The Art of Differentiating Under the Integral Sign

"Feynman's Trick"

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January 2, 2026

Abstract

This article explores the technique of **Leibniz Integration Rule**, famously popularized by physicist Richard Feynman as a powerful tool for solving difficult definite integrals. By introducing a parameter and differentiating with respect to it, we can transform intractable integrals into manageable algebraic or differential equations. Below, we outline the theory and present 10 solved examples.

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1 Introduction

Richard Feynman, in his memoir *Surely You're Joking, Mr. Feynman!*, famously recounted how he gained a reputation for solving integrals that stumped his colleagues. His secret was not complex integration or contour integrals, but a technique he learned from an advanced calculus text: **differentiation under the integral sign**.

While standard calculus curricula often overlook this method in favor of substitution or integration by parts, it remains one of the most elegant and efficient ways to evaluate definite integrals.

2 The Leibniz Integral Rule

The core of the method relies on the Leibniz Integral Rule. For a definite integral with constant limits a and b:

Leibniz Integral Rule

Let $f(x, \alpha)$ be a function of x and a parameter α. If f and $\frac{\partial f}{\partial \alpha}$ are continuous, then:

$$\frac{d}{d\alpha}\int_a^b f(x, \alpha)\, dx = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha)\, dx \tag{1}$$

The General Strategy

1. **Identify** a parameter in the integrand, or introduce a new parameter α to generalize the integral. Call this $I(\alpha)$.
2. **Differentiate** $I(\alpha)$ with respect to α by moving the derivative inside the integral sign.
3. **Evaluate** the new integral (which is often simpler).
4. **Integrate** the result with respect to α to find $I(\alpha)$.
5. **Determine** the constant of integration by evaluating $I(\alpha)$ at a specific easy value (e.g., $\alpha = 0$ or $\alpha \to \infty$).

3 Ten Examples of Feynman's Trick

Example 1: The Classic Logarithm Integral

Evaluate the integral:

$$I = \int_0^1 \frac{x^2 - 1}{\ln x}\, dx$$

Solution:

We generalize the integral by replacing the power 2 with a parameter α:

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\, dx \quad \text{for } \alpha > -1$$

Differentiate with respect to α:

$$I'(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\left(\frac{x^\alpha - 1}{\ln x}\right) dx = \int_0^1 \frac{x^\alpha \ln x}{\ln x}\, dx = \int_0^1 x^\alpha\, dx$$

This is a standard power rule integral:

$$I'(\alpha) = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

Now, integrate with respect to α:

$$I(\alpha) = \int \frac{1}{\alpha+1}\, d\alpha = \ln(\alpha+1) + C$$

To find C, we evaluate at $\alpha = 0$:

$$I(0) = \int_0^1 \frac{x^0 - 1}{\ln x}\, dx = 0 \implies \ln(1) + C = 0 \implies C = 0$$

Thus, $I(\alpha) = \ln(\alpha + 1)$. Setting $\alpha = 2$ for our original problem:

$$I = \ln(3)$$

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Example 2: The Dirichlet Integral

Evaluate the integral:

$$I = \int_0^\infty \frac{\sin x}{x}\, dx$$

Solution:

Introduce a damping parameter $\alpha \geq 0$:

$$I(\alpha) = \int_0^\infty e^{-\alpha x} \frac{\sin x}{x}\, dx$$

Differentiate with respect to α:

$$I'(\alpha) = \int_0^\infty -x e^{-\alpha x} \frac{\sin x}{x}\, dx = -\int_0^\infty e^{-\alpha x} \sin x\, dx$$

Using integration by parts or standard tables, $\int e^{-\alpha x} \sin x \, dx = \frac{1}{\alpha^2+1}$. Thus:

$$I'(\alpha) = -\frac{1}{\alpha^2 + 1}$$

Integrate back to find $I(\alpha)$:

$$I(\alpha) = -\arctan(\alpha) + C$$

As $\alpha \to \infty$, the integral $I(\alpha)$ vanishes because of the $e^{-\alpha x}$ term.

$$\lim_{\alpha \to \infty} I(\alpha) = 0 \implies -\frac{\pi}{2} + C = 0 \implies C = \frac{\pi}{2}$$

Thus, $I(\alpha) = \frac{\pi}{2} - \arctan(\alpha)$. For the original integral, set $\alpha = 0$:

$$I(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

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Example 3: Frullani's Integral

Evaluate the integral for $a, b > 0$:

$$I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx$$

Solution:

Let us treat a as the variable parameter $I(a)$.

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a} \left(\frac{e^{-ax} - e^{-bx}}{x} \right) dx = \int_0^\infty \frac{-xe^{-ax}}{x} \, dx = -\int_0^\infty e^{-ax} \, dx$$

$$I'(a) = -\left[\frac{e^{-ax}}{-a} \right]_0^\infty = -\frac{1}{a}$$

Integrating with respect to a:

$$I(a) = -\ln(a) + C(b)$$

Note that if $a = b$, the original integral is exactly 0.

$$I(b) = -\ln(b) + C(b) = 0 \implies C(b) = \ln(b)$$

Therefore:

$$I = \ln(b) - \ln(a) = \ln\left(\frac{b}{a}\right)$$

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Example 4: Gaussian Variance

Evaluate:

$$I = \int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx$$

Solution:

Start with the standard Gaussian integral with a parameter α:

$$J(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}} = \sqrt{\pi}\alpha^{-1/2}$$

Differentiate $J(\alpha)$ with respect to α:

$$J'(\alpha) = \int_{-\infty}^{\infty} -x^2 e^{-\alpha x^2}\, dx$$

Compute the derivative of the right-hand side:

$$\frac{d}{d\alpha}(\sqrt{\pi}\alpha^{-1/2}) = -\frac{1}{2}\sqrt{\pi}\alpha^{-3/2}$$

Equating them:

$$-\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\, dx = -\frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

Set $\alpha = 1$:

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

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Example 5: Inverse Tangent Difference

Evaluate:

$$I = \int_0^{\infty} \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx$$

Solution:

Consider $I(\alpha) = \int_0^\infty \frac{\arctan(\alpha x) - \arctan(x)}{x}\, dx$.

$$I'(\alpha) = \int_0^{\infty} \frac{1}{x} \cdot \frac{x}{1 + (\alpha x)^2}\, dx = \int_0^{\infty} \frac{1}{1 + \alpha^2 x^2}\, dx$$

Substitute $u = \alpha x$, $du = \alpha dx$:

$$I'(\alpha) = \frac{1}{\alpha} \int_0^{\infty} \frac{1}{1+u^2}\, du = \frac{1}{\alpha}\,[\arctan(u)]_0^{\infty} = \frac{1}{\alpha} \cdot \frac{\pi}{2}$$

Integrate w.r.t α:

$$I(\alpha) = \frac{\pi}{2}\ln(\alpha) + C$$

Since $I(1) = 0$, we have $C = 0$. Thus $I(\alpha) = \frac{\pi}{2}\ln\alpha$.

$$I(\pi) = \frac{\pi}{2}\ln\pi$$

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Example 6: Higher Power Denominators

Evaluate:

$$I = \int_0^\infty \frac{1}{(x^2+1)^2}\,dx$$

Solution:

Consider the simpler integral:

$$J(\alpha) = \int_0^\infty \frac{1}{x^2+\alpha}\,dx = \left[\frac{1}{\sqrt{\alpha}}\arctan\left(\frac{x}{\sqrt{\alpha}}\right)\right]_0^\infty = \frac{\pi}{2\sqrt{\alpha}}$$

Differentiate both sides with respect to α:

$$\frac{d}{d\alpha}J(\alpha) = \int_0^\infty \frac{-1}{(x^2+\alpha)^2}\,dx$$

$$\frac{d}{d\alpha}\left(\frac{\pi}{2}\alpha^{-1/2}\right) = -\frac{\pi}{4}\alpha^{-3/2}$$

Equating the two:

$$\int_0^\infty \frac{1}{(x^2+\alpha)^2}\,dx = \frac{\pi}{4\alpha\sqrt{\alpha}}$$

Set $\alpha = 1$:

$$I = \frac{\pi}{4}$$

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Example 7: Damped Cosine Integral

Evaluate:

$$I = \int_0^\infty x e^{-x}\sin x\,dx$$

Solution:

This looks like a Laplace transform derivative. Consider:

$$J(\alpha) = \int_0^\infty e^{-ax}\cos(\alpha x)\,dx = \frac{a}{a^2+\alpha^2}$$

(Note: standard result $\int e^{-ax}\cos(bx) = \frac{a}{a^2+b^2}$). Differentiate $J(\alpha)$ with respect to α:

$$\frac{\partial}{\partial\alpha}J(\alpha) = \int_0^\infty -xe^{-ax}\sin(\alpha x)\,dx$$

Derivative of RHS:

$$\frac{\partial}{\partial\alpha}\left(a(a^2+\alpha^2)^{-1}\right) = -a(a^2+\alpha^2)^{-2}(2\alpha) = \frac{-2a\alpha}{(a^2+\alpha^2)^2}$$

Thus:

$$\int_0^\infty xe^{-ax}\sin(\alpha x)\,dx = \frac{2a\alpha}{(a^2+\alpha^2)^2}$$

Set $a = 1$ and $\alpha = 1$:

$$I = \frac{2(1)(1)}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$$

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Example 8: Cosine Denominator

Evaluate ($0 < a < 1$):

$$I = \int_0^\infty \frac{1 - \cos(ax)}{x^2}\,dx$$

Solution:

Let $I(a) = \int_0^\infty \frac{1-\cos(ax)}{x^2}\,dx$. Differentiate w.r.t a:

$$I'(a) = \int_0^\infty \frac{\sin(ax)\cdot x}{x^2}\,dx = \int_0^\infty \frac{\sin(ax)}{x}\,dx$$

Let $u = ax$. Then $\int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2}$.

$$I'(a) = \frac{\pi}{2}$$

Integrate w.r.t a:

$$I(a) = \frac{\pi}{2}a + C$$

At $a = 0$, $I(0) = \int 0\,dx = 0$, so $C = 0$.

$$I(a) = \frac{\pi a}{2}$$

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Example 9: A Quadratic Exponential

Evaluate:

$$I = \int_0^\infty e^{-x^2 - \frac{1}{x^2}}\,dx$$

Solution:

Let $I(\alpha) = \int_0^\infty e^{-x^2 - \frac{\alpha}{x^2}}\,dx$.

$$I'(\alpha) = \int_0^\infty -\frac{1}{x^2} e^{-x^2 - \frac{\alpha}{x^2}}\,dx$$

Substitute $u = \frac{\sqrt{\alpha}}{x}$, then $x = \frac{\sqrt{\alpha}}{u}$, $dx = -\frac{\sqrt{\alpha}}{u^2} du$. This substitution reveals a symmetry such that $I'(\alpha) = -\frac{1}{\sqrt{\alpha}} I(\alpha)$. Solving the differential equation $\frac{I'}{I} = -\alpha^{-1/2}$:

$$\ln I(\alpha) = -2\sqrt{\alpha} + C \implies I(\alpha) = Ae^{-2\sqrt{\alpha}}$$

At $\alpha = 0$, $I(0) = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. So $A = \frac{\sqrt{\pi}}{2}$.

$$I(1) = \frac{\sqrt{\pi}}{2} e^{-2}$$

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Example 10: Factorial from Gamma

Evaluate for integer n:

$$I_n = \int_0^1 x^a (\ln x)^n \, dx$$

Solution:

Start with $J(a) = \int_0^1 x^a \, dx = \frac{1}{a+1}$. Differentiate n times with respect to a:

$$\frac{d^n}{da^n} J(a) = \int_0^1 x^a (\ln x)^n \, dx$$

RHS derivative:

$$\frac{d^n}{da^n}(a+1)^{-1} = (-1)^n n! (a+1)^{-(n+1)}$$

Therefore:

$$\int_0^1 x^a (\ln x)^n \, dx = \frac{(-1)^n n!}{(a+1)^{n+1}}$$

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4 Conclusion

Feynman's trick transforms the art of integration from a search for obscure primitives into a structured process of differentiation and algebra. While it requires creativity to choose the correct parameter, the payoff is often a direct path to the solution of integrals that are otherwise impossible to evaluate using elementary methods.