Solutions of Gaussian and Gauss-like integrals in Real and Complex Fields

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Abstract. The definite integrals play a vital role in many branches of natural science. Among them, the Gaussian integrals and Gaussian-like integrals are widely used in many fields, especially in statistics. Many times, the range of integration extends to the complex field. When people need to calculate these more complex integrals, they need some simpler methods to help them quickly calculate the integral value. This article describes how to solve Gaussian integrals and Gaussian-like integrals in real and complex domains. When calculating some complex integrals, integral transformations can be used, such as using Cauchy's remainder theorem to construct closed loops of sectors and parallelograms in complex fields. These methods greatly simplify some integration problems in the real number field. At the same time, some proven formulas such as Fresnel's theorem and the covariant theorem of the Γ function can also be used to solve some Gaussian integrals with trigonometric functions or higher order terms.

Keywords: Gauss-type integral; Gamma Function; Complex analysis; Cauchy's residue theorem.

1. Introduction

The concept of normal distribution was first proposed by the German mathematician and astronomer Moivre in 1733. However, since the German mathematician Gauss first applied it to astronomer research, normal distribution is also known as Gaussian distribution. Gaussian distribution is a very common and important probability distribution in physics, mathematics and engineering. The general form of one-dimensional Gaussian distribution is as follows $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$, recorded as $x \sim N(\mu, \sigma^2)$ [1]. When $\mu = 0, \sigma = 1$, the normal distribution becomes the standard normal distribution.$f(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$. Gaussian distribution is closely related to physics, engineering, and many disciplines related to statistics. For example, in educational statistics, statistical laws indicate a normal distribution of students' learning abilities. Therefore, the normal distribution of exam scores should basically follow the normal distribution. In quantum mechanics of physics, when researchers need to use Feynman path integrals to calculate wave functions in momentum representations, Gaussian like integrals are commonly used together [2].The Fresnel integrals commonly used in optics are also related to Gaussian like integrals.In medicine science, such as the height, hemoglobin, cholesterol, and other random errors in qualitative groups, as well as in experiments, also exhibit a normal or approximately normal distribution [3].

Gauss type integrals are used in the calculation of quantities involving Gaussian distribution. More general forms $f(x) = e^{-ax^2}$ and Gauss-like integrals as f(x) $= x^n e^{-ax^2}$, which will be used to explain Maxwell's velocity distribution law and energy equipartition theorem in thermodynamics and statistical physics, such as $\overline{v^n} = \int_0^\infty v^n f(v)dv$ and f(x) $= e^{-ax^2}cosbx$ which will be used in quantum optics [4]. This article will discuss the calculation of Gauss type integrals and Gauss-like integrals in real and complex fields. The purpose of this paper is to systematically elaborate the commonly used Gauss and Gauss-like integrals based on the Γ function and Cauchy's residue theorem, $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$, when $f\ (z)$ shall be an analytic function, C is a closed contour in a complex plane [5]. This paper also give the corresponding solution methods and general integration results of these integrations.

2. The Gaussian Integral

2.1. The Real Methods

The gaussian integral is expressed as

$$f(x) = e^{-ax^2} \ (a > 0) \tag{1}$$

Before calculating this integral, the convergence of this generalized integral should first be pointed out, because $0 \le e^{-ax^2} \le \frac{1}{1+ax^2}$. Therefore,

$$\int_0^\infty e^{-ax^2}\,dx \le \int_0^\infty \frac{1}{1+ax^2}\,dx = \frac{\pi}{\sqrt{4a}}. \tag{2}$$

By the comparison and discrimination method of the generalized integral of non-negative function, this integral converges. In the range of real numbers, this paper provides two solutions. To begin with, Noticing that $\int_0^\infty e^{-ax^2}\,dx = \int_0^\infty e^{-ay^2}\,dy$, then

$$I^2 = \int_0^\infty \int_0^\infty e^{-a(x^2+y^2)}\,dx\,dy. \tag{3}$$

To conquer this problem, this paper tranfer the variables by using polar coordinate system with $x = r\cos\theta,\ y = r\sin\theta$ and $r^2 = a(x^2+y^2)$. Therefore,

$$I^2 = \int_0^{2\pi}\int_0^\infty e^{-r^2} r\,dr\,d\theta = \left(\int_0^\infty e^{-r^2} r\,dr\right)\cdot\left(\int_0^{2\pi} d\theta\right) = \frac{\pi}{4a}. \tag{4}$$

and it directly implies that $I = \sqrt{\frac{\pi}{4a}}$. Similarly, the following integral is

$$\int_0^\infty e^{-ax^2+bx}\,dx = \int_0^\infty e^{-a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}}\,dx = \sqrt{\frac{\pi}{4a}}\,e^{\frac{b^2}{4a}}. \tag{5}$$

On the other hand, another method to solve the Gaussian integral is by using the Gamma function, which is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \qquad (t \ge 0). \tag{6}$$

Using the Covariant formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sinz\pi}$ with $z \in (0,1)$, it is concluded that when $z=\frac{1}{2}$, $\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$ and thus $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. In fact, let $t=x^2$ in Eq. (8), it is found that

$$I = \int_0^\infty e^{-at}\cdot\frac{1}{2\sqrt{t}}dt = \frac{1}{\sqrt{4a}}\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{4a}}. \tag{7}$$

2.2. The Complex Method

When the domain is expanded to the complex field, sometimes it is difficult to calculate integrals using only real number methods. In this sense, it is necessary to solve it by using Cauchy residue theorem [6]. However, The method of finding generating functions for certain special functions will not be discussed in detail in this article [7].

The first step to calculate the Gaussian integral is by constructing a function of the form

$$f(z) = e^{i\pi z^2}\tan(\pi z) \tag{8}$$

where $\tan(\pi z) = \frac{sin\,(\pi z)}{cos\,(\pi z)} = -i\frac{e^{i2\pi z}-1}{e^{i2\pi z}+1}$. The integral contour L is shown in Fig. 1. By Cauchy integral theorem, it is arrived that

$$\oint_L e^{i\pi z^2}\tan(\pi z)\,dz = \int_{L_1} f(z)dz + \int_{L_2} f(z)dz + \int_{L_3} f(z)dz + \int_{L_4} f(z)dz. \tag{9}$$

There is only one pole at $z = 1/2$ in the complex plane surrounded by the contour, then

$$\oint_L f(z)dz = 2\pi i \,\mathrm{Res}\, f\left(\frac{1}{2}\right) = 2\pi e^{i\pi z^2} \frac{e^{i2\pi z}-1}{\left(e^{i2\pi z}+1\right)'}\bigg|_{z=\frac{1}{2}} = -2ie^{i\frac{\pi}{4}}. \tag{10}$$

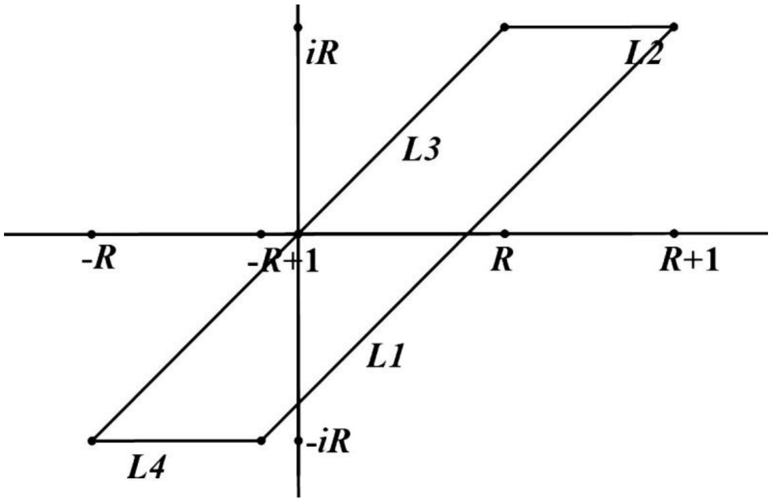

Fig 1. The integration contour of integral in Eq. (10)

When $R \to \infty$, $\lim\limits_{R\to\infty} 1 \pm (1+i)R = \lim\limits_{R\to\infty} \pm R \pm iR = \pm(1+i)\infty$. With the increase of R, its two ends will be closer, when m approaches 0, the two endpoints will coincide. At this point, the integral is zero. So $I_2 + I_4 = \int_{(1+i)\infty}^{(1+i)\infty} f(z)dz + \int_{-(1+i)\infty}^{-(1+i)\infty} f(z)dz = 0$. For path $L_1$and L_3,

$$I_1 = \int_{-(1+i)R}^{(1+i)R} e^{i\pi(z+1)^2} \tan\big(\pi(z+1)\big)\, dz,\ I_3 = \int_{-(1+i)R}^{(1+i)R} e^{i\pi z^2} \tan\big(\pi(z+1)\big)\, dz \tag{11}$$

So,

$$\begin{aligned} -2ie^{i\frac{\pi}{4}} &= \lim_{R\to\infty} \int_{-(1+i)R}^{(1+i)R} \left(e^{i\pi(z+1)^2} - e^{i\pi z^2}\right) \tan\big(\pi(z+1)\big)\, dz \\ &= -i \lim_{R\to\infty} \left(\int_{-(1+i)R}^{(1+i)R} e^{i\pi z^2} dz + \int_{1-(1+i)R}^{1+(1+i)R} e^{i\pi z^2} dz\right) = -2i \int_{-(1+i)\infty}^{(1+i)\infty} e^{i\pi z^2} dz. \end{aligned} \tag{12}$$

which means, $\int_{-(1+i)\infty}^{(1+i)\infty} \boldsymbol{e^{i\pi z^2}} dz = e^{i\frac{\pi}{4}}$. Let $z = e^{i\frac{\pi}{4}}t$, The upper and lower limit of the integral becomes $\pm\infty$. Take it in, find that $\int_{-\infty}^{+\infty} e^{-\pi t^2}\, dt = 1$. Let $-ax^2 = -\pi t^2$, then

$$\int_{-\infty}^{+\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}},\ \int_0^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{4a}} \tag{13}$$

In addition, if the conclusion of Fresnel integral is used, the proof will be simpler. For complex values shown in Eq. (1), writing a = $\alpha + \beta i$, for all $\alpha > 0$, α and β are both real number [8]:

$$e^{-ax^2} = e^{-\alpha x^2} + e^{-i\beta x^2} \tag{14}$$

For $e^{-\alpha x^2}$ several solutions have been provided. Here, this paper now mainly discuss $e^{-i\beta x^2}$, whiich can be solved by using Fresnel integral $\int_0^\infty \cos x^2 dx = \sqrt{\frac{\pi}{8}}$ and $\int_0^\infty \sin x^2 dx = \sqrt{\frac{\pi}{8}}$.

For the integral of the form $f(x) = e^{i\beta x^2}$, it is found that $\int_0^\infty e^{i\beta x^2} = \int_0^\infty \cos\beta x^2 dx + i\int_0^\infty \sin\beta x^2 dx = \sqrt{\frac{\pi}{4\beta e^{-i\frac{\pi}{2}}}} = \sqrt{\frac{i\pi}{4\beta}}$. By complex conjugating, this paper establishes the formula, $I = \int_0^\infty e^{-i\beta x^2} = \sqrt{\frac{\pi}{4i\beta}}$.

3. The Gaussian-like Integrals

3.1. $\mathbf{f}(\mathbf{x}) = \boldsymbol{x^n e^{-ax^2}}$

The famous Γ functions are directly related to Gaussian-like integrals, and the Γ function has the following properties $\Gamma(x+1) = x\Gamma(x)$ [9]. With Section 2.1, The previous formula can obtains that $I = \int_0^\infty e^{-at} \cdot \frac{1}{2\sqrt{t}} dt = \frac{1}{\sqrt{4a}} \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{4a}}$. When $t = ax^2 = u^2$, $\Gamma(z) = 2\int_0^\infty t^{2u-1} e^{-u^2} du$. Make a substitution,

$$\frac{1}{2}\Gamma\left(\frac{z+1}{2}\right) = \int_0^\infty u^z e^{-u^2} du. \tag{15}$$

Discussion on dividing the power n of x into odd and even numbers. When n is an even number and let $n = 2k$,

$$I(2k) = \int_0^\infty x^{2k} e^{-ax^2} dx = \frac{1}{a^k\sqrt{a}} \int_0^\infty u^{2k} e^{-u^2} dx = \frac{1}{2a^k\sqrt{a}} \Gamma\left(\frac{2k+1}{2}\right) \tag{16}$$

Where

$$\frac{1}{2}\Gamma\left(\frac{2k+1}{2}\right) = \sqrt{\frac{\pi}{4}} \prod_{k=0}^{\frac{1}{2}n-1} \frac{2k+1}{2}. \tag{17}$$

So it is deduced that

$$I(2k) = \frac{1}{2a^k\sqrt{a}} \Gamma\left(\frac{2k+1}{2}\right) = \frac{1}{a^k} \sqrt{\frac{\pi}{4a}} \prod_{k=0}^{\frac{1}{2}n-1} \frac{2k+1}{2} \tag{18}$$

when n is an odd number and let $n = 2k+1$,

$$I(2k+1) = \frac{1}{a^{k+1}} \int_0^\infty u^{2k+1} e^{-u^2} dx = \frac{1}{2a^{k+1}} \Gamma(k+1) = \frac{k!}{2a^{k+1}}. \tag{19}$$

In the end, consider the case where the upper and lower limits of the integral are positive infinity and negative infinity. If n is an even number and $f(x)$ is an even function, it means that

$$\int_{-\infty}^{+\infty} x^{2k} e^{-ax^2} dx = 2\int_0^\infty x^{2k} e^{-ax^2} dx = \frac{1}{2a^k} \sqrt{\frac{\pi}{4a}} \prod_{k=0}^{\frac{1}{2}n-1} \frac{2k+1}{2}. \tag{20}$$

If n is an odd number and $f(x)$ is an odd function, it means that

$$\int_{-\infty}^{+\infty} x^{2k+1} e^{-ax^2} dx = 0. \tag{21}$$

3.2. $\boldsymbol{f(x) = e^{-ax^2} \cos b\, x}$ $(\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R})$

The first approach is a general method in real field. Let $I_{(b)} = \int_0^\infty e^{-ax^2} cosbxdx$, where $I_{(b)}$Take partial derivative for b,

$$I'_{(b)} = \frac{\partial}{\partial b} I_{(b)} = \int_0^\infty -xe^{-ax^2} sinbxdx = \frac{1}{2} e^{-ax^2} \sin bx|_0^{+\infty} - \frac{b}{2} \int_0^\infty e^{-ax^2} cosbxdx. \tag{22}$$

Because $cosbx$ and $sinbx$ both in $[-1,1]$,

$$\left|e^{-ax^2} cosbx\right| \le e^{-ax^2}, \ \left|-xe^{-ax^2} sinbx\right| \le xe^{-ax^2} \tag{23}$$

$$\int_0^\infty e^{-ax^2} cosbxdx \le \int_0^\infty e^{-ax^2} dx = \sqrt{\frac{\pi}{4a}} \qquad \int_0^\infty -xe^{-ax^2} sinbxdx \le \int_0^\infty xe^{-ax^2} dx = \frac{1}{2a} \tag{24}$$

So these two integrals converge uniformly. Solving differential equations of $I'_{(b)} = -\frac{b}{2} I_{(b)}$, it is found that $I_{(b)} = ce^{-\frac{b^2}{4a}}$. When b is equal to 0, then according to the discussion in 2.1, $I_{(0)} = \sqrt{\frac{\pi}{4a}} = c$. Then $I_{(b)} = \sqrt{\frac{\pi}{4a}} e^{-\frac{b^2}{4a}}$.

The second approach is a method in complex field. To bein with, by using of Euler formula [10],

$$I = \int_0^{\infty} e^{-ax^2} cosbxdx = \int_0^{\infty} \frac{1}{2} e^{-ax^2} \left(e^{ibx} + e^{-ibx}\right) dx = \int_0^{+\infty} e^{-ax^2+ibx} dx. \tag{25}$$

After formulation, it is equivalent to $e^{-\frac{b^2}{4a}} \int_0^{+\infty} e^{-a\left(x-\frac{ib}{2a}\right)^2} dx$. The part of $\int_0^{+\infty} e^{-a(x-\frac{ib}{2a})^2} dx$ is similar to the Eq. (4), so it equals to $\sqrt{\frac{\pi}{4a}}$. Hence, $I = \sqrt{\frac{\pi}{4a}} e^{-\frac{b^2}{4a}}$.

4. Conclusion

This paper mainly studies the solutions of Gauss and Gauss-like integrals in real and complex fields. In the real field, one can get the expression of different Gaussian-like integrals by using the function and the substitution method. In the complex number field, it is necessary to construct the integrand function and skillfully construct the integral path, and then use the Cauchy residue theorem to make the path integral easy to calculate and change to the real variable integral. In additon, using some known integration formulas and methods, such as Fresnel integral and double integral, it can greatly simplify the calculation process. Before calculating these integrals, this article first gives a proof of uniform convergence of the integrals. When calculating the Gaussian-like integral which $x^n e^{-ax^2}$, this article uses Γ functions and classifies the original function into odd and even cases, respectively, to obtain the results of the integral when n is odd and n is even. When calculating the Gaussian-like integral of the form $e^{-ax^2} \cos b\, x$, this paper simply proves its results using differential equations and Euler's formula. Finally, when the coefficients are in the complex domain, this calculation is too complex, so this article does not fully give the solution of the Gaussian-like integral. In the future, people can discuss the integration methods when the above coefficients are in the complex domain, so as to further study Gaussian like integrals and Gaussian like integrals.

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