorname{Li}_{n+1}(x).$$

Proof. Expand $\frac{1}{1-xy}$ in series and use the definition of the polylogarithm function. $\square$

Lemma 5. *If $p > 0$, $q > 0$ then*

$$\int_0^1 \frac{x^{p-1}\ln^n(x)}{1-x^q}dx = -\frac{1}{q^{n+1}}\psi^{(n)}\left(\frac{p}{q}\right).$$

This is entry 4.271.15 in [5].

Lemma 6. *If $x > 0$ then*

$$(a)\quad \int_0^1 \frac{(1+2xy+x^2)\ln^{2a-1}(y)}{(1+xy)(x+y)}dy = -2(2a-1)!\sum_{k=0}^{a}\frac{\eta(2k)}{(2a-2k)!}\ln^{2a-2k}(x);$$

$$(b)\quad \int_0^1 \frac{(1-x^2)\ln^{2a}(y)}{(1+xy)(x+y)}dy = -2(2a)!\sum_{k=0}^{a}\frac{\eta(2k)}{(2a-2k+1)!}\ln^{2a-2k+1}(x).$$

Proof. In Lewin's book [6, p. 192], we have the polylogarithm inversion formula,

$$\operatorname{Li}_n(-x) + (-1)^n\operatorname{Li}_n\left(-\frac{1}{x}\right) = -\frac{\ln^n(x)}{n!} - 2\sum_{k=1}^{\lfloor \frac{n}{2}\rfloor}\frac{\eta(2k)}{(n-2k)!}\ln^{n-2k}(x), \quad x > 0,$$

where $\lfloor\cdot\rfloor$ is the floor function. Add and subtract the term $(k=0)$ using $\eta(0) = \frac{1}{2}$ given in [7, p. 24], we write

$$\operatorname{Li}_n(-x) + (-1)^n\operatorname{Li}_n\left(-\frac{1}{x}\right) = -2\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{\eta(2k)}{(n-2k)!}\ln^{n-2k}(x).$$

Set $n = 2a$ and write the integral form of the polylogarithm function given in Lemma 4, we complete the proof of part (a) while the proof of part (b) completes on setting $n = 2a+1$. $\square$

Lemma 7. *If $x<1$ then*

$$(a)\quad \mathrm{P.V.}\int_0^1 \frac{(1-2xy+x^2)\ln^{2a-1}(y)}{(1-xy)(x-y)}dy = -2(2a-1)!\sum_{k=0}^{a}\frac{\zeta(2k)}{(2a-2k)!}\ln^{2a-2k}(x);$$

$$(b)\quad \mathrm{P.V.}\int_0^1 \frac{(1-x^2)\ln^{2a}(y)}{(1-xy)(x-y)}dy = -2(2a)!\sum_{k=0}^{a}\frac{\zeta(2k)}{(2a-2k+1)!}\ln^{2a-2k+1}(x),$$

where P.V. represents the Cauchy principal value.

Proof. In Lewin's book [6, p. 5], we have the dilogarithm inversion formula,

$$\mathrm{Li}_2(x)+\mathrm{Li}_2\left(\frac{1}{x}\right) = -i\pi\ln(x)-\frac{1}{2}\ln^2(x)+2\zeta(2), \quad x>1.$$

Let $x\to\frac{1}{x}$ to change the range of x, and then consider the real parts of both sides,

$$\Re\left\{\mathrm{Li}_2(x)+\mathrm{Li}_2\left(\frac{1}{x}\right)\right\} = -\frac{1}{2}\ln^2(x)+2\zeta(2), \quad x<1.$$

Now divide both sides by x, and then integrate from $x=1$ to x repeatedly, we find

$$\Re\left\{\mathrm{Li}_n(x)+(-1)^n\,\mathrm{Li}_n\left(\frac{1}{x}\right)\right\} = 2\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{\zeta(2k)}{(n-2k)!}\ln^{n-2k}(x), \quad x<1.$$

Set $n=2a$ and write the integral form of the polylogarithm function, and then consider the Cauchy principal value (P.V.) due to the singularities generated by the values of x, we complete the proof of part (a) while the proof of part (b) completes on setting $n=2a+1$. □

4. The main theorems and their proofs

Theorem 1. *Let a be a positive integer. Then the following equality holds:*

$$\int_0^1 \frac{\ln(1-x)\ln^{2a-1}(x)}{1+x}dx = -(2a-1)!\sum_{k=0}^{a-1}\zeta(2k)\eta(2a-2k+1)$$
$$+(2a-1)!\left(\frac{1}{2}\zeta(2a+1)+a\eta(2a+1)-\ln(2)\eta(2a)-\ln(2)\zeta(2a)\right).$$

Proof. Divide both sides of part (a) of Lemma 7 by $1+x$, and then integrate from $x=0$ to 1 using Lemma 2, we obtain

$$-2(2a-1)!\sum_{k=0}^{a}\zeta(2k)\eta(2a-2k+1) = \mathrm{P.V.}\int_0^1\int_0^1 \frac{(1-2xy+x^2)\ln^{2a-1}(y)}{(1+x)(1-xy)(x-y)}\mathrm{d}y\mathrm{d}x$$

change the order of integration

$$= \int_0^1 \ln^{2a-1}(y) \left(\text{P.V.} \int_0^1 \frac{1-2xy+x^2}{(1+x)(1-xy)(x-y)} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^1 \ln^{2a-1}(y) \left(\frac{2\ln(1-y)}{1+y} - \frac{\ln(1-y)}{y} - \frac{\ln(y)}{1+y} - \frac{2\ln(2)}{1+y} \right) \mathrm{d}y.$$

By utilizing (7) and Lemma 2, the last three integrals evaluate to

$$-2(2a-1)! \left(\frac{1}{2}\zeta(2a+1) + a\eta(2a+1) - \ln(2)\eta(2a) \right).$$

The proof completes on writing

$$\sum_{k=0}^{a} \zeta(2k)\eta(2a-2k+1) = \ln(2)\zeta(2a) + \sum_{k=0}^{a-1} \zeta(2k)\eta(2a-2k+1),$$

which follows from separating the last term ($k = a$). □

A different proof can be found in [11, p. 37].

Example 1. Setting $a = 1, 2,$ and 3 in Theorem 1 the first three integrals are:

$$\int_0^1 \frac{\ln(x)\ln(1-x)}{1+x} \mathrm{d}x = \frac{13}{8}\zeta(3) - \frac{3}{2}\ln(2)\zeta(2);$$

$$\int_0^1 \frac{\ln^3(x)\ln(1-x)}{1+x} \mathrm{d}x = \frac{273}{16}\zeta(5) - \frac{9}{2}\zeta(2)\zeta(3) - \frac{45}{4}\ln(2)\zeta(4);$$

$$\int_0^1 \frac{\ln^5(x)\ln(1-x)}{1+x} \mathrm{d}x = \frac{7575}{16}\zeta(7) - \frac{225}{2}\zeta(2)\zeta(5) - 90\zeta(3)\zeta(4) - \frac{945}{4}\ln(2)\zeta(6).$$

Theorem 2. *Let a be a positive integer. Then the following equality holds:*

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)\ln^{2a-2}(x)}{x} dx = (2a-2)! \left(a - 2^{-2a-1}\right) \zeta(2a+1)$$

$$+ (2a-2)! \sum_{k=0}^{a-1} 2^{1-2k}\zeta(2k)\eta(2a-2k+1).$$

Proof. By forcing integration by parts, one has

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)\ln^{2a-2}(x)}{x} \mathrm{d}x = \frac{1}{2a-1} \int_0^1 \ln^{2a-1}(x) \left(\frac{\ln(1+x)}{1-x} - \frac{\ln(1-x)}{1+x} \right) \mathrm{d}x.$$

The proof completes on substituting the result of the first integral given in [7, pp. 181–182]:

$$\int_0^1 \frac{\ln(1+x)\ln^{2a-1}(x)}{1-x} \mathrm{d}x = -(2a-1)! \sum_{k=0}^{a-1} \eta(2k)\eta(2a-2k+1)$$

$$+ (2a-1)! \left(\frac{1}{2}\eta(2a+1) + a\zeta(2a+1) - \ln(2)\zeta(2a) - \ln(2)\eta(2a) \right),$$

and the result of Theorem 1, and then applying $\eta(s) = (1 - 2^{1-s})\zeta(s)$ given in (6). □

An alternative proof can be found in [10, p. 6].

Example 2. Setting $a = 1, 2$, and 3 in Theorem 2 the first three integrals are:

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}\mathrm{d}x = -\frac{5}{8}\zeta(3);$$
$$\int_0^1 \frac{\ln(1-x)\ln(1+x)\ln^2(x)}{x}\mathrm{d}x = \frac{3}{4}\zeta(2)\zeta(3) - \frac{27}{16}\zeta(5);$$
$$\int_0^1 \frac{\ln(1-x)\ln(1+x)\ln^4(x)}{x}\mathrm{d}x = \frac{9}{4}\zeta(3)\zeta(4) + \frac{45}{4}\zeta(2)\zeta(5) - \frac{363}{16}\zeta(7).$$

Theorem 3. *Let a be a positive integer. Then the following equality holds:*

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln^{2a-1}(x)}{1-x^2}dx = (2a-1)!\sum_{k=0}^{a-1}\left(2^{-2k} - 2^{-2a-1}\right)\zeta(2k)\zeta(2a-2k+1).$$

Proof. Replace x with x^2 in part (a) of Lemma 7, and then divide both sides by $1+x$ and integrate from $x = 0$ to 1 using Lemma 2, we obtain

$$-2(2a-1)!\sum_{k=0}^{a} 2^{2a-2k}\zeta(2k)\eta(2a-2k+1) = \mathrm{P.V.}\int_0^1\int_0^1 \frac{(1-2x^2y+x^4)\ln^{2a-1}(y)}{(1+x)(1-x^2y)(x^2-y)}\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \ln^{2a-1}(y)\left(\mathrm{P.V.}\int_0^1 \frac{1-2x^2y+x^4}{(1+x)(1-x^2y)(x^2-y)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \ln^{2a-1}(y)\left(\frac{-2\operatorname{arctanh}(\sqrt{y})}{(1-y)\sqrt{y}} - \frac{\ln(1-y)}{y(1-y)} + \frac{\ln(1-y)}{2y} + \frac{\ln(y)}{2(1-y)} + \frac{2\ln(2)}{1-y}\right)\mathrm{d}y$$

let $\sqrt{y} \to y$ in the first integral

$$= \int_0^1 \ln^{2a-1}(y)\left(\frac{-2^{2a+1}\operatorname{arctanh}(y)}{1-y^2} - \frac{\ln(1-y)}{y(1-y)} + \frac{\ln(1-y)}{2y} + \frac{\ln(y)}{2(1-y)} + \frac{2\ln(2)}{1-y}\right)\mathrm{d}y.$$

By using (7) and Lemma 1, the last three integrals reduce to

$$(2a-1)!\left(\frac{1}{2}(2a+1)\zeta(2a+1) - 2\ln(2)\zeta(2a)\right).$$

For the second integral, we get the generating function given in [5, p. 54]:

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}, \quad -1 \le x < 1. \tag{9}$$

Employing this allows one to rewrite the integral as

$$\int_0^1 \frac{\ln(1-y)\ln^{2a-1}(y)}{y(1-y)}\mathrm{d}y = -\sum_{n=1}^{\infty} H_n\int_0^1 y^{n-1}\ln^{2a-1}(y)\mathrm{d}y = (2a-1)!\sum_{n=1}^{\infty}\frac{H_n}{n^{2a}}.$$

Reorder the terms,

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln^{2a-1}(x)}{1-x^2}\mathrm{d}x = \frac{(2a-1)!}{2^{2a+1}}\left(\frac{1}{2}(2a+1)\zeta(2a+1) - 2\ln(2)\zeta(2a)\right)$$

$$+(2a-1)!\sum_{k=0}^{a} 2^{-2k}\zeta(2k)\eta(2a-2k+1) - \frac{(2a-1)!}{2^{2a+1}}\sum_{n=1}^{\infty}\frac{H_n}{n^{2a}}. \tag{10}$$

For the first sum, separate the last term, and then apply $\eta(s) = (1-2^{1-s})\zeta(s)$,

$$\sum_{k=0}^{a} 2^{-2k}\zeta(2k)\eta(2a-2k+1) = \frac{\ln(2)}{2^{2a}}\zeta(2a) + \sum_{k=0}^{a-1}(2^{-2k}-2^{-2a})\zeta(2k)\zeta(2a-2k+1). \tag{11}$$

For the second sum, set $q = 2a$ in the classical Euler's sum given in [3, p. 20, Thm 2.2]:

$$\sum_{n=1}^{\infty}\frac{H_n}{n^q} = \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(q-k)\zeta(k+1), \quad q \geq 2$$

to have

$$\sum_{n=1}^{\infty}\frac{H_n}{n^{2a}} = (a+1)\zeta(2a+1) - \frac{1}{2}\sum_{k=1}^{2a-2}\zeta(2a-k)\zeta(k+1)$$

use $\sum_{k=1}^{2a-2} a_k = \sum_{k=1}^{a-1} a_{2k-1} + \sum_{k=1}^{a-1} a_{2k}$

$$= (a+1)\zeta(2a+1) - \frac{1}{2}\sum_{k=1}^{a-1}\zeta(2a-2k+1)\zeta(2k) - \frac{1}{2}\sum_{k=1}^{a-1}\zeta(2a-2k)\zeta(2k+1)$$

reverse the terms order of the second sum by letting $k \to a-k$

$$= (a+1)\zeta(2a+1) - \sum_{k=1}^{a-1}\zeta(2k)\zeta(2a-2k+1).$$

Add and subtract the the term $(k=0)$ using $\zeta(0) = -\frac{1}{2}$ given in [7, p. 26], we get

$$\sum_{n=1}^{\infty}\frac{H_n}{n^{2a}} = \frac{1}{2}(2a+1)\zeta(2a+1) - \sum_{k=0}^{a-1}\zeta(2k)\zeta(2a-2k+1). \tag{12}$$

The proof completes on plugging (11) and (12) in (10). □

We also have, by integration by parts, that

$$\int_0^1 \frac{\operatorname{arctanh}^2(x)\ln^{2a-2}(x)}{x}\mathrm{d}x = -\frac{2}{2a-1}\int_0^1 \frac{\operatorname{arctanh}(x)\ln^{2a-1}(x)}{1-x^2}\mathrm{d}x.$$

Substituting the result of Theorem 3 gives

$$\int_0^1 \frac{\operatorname{arctanh}^2(x)\ln^{2a-2}(x)}{x}\mathrm{d}x = \sum_{k=0}^{a-1}\left(2^{-2a} - 2^{1-2k}\right)\zeta(2k)\zeta(2a-2k+1). \tag{13}$$

Example 3. Setting $a = 1, 2$, and 3 in Theorem 3 the first three integrals are:

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln(x)}{1-x^2}\mathrm{d}x = -\frac{7}{16}\zeta(3);$$

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln^3(x)}{1-x^2}\mathrm{d}x = -\frac{93}{32}\zeta(5) + \frac{21}{16}\zeta(2)\zeta(3);$$

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln^5(x)}{1-x^2}\mathrm{d}x = -\frac{1905}{32}\zeta(7) + \frac{465}{16}\zeta(2)\zeta(5) + \frac{105}{16}\zeta(3)\zeta(4).$$

Theorem 4. *Let a be a positive integer. Then the following equality holds:*

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^{2a}} = -\frac{1}{2}(2a+1)\eta(2a+1) + \sum_{k=0}^{a-1} \eta(2k)\zeta(2a-2k+1).$$

Proof. We get the Cauchy product of $\frac{\operatorname{arctanh}(x)}{1-x^2}$ given in [7, p. 100]:

$$\frac{\operatorname{arctanh}(x)}{1-x^2} = \frac{1}{2}\sum_{n=1}^{\infty}(2H_{2n} - H_n)x^{2n-1}. \tag{14}$$

Using this, one has

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln^{2a-1}(x)}{1-x^2}\mathrm{d}x = \frac{1}{2}\sum_{n=1}^{\infty}(2H_{2n} - H_n)\int_0^1 x^{2n-1}\ln^{2a-1}(x)\mathrm{d}x$$

$$= -\frac{1}{2}(2a-1)!\sum_{n=1}^{\infty}\frac{2H_{2n} - H_n}{(2n)^{2a}}$$

apply $\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n$ for the first sum

$$= -\frac{1}{2}(2a-1)!\left(\left(1-2^{-2a}\right)\sum_{n=1}^{\infty}\frac{H_n}{n^{2a}} + \sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2a}}\right).$$

Recall the relation (10), and then reorganize the terms,

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2a}} = 2^{1-2a}\ln(2)\zeta(2a) - \frac{2a+1}{2^{2a+1}}\zeta(2a+1)$$
$$-2\sum_{k=0}^{a} 2^{-2k}\zeta(2k)\eta(2a-2k+1) - \left(1-2^{1-2a}\right)\sum_{n=1}^{\infty}\frac{H_n}{n^{2a}}.$$

Substitute the sums (11) and (12), and then use $\eta(s) = (1-2^{1-s})\zeta(s)$ to finish the proof. $\square$

Different proofs can be found in [2, II.1, pp. 4–7] and [11, p. 421]. In addition, Cornel Vălean evaluated this sum in a 2019 preprint published on the ResearchGate platform titled *"A New Powerful Strategy for Calculating a Class of Alternating Euler Sums"*.

Theorem 5. *Let a be a positive integer. Then the following equality holds:*

$$\int_0^1 \frac{\arctan(x)\ln^{2a-1}(x)}{1+x^2}dx = (2a-1)!\left(-\frac{\pi}{4}\beta(2a) + \sum_{k=0}^{a-1}\left(2^{-2k} - 2^{-2a-1}\right)\eta(2k)\zeta(2a-2k+1)\right).$$

Proof. Replace x with x^2 in part (a) of Lemma 6, and then divide both sides by $1+x$ and integrate from $x=0$ to 1 using Lemma 2, we obtain

$$-2(2a-1)!\sum_{k=0}^{a} 2^{2a-2k}\eta(2k)\eta(2a-2k+1) = \int_0^1\int_0^1 \frac{(1+2x^2y+x^4)\ln^{2a-1}(y)}{(1+x)(1+x^2y)(x^2+y)}\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \ln^{2a-1}(y)\left(\int_0^1 \frac{1+2x^2y+x^4}{(1+x)(1+x^2y)(x^2+y)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \ln^{2a-1}(y)\left(\frac{-2\arctan(\sqrt{y})}{(1+y)\sqrt{y}} + \frac{\ln(1+y)}{y(1+y)} - \frac{\ln(1+y)}{2y} + \frac{\ln(y)}{2(1+y)} + \frac{2\ln(2)}{1+y} + \frac{\pi}{2(1+y)\sqrt{y}}\right)\mathrm{d}y$$

let $\sqrt{y} \to y$ in the first and last integrals

$$= \int_0^1 \ln^{2a-1}(y)\left(\frac{-2^{2a+1}\arctan(y)}{1+y^2} + \frac{\ln(1+y)}{y(1+y)} - \frac{\ln(1+y)}{2y} + \frac{\ln(y)}{2(1+y)} + \frac{2\ln(2)}{1+y} + \frac{2^{2a-1}\pi}{1+y^2}\right)\mathrm{d}y.$$

By employing (8) and Lemmas 2 and 3, the last four integrals evaluate to

$$(2a-1)!\left(\frac{1}{2}(2a+1)\eta(2a+1) - 2\ln(2)\eta(2a) - 2^{2a-1}\pi\beta(2a)\right).$$

For the second integral, replacing x with $-x$ in (9) gives

$$\frac{\ln(1+x)}{1+x} = -\sum_{n=1}^{\infty} H_n(-x)^n,$$

from which, it follows that

$$\int_0^1 \frac{\ln(1+x)\ln^{2a-1}(x)}{x(1+x)}\mathrm{d}x = -\sum_{n=1}^{\infty}(-1)^n H_n \int_0^1 y^{n-1}\ln^{2a-1}(y)\mathrm{d}y = (2a-1)!\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2a}}.$$

Reorganize the terms,

$$\int_0^1 \frac{\arctan(x)\ln^{2a-1}(x)}{1+x^2}\mathrm{d}x$$

$$= \frac{(2a-1)!}{2^{2a+1}}\left(\frac{1}{2}(2a+1)\eta(2a+1) - 2\ln(2)\eta(2a) - 2^{2a-1}\pi\beta(2a)\right)$$

$$+\frac{(2a-1)!}{2^{2a+1}}\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2a}}+(2a-1)!\sum_{k=0}^{a}2^{-2k}\eta(2k)\eta(2a-2k+1). \tag{15}$$

The proof completes on substituting the result of Theorem 4 and writing

$$\sum_{k=0}^{a}2^{-2k}\eta(2k)\eta(2a-2k+1)=2^{-2a}\ln(2)\eta(2a)+\sum_{k=0}^{a-1}(2^{-2k}-2^{-2a})\eta(2k)\zeta(2a-2k+1), \tag{16}$$

which follows from separating the last term, and then using $\eta(s)=(1-2^{1-s})\zeta(s)$. □

We also have, by integration by parts, that

$$\int_0^1\frac{\arctan^2(x)\ln^{2a-2}(x)}{x}\mathrm{d}x=-\frac{2}{2a-1}\int_0^1\frac{\arctan(x)\ln^{2a-1}(x)}{1+x^2}\mathrm{d}x.$$

Recalling the result of Theorem 5 yields

$$\int_0^1\frac{\arctan^2(x)\ln^{2a-2}(x)}{x}\mathrm{d}x=(2a-2)!\left(\frac{\pi}{2}\beta(2a)+\sum_{k=0}^{a-1}\left(2^{-2a}-2^{1-2k}\right)\eta(2k)\zeta(2a-2k+1)\right). \tag{17}$$

Example 4. Setting $a=1,2,$ and 3 in Theorem 5 the first three integrals are:

$$\int_0^1\frac{\arctan(x)\ln(x)}{1+x^2}\mathrm{d}x=\frac{7}{16}\zeta(3)-\frac{\pi}{4}\beta(2);$$

$$\int_0^1\frac{\arctan(x)\ln^3(x)}{1+x^2}\mathrm{d}x=\frac{93}{32}\zeta(5)+\frac{21}{32}\zeta(2)\zeta(3)-\frac{3\pi}{2}\beta(4);$$

$$\int_0^1\frac{\arctan(x)\ln^5(x)}{1+x^2}\mathrm{d}x=\frac{1905}{32}\zeta(7)+\frac{465}{32}\zeta(2)\zeta(5)+\frac{735}{128}\zeta(3)\zeta(4)-30\pi\beta(6).$$

Theorem 6. *Let a be a positive integer. Then the following equality holds:*

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{n^{2a}}=-\frac{1}{4}(2a+1)\eta(2a+1)-2^{2a-2}\pi\beta(2a)+\sum_{k=0}^{a-1}2^{2a-2k}\eta(2k)\zeta(2a-2k+1).$$

Proof. Replace x with ix in (14) to get

$$\frac{\arctan(x)}{1+x^2}=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(2H_{2n}-H_n)x^{2n-1}, \tag{18}$$

which gives

$$\int_0^1\frac{\arctan(x)\ln^{2a-1}(x)}{1+x^2}\mathrm{d}x=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(2H_{2n}-H_n)\int_0^1 x^{2n-1}\ln^{2a-1}(x)\mathrm{d}x$$

$$= \frac{1}{2}(2a-1)! \sum_{n=1}^{\infty} \frac{(-1)^n (2H_{2n} - H_n)}{(2n)^{2a}}$$

$$= \frac{(2a-1)!}{2^{2a+1}} \left(2 \sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{n^{2a}} - \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^{2a}} \right).$$

Recall the relation (15), and then reorganize the terms,

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{n^{2a}} = \frac{1}{4}(2a+1)\eta(2a+1) - \ln(2)\eta(2a) - 2^{2a-2}\pi\beta(2a)$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^{2a}} + 2^{2a} \sum_{k=0}^{a} 2^{-2k}\eta(2k)\eta(2a-2k+1).$$

The proof completes on substituting the results of Theorem 4 and the sum (16). $\square$

Theorem 7. *Let a be a positive integer. Then the following equality holds:*

$$\sum_{n=1}^{\infty} \frac{H_{4n}}{n^{2a}} = \frac{1}{8}(2a+1)\zeta(2a+1) - \frac{\pi}{8}2^{4a}\beta(2a) - \sum_{k=0}^{a-1} 2^{4a-4k}\zeta(2k)\zeta(2a-2k+1).$$

Proof. Applying $\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n a_n$ gives

$$\sum_{n=1}^{\infty} \frac{H_{4n}}{(4n)^{2a}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^{2a}} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{(2n)^{2a}}$$

$$= \frac{1}{2}\left(\frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n}{n^{2a}} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^{2a}} \right) + \frac{1}{2^{2a+1}} \sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}}{n^{2a}}.$$

The first sum is given in (12) while the other two sums are given in Theorems 4 and 6 respectively. Combining them finishes the proof. $\square$

Theorem 8. *Let a be a positive integer. Then the following equality holds:*

$$\int_0^1 \frac{x\ln(1+x)\ln^{2a-1}(x)}{1+x^2}dx = \frac{(2a-1)!}{2^{2a+1}}\big((a-2^{2a})\eta(2a+1) - \ln(2)\eta(2a)\big)$$

$$+ \frac{(2a-1)!}{2^{2a+1}} \sum_{k=0}^{a} 2^{2k}\eta(2k)\eta(2a-2k+1).$$

Proof. Replace x with $\sqrt{x}$ in part (a) of Lemma 6, and then divide both sides by $1+x$ and integrate from $x=0$ to 1 using Lemma 2, we obtain

$$-\frac{(2a-1)!}{2^{2a-1}} \sum_{k=0}^{a} 2^{2k}\eta(2k)\eta(2a-2k+1) = \int_0^1 \int_0^1 \frac{(1+2\sqrt{x}y+x)\ln^{2a-1}(y)}{(1+x)(1+\sqrt{x}y)(\sqrt{x}+y)} \mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \ln^{2a-1}(y) \left(\int_0^1 \frac{1+2\sqrt{x}y+x}{(1+x)(1+\sqrt{x}y)(\sqrt{x}+y)} \mathrm{d}x \right) \mathrm{d}y$$
$$= \int_0^1 \ln^{2a-1}(y) \left(\frac{-4y\ln(1+y)}{1+y^2} + \frac{2\ln(1+y)}{y} + \frac{2y\ln(y)}{1+y^2} + \frac{2y\ln(2)}{1+y^2} \right) \mathrm{d}y.$$

The second integral is given in (8). For the last two integrals, make the change of variable $y = \sqrt{x}$, and then use Lemma 2. The three integrals reduce to

$$\frac{(2a-1)!}{2^{2a-1}} \left((a - 2^{2a})\eta(2a+1) - \ln(2)\eta(2a) \right),$$

and the proof is completed. ☐

A different proof can be found in [11, p. 64-65].

Example 5. Setting $a = 1, 2,$ and 3 in Theorem 8 the first three integrals are:

$$\int_0^1 \frac{x\ln(1+x)\ln(x)}{1+x^2} \mathrm{d}x = \frac{3}{16}\ln(2)\zeta(2) - \frac{15}{64}\zeta(3);$$
$$\int_0^1 \frac{x\ln(1+x)\ln^3(x)}{1+x^2} \mathrm{d}x = \frac{315}{128}\ln(2)\zeta(4) + \frac{9}{32}\zeta(2)\zeta(3) - \frac{1215}{512}\zeta(5);$$
$$\int_0^1 \frac{x\ln(1+x)\ln^5(x)}{1+x^2} \mathrm{d}x = \frac{29295}{512}\ln(2)\zeta(6) + \frac{315}{32}\zeta(3)\zeta(4) + \frac{225}{128}\zeta(2)\zeta(5) - \frac{114345}{2048}\zeta(7).$$

Theorem 9. *Let a be a positive integer. Then the following equality holds:*

$$\int_0^1 \frac{x\ln(1-x)\ln^{2a-1}(x)}{1+x^2} dx = \frac{(2a-1)!}{2^{2a+1}} \left(a\eta(2a+1) + 2^{-2a}\zeta(2a+1) - \ln(2)\eta(2a) \right)$$
$$- \frac{(2a-1)!}{2^{2a+1}} \sum_{k=0}^{a} 2^{2k}\zeta(2k)\eta(2a-2k+1).$$

Proof. Set $z = \sqrt{x}$ in part (a) of Lemma 7, and then divide both sides by $1+x$ and integrate from $x = 0$ to $x = 1$ using Lemma 2, we obtain

$$-\frac{(2a-1)!}{2^{2a-1}} \sum_{k=0}^{a} 2^{2k}\zeta(2k)\eta(2a-2k+1) = \mathrm{P.V.} \int_0^1 \int_0^1 \frac{(1-2\sqrt{x}y+x)\ln^{2a-1}(y)}{(1+x)(1-\sqrt{x}y)(\sqrt{x}-y)} \mathrm{d}y\mathrm{d}x$$
$$= \int_0^1 \ln^{2a-1}(y) \left(\mathrm{P.V.} \int_0^1 \frac{1-2\sqrt{x}y+x}{(1+x)(1-\sqrt{x}y)(\sqrt{x}-y)} \mathrm{d}x \right) \mathrm{d}y$$
$$= \int_0^1 \ln^{2a-1}(y) \left(\frac{4y\ln(1-y)}{1+y^2} - \frac{2\ln(1-y)}{y} - \frac{2y\ln(y)}{1+y^2} - \frac{2y\ln(2)}{1+y^2} \right) \mathrm{d}y.$$

The second integral is given in (7). For the last two integrals, make the change of variable $y = \sqrt{x}$, and then use Lemma 2. The three integrals reduce to

$$-\frac{(2a-1)!}{2^{2a-1}} \left(a\eta(2a+1) + 2^{-2a}\zeta(2a+1) - \ln(2)\eta(2a) \right),$$

and the proof is completed. ☐

Also check [11, p. 64] for another proof.

Example 6. Setting $a = 1, 2$, and 3 in Theorem 9 the first three integrals are:

$$\int_0^1 \frac{x\ln(1-x)\ln(x)}{1+x^2}\mathrm{d}x = \frac{41}{64}\zeta(3) - \frac{9}{16}\ln(2)\zeta(2);$$

$$\int_0^1 \frac{x\ln(1-x)\ln^3(x)}{1+x^2}\mathrm{d}x = \frac{1761}{512}\zeta(5) - \frac{9}{16}\zeta(2)\zeta(3) - \frac{405}{128}\ln(2)\zeta(4);$$

$$\int_0^1 \frac{x\ln(1-x)\ln^5(x)}{1+x^2}\mathrm{d}x = \frac{129495}{2048}\zeta(7) - \frac{225}{64}\zeta(2)\zeta(5) - \frac{45}{4}\zeta(3)\zeta(4) - \frac{31185}{512}\ln(2)\zeta(6).$$

Theorem 10. *Let a be a positive integer. Then the following equality holds:*

$$\sum_{n=1}^{\infty} \frac{\overline{O}_n}{n^{2a+1}} = \frac{\pi}{4}(2^{2a+1}-1)\zeta(2a+1) - 2\sum_{k=0}^{a} 2^{2k}\eta(2a-2k)\beta(2k+2),$$

where $\overline{O}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}$.

Proof. By applying Cauchy product, one has

$$\frac{\arctan(x)}{1-x^2} = \sum_{n=1}^{\infty} \overline{O}_n x^{2n-1}. \tag{19}$$

Employing this, we get

$$\int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x^2}\mathrm{d}x = \sum_{n=1}^{\infty} \overline{O}_n \int_0^1 x^{2n-1}\ln^{2a}(x)\mathrm{d}x = \frac{(2a)!}{2^{2a+1}}\sum_{n=1}^{\infty}\frac{\overline{O}_n}{n^{2a+1}}.$$

The proof completes on substituting the result of the integral given in [7, p. 191]:

$$\int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x^2}\mathrm{d}x = (2a)!\left(1-\frac{1}{2^{2a+1}}\right)\frac{\pi}{4}\zeta(2a+1)$$
$$-(2a)!\sum_{k=0}^{a} 2^{2k-2a}\eta(2a-2k)\beta(2k+2).$$

□

Theorem 11. *Let a be a positive integer. Then the following equality holds:*

$$\sum_{n=1}^{\infty}(-1)^n\frac{\overline{O}_n}{n^{2a+1}} = 2\sum_{k=0}^{a} 2^{2k}\zeta(2a-2k)\beta(2k+2).$$

Proof. Replacing x with ix in (19) yields

$$\frac{\operatorname{arctanh}(x)}{1+x^2}=\sum_{n=1}^{\infty}(-1)^{n-1}\overline{O}_n x^{2n-1}.$$

Making use of this series, we obtain

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln^{2a}(x)}{1+x^2}\mathrm{d}x=\sum_{n=1}^{\infty}(-1)^{n-1}\overline{O}_n\int_0^1 x^{2n-1}\ln^{2a}(x)\mathrm{d}x=-\frac{(2a)!}{2^{2a+1}}\sum_{n=1}^{\infty}(-1)^n\frac{\overline{O}_n}{n^{2a+1}}.$$

The proof completes on substituting the result of the integral given in [7, p. 190]:

$$\int_0^1 \frac{\operatorname{arctanh}(x)\ln^{2a}(x)}{1+x^2}\mathrm{d}x=\frac{(2a)!}{2^{2a-1}}\sum_{k=0}^{a}2^{2k}\zeta(2a-2k)\beta(2k+2).$$

□

In addition, by employing $\sum_{n=1}^{\infty}a_{2n}=\frac{1}{2}\sum_{n=1}^{\infty}a_n+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n$, one obtains

$$\sum_{n=1}^{\infty}\frac{\overline{O}_{2n}}{(2n)^{2a+1}}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\overline{O}_n}{n^{2a+1}}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{\overline{O}_n}{n^{2a+1}}.$$

Combining the results of Theorems 10 and 11 yields

$$\sum_{n=1}^{\infty}\frac{\overline{O}_{2n}}{n^{2a+1}}=2^{2a-2}(2^{2a+1}-1)\pi\zeta(2a+1)+4\sum_{k=0}^{a}2^{4k}\zeta(2a-2k)\beta(2k+2). \tag{20}$$

Theorem 12. *Let a be an integer ≥ 0. Then the following equality holds:*

$$\int_0^1 \frac{x\ln(1+x)\ln^{2a}(x)}{1+x^4}dx=\frac{(2a)!}{2^{2a+3}}\ln(2)\beta(2a+1)-\frac{(2a+1)!}{2^{2a+3}}\beta(2a+2)$$
$$+\frac{\sqrt{2}\pi}{2^{6a+5}}\operatorname{arcsinh}(1)\left(\lim_{n\to 1/8}+\lim_{n\to 3/8}\right)\frac{d^{2a}}{dn^{2a}}\cot(\pi n)-\frac{(2a)!}{2^{2a+2}}\sum_{k=0}^{a}2^{2k}\eta(2k)\beta(2a-2k+2).$$

Proof. Replace x with $\sqrt{x}$ in part (b) of Lemma 6, and then divide both sides by $1+x^2$ and integrate from $x=0$ to $x=1$ using Lemma 3, we obtain

$$(2a)!\sum_{k=0}^{a}2^{2k-2a}\eta(2k)\beta(2a-2k+2)=\int_0^1\int_0^1\frac{(1-x)\ln^{2a}(y)}{(1+x^2)(1+\sqrt{x}y)(\sqrt{x}+y)}\mathrm{d}y\mathrm{d}x$$
$$=\int_0^1\ln^{2a}(y)\left(\int_0^1\frac{1-x}{(1+x^2)(1+\sqrt{x}y)(\sqrt{x}+y)}\mathrm{d}x\right)\mathrm{d}y$$
$$=\int_0^1\ln^{2a}(y)\left(-\frac{4y\ln(1+y)}{1+y^4}+\sqrt{2}\operatorname{arcsinh}(1)\frac{1+y^2}{1+y^4}+\frac{2y\ln(y)}{1+y^4}+\frac{\ln(2)y}{1+y^4}\right)\mathrm{d}y.$$

Let $y=\sqrt{x}$ in the last two integrals, and then use Lemma 3, they evaluate to

$$-\frac{(2a+1)!}{2^{2a+1}}\beta(2a+2)+\frac{(2a)!}{2^{2a+1}}\ln(2)\beta(2a+1).$$

For the second integral, multiply and divide the integrand by $1-y^4$,

$$\int_0^1 \frac{1+y^2}{1+y^4}\ln^{2a}(y)\mathrm{d}y=\int_0^1 \frac{(1-y^4)(1+y^2)}{1-y^8}\ln^{2a}(y)\mathrm{d}y$$

make use of the integral form of the polygamma function given in Lemma 5

$$=\frac{1}{8^{2a+1}}\left[\psi^{(2a)}\left(\frac{7}{8}\right)-\psi^{(2a)}\left(\frac{1}{8}\right)+\psi^{(2a)}\left(\frac{5}{8}\right)-\psi^{(2a)}\left(\frac{3}{8}\right)\right]$$

$$=\frac{1}{8^{2a+1}}\left[\lim_{n\to 1/8}\left(\psi^{(2a)}(1-n)-\psi^{(2a)}(n)\right)+\lim_{n\to 3/8}\left(\psi^{(2a)}(1-n)-\psi^{(2a)}(n)\right)\right]$$

recall the definition of the polygamma given in (5)

$$=\frac{1}{8^{2a+1}}\left[\lim_{n\to 1/8}\frac{d^{2a}}{dn^{2a}}\left(\psi(1-n)-\psi(n)\right)+\lim_{n\to 3/8}\frac{d^{2a}}{dn^{2a}}\left(\psi(1-n)-\psi(n)\right)\right]$$

substitute $\psi(1-n)-\psi(n)=\pi\cot(n\pi)$ given in [9, p. 25]

$$=\frac{1}{8^{2a+1}}\left[\lim_{n\to 1/8}\frac{d^{2a}}{dn^{2a}}\pi\cot(n\pi)+\lim_{n\to 3/8}\frac{d^{2a}}{dn^{2a}}\pi\cot(n\pi)\right]$$

$$=\frac{\pi}{8^{2a+1}}\left(\lim_{n\to 1/8}+\lim_{n\to 3/8}\right)\frac{d^{2a}}{dn^{2a}}\cot(n\pi).$$

Collect all integrals to complete the proof. □

Example 7. Setting $a=1,2,$ and 3 in Theorem 12 the first three integrals are:

$$\int_0^1 \frac{x\ln(1+x)\ln^2(x)}{1+x^4}\mathrm{d}x=\frac{3\pi^3}{128}\operatorname{arcsinh}(1)-\frac{1}{4}\zeta(2)\beta(2)+\frac{1}{16}\ln(2)\beta(3)-\frac{1}{4}\beta(4);$$

$$\int_0^1 \frac{x\ln(1+x)\ln^4(x)}{1+x^4}\mathrm{d}x=\frac{57\pi^5}{2048}\operatorname{arcsinh}(1)-\frac{21}{4}\zeta(4)\beta(2)-\frac{3}{4}\zeta(2)\beta(4)+\frac{3}{16}\ln(2)\beta(5)-\frac{9}{8}\beta(6);$$

$$\int_0^1 \frac{x\ln(1+x)\ln^6(x)}{1+x^4}\mathrm{d}x=\frac{2763\pi^7}{32768}\operatorname{arcsinh}(1)-\frac{1395}{8}\zeta(6)\beta(2)-\frac{315}{8}\zeta(4)\beta(4)$$
$$-\frac{45}{8}\zeta(2)\beta(6)+\frac{45}{32}\ln(2)\zeta(7)-\frac{45}{4}\beta(8).$$

Theorem 13. *Let a be an integer ≥ 0. Then the following equality holds:*

$$\int_0^1 \frac{x\ln(1-x)\ln^{2a}(x)}{1+x^4}dx=\frac{(2a)!}{2^{2a+3}}\ln(2)\beta(2a+1)-\frac{(2a+1)!}{2^{2a+3}}\beta(2a+2)$$
$$-\frac{\sqrt{2}\pi}{2^{6a+5}}\operatorname{arcsinh}(1)\left(\lim_{n\to 1/8}+\lim_{n\to 3/8}\right)\frac{d^{2a}}{dn^{2a}}\cot(\pi n)+\frac{(2a)!}{2^{2a+2}}\sum_{k=0}^{a}2^{2k}\zeta(2k)\beta(2a-2k+2).$$

Proof. Replace x with $\sqrt{x}$ in part (b) of Lemma 7, and then divide both sides by $1+x^2$ and integrate from $x=0$ to $x=1$ using Lemma 3, we obtain

$$(2a)!\sum_{k=0}^{a} 2^{2k-2a}\zeta(2k)\beta(2a-2k+2) = \mathrm{P.V.}\int_0^1\int_0^1 \frac{(1-x)\ln^{2a}(y)}{(1+x^2)(1-\sqrt{x}y)(\sqrt{x}-y)}\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \ln^{2a}(y)\left(\mathrm{P.V.}\int_0^1 \frac{1-x}{(1+x^2)(1-\sqrt{x}y)(\sqrt{x}-y)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \ln^{2a}(y)\left(\frac{4y\ln(1-y)}{1+y^4} + \sqrt{2}\operatorname{arcsinh}(1)\frac{1+y^2}{1+y^4} - \frac{2y\ln(y)}{1+y^4} - \frac{\ln(2)y}{1+y^4}\right)\mathrm{d}y.$$

The proof completes on recalling the results of the last three integrals from the previous proof. □

For a different approach, write $\ln(1-x) = \ln(1-x^2) - \ln(1+x)$,

$$\int_0^1 \frac{x\ln(1-x)\ln^{2a}(x)}{1+x^4}\mathrm{d}x = \int_0^1 \frac{x\ln(1-x^2)\ln^{2a}(x)}{1+x^4}\mathrm{d}x - \int_0^1 \frac{x\ln(1+x)\ln^{2a}(x)}{1+x^4}\mathrm{d}x$$

let $x=\sqrt{y}$ in the first integral

$$= 2^{-2a-1}\int_0^1 \frac{\ln(1-y)\ln^{2a}(y)}{1+y^2}\mathrm{d}y - \int_0^1 \frac{x\ln(1+x)\ln^{2a}(x)}{1+x^4}\mathrm{d}x.$$

Substitute the result of the first integral given in [7, pp. 188-189]:

$$\int_0^1 \frac{\ln(1-x)\ln^{2a}(x)}{1+x^2}\mathrm{d}x = \frac{(2a)!}{2}\ln(2)\beta(2a+1) - \frac{(2a+1)!}{2}\beta(2a+2)$$
$$+(2a)!\sum_{k=0}^{a}\zeta(2k)\beta(2a-2k+2),$$

and the result of Theorem 12 to finalize the proof.

Example 8. Setting $a=1,2,$ and 3 in Theorem 13 the first three integrals are:

$$\int_0^1 \frac{x\ln(1-x)\ln^2(x)}{1+x^4}\mathrm{d}x = -\frac{3\pi^3}{128}\operatorname{arcsinh}(1) + \frac{1}{2}\zeta(2)\beta(2) + \frac{1}{16}\ln(2)\beta(3) - \frac{1}{4}\beta(4);$$

$$\int_0^1 \frac{x\ln(1-x)\ln^4(x)}{1+x^4}\mathrm{d}x = -\frac{57\pi^5}{2048}\operatorname{arcsinh}(1) + 6\zeta(4)\beta(2) + \frac{3}{2}\zeta(2)\beta(4) + \frac{3}{16}\ln(2)\beta(5) - \frac{9}{8}\beta(6);$$

$$\int_0^1 \frac{x\ln(1-x)\ln^6(x)}{1+x^4}\mathrm{d}x = -\frac{2763\pi^7}{32768}\operatorname{arcsinh}(1) + 180\zeta(6)\beta(2) + 45\zeta(4)\beta(4)$$
$$+\frac{45}{4}\zeta(2)\beta(6) + \frac{45}{32}\ln(2)\zeta(7) - \frac{45}{4}\beta(8).$$

Theorem 14. *Let a be an integer ≥ 0. Then the following equality holds:*

$$\int_0^1 \frac{\arctan(x)\ln^a(x)}{1+x}dx = \frac{(-1)^a a!}{4}\left(\pi\,\eta(a+1) - 2\sum_{k=0}^{a} 2^{k-a}\eta(a-k+1)\beta(k+1)\right).$$

In the following proof, we apply the same strategy in [4].

Proof. Using the special case of the Leibniz integral rule:

$$\frac{d}{dx}\int_\alpha^{f(x)} g(t)\mathrm{d}t = g(f(x))\cdot\frac{d}{dx}f(x),$$

where α is a constant, we can write

$$\frac{d}{dx}\int_0^x \frac{\ln^a(t)}{1+t}\mathrm{d}t = \frac{\ln^a(x)}{1+x}.$$

Multiply both sides by $\arctan(x)$, and then integrate from $x = 0$ to 1,

$$\int_0^1 \frac{\arctan(x)\ln^a(x)}{1+x}\mathrm{d}x = \int_0^1 \arctan(x)\left(\frac{d}{dx}\int_0^x \frac{\ln^a(t)}{1+t}\mathrm{d}t\right)\mathrm{d}x$$

force integration by parts

$$= \arctan(x)\int_0^x \frac{\ln^a(t)}{1+t}\mathrm{d}t\Big|_{x=0}^{x=1} - \underbrace{\int_0^1\int_0^x \frac{\ln^a(t)}{(1+x^2)(1+t)}\mathrm{d}t\mathrm{d}x}_{t=xy}$$

$$= \frac{\pi}{4}\int_0^1 \frac{\ln^a(t)}{1+t}\mathrm{d}t - \int_0^1\int_0^1 \frac{x\ln^a(xy)}{(1+x^2)(1+xy)}\mathrm{d}y\mathrm{d}x \tag{21}$$

substitute $\frac{x}{(1+x^2)(+xy)} = \frac{x}{(1+x^2)(1+y^2)} + \frac{y}{(1+x^2)(1+y^2)} - \frac{y}{(1+y^2)(1+xy)}$ in the second integral

$$= \frac{\pi}{4}\int_0^1 \frac{\ln^a(t)}{1+t}\mathrm{d}t - \int_0^1\int_0^1 \frac{x\ln^a(xy)}{(1+x^2)(1+y^2)}\mathrm{d}y\mathrm{d}x$$

$$-\underbrace{\int_0^1\int_0^1 \frac{y\ln^a(xy)}{(1+x^2)(1+y^2)}\mathrm{d}y\mathrm{d}x}_{x\leftrightarrow y} + \underbrace{\int_0^1\int_0^1 \frac{y\ln^a(xy)}{(1+y^2)(1+xy)}\mathrm{d}y\mathrm{d}x}_{x\leftrightarrow y}$$

$$= \frac{\pi}{4}\int_0^1 \frac{\ln^a(t)}{1+t}\mathrm{d}t - 2\int_0^1\int_0^1 \frac{x\ln^a(xy)}{(1+x^2)(1+y^2)}\mathrm{d}y\mathrm{d}x + \int_0^1\int_0^1 \frac{x\ln^a(xy)}{(1+x^2)(1+xy)}\mathrm{d}y\mathrm{d}x. \tag{22}$$

Adding (21) and (22) yields

$$\int_0^1 \frac{\arctan(x)\ln^a(x)}{1+x}\mathrm{d}x = \frac{\pi}{4}\int_0^1 \frac{\ln^a(t)}{1+t}\mathrm{d}t - \int_0^1\int_0^1 \frac{x\ln^a(xy)}{(1+x^2)(1+y^2)}\mathrm{d}y\mathrm{d}x.$$

The first integral is given in Lemma 2. For the second, use the binomial theorem,

$$\int_0^1\int_0^1 \frac{x\ln^a(xy)}{(1+x^2)(1+y^2)}\mathrm{d}y\mathrm{d}x = \int_0^1\int_0^1 \frac{x(\ln(x)+\ln(y))^a}{(1+x^2)(1+y^2)}\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1\int_0^1 \frac{x\sum_{k=0}^{a}\binom{a}{k}\ln^{a-k}(x)\ln^k(y)}{(1+x^2)(1+y^2)}\mathrm{d}y\mathrm{d}x$$

$$= \sum_{k=0}^{a}\binom{a}{k}\left(\int_0^1 \frac{x\ln^{a-k}(x)}{1+x^2}\mathrm{d}x\right)\left(\int_0^1 \frac{\ln^k(y)}{1+y^2}\mathrm{d}y\right)$$

the first integral in given in Lemma 2 after making the change of variable $x=\sqrt{y}$, and the second integral is given in Lemma 3

$$= \sum_{k=0}^{a}\binom{a}{k}\left(\frac{(-1)^{a-k}}{2^{a-k+1}}(a-k)!\eta(a-k+1)\right)\left((-1)^k k!\beta(k+1)\right)$$

$$= \frac{(-1)^a a!}{2^{a+1}}\sum_{k=0}^{a}2^k\eta(a-k+1)\beta(k+1).$$

Combine the two integrals to finish the proof. □

It's worth noting Cornel's statement from his first book [10, p. 146]: " *The curious reader interested in a larger generalization, with the logarithm raised at any positive integer, may use the strategy in...*". He was referring to the strategy in [4].

Example 9. Setting $a=1,2$, and 3 in Theorem 14 the first three integrals are:

$$\int_0^1 \frac{\arctan(x)\ln(x)}{1+x}\mathrm{d}x = -\frac{3\pi}{32}\zeta(2)+\frac{1}{2}\ln(2)\beta(2);$$

$$\int_0^1 \frac{\arctan(x)\ln^2(x)}{1+x}\mathrm{d}x = \frac{21\pi}{64}\zeta(3)-\frac{1}{4}\zeta(2)\beta(2)-\ln(2)\beta(3);$$

$$\int_0^1 \frac{\arctan(x)\ln^3(x)}{1+x}\mathrm{d}x = -\frac{315\pi}{256}\zeta(4)+\frac{9}{16}\zeta(3)\beta(2)+\frac{3}{4}\zeta(2)\beta(3)+3\ln(2)\beta(4).$$

Theorem 15. *Let a be an integer ≥ 0. Then the following equality holds:*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\psi^{(a)}(n+1)-\psi^{(a)}\left(n+\frac{1}{2}\right)\right) = 2^{a+1}(-1)^a a!\left(\beta(a+2)-\frac{\pi}{4}\eta(a+1)\right)$$

$$+(-1)^a a!\sum_{k=0}^{a}2^k\eta(a-k+1)\beta(k+1).$$

Proof. Write $\frac{1}{1+x} = \frac{1}{x} - \frac{1}{x(1+x)}$,

$$\int_0^1 \frac{\arctan(x)\ln^a(x)}{1+x}\mathrm{d}x = \int_0^1 \frac{\arctan(x)\ln^a(x)}{x}\mathrm{d}x - \int_0^1 \frac{\arctan(x)\ln^a(x)}{x(1+x)}\mathrm{d}x.$$

By integration by parts, the first integral reduces to

$$\int_0^1 \frac{\arctan(x)\ln^a(x)}{x}\mathrm{d}x = -\frac{1}{a+1}\int_0^1 \frac{\ln^{a+1}(x)}{1+x^2}\mathrm{d}x = (-1)^a a!\beta(a+2).$$

For the second integral, employ the Taylor series of $\arctan(x)$,

$$\int_0^1 \frac{\arctan(x)\ln^a(x)}{x(1+x)}\mathrm{d}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 \frac{x^{2n}\ln^a(x)}{1+x}\mathrm{d}x$$

multiply and divide the integrand by $1-x$,

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 \frac{(x^{2n}-x^{2n+1})\ln^a(x)}{1-x^2}\mathrm{d}x$$

make use of the integral form of the polygamma function given in Lemma 5

$$= \frac{1}{2^{a+1}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(-\psi^{(a)}\left(n+\frac{1}{2}\right)+\psi^{(a)}(n+1)\right).$$

Combine the two integrals to finish the proof. □

Theorem 16. *Let a be a positive integer. Then the following equality holds:*

$$\int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x}dx = \frac{(2a)!}{4}\pi\zeta(2a+1) - \frac{(2a)!}{2}\beta(2a+2)$$
$$+\frac{(2a)!}{2^{2a+1}}\sum_{k=0}^{a}2^{2k}\{\eta(2a-2k+1)\beta(2k+1)-2\eta(2a-2k)\beta(2k+2)\}.$$

Proof. Using $\frac{1}{1-x} = \frac{2}{1-x^2} - \frac{1}{1+x}$, we get

$$\int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x}\mathrm{d}x = 2\int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x^2}\mathrm{d}x - \int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1+x}\mathrm{d}x.$$

Substituting the result of the first integral given in [7, pp. 190–191]:

$$\int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x^2}\mathrm{d}x = \frac{\pi}{4}(2a)!\left(1-\frac{1}{2^{2a+1}}\right)\zeta(2a+1)$$
$$-(2a)!\sum_{k=0}^{a}2^{2k-2a}\eta(2a-2k)\beta(2k+2),$$

and the result of Theorem 14 after replacing a with $2a$, we get

$$\int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x}\mathrm{d}x = \frac{\pi}{4}(2a)!\zeta(2a+1) - 2(2a)!\sum_{k=0}^{a} 2^{2k-2a}\eta(2a-2k)\beta(2k+2)$$
$$+\frac{(2a)!}{2^{2a+1}}\sum_{k=0}^{2a} 2^k\eta(2a-k+1)\beta(k+1).$$

Use $\sum_{k=0}^{2a} a_k = -a_{2a+1} + \sum_{k=0}^{a} a_{2k} + \sum_{k=0}^{a} a_{2k+1}$ in the second sum to finish the proof. $\square$

Example 10. Setting $a = 1, 2,$ and 3 in Theorem 16 the first three integrals are:

$$\int_0^1 \frac{\arctan(x)\ln^2(x)}{1-x}\mathrm{d}x = \frac{35\pi}{64}\zeta(3) - \frac{1}{4}\zeta(2)\beta(2) + \ln(2)\beta(3) - 2\beta(4);$$

$$\int_0^1 \frac{\arctan(x)\ln^4(x)}{1-x}\mathrm{d}x = \frac{1581\pi}{256}\zeta(5) - \frac{21}{16}\zeta(4)\beta(2) + \frac{9}{4}\zeta(3)\beta(3) - 3\zeta(2)\beta(4)$$
$$+12\ln(2)\beta(5) - 24\beta(6);$$

$$\int_0^1 \frac{\arctan(x)\ln^6(x)}{1-x}\mathrm{d}x = \frac{371475\pi}{2048}\zeta(7) - \frac{1395}{128}\zeta(6)\beta(2) + \frac{675}{32}\zeta(5)\beta(3) - \frac{315}{8}\zeta(4)\beta(4)$$
$$+\frac{135}{2}\zeta(3)\beta(5) - 90\zeta(2)\beta(6) + 360\ln(2)\beta(7) - 720\beta(8).$$

Theorem 17. *Let a be a positive integer. Then the following equality holds:*

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2a+1)}}{2n+1} = \frac{1}{2}\beta(2a+2)$$
$$-\frac{1}{2}\sum_{k=0}^{a} 2^{2k-2a}\left\{\eta(2a-2k+1)\beta(2k+1) - 2\eta(2a-2k)\beta(2k+2)\right\}.$$

Proof. Start with the Taylor series of $\arctan(x)$,

$$\int_0^1 \frac{\arctan(x)\ln^{2a}(x)}{1-x}\mathrm{d}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 \frac{x^{2n+1}\ln^{2a}(x)}{1-x}\mathrm{d}x$$

employ the integral form of the polygamma function given in Lemma 5

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(-\psi^{(2a)}(2n+2)\right)$$

make use of $\psi^{(a)}(n) = (-1)^a a!(H_{n-1}^{(a+1)} - \zeta(a+1))$ given in [7, p. 60]

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(2a)!\left(\zeta(2a+1) - H_{2n+1}^{(2a+1)}\right) = (2a)!\frac{\pi}{4}\zeta(2a+1) - (2a)!\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}^{(2a+1)}}{2n+1}.$$

The proof completes on recalling the result of Theorem 16. $\square$

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