# The Derivative of the Riemann Zeta Function and Its Values at Integers

Mehmet Kirdar

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**Abstract**

We find some identities for the values of the derivative of the Riemann zeta function at integers. As an application we calculate the Apery's constant approximately.

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## 1 Introduction

Finding the specials values of the Riemann-zeta function (RZF) has been of great interest in mathematics. For example, finding closed or useful expressions for the values of the RZF at odd positive integers greater than 2 are all open problems.

Finding the values of the derivative of the RZF at integers is also a very interesting problem. Since they depend on the values of RZF, finding expressions for these values is also extremely difficult.

First, we will find the classical expression for $\zeta'(s)$, the derivative of RZF, directly from the contour integral of the derivative. In this way, we actually proved the derivative version of the functional equation by the derivative of the contour integral. We also proved the functional equation. These computations are repeated many times in the history, [1].

Next, we found the classical identity for $\zeta'(-2m)$ for $m \in \mathbb{Z}^+$. You may see [2] for comparison. But this expression depends on $\zeta(2m+1)$. We also found an identity for $\zeta'(1-2m)$ for $m \in \mathbb{Z}^+$, in agreement with the given values in [2]. But this expression depends on $\zeta'(2m)$.

Because of these dependences, we started from the contour integral of the derivative again and obtained the following identities:

$$
\begin{aligned}
\zeta'(1-2m) \;=\; & \frac{1}{2m}\psi(2m)B_{2m} - (2m-1)!\int_1^\infty \frac{z^{-2m}}{e^z-1}dz - \frac{(2m-2)!}{2} \\
& -(2m-1)!\sum_{\substack{t=0 \\ t\neq m}}^{\infty} \frac{B_{2t}}{(2t)!}\frac{1}{(2t-2m)} \qquad\text{(A)}
\end{aligned}
$$

and

$$\zeta'(-2m) = (2m)! \left( \int\limits_{1}^{\infty} \frac{z^{-2m-1}}{e^z - 1} dz + \sum_{k=0}^{\infty} \frac{B_k}{k!} \frac{1}{(k - 2m - 1)} \right) \tag{B}$$

where $m \in \mathbb{Z}^+$, $\psi$ is the digamma function and $B_n$ is the $n$-th Bernoulli number. Note that in the equation (B), $B_{2m+1} = 0$, there is no $k = 2m + 1$ term. As an amusing application, we calculated the Apery's constant approximately.

## 2 Preliminaries

Let us recall some important definitions and values which are frequently used in the calculations related to RZF. The RZF, denoted by $\zeta(s)$ can be defined by a counter integral for all $s \in \mathbb{C} - \{1\}$ in the following way, [3]:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} I(s) \tag{1}$$

where

$$I(s) = \oint_C \frac{z^{s-1}}{e^{-z} - 1} dz. \tag{2}$$

The contour $C$ is the counter-clockwise contour coming from minus infinity just below negative side of the $x$-axis, enclosing the origin and then going back just above $x$-axis to minus infinity and not including any other singularities of the integrand except 0 in its domain of definition.

From (1) and (2), by using the series $\frac{1}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^{k-1}$ and Residue theorem, at the only singularity $z = 0$ of the integrand now, one can immediately find the special values

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \tag{3}$$

for all $n \in \mathbb{N}$.

The derivative of the natural logarithm $\ln \Gamma(s)$ of the Gamma function is the digamma function $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ and they have the special values

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \tag{4}$$

for all positive integers $n$ where $\gamma$ is the Euler-Mascheroni constant.

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# 3 Derivative of the RZF

Taking the derivative with respect to $s$ of $\zeta(s)$ by using (1), (2) and the definition of the digamma function, we get

$$\zeta'(s) = -\psi(1-s)\zeta(s) + \frac{\Gamma(1-s)}{2\pi i}J(s) \qquad (5)$$

where

$$J(s) = \oint_C \frac{(\ln z)z^{s-1}}{e^{-z}-1}dz. \qquad (6)$$

Let $-C_R$ be the anti-clockwise version of the contour in the introduction cutted at $z = -R$ vertically. Now let us complete the contour $-C_R$ into a closed contour, by using the the counter $D_R$ which is almost the circle $|z| = R$, clockwise. Then, we have a closed clockwise contour which includes the singularities $\pm 2\pi in$ of the integrands of both $I(s)$ and $J(s)$ for $n = 1, 2, 3....$ When $R \to \infty$ it will include all singularities in this form. Also, we notice that the integrals $I(s)$ and $J(s)$ will vanish on the contour $D_R$ when $R \to \infty$. For this reason, the integrals $I(s)$ and $J(s)$ are just $-2\pi i$ times the sum of their residues.

By Residue theorem, we calculate $I(s)$ as

$$I(s) = -2\pi i \sum_{n=1}^{\infty} -(2\pi ni)^{s-1} - (-2\pi ni)^{s-1} = i2^{s+1}\pi^s \sin(\frac{\pi s}{2})\zeta(1-s).$$

Note that this is equivalent to the functional equation of the RZF by the help of the dublication and reflection formulas of the Gamma function.

By residue theorem again, we can calculate $J(s)$:

$$J(s) = -2\pi i \sum_{n=1}^{\infty} -\ln(2\pi ni)(2\pi ni)^{s-1} - \ln(-2\pi ni)(-2\pi ni)^{s-1}$$

and hence

$$J(s) = i2^{s+1}\pi^s \sin(\frac{\pi s}{2}) \sum_{n=1}^{\infty} \frac{\ln(2\pi n)}{n^{1-s}} + i2^s\pi^{s+1}\cos(\frac{\pi s}{2})\zeta(1-s).$$

By collecting all in (5), we obtain

$$\begin{aligned}
\zeta'(s) &= -\psi(1-s)\zeta(s) + 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\sum_{n=1}^{\infty}\frac{\ln(2\pi) + \ln(n)}{n^{1-s}} \\
&\quad + 2^{s-1}\pi^s\cos(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) \qquad (7)
\end{aligned}$$

for each $s \in \mathbb{C}$ whenever each term of the three is valid when they are evaluated at $s$.

**Theorem 1:**

$$\begin{aligned}
\zeta'(s) &= -\psi(1-s)\zeta(s) \\
&\quad +2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)(\ln(2\pi)\zeta(1-s)-\zeta'(1-s)) \\
&\quad +2^{s-1}\pi^s\cos(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)
\end{aligned}$$

for each $s \in \mathbb{C}$ whenever each term of the three is valid when they are evaluated at $s$.

*Proof :* The expression (7) makes sense by analytic continuation even if it is not valid at some terms. In particular, we can use the derivative of the RZF, $\zeta'(s) = -\sum_{n=1}^{\infty}\frac{\ln n}{n^s}$ in (7). Hence we obtain the expression above.■

By using the functional equation $\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$, we can further change (7) into

$$\frac{\zeta'(s)}{\zeta(s)} = -\psi(1-s) + \ln(2\pi) - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\pi}{2}\cot(\frac{\pi s}{2}). \qquad (8)$$

Note that (8) is actually the logarithmic derivative of the functional equation! But, here we derived it from Theorem 1. The expression (7) is more useful for computations.

# 4  Values of $\zeta'(s)$ at Negative Integers

**Corollary 2:** For $m \in \mathbb{Z}^+$,

$$\zeta'(-2m) = \frac{(-1)^m(2m)!}{2^{2m+1}\pi^{2m}}\zeta(2m+1) \text{ for all } m \in \mathbb{Z}^+.$$

*Proof :* Let $s = -2m$, $m \in \mathbb{Z}^+$ in (7). The first two terms vanish and the third term is valid. We immediately get the expression.■

Although zero is not negative integer, we want to calculate $\zeta'(-0)$ here by the equation (7).

**Corollary 3:** $\zeta'(-0) = -\dfrac{\ln(2\pi)}{2}$.

*Proof :* For $s = 0$, the second and the third terms are not valid. We will make some trick here. Taking the limit $s \to 0$ in (7) and noting that $\psi(1) = -\gamma$ and $\zeta(0) = -\frac{1}{2}$, we get

$$\zeta'(-0) = -\frac{\gamma}{2} + \frac{\ln(2\pi)}{2}\lim_{s\to 0}s\zeta(1-s) + \frac{1}{2}\lim_{s\to 0}\frac{d}{ds}(s\zeta(1-s)).$$

But, $s\zeta(1-s)$ is analytic at $s = 0$ and it has the expansion $-1 + \gamma s + h.o.t.$ Hence we are done.■

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**Corollary 4:** For $m \in \mathbb{Z}^+$,

$$\zeta'(1 - 2m) = \frac{1}{2m}\psi(2m)B_{2m} - \frac{\ln(2\pi)}{2m}B_{2m} + \frac{(-1)^{m+1}(2m)!}{m(2\pi)^{2m}}\zeta'(2m)$$

*Proof :* Now, lets substitute $s = -(2m - 1)$ in (8) where $m \in \mathbb{Z}^+$. The cotangent vanishes. We get

$$\frac{\zeta'(1 - 2m)}{\zeta(1 - 2m)} = -\psi(2m) + \ln(2\pi) - \frac{\zeta'(2m)}{\zeta(2m)}$$

and putting $\zeta(1-2m) = -\dfrac{B_{2m}}{2m}$ and $\zeta(2m) = (-1)^{m+1}\dfrac{2^{2m-1}\pi^{2m}}{(2m)!}B_{2m}$ we obtain the expression.∎

**Example:** $\zeta'(-1) = \dfrac{1}{2}\psi(2)B_2 - \dfrac{\ln(2\pi)}{2}B_2 + \dfrac{1}{2\pi^2}\zeta'(2) = \dfrac{1 - \gamma}{12} - \dfrac{\ln(2\pi)}{12} +$

$\dfrac{1}{2\pi^2}\zeta'(2)$. And $\zeta'(2)$ can be computed approximately by $\zeta'(2) = -\displaystyle\sum_{n=1}^{\infty}\dfrac{\ln n}{n^2}$. We

get $\zeta'(-1) \approx -0,16542$ which agrees with the value in [2].

We found the values at negative odd integers depending on the values $\zeta'(2m)$. Can we drop this dependence? Yes we can. But, in the expense of a complicated expression.

**Theorem 5:** For $m \in \mathbb{Z}^+$, the identity (A) in the introduction is true.

*Proof :* Let us take $s = 1 - 2m$ in (6) then $J(1 - 2m) \approx \displaystyle\oint_{C_R} \frac{(\ln z)z^{-2m}}{e^{-z} - 1}dz$.

Now we will make integration by parts and evaluate the integral from $Re^{-i\pi}$ to $Re^{i\pi}$ and then will take the limit $R \to \infty$ to find $J(1 - 2m)$. The following integral is the integral of the $dv$ part of the integration by parts process:

$$\int \frac{z^{-2m}}{e^{-z} - 1}dz = \sum_{\substack{k=0 \\ k \neq 2m}}^{\infty} \frac{(-1)^k B_k}{k!}\frac{z^{k-2m}}{k - 2m} - \frac{B_{2m}}{(2m)!}\ln z$$

up to a constant. Noting that $\ln(Re^{\pm i\pi}) = \ln(R) \pm i\pi$, a routine and tedious calculation gives us the following approximation for $J(1 - 2m)$:

$$J(1 - 2m) \approx -2\pi i \sum_{\substack{t=0 \\ t \neq m}}^{\infty} \frac{B_{2t}}{(2t)!}\frac{R^{2t-2m}}{2t - 2m} + \pi i\frac{1}{(1 - 2m)R^{2m-1}} - 2\pi i\frac{B_{2m}}{(2m)!}\ln R.$$

On the other hand, starting from the equation

$$\sum_{t=0}^{\infty} \frac{B_{2t}}{(2t)!}z^{2t-2m-1} = \frac{z^{-2m}}{e^z - 1} + \frac{z^{-2m}}{2}$$

and integrating it from 1 to $R$, one can show that

$$\sum_{\substack{t=0 \\ t \neq m}}^{\infty} \frac{B_{2t}}{(2t)!}\frac{R^{2t-2m}}{2t - 2m} = \int_1^R \frac{z^{-2m}}{e^z - 1}dz + \frac{1}{(2 - 4m)R^{2m-1}} + \frac{1}{4m - 2} + \sum_{\substack{t=0 \\ t \neq m}}^{\infty} \frac{B_{2t}}{(2t)!}\frac{1}{(2t - 2m)} - \frac{B_{2m}}{(2m)!}\ln R.$$

If we substitute this last equality in the above approximation of $J(1 - 2m)$ and take the limit $R \to \infty$ we get

$$J(1 - 2m) = -2\pi i \int\limits_{1}^{\infty} \frac{z^{-2m}}{e^z - 1} dz - \frac{\pi i}{2m - 1} - 2\pi i \sum_{\substack{t=0 \\ t \neq m}}^{\infty} \frac{B_{2t}}{(2t)!} \frac{1}{(2t - 2m)}.$$

Finally, we substitute $J(1 - 2m)$ in the equation (5) with $s = 1 - 2m$, we obtain the identity (A).■

**Theorem 6:** For $m \in \mathbb{Z}^+$, the identity (B) in the introduction is true.

*Proof:* It is very similar to and easier than the proof Theorem 5. It is left to the reader.■

Combining Theorem 6 with Corollary 2 we get

**Corollary 7:** For $m \in \mathbb{Z}^+$,

$$\zeta(2m + 1) = (-1)^m 2^{2m+1} \pi^{2m} \left( \int\limits_{1}^{\infty} \frac{z^{-2m-1}}{e^z - 1} dz + \sum_{k=0}^{\infty} \frac{B_k}{k!} \frac{1}{(k - 2m - 1)} \right).$$

**Example:** Taking $m = 1$, we can calculate Apery's constant $\zeta(3)$, [2], approximately. We took only the first 7 non-zero terms of the series, and with computer-help calculation for the integral, we found approximately $\zeta(3) \approx 1,2025$.

# 5   Values of $\zeta'(s)$ at Positive Integers

Corollary 4 and Theorem 5 give an expression for $\zeta'(2m)$, $m \in \mathbb{Z}^+$ and $m \geq 2$. So, the only remaining case is the positive odd integers.

Because digamma function is not defined for negative integers, we can not use the technique of the previous section which was using the identities (5) and (6). We have another resource which is the functional equation for the derivative in Theorem 1. But, unfortunately it seems that it doesn't work too.

# References

[1] Riemann Bernhard, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsberichte der Beriner Akademie, November 1859, 671-680.

[2] https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function

[3] https://mathworld.wolfram.com/RiemannZetaFunction.html