

EULER SUMS OF GENERALIZED HARMONIC NUMBERS AND CONNECTED EXTENSIONS

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This paper presents the evaluation of the Euler sums of generalized hyperharmonic numbers $H_n^{(p,q)}$

$$\zeta_{H^{(p,q)}}(r) = \sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{n^r}$$

in terms of the famous Euler sums of generalized harmonic numbers. Moreover, several infinite series, whose terms consist of certain harmonic numbers and reciprocal binomial coefficients, are evaluated in terms of the Riemann zeta values.

1. INTRODUCTION

The classical Euler sum $\zeta_H(r)$ is the following Dirichlet series

$$\zeta_H(r) = \sum_{n=1}^{\infty} \frac{H_n}{n^r},$$

where H_n is the n th harmonic number. This series is also known as the harmonic zeta function. The famous Euler's identity for this sum is [14, 22, 30]

$$(1) \quad 2\zeta_H(r) = (r+2)\zeta(r+1) - \sum_{j=1}^{r-2} \zeta(r-j)\zeta(j+1), \quad r \in \mathbb{N} \setminus \{1\},$$

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where $\zeta(r)$ is the classical Riemann zeta function (for more details, see for instance [40]). Many generalizations of the Euler sums (the so called Euler-type sums) are given using generalizations of harmonic numbers (see [2–4, 7, 8, 10, 21, 26, 36, 37, 41, 43, 45–49]). Evaluation of the Euler-type sums and construction of closed forms are active fields of study in analytical number theory. Furthermore [3, 7, 10, 16, 17] are some of the studies that make this area interesting in the sense that the Euler sums have potential applications in quantum field theory and knot theory, especially in evaluation of the Feynman diagrams.

Euler actually considered also the more general form [5, 19, 22, 30]

$$(2) \quad \zeta_{H^{(p)}}(m) = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^m},$$

where $H_n^{(p)}$ defined by

$$H_n^{(p)} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}, \quad (p \in \mathbb{Z}, n \in \mathbb{N}),$$

is the n th partial sum of $\zeta(p)$ and is called the n th generalized harmonic number for $p > 1$. In particular, $H_0^{(p)} = 0$ and $H_n^{(1)} = H_n$, the n th harmonic number. When $p \leq 0$ it is called the sum of powers of integers.

One of the most important issues here is to write the Euler-type sums as combinations of the Riemann zeta function as in (1). This problem has remained important for various Euler-type sums from the era of Euler to the present day. It's shown by Euler himself that, the cases of $p = 1$, $p = q$, $p + q$ odd, and for special pairs $(p, q) \in \{(2, 4), (4, 2)\}$, the sums of the form (2) have evaluations in terms of the Riemann zeta function (see [5, 19, 22, 30]). There is a very comprehensive literature on this subject, both theoretical and numerical ([1, 6, 7, 10, 14, 19–21, 29, 36, 41–45, 48]). One of these results; the Euler identity (1) was further extended in the works of Borwein et al. [6] and Huard et al. [24]. For odd weight $N \geq 3$ and $p = 1, 2, \dots, N - 2$, we have [24, Theorem 1] (or [6, p. 278])

$$(3) \quad \begin{aligned} \zeta_{H^{(p)}}(N-p) &= (-1)^p \sum_{j=0}^{[(N-p-1)/2]} \binom{N-2j-1}{p-1} \zeta(N-2j) \zeta(2j) \\ &\quad + (-1)^p \sum_{j=0}^{[p/2]} \binom{N-2j-1}{N-p-1} \zeta(N-2j) \zeta(2j) - \zeta(0) \zeta(N). \end{aligned}$$

Moreover, these so called "linear Euler sums" satisfy a simple reflection formula

$$(4) \quad \zeta_{H^{(p)}}(r) + \zeta_{H^{(r)}}(p) = \zeta(p+r) + \zeta(p)\zeta(r).$$

Considering nested partial sums of the harmonic numbers, Conway and Guy [18] introduced hyperharmonic numbers for an integer $r > 1$ as

$$h_n^{(r)} = \sum_{k=1}^n h_k^{(r-1)}, \quad n \in \mathbb{N},$$

with $h_n^{(0)} = 1/n$, $h_n^{(1)} = H_n$ and $h_0^{(r)} = 0$. Hyperharmonic numbers are also important because they build a step in the transition to the multiple zeta functions (see [26, 42]). Dil and Boyadzhiev [20] extended Euler's identity (1) to the Euler sums of the hyperharmonic numbers:

$$(5) \quad \zeta_{h^{(q)}}(r) = \sum_{n=1}^{\infty} \frac{h_n^{(q)}}{n^r}, \quad (r > q),$$

as

$$(6) \quad \zeta_{h^{(q)}}(r) = \frac{1}{(q-1)!} \sum_{k=1}^q \begin{bmatrix} q \\ k \end{bmatrix} \times \left\{ \zeta_H(r-k+1) - H_{q-1} \zeta(r-k+1) + \sum_{j=1}^{q-1} \mu(r-k+1, j) \right\},$$

where $\begin{bmatrix} q \\ k \end{bmatrix}$ is the Stirling number of the first kind and

$$(7) \quad \mu(r, j) = \sum_{n=1}^{\infty} \frac{1}{n^r (n+j)} = \sum_{k=1}^{r-1} \frac{(-1)^{k-1}}{j^k} \zeta(r+1-k) + (-1)^{r-1} \frac{H_j}{j^r}.$$

Formula (6) was the general form of the results obtained for some special values of q and r in the study of [29].

Studies on evaluating the Euler sums (2) and (5) in terms of the Riemann zeta values $\zeta(m)$ have motivated researchers to find representations harmonic number series of the forms

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{(n+m)^r \binom{n+m+l}{l}}, \quad \sum_{n=1}^{\infty} \frac{h_n^{(q)}}{n \binom{n+q}{q}}.$$

It has been shown that some families of these type of series can be evaluated in terms of the Euler sums and Riemann zeta values (see for example for $m = 0$, $p = 1$, $r \in \{0, 1\}$ [32, 34, 38], for $m = r = 0$ [33], for $m = 0$ [32, 44], for $m > 0$, $r = 1$, $p \in \{1, 2\}$ [35, 39] and for the series involving hyperharmonic numbers [20]). We would like to emphasize that in some studies these type of series have been expressed in terms of hypergeometric series [13, 14, 29, 35, 36, 39].

In this work we mainly concentrate on the generalized hyperharmonic numbers defined as (see [21])

$$(8) \quad H_n^{(p,r)} = \sum_{k=1}^n H_k^{(p,r-1)}, \quad (p \in \mathbb{Z}, r \in \mathbb{N}),$$

with $H_n^{(p,0)} = 1/n^p$. These are a unified extension of the generalized harmonic numbers and hyperharmonic numbers:

$$H_n^{(p,1)} = H_n^{(p)} \text{ and } H_n^{(1,r)} = h_n^{(r)}.$$

The main objective of this study is the evaluation of Euler sums of generalized hyperharmonic numbers

$$\zeta_{H^{(p,q)}}(r) = \sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{n^r}.$$

A step towards the solution of this problem is taken in [21]. However, the recurrence used by the authors did not return a closed formula. Without an available closed formula, they listed only the following few special cases

$$\begin{aligned} \zeta_{H^{(p,2)}}(r) &= \zeta_{H^{(r-1)}}(p) - \zeta_{H^{(r)}}(p-1) + \zeta_{H^{(r)}}(p), \\ 2\zeta_{H^{(p,3)}}(r) &= 2\zeta_{H^{(r)}}(p) + 3\zeta_{H^{(r-1)}}(p) + \zeta_{H^{(r-2)}}(p) - 3\zeta_{H^{(r)}}(p-1) \\ &\quad + \zeta_{H^{(r)}}(p-2) - 2\zeta_{H^{(r-1)}}(p-1). \end{aligned}$$

Later, Göral and Sertbaş [23] showed that the Euler sums of generalized hyperharmonic numbers can be evaluated in terms of the Euler sums of generalized harmonic numbers and special values of the Riemann zeta function. However, their method does not determine the coefficients explicitly. This gap is filled in this study. The following recurrence relation for $H_n^{(p,q)}$ depending on the index q ,

$$(q-1) H_n^{(p,q)} = (n+q-1) H_n^{(p,q-1)} - H_n^{(p-1,q-1)}$$

is obtained. Thanks to this recurrence relation, it is managed to obtain a closed formula for $H_n^{(p,q)}$ in terms of $H_n^{(p)}$ in Theorem 2. This enables the evaluation of the Euler sums of generalized hyperharmonic numbers in terms of the Euler sums of generalized harmonic numbers as

$$\zeta_{H^{(p,q+1)}}(r) = \frac{1}{q!} \sum_{m=0}^q \sum_{k=0}^m (-1)^k \binom{q+1}{m+1} \binom{m}{k} \zeta_{H^{(p-k)}}(r+k-m).$$

A demonstration of this formula is the following example:

$$\begin{aligned} 2\zeta_{H^{(6,4+1)}}(6) &= -1925\zeta(11) + \left(175\pi^2 - \frac{905}{4} - \frac{3937\pi^8}{544320}\right)\zeta(9) \\ &\quad + \left(\frac{245\pi^2}{12} + \frac{35\pi^4}{18} + \frac{31\pi^{10}}{46656}\right)\zeta(7) + \left(\frac{\pi^4}{4} + \frac{5\pi^6}{1134} + \frac{31\pi^{12}}{6123600}\right)\zeta(5) \\ &\quad - \frac{35}{12}\zeta^2(5) - \frac{1}{3}\zeta(3)\zeta(5) + \frac{\pi^6}{1134}\zeta(3) + \frac{\pi^{10}}{29160} + \frac{1406\pi^{12}}{638512875}. \end{aligned}$$

In addition, a counterpart of the reflection formula (4) is obtained in the following form:

$$\zeta_{H^{(p,q+1)}}(r) + \zeta_{H^{(r,q+1)}}(p).$$

Section 2 completes with this formula which serves to calculate sums similar to the foregoing example with less computational cost.

In the last section we further extend our results. In this direction we establish new and more general identities for the series whose terms are generalizations of harmonic numbers and reciprocal binomial coefficients. For instance,

$$\sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{(n+m) \binom{n+m+l}{l}}$$

is evaluated in terms of the Riemann zeta values. This leads to several new evaluation formulas for particular series involving the generalized harmonic and hyperharmonic numbers. We point out special cases of these formulas which match with several known results in the literature.

2. EULER SUMS OF GENERALIZED HYPERHARMONIC NUMBERS

In this section we present an evaluation formula for Euler sums $\zeta_{H^{(p,q)}}(r)$ under certain conditions. To state and prove our result we need some preliminaries.

Firstly, recall the polylogarithm defined by

$$Li_p(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^p}, \quad (|t| \leq 1 \text{ if } p > 1, \text{ and } |t| < 1 \text{ if } p \leq 1).$$

The generating function of the numbers $H_n^{(p,q)}$ in terms of the polylogarithm is [21]

$$(9) \quad \sum_{n=0}^{\infty} H_n^{(p,q)} t^n = \frac{Li_p(t)}{(1-t)^q}, \quad |t| < 1, \quad p, q \in \mathbb{Z}.$$

Our first result presents the following reduction formula for $H_n^{(p,q)}$.

Lemma 1. *Let p and q be integers with $q \geq 1$. Reduction relation for $H_n^{(p,q)}$ in the index q is*

$$(10) \quad (q-1) H_n^{(p,q)} = (n+q-1) H_n^{(p,q-1)} - H_n^{(p-1,q-1)}.$$

Proof. We define the polynomial $H_n^{(p,q)}(z)$ as

$$H_n^{(p,q)}(z) = \sum_{k=0}^n H_k^{(p,q)} z^k.$$

Considering (9), we obtain the ordinary generating function of $H_n^{(p,q)}(z)$ as

$$(11) \quad \sum_{n=0}^{\infty} H_n^{(p,q)}(z) t^n = \frac{Li_p(zt)}{(1-t)(1-zt)^q}.$$

From (11), it can be seen that

$$(12) \quad z \frac{d}{dz} H_n^{(p,q)}(z) = H_n^{(p-1,q)}(z) + qz H_{n-1}^{(p,q+1)}(z).$$

On the other hand, we utilize (8) twice to find that

$$\begin{aligned} H_n^{(p,q)} &= \sum_{k=1}^n H_k^{(p,q-1)} = \sum_{k=1}^n \sum_{j=1}^k H_j^{(p,q-2)} \\ &= (n+1) H_n^{(p,q-1)} - \sum_{j=1}^n j H_j^{(p,q-2)} \\ &= (n+1) H_n^{(p,q-1)} - \left. \frac{d}{dz} H_n^{(p,q-2)}(z) \right|_{z=1}, \end{aligned}$$

or equivalently

$$(13) \quad \left. \frac{d}{dz} H_n^{(p,q-2)}(z) \right|_{z=1} = (n+1) H_n^{(p,q-1)} - H_n^{(p,q)} \stackrel{(8)}{=} n H_n^{(p,q-1)} - H_{n-1}^{(p,q)}.$$

Therefore, (12) and (13) yield the desired formula. \square

The objective here is to express $H_n^{(p,q)}$ in terms of $H_n^{(p)}$. In [21] this relation is listed for at most $q = 4$ due to the complexity of the process. However, the next result provides a general solution to this problem where the numbers $H_n^{(p,q)}$ are expressed in terms of the numbers $H_n^{(p)}$ and $[q]_r$. Here $[q]_r$ denotes the r -Stirling number of the first kind defined by the "horizontal" generating function [11, 12, 28]

$$(14) \quad (x+r)(x+r+1)\cdots(x+r+q-1) = \sum_{j=0}^q [q]_r x^j.$$

The essence of the proof is based on the relationship between r -Stirling numbers and symmetric polynomials. The k th elementary symmetric polynomial $e_k(X_1, \dots, X_q)$ in variables X_1, \dots, X_q is defined by (see for example [27])

$$\begin{aligned} e_0(X_1, \dots, X_q) &= 1, \\ e_k(X_1, \dots, X_q) &= \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq q} \prod_{i=1}^k X_{j_i}, \quad 1 \leq k \leq q, \\ e_k(X_1, \dots, X_q) &= 0, \quad k > q, \end{aligned}$$

and possesses the identity

$$(15) \quad \prod_{j=1}^q (x - X_j) = \sum_{j=0}^q (-1)^{r-j} e_{q-j}(X_1, \dots, X_q) x^j.$$

The comparison of (14) with (15) obviously leads to the following relationship [28, Theorem 4.1]

$$(16) \quad e_{q-j} (n+1, n+2, \dots, n+q) = \begin{bmatrix} q \\ j \end{bmatrix}_{n+1}.$$

Theorem 2. Let p and q be integers with $q \geq 0$. Then,

$$(17) \quad q! H_n^{(p,q+1)} = \sum_{k=0}^q (-1)^k \begin{bmatrix} q \\ k \end{bmatrix}_{n+1} H_n^{(p-k)}.$$

Proof. We employ (10) on the right-hand side of

$$(q-1) q H_n^{(p,q+1)} = (n+q) (q-1) H_n^{(p,q)} - (q-1) H_n^{(p-1,q)},$$

and see that

$$\begin{aligned} (q-1) q H_n^{(p,q+1)} &= H_n^{(p,q-1)} \{(n+q)(n+q-1)\} \\ &\quad - H_n^{(p-1,q-1)} \{(n+q)+(n+q-1)\} + H_n^{(p-2,q-1)} \\ &= \sum_{k=0}^2 (-1)^k e_{2-k} (n+q-1, n+q) H_n^{(p-k,q+1-2)}. \end{aligned}$$

These initial steps suggest that the following equality should hold:

$$\begin{aligned} (18) \quad (q+1-r) (q+1-(r-1)) \cdots (q-1) q H_n^{(p,q+1)} \\ &= \sum_{k=0}^r (-1)^k e_{r-k} (n+q-(r-1), n+q-(r-2), \dots, n+q) H_n^{(p-k,q+1-r)}. \end{aligned}$$

To prove this by induction we show that it is also true for $r+1 \leq q$. We multiply (18) by $(q-r)$ and then use (10). Hence we find that

$$\begin{aligned} &(q-r) (q+1-r) (q+1-(r-1)) \cdots (q-1) q H_n^{(p,q+1)} \\ &= \sum_{k=0}^r (-1)^k e_{r-k} (n+q-(r-1), \dots, n+q) (n+q-r) H_n^{(p-k,q-r)} \\ &\quad + \sum_{k=0}^r (-1)^{k+1} e_{r-k} (n+q-(r-1), \dots, n+q) H_n^{(p-k-1,q-r)} \\ &= e_r (n+q-(r-1), \dots, n+q) (n+q-r) H_n^{(p,q-r)} \\ &\quad + \sum_{k=1}^r (-1)^k H_n^{(p-k,q-r)} \{(n+q-r) e_{r-k} (n+q-(r-1), \dots, n+q) \\ &\quad \quad \quad + e_{r+1-k} (n+q-(r-1), \dots, n+q)\} \\ &\quad + (-1)^{r+1} e_0 (n+q-r, \dots, n+q) H_n^{(p-(r+1),q-r)} \end{aligned}$$

$$= \sum_{k=0}^{r+1} (-1)^k e_{r+1-k} (n+q-r, \dots, n+q) H_n^{(p-k, q+1-(r+1))}.$$

The case $r = q$ in (18) gives

$$q! H_n^{(p, q+1)} = \sum_{k=0}^q (-1)^k e_{q-k} (n+1, \dots, n+q) H_n^{(p-k)},$$

which combines with (16) to give the statement. \square

Now, we are ready to state and prove our evaluation formula for $\zeta_{H^{(p, q)}}(r)$. Thanks to this formula the evaluation of the Euler sums of generalized hyperharmonic numbers reduces to the evaluation of the Euler sums of generalized harmonic numbers.

Theorem 3. *For $p, q \geq 1$ and $r > q + 1$, we have*

$$\zeta_{H^{(p, q+1)}}(r) = \frac{1}{q!} \sum_{m=0}^q \sum_{k=0}^m (-1)^k \begin{bmatrix} q+1 \\ m+1 \end{bmatrix} \binom{m}{k} \zeta_{H^{(p-k)}}(r+k-m).$$

Proof. From (17) and the following identity [31, p. 1661]

$$\begin{bmatrix} n \\ k \end{bmatrix}_{r+1} = \sum_{m=k}^n \begin{bmatrix} n+1 \\ m+1 \end{bmatrix} \binom{m}{k} r^{m-k},$$

we have

$$\begin{aligned} H_n^{(p, q+1)} &= \frac{1}{q!} \sum_{k=0}^q (-1)^k \begin{bmatrix} q \\ k \end{bmatrix}_{n+1} H_n^{(p-k)} \\ &= \frac{1}{q!} \sum_{k=0}^q \sum_{m=k}^q (-1)^k \begin{bmatrix} q+1 \\ m+1 \end{bmatrix} \binom{m}{k} n^{m-k} H_n^{(p-k)}. \end{aligned}$$

Multiplying both sides with n^{-r} and summing over the positive integer values on n , we deduce the desired result. \square

As mentioned in the introductory section the sums $\zeta_{H^{(p, q)}}(r)$ were listed up to $q = 3$ in [21]. With the help of Theorem 3 these sums can be evaluated for further choices of q . For instance for $q = 4$ one can obtain:

$$\begin{aligned} \zeta_{H^{(p, 4)}}(r) &= \zeta_{H^{(p)}}(r) + \frac{11}{6} \zeta_{H^{(p)}}(r-1) + \zeta_{H^{(p)}}(r-2) + \frac{1}{6} \zeta_{H^{(p)}}(r-3) \\ &\quad - \frac{11}{6} \zeta_{H^{(p-1)}}(r) - 2 \zeta_{H^{(p-1)}}(r-1) - \frac{1}{2} \zeta_{H^{(p-1)}}(r-2) + \zeta_{H^{(p-2)}}(r) \\ &\quad + \frac{1}{2} \zeta_{H^{(p-2)}}(r-1) - \frac{1}{6} \zeta_{H^{(p-3)}}(r). \end{aligned}$$

Hence, with the use of some values of $\zeta_{H(p)}(r)$ listed in forthcoming Remark 5, a few concrete expressions of $\zeta_{H(p,4)}(r)$ are:

- $\zeta_{H^{(1,4)}}(5) = \frac{11}{2}\zeta(5) - \left(1 - \frac{11}{36}\pi^2\right)\zeta(3) - \frac{1}{2}(\zeta(3))^2 - \frac{11}{216}\pi^2 - \frac{\pi^4}{810} + \frac{\pi^6}{540},$
- $\zeta_{H^{(2,4)}}(5) = -10\zeta(7) + \left(\frac{5}{6}\pi^2 - \frac{21}{2}\right)\zeta(5) + \left(\frac{\pi^4}{45} + \frac{5}{6}\pi^2 + \frac{5}{12}\right)\zeta(3)$
 $+ \frac{11}{4}(\zeta(3))^2 + \frac{7\pi^4}{1080} - \frac{55\pi^6}{13608},$
- $\zeta_{H^{(3,4)}}(5) = \zeta_{H^{(3)}}(5) + \frac{154}{3}\zeta(7) + \left(\frac{14}{3} - \frac{55}{12}\pi^2\right)\zeta(5) - 2(\zeta(3))^2$
 $- \left(\frac{7\pi^2}{18} + \frac{11\pi^4}{270}\right)\zeta(3) - \frac{\pi^4}{540} + \frac{\pi^6}{324},$
- $\zeta_{H^{(4,4)}}(5) = -\frac{11}{6}\zeta_{H^{(3)}}(5) - \frac{125}{2}\zeta(9) + \left(\frac{35}{6}\pi^2 - 63\right)\zeta(7)$
 $+ \left(\frac{35}{6}\pi^2 + \frac{\pi^4}{18}\right)\zeta(5) + \frac{\pi^4}{30}\zeta(3) - \frac{\pi^6}{1944} + \frac{143\pi^8}{680400},$
- $\zeta_{H^{(5,4)}}(5) = 231\zeta(9) + \left(21 - \frac{385}{18}\pi^2\right)\zeta(7) - \left(\frac{11}{60}\pi^4 + \frac{23}{12}\pi^2\right)\zeta(5)$
 $+ \frac{1}{2}(\zeta(5))^2 + \zeta(3)\zeta(5) - \frac{7\pi^4}{540}\zeta(3) - \frac{\pi^8}{8100} + \frac{\pi^{10}}{187110}.$

The following corollary gives the reflection formula for the Euler sums of generalized hyperharmonic numbers. Combined with (3), this corollary shows that $\zeta_{H(p,q+1)}(r) + \zeta_{H(r,q+1)}(p)$ can be written as a combination of the Riemann zeta values. In this way, particular the Euler sums of type $\zeta_{H(p,q)}(p)$ can be evaluated with less computation.

Corollary 4. *Let $p > q + 1$, $r > q + 1$ and $p + r$ be even. Then*

$$\begin{aligned} & \zeta_{H(p,q+1)}(r) + \zeta_{H(r,q+1)}(p) \\ &= \zeta(p+r) + \frac{2}{q!} \sum_{\substack{m=0 \\ m \text{ odd}}}^q \sum_{k=0}^m (-1)^k \begin{bmatrix} q+1 \\ m+1 \end{bmatrix} \binom{m}{k} \zeta_{H^{(p-k)}}(r+k-m) \\ &+ \frac{1}{q!} \sum_{m=0}^q \sum_{k=0}^m (-1)^{m+k} \begin{bmatrix} q+1 \\ m+1 \end{bmatrix} \binom{m}{k} \zeta(p-k) \zeta(r+k-m). \end{aligned}$$

Proof. Let $(p+r)$ be even. It is obvious from Theorem 3 that

$$\begin{aligned} & \zeta_{H(p,q+1)}(r) + \zeta_{H(r,q+1)}(p) \\ &= \frac{1}{q!} \sum_{m=0}^q \sum_{k=0}^m (-1)^k \begin{bmatrix} q+1 \\ m+1 \end{bmatrix} \binom{m}{k} \left\{ \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-m}} + (-1)^m \sum_{n=1}^{\infty} \frac{H_n^{(r+k-m)}}{n^{p-k}} \right\}. \end{aligned}$$

We write the right-hand side as

$$\begin{aligned} & \sum_{\substack{m=0 \\ m \text{ odd}}}^q \sum_{k=0}^m (-1)^k \begin{bmatrix} q+1 \\ m+1 \end{bmatrix} \binom{m}{k} \left\{ \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-m}} - \sum_{n=1}^{\infty} \frac{H_n^{(r+k-m)}}{n^{p-k}} \right\} \\ & + \sum_{0 \leq m \leq q/2} \sum_{k=0}^{2m} (-1)^k \begin{bmatrix} q+1 \\ 2m+1 \end{bmatrix} \binom{2m}{k} \left\{ \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-2m}} + \sum_{n=1}^{\infty} \frac{H_n^{(r+k-2m)}}{n^{p-k}} \right\}. \end{aligned}$$

By the reflection formula (4) we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-2m}} + \sum_{n=1}^{\infty} \frac{H_n^{(r+k-2m)}}{n^{p-k}} = \zeta(p+r-2m) + \zeta(p-k)\zeta(r+k-2m).$$

Moreover, for odd m , it can be seen from (3) that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-m}} - \sum_{n=1}^{\infty} \frac{H_n^{(r+k-m)}}{n^{p-k}} &= 2\zeta_{H^{(p-k)}}(r+k-m) - \zeta(p+r-m) \\ &\quad - \zeta(p-k)\zeta(r+k-m). \end{aligned}$$

Hence, we obtain the desired equation. \square

Remark 5. For interested readers we would like to list some values of $\zeta_{H^{(p)}}(r)$, used in the evaluations of $\zeta_{H^{(p,4)}}(5)$, $1 \leq p \leq 5$, and $\zeta_{H^{(6,5)}}(6)$. These are calculated with the help of (1), (3) and (4).

- $\zeta_{H^{(1)}}(2) = 2\zeta(3)$,
- $\zeta_{H^{(1)}}(3) = \frac{\pi^4}{72}$,
- $\zeta_{H^{(1)}}(4) = 3\zeta(5) - \frac{\pi^2}{6}\zeta(3)$,
- $\zeta_{H^{(1)}}(5) = -\frac{1}{2}\zeta^2(3) + \frac{\pi^6}{540}$,
- $\zeta_{H^{(2)}}(2) = \frac{7\pi^4}{360}$,
- $\zeta_{H^{(2)}}(3) = -\frac{9}{2}\zeta(5) + \frac{\pi^2}{2}\zeta(3)$,
- $\zeta_{H^{(2)}}(4) = \zeta^2(3) - \frac{\pi^6}{2835}$,
- $\zeta_{H^{(2)}}(5) = -10\zeta(7) + \frac{5\pi^2}{6}\zeta(5) + \frac{\pi^4}{45}\zeta(3)$,
- $\zeta_{H^{(3)}}(2) = \frac{11}{2}\zeta(5) - \frac{\pi^2}{3}\zeta(3)$,
- $\zeta_{H^{(3)}}(3) = \frac{1}{2}\zeta^2(3) + \frac{\pi^6}{1890}$,
- $\zeta_{H^{(3)}}(4) = 18\zeta(7) - \frac{5\pi^2}{3}\zeta(5)$,
- $\zeta_{H^{(3)}}(6) = \frac{85}{2}\zeta(9) - \frac{7\pi^2}{2}\zeta(7) - \frac{\pi^4}{15}\zeta(5)$,
- $\zeta_{H^{(4)}}(2) = -\zeta^2(3) + \frac{37\pi^6}{11340}$,
- $\zeta_{H^{(4)}}(3) = -17\zeta(7) + \frac{5\pi^2}{3}\zeta(5) + \frac{\pi^4}{90}\zeta(3)$,
- $\zeta_{H^{(4)}}(4) = \frac{13\pi^8}{113400}$,
- $\zeta_{H^{(4)}}(5) = -\frac{125}{2}\zeta(9) + \frac{35\pi^2}{6}\zeta(7) + \frac{\pi^4}{18}\zeta(5)$,
- $\zeta_{H^{(5)}}(2) = 11\zeta(7) - \frac{2\pi^2}{3}\zeta(5) - \frac{\pi^4}{45}\zeta(3)$,
- $\zeta_{H^{(5)}}(4) = \frac{127}{2}\zeta(9) - \frac{35\pi^2}{6}\zeta(7) - \frac{2\pi^4}{45}\zeta(5)$,
- $\zeta_{H^{(5)}}(5) = \frac{1}{2}\zeta^2(5) + \frac{\pi^{10}}{187110}$,
- $\zeta_{H^{(5)}}(6) = \frac{463}{2}\zeta(11) - 21\pi^2\zeta(9) - \frac{7}{30}\pi^4\zeta(7)$,
- $\zeta_{H^{(6)}}(3) = \frac{7\pi^2}{2}\zeta(7) - \frac{83}{2}\zeta(9) + \frac{\pi^4}{15}\zeta(5) + \frac{\pi^6}{945}\zeta(3)$,
- $\zeta_{H^{(6)}}(5) = 21\pi^2\zeta(9) - \frac{461}{2}\zeta(11) + \frac{7\pi^4}{30}\zeta(7) + \frac{\pi^6}{945}\zeta(5)$.

3. SERIES INVOLVING HARMONIC NUMBERS AND RECIPROCAL BINOMIAL COEFFICIENTS

In this section we introduce evaluation formulas for some series involving the harmonic numbers and their generalizations.

Theorem 6. *Let $p \geq 1$ and $q, l \geq 0$ be integers with $l \geq q$. For $m \geq 1$,*

$$(19) \quad \sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{(n+m) \binom{n+m+l}{l}} = \sum_{j=0}^{l-q} \binom{l-q}{j} \left\{ \frac{(-1)^{j+p-1} H_{m+j}}{(m+j)^p} + \sum_{k=1}^{p-1} \frac{(-1)^{j+k-1}}{(m+j)^k} \zeta(p+1-k) \right\}$$

and

$$(20) \quad \sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{n \binom{n+l}{l}} = \zeta(p+1) - \sum_{j=1}^{l-q} \binom{l-q}{j} \left\{ \frac{(-1)^{j+p} H_j}{j^p} + \sum_{k=1}^{p-1} \frac{(-1)^{j+k}}{j^k} \zeta(p+1-k) \right\}.$$

Proof. Using the formula (see [25, p.909])

$$\int_0^1 t^{n+m-1} (1-t)^l dt = \frac{1}{(n+m) \binom{n+m+l}{l}},$$

we can write

$$\frac{H_n^{(p,q)}}{(n+m) \binom{n+m+l}{l}} = \int_0^1 H_n^{(p,q)} t^{n+m-1} (1-t)^l dt.$$

With the help of (9), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{(n+m) \binom{n+m+l}{l}} &= \int_0^1 t^{m-1} (1-t)^{l-q} Li_p(t) dt \\ &= \sum_{j=0}^{l-q} \binom{l-q}{j} (-1)^j \sum_{n=1}^{\infty} \frac{1}{n^p (n+m+j)}. \end{aligned}$$

Then (19) follows from (7). If $m = 0$, then we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{n \binom{n+l}{l}} = \zeta(p+1) + \sum_{j=1}^{l-q} \binom{l-q}{j} (-1)^j \sum_{n=1}^{\infty} \frac{1}{n^p (n+j)},$$

which is equivalent to (20). \square

Now, we deal with some special cases of Theorem 6. Setting $q = l$ gives

$$(21) \quad \sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{(n+m) \binom{n+m+q}{q}} = \frac{(-1)^{p-1} H_m}{m^p} + \sum_{i=1}^{p-1} \frac{(-1)^{i-1}}{m^i} \zeta(p+1-i)$$

and

$$(22) \quad \sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{n \binom{n+q}{q}} = \zeta(p+1).$$

Note that the variable q does not appear in the right-hand sides and all these series converge very slowly.

For $p = 1$, (21) and (22) give Proposition 5 and Proposition 6 in [20]

$$\sum_{n=1}^{\infty} \frac{h_n^{(q)}}{n \binom{n+q}{q}} = \frac{1}{6} \pi^2 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{h_{n-1}^{(q)}}{n \binom{n+q}{q}} = 1,$$

respectively. (21) also yields [44, Eq. (2.30)] for $q = 1$. Additionally, when $p = 1$ in Theorem 6, we reach at

$$\sum_{n=1}^{\infty} \frac{h_n^{(q)}}{(n+m) \binom{n+m+l}{l}} = \sum_{j=0}^{l-q} \binom{l-q}{j} (-1)^j \frac{H_{m+j}}{m+j}$$

and

$$\sum_{n=1}^{\infty} \frac{h_n^{(q)}}{n \binom{n+l}{l}} = \frac{1}{6} \pi^2 + \sum_{j=1}^{l-q} (-1)^j \binom{l-q}{j} \frac{H_j}{j}.$$

Now employing [15, Eq.(18)]

$$\sum_{k=0}^m (-1)^k \binom{m}{k} \frac{H_{n+k}}{n+k} = \frac{H_{n+m} - H_m}{n \binom{n+m}{m}}$$

and [9, Eq. (9.4b)]

$$\sum_{j=1}^n (-1)^{j+1} \binom{n}{j} \frac{H_j}{j} = H_n^{(2)}$$

gives the following closed forms for series involving hyperharmonic numbers with reciprocal binomial coefficients.

Corollary 7. *Let $q, l \geq 0$ be integers with $l \geq q$. For all integers $m \geq 1$*

$$\sum_{n=1}^{\infty} \frac{h_n^{(q)}}{(n+m) \binom{n+m+l}{l}} = \frac{H_{m+l-q} - H_{l-q}}{m \binom{n+m+l-q}{l-q}}$$

and

$$\sum_{n=1}^{\infty} \frac{h_n^{(q)}}{n \binom{n+l}{l}} = \frac{1}{6} \pi^2 - H_{l-q}^{(2)}.$$

For $p = q = 1$, Theorem 6 gives [39, Eq. (2.31)]

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+m) \binom{n+m+q}{q}} = \frac{H_{n+q-1} - H_{q-1}}{n \binom{n+q-1}{q-1}}$$

and [38]

$$\sum_{n=1}^{\infty} \frac{H_n}{n \binom{n+l}{l}} = \frac{1}{6} \pi^2 - H_{l-1}^{(2)}.$$

For $q = 1$, (20) becomes

$$(23) \quad \begin{aligned} & \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n \binom{n+l}{l}} \\ &= \zeta(p+1) - \sum_{j=1}^{l-1} \binom{l-1}{j} \left\{ \frac{(-1)^{j+p} H_j}{j^p} + \sum_{k=1}^{p-1} \frac{(-1)^{j+k}}{j^k} \zeta(p+1-k) \right\}, \end{aligned}$$

which is also given by Sofo [33, Theorem 2.2] in a slightly different form.

Setting $q = 1$ in (19) yields the following corollary involving generalized harmonic numbers.

Corollary 8. *For all integers $m, p, l \geq 1$,*

$$(24) \quad \begin{aligned} & \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{(n+m) \binom{n+m+l}{l}} \\ &= \sum_{j=0}^{l-1} \binom{l-1}{j} (-1)^j \left\{ \frac{(-1)^{p-1} H_{m+j}}{(m+j)^p} + \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{(m+j)^k} \zeta(p+1-k) \right\}. \end{aligned}$$

The following particular cases can be deduced setting $p = 2$ and $p = 3$:

$$(25) \quad \begin{aligned} & \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(n+m) \binom{n+m+l}{l}} = \frac{\pi^2}{6m \binom{m+l-1}{l-1}} - \sum_{j=0}^{l-1} (-1)^j \binom{l-1}{j} \frac{H_{m+j}}{(m+j)^2}, \\ & \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(n+m) \binom{n+m+l}{l}} = \sum_{j=0}^{l-1} (-1)^j \binom{l-1}{j} \frac{H_{m+j}}{(m+j)^3} \\ & \quad + \frac{1}{m \binom{m+l-1}{l-1}} \left\{ \zeta(3) - \frac{\pi^2}{6} (H_{m+l-1} - H_{m-1}) \right\}. \end{aligned}$$

It is worth noting that the special case with choices $m = 6$ and $l = 3$ in (25) is recorded in [35, Remark 1] despite a misprint. The correct form is as follows

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{(n+6) \binom{n+9}{3}} = \frac{1}{168} \zeta(2) - \frac{37073}{7902720}.$$

For our final results, we deal with the special case $q = 2$ of Theorem 6. By the aid of (17), $\begin{bmatrix} q \\ 0 \end{bmatrix}_r = r(r+1)\cdots(r+q-1)$ and $\begin{bmatrix} q \\ q \end{bmatrix}_r = 1$, we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{\binom{n+l}{l}} = \sum_{n=1}^{\infty} \frac{H_n^{(p,2)}}{n\binom{n+l}{l}} + \sum_{n=1}^{\infty} \frac{H_n^{(p-1)}}{n\binom{n+l}{l}} - \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n\binom{n+l}{l}},$$

where l is any integer greater than 1. From (20), (23) and some arrangements we obtain

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{\binom{n+l}{l}} = \zeta(p) + \sum_{j=1}^{l-1} (-1)^j \left\{ \binom{l-1}{j} \mu(p-1, j) - \binom{l-2}{j-1} \mu(p, j) \right\},$$

where $\mu(p, j)$ is given in (7). A slightly different form of the equation above is given in [33, Theorem 2.1]. Similarly

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{nH_n^{(p)}}{(n+m)\binom{n+m+l}{l}} &= \sum_{n=1}^{\infty} \frac{H_n^{(p,2)}}{(n+m)\binom{n+m+l}{l}} + \sum_{n=1}^{\infty} \frac{H_n^{(p-1)}}{(n+m)\binom{n+m+l}{l}} \\ &\quad - \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{(n+m)\binom{n+m+l}{l}}. \end{aligned}$$

Then exploiting (20) and (24) in the last equation yields the following corollary.

Corollary 9. *Let $l > 1$ be an integer. Then*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{nH_n^{(p)}}{(n+m)\binom{n+m+l}{l}} \\ = \mu(p-1, m) + \sum_{j=1}^{l-1} (-1)^j \left\{ \binom{l-1}{j} \mu(p-1, m+j) - \binom{l-2}{j-1} \mu(p, m+j) \right\}, \end{aligned}$$

where $\mu(p, j)$ is given in (7).

For $l = 2$ this formula can be read as

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{nH_n^{(p)}}{(n+m)(n+m+1)(n+m+2)} \\ = (-1)^{p+1} \frac{m+2}{2(m+1)^p} H_{m+1} + (-1)^p \frac{H_m}{2m^{p-1}} \\ + \sum_{k=1}^{p-2} (-1)^{k-1} \left\{ \frac{1}{2m^k} - \frac{1}{2(m+1)^k} \right\} \zeta(p-k) + \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{2(m+1)^k} \zeta(p+1-k). \end{aligned}$$

The first few cases of this formula are listed below:

$$\sum_{n=1}^{\infty} \frac{nH_n}{(n+m)(n+m+1)(n+m+2)} = \frac{1}{2(m+1)} (H_{m+1} + 1),$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{nH_n^{(2)}}{(n+m)(n+m+1)(n+m+2)} &= \frac{\pi^2}{12(m+1)} - \frac{(m+2)}{2(m+1)^2} H_{m+1} + \frac{1}{2m} H_m, \\ \sum_{n=1}^{\infty} \frac{nH_n^{(3)}}{(n+m)(n+m+1)(n+m+2)} &= \frac{\zeta(3)}{2(m+1)} + \frac{\pi^2}{12m(m+1)^2} \\ &\quad + \frac{m+2}{2(m+1)^3} H_{m+1} - \frac{1}{2m^2} H_m. \end{aligned}$$

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