

Differentiation Under the Integral Sign

21.1 Introduction. If the integrand of a definite integral is a function of one or more parameters in addition to the variable of integration, then the given definite integral between the limits, which may be constants or functions of the parameters, is a function of these parameters as illustrated in the following examples:

$$\int_0^1 \cos(\alpha x) dx = \left[(1/\alpha) \times \sin(\alpha x) \right]_0^1 = (\sin \alpha) / \alpha$$

$$\int_0^1 (\cos \alpha x + \cos \beta x) dx = \left[(1/\alpha) \times \sin \alpha x + (1/\beta) \times \sin \beta x \right]_0^1 = (\sin \alpha) / \alpha + (\sin \beta) / \beta$$

$$\text{Thus, in general, } \int_a^b f(x, \alpha) dx = F(\alpha) \quad \text{and} \quad \int_a^b g(x, \alpha, \beta) dx = G(\alpha, \beta) \quad \dots (1)$$

where α and β are parameters and a, b are constants or functions of parameters.

In some problems functions $f(x, \alpha)$ and $g(x, \alpha, \beta)$ are such that the evaluation of the corresponding integrals given by (1) are either very complicated or impossible. In such problems sometimes the integrals

$$\int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx \quad \text{or} \quad \int_a^b \frac{\partial g(x, \alpha, \beta)}{\partial \alpha} dx \quad \text{or} \quad \int_a^b \frac{\partial g(x, \alpha, \beta)}{\partial \beta} dx$$

may be easily evaluated. In view of this fact, we propose to discuss the technique of differentiation under the integral sign.

21.2. Leibnitz's rule for differentiation under the integral sign. (Kanpur 2011)

Theorem. If $f(x, \alpha)$ and $\partial f / \partial \alpha$ are continuous functions of x and α for $a \leq x \leq b$, $c \leq \alpha \leq d$, a, b being independent of α , then

$$\frac{d}{d\alpha} \int_a^b f(x, \alpha) dx = \int_a^b \frac{\partial f}{\partial \alpha} dx$$

$$\text{Proof. Let} \quad F(\alpha) = \int_a^b f(x, \alpha) dx \quad \dots (1)$$

Let α change to $\alpha + \delta\alpha$ (α and $\alpha + \delta\alpha$ both lying in the closed interval $[c, d]$), then a, b and x being independent of α , remain unaltered and $F(\alpha)$ changes to $F(\alpha + \delta\alpha)$. Hence, we have

$$F(\alpha + \delta\alpha) = \int_a^b f(x, \alpha + \delta\alpha) dx. \quad \dots (2)$$

From (1) and (2),
$$F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b \{f(x, \alpha + \delta\alpha) - f(x, \alpha)\} dx \quad \dots(3)$$

Using the Lagrange's mean value theorem for derivatives, we get

$$f(x, \alpha + \delta\alpha) - f(x, \alpha) = \delta\alpha \cdot \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha), \text{ where } 0 < \theta < 1 \quad \dots(4)$$

Using (4), (3) reduces to

$$F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b \delta\alpha \cdot \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha) dx = \delta\alpha \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha) dx$$

Thus,
$$\frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha) dx \quad \dots(5)$$

By definition, we have
$$\lim_{\delta\alpha \rightarrow 0} \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \frac{dF(\alpha)}{d\alpha} \quad \dots(6)$$

Taking limits as $\delta\alpha \rightarrow 0$ on both sides of (5) and using (6), we get

$$\frac{dF(\alpha)}{d\alpha} = \lim_{\delta\alpha \rightarrow 0} \int_a^b \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha) dx = \int_a^b \lim_{\delta\alpha \rightarrow 0} \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha) dx, \quad \dots(7)$$

where we have assumed that the limit of integral is equal to the integral of limit. Again, since $\partial f / \partial\alpha$ is continuous, we have

$$\lim_{\delta\alpha \rightarrow 0} \frac{\partial}{\partial\alpha} f(x, \alpha + \theta\delta\alpha) = \frac{\partial}{\partial\alpha} f(x, \alpha) \quad \dots(8)$$

Using (8), (7) reduces to

$$\frac{dF(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial\alpha} dx \quad \text{or} \quad \frac{d}{d\alpha} \int_a^b f(x, \alpha) dx = \int_a^b \frac{\partial f(x, \alpha)}{\partial\alpha} dx, \text{ using (1)}$$

Remark Let
$$G(\alpha, \beta) = \int_a^b g(x, \alpha, \beta) dx, \quad \dots(i)$$

where a and b are independent of parameters α and β . Then, proceeding as in the above theorem, we may show that

$$\frac{\partial G(\alpha, \beta)}{\partial\alpha} = \int_a^b \frac{\partial g(x, \alpha, \beta)}{\partial\alpha} dx \quad \text{and} \quad \frac{\partial G(\alpha, \beta)}{\partial\beta} = \int_a^b \frac{\partial g(x, \alpha, \beta)}{\partial\beta} dx \quad \dots(ii)$$

While dealing with function $g(x, \alpha, \beta)$ of two parameters, we make a choice of appropriate parameter α or β in the above results (ii). The selected parameter must lead us to new integral which can be easily evaluated. For more discussion, refer Art. 21.5 A.

21.3. General form of Leibnitz's rule of differentiation under the integral sign when the limits of integration are functions of the parameters.

Theorem I. $f(x, \alpha)$ and $\partial f / \partial\alpha$ are continuous functions of x and α for $g(\alpha) \leq x \leq h(\alpha)$, $c \leq \alpha \leq d$ and $g(\alpha)$, $h(\alpha)$ are themselves functions of α , possessing continuous first order derivatives, then

$$\frac{d}{d\alpha} \int_{g(\alpha)}^{h(\alpha)} f(x, \alpha) dx = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x, \alpha)}{\partial\alpha} dx + \frac{dh(\alpha)}{d\alpha} f(h(\alpha), \alpha) - \frac{dg(\alpha)}{d\alpha} f(g(\alpha), \alpha)$$

Proof. Let
$$F(\alpha) = \int_{g(\alpha)}^{h(\alpha)} f(x, \alpha) dx \quad \dots(1)$$

Since $g(\alpha)$ and $h(\alpha)$ are functions of α , when α changes to $\alpha + \delta\alpha$, let $g(\alpha)$ change to $g(\alpha + \delta\alpha)$, $h(\alpha)$ change to $h(\alpha + \delta\alpha)$ and $F(\alpha)$ change to $F(\alpha + \delta\alpha)$. Hence, we get

$$F(\alpha + \delta\alpha) = \int_{g(\alpha + \delta\alpha)}^{h(\alpha + \delta\alpha)} f(x, \alpha + \delta\alpha) dx. \quad \dots(2)$$

$$\begin{aligned} (1) \text{ and } (2) \Rightarrow F(\alpha + \delta\alpha) - F(\alpha) &= \int_{g(\alpha + \delta\alpha)}^{h(\alpha + \delta\alpha)} f(x, \alpha + \delta\alpha) dx - \int_{g(\alpha)}^{h(\alpha)} f(x, \alpha) dx \\ &= \int_{g(\alpha + \delta\alpha)}^{g(\alpha)} f(x, \alpha + \delta\alpha) dx + \int_{g(\alpha)}^{h(\alpha)} f(x, \alpha + \delta\alpha) dx + \int_{h(\alpha)}^{h(\alpha + \delta\alpha)} f(x, \alpha + \delta\alpha) dx - \int_{g(\alpha)}^{h(\alpha)} f(x, \alpha) dx \end{aligned}$$

$$\begin{aligned} \text{Thus, } F(\alpha + \delta\alpha) - F(\alpha) &= \int_{g(\alpha)}^{h(\alpha)} \{f(x, \alpha + \delta\alpha) - f(x, \alpha)\} dx \\ &\quad + \int_{h(\alpha)}^{h(\alpha + \delta\alpha)} f(x, \alpha + \delta\alpha) dx - \int_{g(\alpha)}^{g(\alpha + \delta\alpha)} f(x, \alpha + \delta\alpha) dx \quad \dots(3) \end{aligned}$$

Using the Lagrange's mean theorem for derivatives, we get

$$f(x, \alpha + \delta\alpha) - f(x, \alpha) = \delta\alpha \frac{\partial}{\partial \alpha} f(x, \alpha + \theta \delta\alpha), \quad 0 < \theta < 1 \quad \dots(4)$$

Again, using the mean value theorem for integrals, we get

$$\int_{h(\alpha)}^{h(\alpha + \delta\alpha)} f(x, \alpha + \delta\alpha) dx = \{h(\alpha + \delta\alpha) - h(\alpha)\} \times f(\xi, \alpha + \delta\alpha) \quad \dots(5)$$

$$\text{and} \quad \int_{g(\alpha)}^{g(\alpha + \delta\alpha)} f(x, \alpha + \delta\alpha) dx = \{g(\alpha + \delta\alpha) - g(\alpha)\} \times f(\eta, \alpha + \delta\alpha), \quad \dots(6)$$

where ξ lies between $h(\alpha)$ and $h(\alpha + \delta\alpha)$ and η lies between $g(\alpha)$ and $g(\alpha + \delta\alpha)$.

Using (4) (5) and (6), (3) reduces to

$$\begin{aligned} F(\alpha + \delta\alpha) - F(\alpha) &= \int_{g(\alpha)}^{h(\alpha)} \delta\alpha \cdot \frac{\partial}{\partial \alpha} f(x, \alpha + \theta \delta\alpha) dx \\ &\quad + \{h(\alpha + \delta\alpha) - h(\alpha)\} f(\xi, \alpha + \delta\alpha) - \{g(\alpha + \delta\alpha) - g(\alpha)\} f(\eta, \alpha + \delta\alpha) \end{aligned}$$

$$\begin{aligned} \text{or} \quad \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} &= \int_{g(\alpha)}^{h(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha + \theta \delta\alpha) dx \\ &\quad + \frac{h(\alpha + \delta\alpha) - h(\alpha)}{\delta\alpha} f(\xi, \alpha + \delta\alpha) - \frac{g(\alpha + \delta\alpha) - g(\alpha)}{\delta\alpha} f(\eta, \alpha + \delta\alpha) \quad \dots(7) \end{aligned}$$

$$\text{Now, } \lim_{\delta\alpha \rightarrow 0} \frac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \frac{dF}{d\alpha}, \quad \lim_{\delta\alpha \rightarrow 0} \frac{h(\alpha + \delta\alpha) - h(\alpha)}{\delta\alpha} = \frac{dh}{d\alpha}, \quad \lim_{\delta\alpha \rightarrow 0} \frac{g(\alpha + \delta\alpha) - g(\alpha)}{\delta\alpha} = \frac{dg}{d\alpha}$$

Taking limit as $\delta\alpha \rightarrow 0$ on both sides of (7) and using the above three results, we obtain

$$\frac{dF}{d\alpha} = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x, \alpha)}{\partial \alpha} dx + \frac{dh}{d\alpha} f(h(\alpha), \alpha) - \frac{dg}{d\alpha} f(g(\alpha), \alpha)$$

[Since $h(\alpha) \leq \xi \leq h(\alpha + \delta\alpha)$ and $g(\alpha) \leq \eta \leq g(\alpha + \delta\alpha)$, hence

$$\lim_{\delta\alpha \rightarrow 0} f(\xi, \alpha + \delta\alpha) = f(h(\alpha), \alpha) \text{ and } \lim_{\delta\alpha \rightarrow 0} f(\eta, \alpha + \delta\alpha) = f(g(\alpha), \alpha)]$$

Thus,
$$\frac{d}{d\alpha} \int_{g(\alpha)}^{h(\alpha)} f(x, \alpha) dx = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x, \alpha)}{\partial \alpha} dx + \frac{dh(\alpha)}{d\alpha} f(h(\alpha), \alpha) - \frac{dg(\alpha)}{d\alpha} f(g(\alpha), \alpha) \quad \dots(8)$$

Remark 1. If $g(\alpha) = a$ and $h(\alpha) = b$ are independent of α , then the last two terms in result (8) are zero and we get

$$\frac{d}{d\alpha} \int_a^b f(x, \alpha) dx = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} d\alpha,$$

which is what we have already proved in Art 21.2

Remark 2. *Differentiation under the integral sign in the case of improper integrals.*

The results obtained in Art. 21.2 and Art. 21.3 may not be applicable in the case of improper integrals, and the question of validity of the results to improper integral requires further investigation. However, in our discussion in this chapter we shall omit this investigation. Accordingly, whenever we shall deal with any improper integral, we shall assume that the necessary conditions for validity of the results are satisfied.

21.4A. Evaluation of integral $\int_a^b f(x, \alpha) dx$, where a and b are independent of parameter α . WORKING RULE.

Step 1: Let
$$F(\alpha) = \int_a^b f(x, \alpha) dx \quad \dots(1)$$

Step 2: Differentiating both sides of (1) w.r.t. ' α ' and using Leibnitz's rule for differentiation under the integral sign (refer Art 21.2), we have

$$\frac{dF}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx \quad \dots(2)$$

Step 3: Evaluate the integral on R.H.S of (2) as usual.

Step 4: Solve the resulting differential equation obtained in step 3. The solution so obtained will involve a constant of integration C .

Step 5: Using (1), compute the value of constant C (obtained in step 4) by giving a suitable value to the parameter. Substitute this value of C in the result of step 4 and thus get the value of the integral (1)

21.4 B. Solved examples of type 1 based on Art 21.4 A

Ex. 1. Assuming the validity of differentiation under the integral sign, show that

$$(i) \int_0^{\pi/2} \frac{\log(1 + y \sin^2 x)}{\sin^2 x} dx = \pi(\sqrt{1+y} - 1), \text{ where } y > -1$$

$$(ii) \int_0^{\pi/2} \log(1 - x^2 \cos^2 \theta) d\theta = \pi \left\{ \log(1 + \sqrt{1 - x^2}) - \log 2 \right\}, \quad x^2 \leq 1$$

$$(iii) \int_0^{\pi/2} \log(1 - x^2 \sin^2 \theta) d\theta = \pi \left\{ \log(1 + \sqrt{1 - x^2}) - \log 2 \right\}, \quad x^2 \leq 1$$

Sol. (i). Let
$$F(y) = \int_0^{\pi/2} \frac{\log(1 + y \sin^2 x)}{\sin^2 x} dx \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. 'y' and using the Leibnitz's rule of differentiation under the integral sign, we get

$$\begin{aligned} \frac{dF}{dy} &= \int_0^{\pi/2} \frac{\partial}{\partial y} \left[\frac{\log(1 + y \sin^2 x)}{\sin^2 x} \right] dx = \int_0^{\pi/2} \frac{1}{\sin^2 x} \times \frac{1}{1 + y \sin^2 x} \times \sin^2 x dx \\ &= \int_0^{\pi/2} \frac{dx}{1 + y \sin^2 x} = \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + y \tan^2 x} dx = \int_0^{\pi/2} \frac{\sec^2 x}{1 + (1 + y) \tan^2 x} dx \\ &\quad \text{[on dividing the numerator and denominator by } \cos^2 x \text{]} \\ &= \int_0^{\infty} \frac{dt}{1 + (1 + y)t^2}, \text{ putting } \tan x = t \text{ and } \sec^2 x dx = dt \\ &= \frac{1}{1 + y} \int_0^{\infty} \frac{dt}{t^2 + \{1/(1 + y)\}} = \frac{1}{1 + y} \times \frac{1}{(1/\sqrt{1 + y})} \times \left[\tan^{-1} \frac{t}{(1/\sqrt{1 + y})} \right]_0^{\infty} \\ &= (1 + y)^{-1/2} (\tan^{-1} \infty - \tan^{-1} 0) = (1 + y)^{-1/2} \times (\pi/2 - 0) = (\pi/2) \times (1 + y)^{-1/2} \end{aligned}$$

Thus,
$$dF = (\pi/2) \times (1 + y)^{-1/2} dy \quad \dots(2)$$

Integrating (2),
$$F(y) = \pi(1 + y)^{1/2} + C, \quad C \text{ being an arbitrary constant} \quad \dots(3)$$

Putting $y = 0$ in (1), we get $F(0) = 0$. Next, putting $y = 0$ and $F(0) = 0$ in (3), we get $C = -\pi$. Hence (3) reduces to

$$F(y) = \pi(1 + y)^{1/2} - \pi \quad \text{or} \quad \int_0^{\pi/2} \frac{\log(1 + y \sin^2 x)}{\sin^2 x} dx = \pi(\sqrt{1 + y} - 1), \text{ using (1)}$$

(ii) Let
$$F(x) = \int_0^{\pi/2} \log(1 - x^2 \cos^2 \theta) d\theta \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. 'x' and using the Leibnitz's rule of differentiation under the integral sign, we have

$$\begin{aligned} \frac{dF}{dx} &= \int_0^{\pi/2} \frac{\partial}{\partial x} \log(1 - x^2 \cos^2 \theta) d\theta = \int_0^{\pi/2} \frac{1}{1 - x^2 \cos^2 \theta} \times (-2x \cos^2 \theta) d\theta \\ &= \frac{2}{x} \int_0^{\pi/2} \frac{(-x^2 \cos^2 \theta)}{1 - x^2 \cos^2 \theta} d\theta = \frac{2}{x} \int_0^{\pi/2} \frac{(1 - x^2 \cos^2 \theta) - 1}{1 - x^2 \cos^2 \theta} d\theta \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{x} \left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2} \frac{d\theta}{1-x^2 \cos^2 \theta} \right] = \frac{2}{x} \times \frac{\pi}{2} - \frac{2}{x} \int_0^{\pi/2} \frac{d\theta}{1-x^2 \cos^2 \theta} \\
 &= \frac{\pi}{x} - \frac{2}{x} \int_0^{\pi/2} \frac{\sec^2 \theta}{\sec^2 \theta - x^2} d\theta = \frac{\pi}{x} - \frac{2}{x} \int_0^{\pi/2} \frac{\sec^2 \theta}{1 + \tan^2 \theta - x^2} d\theta \\
 &= \frac{\pi}{x} - \frac{2}{x} \int_0^{\infty} \frac{dt}{t^2 + (1-x^2)}, \text{ putting } \tan \theta = t \text{ and } \sec^2 \theta d\theta = dt \\
 &= \frac{\pi}{x} - \frac{2}{x} \left[\frac{1}{\sqrt{1-x^2}} \tan^{-1} \frac{t}{\sqrt{1-x^2}} \right]_0^{\infty} = \frac{\pi}{x} - \frac{2}{x\sqrt{1-x^2}} \times \frac{\pi}{2}
 \end{aligned}$$

Thus,
$$dF = \left(\frac{\pi}{x} - \frac{\pi}{x\sqrt{1-x^2}} \right) dx \quad \dots(2)$$

Integrating (2), $F(x) = \pi \log x - \pi \int \frac{dx}{x\sqrt{1-x^2}} + C$, C being an arbitrary constant $\dots(3)$

Now, putting $x = 1/z$ and $dz = -(1/z^2)dz$, we have

$$\begin{aligned}
 \int \frac{dx}{x\sqrt{1-x^2}} &= \int \frac{(-1/z^2)dz}{(1/z)\sqrt{1-(1/z)^2}} = - \int \frac{dz}{\sqrt{z^2-1}} = -\log(z + \sqrt{z^2-1}) \\
 &= -\log\{1/x + \sqrt{(1/x)^2-1}\} = -\log\{(1 + \sqrt{1-x^2})/x\} \\
 &= -\{\log(1 + \sqrt{1-x^2}) - \log x\} = \log x - \log(1 + \sqrt{1-x^2})
 \end{aligned}$$

Hence (3) yields, $F(x) = \pi \log x - \pi \{\log x - \log(1 + \sqrt{1-x^2})\} + C$

or $F(x) = \pi \log(1 + \sqrt{1-x^2}) + C$, C being an arbitrary constant $\dots(4)$

Putting $x = 0$ in (1), we get $F(0) = 0$. Next, putting $x = 0$ and $F(0) = 0$ in (4), we get $0 = \pi \log 2 + C$ giving $C = -\pi \log 2$. Hence, (4) reduces to

$$F(x) = \pi \log(1 + \sqrt{1-x^2}) - \pi \log 2$$

or
$$\int_0^{\pi/2} \log(1-x^2 \cos^2 \theta) d\theta = \pi \{\log(1 + \sqrt{1-x^2}) - \log 2\}, \text{ using (1)}$$

(iii). Using the property $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ of definite integrals, we have

$$\begin{aligned}
 \int_0^{\pi/2} (1-x^2 \sin^2 \theta) d\theta &= \int_0^{\pi/2} \{1-x^2 \sin^2(\pi/2-\theta)\} d\theta = \int_0^{\pi/2} (1-x^2 \cos^2 \theta) d\theta \\
 &= \pi \{\log(1 + \sqrt{1-x^2}) - \log 2\}, \text{ by, part (ii)}
 \end{aligned}$$

Ex. 2 Assuming the validity of differentiation under the integral sign, show that

$$(i) \int_0^{\pi} \frac{\log(1 + a \cos x)}{\cos x} dx = \pi \sin^{-1} a, \text{ if } |a| < 1 \quad \text{[Delhi Maths (H) 2001, 04, 05]} \\ \text{[Pune 2010]}$$

$$(ii) \int_0^{\pi} \frac{\log(1 + \sin \alpha \cos x)}{\cos x} dx = \pi \alpha$$

$$(iii) \int_0^{\pi/2} \frac{\log(1 + \cos \alpha \sin x)}{\cos x} dx = \frac{1}{2} \left(\frac{\pi^2}{4} - \alpha \right)$$

$$(iv) \int_{-\pi/2}^{\pi/2} \frac{\log(1 + a \sin x)}{\sin x} dx = \pi \sin^{-1} a, \text{ if } |a| < 1$$

Sol. (i). Let
$$F(a) = \int_0^{\pi} \frac{\log(1 + a \cos x)}{\cos x} dx \quad \dots(1)$$

Differentiating both sides of (1) w.r.t 'a' and using the Leibnitz's rule of differentiation under the sign of integral, we get

$$\frac{dF}{da} = \int_0^{\pi} \frac{\partial}{\partial a} \left[\frac{\log(1 + a \cos x)}{\cos x} \right] dx = \int_0^{\pi} \frac{1}{\cos x} \cdot \frac{\cos x}{1 + a \cos x} dx = \int_0^{\pi} \frac{dx}{1 + a \cos x} \quad \dots(2)$$

From Integral Calculus,
$$\int \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1} \frac{b + a \cos x}{a + b \cos x}, \text{ if } b^2 < a^2 \quad \dots(3)$$

Here, given that $|a| < 1$ so that $a^2 < 1$ Hence, using (3), (2) yields

$$\begin{aligned} \frac{dF}{da} &= \frac{1}{\sqrt{1-a^2}} \left[\cos^{-1} \frac{a + \cos x}{1 + a \cos x} \right]_0^{\pi} = \frac{1}{\sqrt{1-a^2}} \left[\cos^{-1} \frac{a-1}{1-a} - \cos^{-1} \frac{a+1}{1+a} \right] \\ &= \{ \cos^{-1}(-1) - \cos^{-1}(1) \} / (1-a^2)^{1/2} = (\pi - 0) / (1-a^2)^{1/2} = \pi / (1-a^2)^{1/2} \end{aligned}$$

Thus,
$$dF = \{ \pi / (1-a^2)^{1/2} \} da$$

Integrating it,
$$F(a) = \pi \sin^{-1} a + C, \text{ C being an arbitrary constant} \quad \dots(4)$$

Putting $a = 0$ in (1), we get $F(0) = 0$. Next, putting $a = 0$ and $F(0) = 0$ in (4), we get $C = 0$. Hence, (4) reduces to

$$F(a) = \pi \sin^{-1} a \quad \text{or} \quad \int_0^{\pi} \frac{\log(1 + a \cos x)}{\cos x} dx = \pi \sin^{-1} a, \text{ using (1)}$$

(ii). Taking $a = \sin \alpha$, proceed as in part (i). Also, note that the condition $|a| < 1 \Rightarrow$

$|\sin \alpha| < 1$, which is true. Thus,
$$\int_0^{\pi} \frac{\log(1 + \sin \alpha \cos x)}{\cos x} dx = \pi \sin^{-1}(\sin \alpha) = \pi \alpha$$

(iii). Let
$$F(\alpha) = \int_0^{\pi/2} \frac{\log(1 + \cos \alpha \cos x)}{\cos x} dx \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. ' α ' and using Leibnitz's rule of differentiation under the integral sign, we have

$$\frac{dF}{d\alpha} = \int_0^{\pi/2} \frac{\partial}{\partial \alpha} \left[\frac{\log(1 + \cos \alpha \cos x)}{\cos x} \right] dx = -\sin \alpha \int_0^{\pi/2} \frac{dx}{1 + \cos \alpha \cos x} \quad \dots(2)$$

From Integral Calculus, $\int \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1} \frac{b + a \cos x}{a + b \cos x}, \text{ if } b^2 < a^2 \quad \dots(3)$

Here $\cos^2 \alpha < 1$ and so using (3), (2) yields

$$\frac{dF}{d\alpha} = -\sin \alpha \left[\frac{1}{\sqrt{1 - \cos^2 \alpha}} \cos^{-1} \frac{\cos \alpha + \cos x}{1 + \cos \alpha \cos x} \right]_0^{\pi/2} = -(\cos \cos^{-1} \alpha - \cos^{-1} 1) = -\alpha$$

Thus, $dF = -\alpha d\alpha$

Integrating, $F(\alpha) = -(\alpha^2 / 2) + C, C$ being an arbitrary constant $\dots(4)$

Putting $\alpha = \pi / 2$ in (1), we get $F(\pi / 2) = 0$. Next, putting $\alpha = \pi / 2$ and $F(\pi / 2) = 0$ in (4), we get $0 = -\pi^2 / 8 + C$ so that $C = \pi^2 / 8$. Hence (4) yields

$$F(\alpha) = -\frac{\alpha^2}{2} + \frac{\pi^2}{8} \quad \text{or} \quad \int_0^{\pi/2} \frac{\cos(1 + \cos \alpha \cos x)}{\cos x} dx = \frac{1}{2} \left(\frac{\pi^2}{4} - \alpha^2 \right), \text{ using (1)}$$

(iv). Let $F(a) = \int_{-\pi/2}^{\pi/2} \frac{\log(1 + a \sin x)}{\sin x} dx \quad \dots(1)$

Differentiating both sides of (1) w.r.t. ' a ' and using Leibnitz's rule of differentiation under the integral sign, we obtain

$$\frac{dF}{da} = \int_{-\pi/2}^{\pi/2} \frac{\partial}{\partial a} \left[\frac{\log(1 + a \sin x)}{\sin x} \right] dx = \int_{-\pi/2}^{\pi/2} \frac{dx}{1 + a \sin x} \quad \dots(2)$$

Putting $x = \pi / 2 - t$ and $dx = -dt$, (2) reduces to

$$\frac{dF}{da} = \int_{\pi}^0 \frac{(-dt)}{1 + a \sin(\pi / 2 - t)} = \int_0^{\pi} \frac{dt}{1 + a \cos t} \quad \dots(3)$$

From Integral Calculus, $\int \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1} \frac{b + a \cos x}{a + b \cos x}, b^2 < a^2 \quad \dots(4)$

Here, given that $|a| < 1$ so that $a^2 < 1$. Hence, using (4), (3) yields

$$\frac{dF}{da} = \left[\frac{1}{\sqrt{1 - a^2}} \cos^{-1} \frac{a + \cos t}{1 + \cos t} \right]_0^{\pi} = \frac{1}{\sqrt{1 - a^2}} \left[\cos^{-1} \frac{a - 1}{1 - a} - \cos^{-1} \frac{a + 1}{1 + a} \right]$$

$$dF/da = \{[\cos^{-1}(-1) - \cos^{-1}1]\} / (1 - a^2)^{1/2} \quad \text{or} \quad dF = \{\pi / (1 - a^2)^{1/2}\} da.$$

Integrating, $F(a) = \pi \sin^{-1} a + C, C$ being an arbitrary constant $\dots(5)$

Putting $a = 0$ in (1) yields $F(0) = 0$. Next, putting $a = 0$ and $F(0) = 0$ in (5) yields $C = 0$. Hence, (5) reduces to

$$F(a) = \pi \sin^{-1} a \quad \text{or} \quad \int_{-\pi/2}^{\pi/2} \frac{\log(1 + a \sin x)}{\sin x} dx = \pi \sin^{-1} a, \text{ using (1)}$$

Ex. 3. Assuming the validity of differentiation under the integral sign, show that

$$\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a), a \geq 0.$$

Also find the value of integral if $a < 0$.

[Delhi B.Sc. (Hons) 2008, 11]

Sol. Let
$$F(a) = \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. ' a ' and using Leibnitz's rule of differentiation under the integral sign, we obtain

$$\begin{aligned} \frac{dF}{da} &= \int_0^{\infty} \frac{\partial}{\partial a} \left[\frac{\tan^{-1} ax}{x(1+x^2)} \right] dx = \int_0^{\infty} \frac{1}{x(1+x^2)} \times \frac{x}{1+a^2x^2} dx = \int_0^{\infty} \frac{dx}{(1+x^2)(1+a^2x^2)} \\ &= \frac{1}{1-a^2} \int_0^{\infty} \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2} \right] dx, \text{ on resolving into partial fractions} \\ &= \frac{1}{1-a^2} \left[\int_0^{\infty} \frac{dx}{1+x^2} - \int_0^{\infty} \frac{dx}{x^2 + (1/a)^2} \right] = \frac{1}{1-a^2} \left[\tan^{-1} x - a \tan^{-1} ax \right]_0^{\infty} \quad \dots(2) \end{aligned}$$

Case (i). Let $a \geq 0$. Then, (2) reduces to

$$\frac{dF}{da} = \frac{1}{1-a^2} \left[\frac{\pi}{2} - \frac{a\pi}{2} \right] = \frac{1}{(1-a)(1+a)} \times \frac{\pi(1-a)}{2} = \frac{\pi}{2(1+a)}$$

Thus,

$$dF = [\pi/2(1+a)] da$$

Integrating, $F(a) = (\pi/2) \times \log(1+a) + C$, C being an arbitrary constant $\dots(3)$

Putting $a = 0$ in (1) yields $F(0) = 0$. Next, putting $a = 0$ and $F(0) = 0$ in (3) yields $C = 0$. Hence, (3) reduces to

$$F(a) = \frac{\pi}{2} (1+a) \quad \text{or} \quad \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} (1+a), \text{ using (1)} \quad \dots(4)$$

Case (ii). Let $a < 0$. Then, $a < 0 \Rightarrow \tan^{-1}(ax) = -\pi/2$. Therefore, (2) yields

$$\frac{dF}{da} = \frac{1}{1-a^2} \left[\frac{\pi}{2} + \frac{a\pi}{2} \right] = \frac{1}{(1-a)(1+a)} \times \frac{\pi(1+a)}{2} = \frac{\pi}{2(1-a)}$$

Then,

$$dF = [\pi/2(1-a)] da$$

Integrating, $F(a) = -(\pi/2) \times \log(1-a) + D$, D being an arbitrary constant $\dots(5)$

Putting $a = 0$ in (1) yields $F(0) = 0$. Next, putting $a = 0$ and $F(0) = 0$ in (5) yields $D = 0$. Hence, (5) reduces to

$$F(a) = -\frac{\pi(1-a)}{2} \quad \text{or} \quad \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx = -\frac{\pi(1-a)}{2}, \text{ using (1)}$$

Ex. 4. If $|a| < 1$, prove that $\int_0^\pi \log(1 + a \cos x) dx = \pi \log(1/2 + \sqrt{1-a^2}/2)$ [Delhi 2007]

Sol. Let
$$F(a) = \int_0^\pi \log(1 + a \cos x) dx \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. ' a ' and using Leibnitz's rule of differentiation under the integral sign, we get

$$\frac{dF}{da} = \int_0^\pi \frac{\partial}{\partial a} [\log(1 + a \cos x)] dx = \int_0^\pi \frac{\cos x}{1 + a \cos x} dx = \frac{1}{a} \int_0^\pi \frac{(1 + a \cos x) - 1}{1 + a \cos x} dx$$

or
$$\frac{dF}{da} = \frac{1}{a} \int_0^\pi \left(1 - \frac{1}{1 + a \cos x} \right) dx = \frac{\pi}{a} - \frac{1}{a} \int_0^\pi \frac{dx}{1 + a \cos x} \quad \dots(2)$$

From Integral Calculus,
$$\int \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1} \frac{b + a \cos x}{a + b \cos x}, \text{ if } b^2 < a^2$$

Here, given that $|a| < 1$ so that $a^2 < 1$. Hence, using the above formula, (2) reduces to

$$\frac{dF}{da} = \frac{\pi}{a} - \frac{1}{a} \left[\frac{1}{\sqrt{1-a^2}} \cos^{-1} \frac{a + \cos x}{1 + a \cos x} \right]_0^\pi = \frac{\pi}{a} - \frac{1}{a\sqrt{1-a^2}} \left[\cos^{-1} \frac{a-1}{1-a} - \cos^{-1} \frac{a+1}{1+a} \right]$$

or
$$\frac{dF}{da} = \frac{\pi}{a} - \frac{\pi}{a\sqrt{1-a^2}} \quad \text{or} \quad dF = \left(\frac{\pi}{a} - \frac{\pi}{a\sqrt{1-a^2}} \right) da$$

[Using the fact that $\cos^{-1}(-1) = \pi$ and $\cos^{-1} 1 = 0$]

Integrating,
$$F(a) = \pi \log a - \pi \int \frac{da}{a\sqrt{1-a^2}} + C, \text{ } C \text{ being an arbitrary constant} \quad \dots(3)$$

Putting $a = 1/t$ and $da = -(1/t^2)dt$, we get

$$\begin{aligned} \int \frac{da}{a\sqrt{1-a^2}} &= -\int \frac{(1/t^2)dt}{(1/t)\sqrt{1-(1/t^2)}} = -\int \frac{dt}{\sqrt{t^2-1}} = -\log(t + \sqrt{t^2-1}) \\ &= -\log\{1/a + \sqrt{1-(1/a^2)}\} = -\log\{(1 + \sqrt{1-a^2})/a\} = \log a - \log(1 + \sqrt{1-a^2}) \end{aligned}$$

Hence, (3) yields
$$F(a) = \pi \log a - \pi \{\log a - \log(1 + \sqrt{1-a^2})\} + C$$

or
$$F(a) = \pi \log(1 + \sqrt{1-a^2}) + C \quad \dots(4)$$

Putting $a = 0$ in (1) yields $F(0) = 0$. Next, putting $a = 0$ and $F(0) = 0$ in (4) yields $0 = \pi \log 2 + C$ so that $C = -\pi \log 2$. Hence, (4) reduces to

$$F(a) = \pi \{\log(1 + \sqrt{1-a^2}) - \log 2\} = \pi \log\{(1 + \sqrt{1-a^2})/2\}$$

or
$$\int_0^\pi \log(1 + a \cos x) dx = \pi \log\{1/2 + (\sqrt{1-a^2})/2\}, \text{ using (1)}$$

Ex. 5. Assuming the validity of differentiation under integral sign, show that

$$\int_0^1 \log \left(\frac{1+ax}{1-ax} \right) \frac{dx}{x\sqrt{1-x^2}} = \pi \sin^{-1} a \quad \text{[Delhi Maths (H) 1994]}$$

Sol. Let
$$F(a) = \int_0^1 \log \left(\frac{1+ax}{1-ax} \right) \frac{dx}{x\sqrt{1-x^2}}. \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. 'a' and using Leibnitz's rule of differentiating under the integral sign, we get

$$\begin{aligned} \frac{dF}{da} &= \int_0^1 \frac{1}{(1+ax)(1-ax)} \times \frac{x(1-ax) - (-x)(1+ax)}{(1-ax)^2} \cdot \frac{dx}{x\sqrt{1-x^2}} \\ &= 2 \int_0^1 \frac{dx}{(1-a^2x^2)\sqrt{1-x^2}} = \int_0^1 \frac{(2/x^3) dx}{(1/x^2 - a^2)\sqrt{(1/x^2) - 1}} \\ &= \int_{\infty}^0 \frac{(-2t) dt}{(t^2 + 1 - a^2)t}, \quad \text{putting } \frac{1}{x^2} - 1 = t^2 \quad \text{and} \quad -\frac{2}{x^3} dx = 2t dt \\ &= 2 \int_0^{\infty} \frac{dt}{t^2 + (\sqrt{1-a^2})^2} = \frac{2}{\sqrt{1-a^2}} \left[\tan^{-1} \frac{t}{\sqrt{1-a^2}} \right]_0^{\infty} = \frac{2}{\sqrt{1-a^2}} \times \frac{\pi}{2}, \quad \text{if } a^2 < 1 \end{aligned}$$

Thus,
$$F(a) = (\pi / \sqrt{1-a^2}) da \quad \dots(2)$$

Integrating (2),
$$F(a) = \pi \sin^{-1} a + C, \quad C \text{ being an arbitrary constant} \quad \dots(3)$$

Putting $a = 0$ in (1) yields $F(0) = 0$. Next, putting $a = 0$ and $F(0) = 0$ in (3) yields $C = 0$. Hence (3) reduces to

$$F(a) = \pi \sin^{-1} a \quad \text{or} \quad \int_0^1 \log \left(\frac{1+ax}{1-ax} \right) \frac{dx}{x\sqrt{1-x^2}} = \pi \sin^{-1} a, \quad \text{using (1)}$$

Ex. 6. If $y > 0$, show that
$$\int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx = \cot^{-1} y = \pi/2 - \tan^{-1} y$$

[Kurukshetra B.C.A (II) 2008]

Sol. Let
$$F(y) = \int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. 'y' and using Leibnitz's rule of differentiation under integral sign, we get

$$\begin{aligned} \frac{dF}{dy} &= \int_0^{\infty} \frac{\partial}{\partial y} \left(e^{-xy} \frac{\sin x}{x} \right) dx = \int_0^{\infty} \frac{\sin x}{x} e^{-xy} (-x) dx = - \int_0^{\infty} e^{-yx} \sin x dx \\ &= - \left[\frac{e^{-yx}}{y^2 + 1} (-y \sin x - \cos x) \right]_0^{\infty}, \quad \text{as } \int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} \end{aligned}$$

or
$$dF/dy = -[1/(1+y^2)] \quad \text{or} \quad dF = -\{1/(1+y^2)\} dy$$

Integrating, $F(y) = \cot^{-1} y + c$, c being an arbitrary constant ... (2)

Letting $y \rightarrow \infty$ in (1) $\Rightarrow F(y) \rightarrow 0$ as $y \rightarrow \infty$. Next, letting $y \rightarrow \infty$ in (3) and using the fact that $F(y) \rightarrow 0$ as $y \rightarrow \infty$, we get $C = 0$. Hence, (2) reduces to

$$F(y) = \cot^{-1} y \quad \text{or} \quad \int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx = \cot^{-1} y, \text{ using (1)}$$

or $\int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx = \pi/2 - \tan^{-1} y$, as $\tan^{-1} y + \cot^{-1} y = \pi/2$

Ex.7. Assuming the validity of differentiation under the integral sign, show that

$$(i) \int_0^{\infty} e^{-x^2} \cos \alpha x dx = \frac{1}{2} \sqrt{\pi} e^{-\alpha^2/4} \quad [\text{Delhi Physics. (H) 1994}]$$

$$(ii) \int_0^{\infty} \exp(-x^2) \cos 2\alpha x dx = \frac{1}{2} \sqrt{\pi} \exp(-\alpha^2) \quad [\text{Delhi Physics (H) 1998}]$$

Sol. (i) Let $F(\alpha) = \int_0^{\infty} e^{-x^2} \cos \alpha x dx$... (1)

Differentiating both sides of (1) w.r.t. ' α ' and using Leibnitz's rule under the integral sign, we have

$$\frac{dF}{d\alpha} = \int_0^{\infty} \frac{\partial(e^{-x^2} \cos \alpha x)}{\partial \alpha} dx = \int_0^{\infty} e^{-x^2} (-x \sin \alpha x) dx = \frac{1}{2} \int_0^{\infty} \sin \alpha x \cdot (-2x e^{-x^2}) dx \dots (2)$$

Putting $-x^2 = t$ and $-2x dx = dt$, we have

$$\int (-2x) e^{-x^2} dx = \int e^t dt = e^t = e^{-x^2}, \quad \text{as} \quad t = -x^2 \quad \dots (3)$$

Integrating by parts the integral on R.H.S. of (2) and using (3) while treating $(-2x e^{-x^2})$ as function to be integrated, (2) reduces to

$$\frac{dF}{d\alpha} = \frac{1}{2} \left\{ \left[\sin \alpha x \cdot e^{-x^2} \right]_0^{\infty} - \int_0^{\infty} \alpha \cos \alpha x e^{-x^2} dx \right\} = -\frac{\alpha}{2} \int_0^{\infty} e^{-x^2} \cos \alpha x dx \quad \dots (4)$$

Now, (1) and (4) $\Rightarrow \frac{dF}{d\alpha} = -\frac{\alpha}{2} F(\alpha)$ or $\frac{F'(\alpha)}{F(\alpha)} d\alpha = -\frac{\alpha}{2} d\alpha$

Integrating, $\log F(\alpha) = -(\alpha^2/4) + C$, C being an arbitrary constant ... (5)

Putting $\alpha = 0$ in (1) yields $F(0) = \int_0^{\infty} e^{-x^2} dx$... (6)

Putting $x^2 = u$, i.e., $x = u^{1/2}$ and $dx = (1/2) \times u^{-1/2} du$, (6) yields

$$F(0) = \frac{1}{2} \int_0^{\infty} e^{-u} u^{-1/2} du = \frac{1}{2} \int_0^{\infty} e^{-u} u^{(1/2)-1} du = (1/2) \times \Gamma(1/2)$$

[Since, by definition (see Art 20.2), $\int_0^{\infty} e^{-x} x^{n-1} dx = \Gamma(n)$]

or $F(0) = (1/2) \times \sqrt{\pi}$, as $\Gamma(1/2) = \sqrt{\pi}$, (see Art. 20.5)

Putting $\alpha = 0$ in (5) and using $F(0) = \sqrt{\pi}/2$, (5) yields $C = \log(\sqrt{\pi}/2)$. So, (5) gives

$$\log F(\alpha) = -(\alpha^2/4) + \log(\sqrt{\pi}/2) \quad \text{or} \quad F(\alpha) = (\sqrt{\pi}/2) \times e^{-\alpha^2/4}$$

or
$$\int_0^\infty e^{-x^2} \cos \alpha x \, dx = (\sqrt{\pi}/2) \times e^{-\alpha^2/4}, \text{ using (1)} \quad \dots(7)$$

(ii). Note that $\exp a$ stands for e^a . Hence, we are to show that

$$\int_0^\infty e^{-x^2} \cos 2\alpha x \, dx = (\sqrt{\pi}/2) \times e^{-\alpha^2} \quad \dots(8)$$

Replacing α by 2α in (7), we get the required result (8).

Ex. 8. Evaluate $\int_0^\infty (e^{-x}/x) \{a - (1/x) + (1/x) \times e^{-ax}\} dx$

Sol. Let
$$F(a) = \int_0^\infty (e^{-x}/x) \{a - (1/x) + (1/x) \times e^{-ax}\} dx \quad \dots (1)$$

Differentiating both sides of (1) w.r.t. 'a' and using Leibnitz's rule of differentiation under the integral sign, we have

$$\frac{dF}{da} = \int_0^\infty \frac{\partial}{\partial a} \left\{ \frac{e^{-x}}{x} \left(a - \frac{1}{x} + \frac{1}{x} e^{-ax} \right) \right\} dx = \int_0^\infty \frac{e^{-x}}{x} (1 - e^{-ax}) dx \quad \dots (2)$$

Again, differentiating both sides of (2) w.r.t 'a' as before yields

$$\frac{d^2 F}{da^2} = \int_0^\infty \frac{\partial}{\partial a} \left\{ \frac{e^{-x}}{x} (1 - e^{-ax}) \right\} dx = \int_0^\infty \frac{e^{-x}}{x} (xe^{-ax}) dx = \int_0^\infty e^{-(1+a)x} dx$$

or
$$d^2 F / da^2 = \left[e^{-(1+a)x} / \{- (1+a)\} \right]_0^\infty = 1/(1+a) \quad \dots (3)$$

Integrating (3), $dF/da = \log(1+a) + C_1$, C_1 being an arbitrary constant $\dots(4)$

Putting $a = 0$ in (2), we get $dF/da = 0$. Next, putting $a = 0$ and $dF/da = 0$ in (4), we get $C_1 = 0$. Hence, (4) reduces to

$$dF/da = \log(1+a) \quad \text{or} \quad dF = \log(1+a) da \quad \dots (5)$$

Integrating (5), $F(a) = \int \log(a+1) da = a \log(a+1) - \int \{a/(1+a)\} da$

or
$$F(a) = a \log(a+1) - \int \{1 - 1/(1+a)\} da = a \log(a+1) - a + \log(a+1) + C_2$$

or
$$F(a) = (a+1) \log(a+1) - a + C_2, C_2 \text{ being an arbitrary constant} \quad \dots (6)$$

Putting $a = 0$ in (1), we get $F(a) = 0$. Next, putting $a = 0$ and $F(a) = 0$ in (6), we get $C_2 = 0$. Hence, (6) reduces to

$$F(a) = (a+1) \log(a+1) - a \quad \dots (7)$$

From (1) and (6),
$$\int_0^\infty (e^{-x}/x) \{a - (1/x) + (1/x) e^{-ax}\} dx = (a+1) \log(a+1) - a$$

EXERCISE 21 (A)

Assuming the validity of differentiation under the integral sign, show that

1. $\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(1+\alpha), \alpha > -1$

[Pune 2010; Delhi Maths (H) 2006]

$$2. \int_0^{\pi/2} \log(1 - e^2 \sin^2 \theta) d\theta = \pi \log \{ (1 + \sqrt{1 - e^2}) / 2 \}, \text{ if } e^2 < 1$$

$$3. \int_0^\infty \frac{1 - \cos mx}{x} e^{-x} dx = \frac{1}{2} \log(1 + m^2)$$

$$4. \int_0^\infty e^{-(x^2 + b^2/x)} dx = (\sqrt{\pi}/2) \times e^{-2b}, b \geq 0$$

21.5A. Evaluation of integral $\int_a^b g(x, \alpha, \beta) dx$, where a and b are independent of parameters α and β . Working rule.

Step 1. Let $G(\alpha, \beta) = \int_a^b g(x, \alpha, \beta) dx \quad \dots(i)$

Step 2. Read carefully remark of Art 21.2 for getting

$$\frac{\partial G(\alpha, \beta)}{\partial \alpha} = \int_a^b \frac{\partial g(x, \alpha, \beta)}{\partial \alpha} dx \quad \text{or} \quad \frac{\partial G(\alpha, \beta)}{\partial \beta} = \int_a^b \frac{\partial g(x, \alpha, \beta)}{\partial \beta} dx \quad \dots(ii)$$

Steps 3 to 5: Read Art 21.4 A with corresponding modifications. The process will be clear by the following solved examples given in Art 21.5 B.

21.5B. Solved examples of type 2 based on Art 21.5 A

Ex.1. Show that $\int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx = \tan^{-1} \frac{\beta}{\alpha}, \alpha \geq 0$

and deduce that $\int_0^\infty \frac{\sin \beta x}{x} dx = \begin{cases} \pi/2, & \text{if } \beta > 0 \\ 0, & \text{if } \beta = 0 \\ -\pi/2, & \text{if } \beta < 0 \end{cases} \quad \text{[Kanpur 2001]}$

Sol. First part. Here, the integrand $(e^{-\alpha x} \sin \beta x)/x$ contains two parameters α and β . In order to get rid of the factor $(1/x)$ in this integrand, we must treat only β as parameter. So, let

$$F(\beta) = \int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. ' β ' and using Leibnitz's rule of differentiation under the integral sign, we get

$$\begin{aligned} \frac{dF}{d\beta} &= \int_0^\infty \frac{\partial}{\partial \beta} \left[\frac{e^{-\alpha x} \sin \beta x}{x} \right] dx = \int_0^\infty \frac{e^{-\alpha x}}{x} \times (x \cos \beta x) dx = \int_0^\infty e^{-\alpha x} \cos \beta x dx \\ &= \left[\frac{e^{-\alpha x}}{\alpha^2 + \beta^2} (-\alpha \cos \beta x + \beta \sin \beta x) \right]_0^\infty, \text{ as } \int_0^\infty e^{ax} \cos bx dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2} \end{aligned}$$

or $\frac{dF}{d\beta} = \frac{\alpha}{\alpha^2 + \beta^2} \quad \text{or} \quad dF = \frac{\alpha}{\alpha^2 + \beta^2} d\beta, \quad \alpha > 0$

Integrating, $F(\beta) = \tan^{-1}(\beta/\alpha) + C, C \text{ being an arbitrary constant.} \quad \dots(2)$

Putting $\beta = 0$ in (1) yields $F(0) = 0$. Next, putting $\beta = 0$ and $F(0) = 0$ in (2) yields $C = 0$. Hence, (2) reduces to

$$F(\beta) = \tan^{-1} \frac{\beta}{\alpha} \quad \text{or} \quad \int_0^{\infty} e^{-\alpha x} \frac{\sin \beta x}{x} dx = \tan^{-1} \frac{\beta}{\alpha}, \text{ using (1)}$$

Letting $\alpha \rightarrow 0$ on both sides of the above result, we get

$$\int_0^{\infty} \frac{\sin \beta x}{x} dx = \lim_{\alpha \rightarrow 0} \tan^{-1} \frac{\beta}{\alpha} = \begin{cases} \pi/2, & \text{if } \beta > 0 \\ 0, & \text{if } \beta = 0 \\ -\pi/2, & \text{if } \beta < 0 \end{cases}$$

Ex. 2. Assuming the validity of differentiation under the integral sign, show that

$$(i) \int_0^{\pi/2} \log \frac{a+b \sin \theta}{a-b \sin \theta} \frac{d\theta}{\sin \theta} = \pi \sin^{-1} \frac{b}{a}, \quad a > b. \quad [\text{Delhi Maths (H) 2002, 03, 07, 09}]$$

$$(ii) \int_0^{\pi/2} \log \frac{1+\lambda \sin \theta}{1-\lambda \sin \theta} \frac{d\theta}{\sin \theta} = \pi \sin^{-1} \lambda, \text{ where } \lambda < 1$$

Sol (i). Here, the integrand contains two parameters a and b . Because of the presence of $(1/\sin \theta)$ as a factor in the integrand, we treat only b as a parameter. So let

$$F(b) = \int_0^{\pi/2} \log \frac{a+b \sin \theta}{a-b \sin \theta} \frac{d\theta}{\sin \theta} \quad \dots(1)$$

Differentiating both sides of (1) w.r.t ' b ' and using Leibnitz's rule of differentiation under the integral sign, we have

$$\begin{aligned} \frac{dF}{db} &= \int_0^{\pi/2} \frac{\partial}{\partial b} \left[\{ \log(a+b \sin \theta) - \log(a-b \sin \theta) \} \times \frac{1}{\sin \theta} \right] d\theta \\ &= \int_0^{\pi/2} \left(\frac{\sin \theta}{a+b \sin \theta} + \frac{\sin \theta}{a-b \sin \theta} \right) \frac{d\theta}{\sin \theta} = 2a \int_0^{\pi/2} \frac{d\theta}{a^2 - b^2 \sin^2 \theta} \\ &= 2a \int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{a^2 \sec^2 \theta - b^2 \tan^2 \theta} = 2a \int_0^{\pi/2} \frac{\sec^2 \theta d\theta}{a^2 (1 + \tan^2 \theta) - b^2 \tan^2 \theta} \\ &= 2a \int_0^{\infty} \frac{dt}{a^2 + t^2 (a^2 - b^2)}, \text{ putting } \tan \theta = t \text{ and } \sec^2 \theta d\theta = dt \\ &= \frac{2a}{a^2 - b^2} \int_0^{\infty} \frac{dt}{t^2 + (a/\sqrt{a^2 - b^2})^2} = \left[\frac{2a}{a^2 - b^2} \times \frac{1}{(a/\sqrt{a^2 - b^2})} \tan^{-1} \frac{t}{a/\sqrt{a^2 - b^2}} \right]_0^{\infty} \end{aligned}$$

$$\text{Thus, } dF/db = (\pi/\sqrt{a^2 - b^2}) \quad \text{or} \quad dF = (\pi/\sqrt{a^2 - b^2}) db$$

$$\text{Integrating, } F(b) = \pi \sin^{-1} (b/a) + c, \quad c \text{ being an arbitrary constant} \quad \dots(2)$$

Putting $b = 0$ in (1) yields $F(0) = 0$. Next, putting $b = 0$ and $F(0) = 0$ in (2), yields $C = 0$. Hence, (2) reduces to

$$F(b) = \pi \sin^{-1} \frac{b}{a} \quad \text{or} \quad \int_0^{\pi/2} \log \frac{a+b \sin \theta}{a-b \sin \theta} \frac{d\theta}{\sin \theta} = \pi \sin^{-1} \frac{b}{a}, \text{ using (1)} \quad \dots(4)$$

(ii). Re-writing (4),
$$\int_0^{\pi/2} \log \frac{1+(b/a) \times \sin \theta}{1-(b/a) \times \sin \theta} \frac{d\theta}{\sin \theta} = \pi \sin^{-1} \frac{b}{a} \quad \dots(5)$$

(4) is true for $a > b$, i.e., $b < a$. Setting $b/a = \lambda$ in (5), we get

$$\int_0^{\pi/2} \log \frac{1+\lambda \sin \theta}{1-\lambda \sin \theta} \frac{d\theta}{\sin \theta} = \pi \sin^{-1} \lambda, \text{ where } \lambda < 1$$

Ex. 3. Assuming the validity of differentiation under the integral sign, show that

(i) $\int_0^{\pi/2} \log(\alpha \cos^2 \theta + \beta \sin^2 \theta) d\theta = \pi \log\{(\sqrt{\alpha} + \sqrt{\beta})/2\}$. **[Delhi-Maths (H) 2000,04]**

(ii) $\int_0^{\pi/2} \log(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \pi \log\{(a+b)/2\}$

Sol (i). Let
$$G(\alpha, \beta) = \int_0^{\pi/2} (\alpha \cos^2 \theta + \beta \sin^2 \theta) d\theta \quad \dots(1)$$

Differentiating both sides of (1) partially w.r.t. ' α ' and using Leibnitz's rule of differentiation under the integral sign, we get

$$\frac{\partial G}{\partial \alpha} = \int_0^{\pi/2} \frac{\cos^2 \theta}{\alpha \cos^2 \theta + \beta \sin^2 \theta} d\theta = \int_0^{\pi/2} \frac{d\theta}{\alpha + \beta \tan^2 \theta} \quad \dots(2)$$

Putting $\tan \theta = t$ so that $\sec^2 \theta d\theta = dt$ i.e., $d\theta = (dt)/(1 + \tan^2 \theta)$ i.e., $d\theta = (dt)/(1 + t^2)$, (1) reduces to

$$\begin{aligned} \frac{\partial G}{\partial \alpha} &= \int_0^{\infty} \frac{dt}{(1+t^2)(\alpha + \beta t^2)} = \frac{1}{\alpha - \beta} \int_0^{\infty} \left(\frac{1}{1+t^2} - \frac{\beta}{\alpha + \beta t^2} \right) dt, \text{ if } \alpha \neq \beta \\ &\quad \text{[on resolving into partial fractions]} \\ &= \frac{1}{\alpha - \beta} \int_0^{\infty} \left\{ \frac{1}{1+t^2} - \frac{1}{t^2 + (\alpha/\beta)} \right\} dt = \frac{1}{\alpha - \beta} \left[\tan^{-1} t - \frac{1}{\sqrt{\alpha/\beta}} \tan^{-1} \frac{t}{\sqrt{\alpha/\beta}} \right]_0^{\infty} \\ &= \frac{1}{\alpha - \beta} \left[\frac{\pi}{2} - \frac{\sqrt{\beta}}{\sqrt{\alpha}} \times \frac{\pi}{2} \right] = \frac{\pi}{(\sqrt{\alpha})^2 - (\sqrt{\beta})^2} \times \frac{\sqrt{\alpha} - \sqrt{\beta}}{2\sqrt{\alpha}} = \frac{\pi}{2\sqrt{\alpha}(\sqrt{\alpha} + \sqrt{\beta})} \quad \dots(3) \end{aligned}$$

For $\beta = \alpha$, (2) reduces to

$$\frac{\partial G}{\partial \alpha} = \int_0^{\pi/2} \frac{\cos^2 \theta}{\alpha} d\theta = \frac{1}{2\alpha} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta = \frac{1}{2\alpha} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

Thus,
$$\partial G / \partial \alpha = \pi / 4\alpha. \quad \dots(4)$$

From (3) and (4), it follows that without exception, we have

$$\frac{\partial G}{\partial \alpha} = \frac{\pi}{2\sqrt{\alpha}(\sqrt{\alpha} + \sqrt{\beta})} = \pi \frac{(1/2\sqrt{\alpha})}{\sqrt{\alpha} + \sqrt{\beta}}$$

Integrating w.r.t. ' α ',
$$G(\alpha, \beta) = \pi \int \frac{d(\sqrt{\alpha} + \sqrt{\beta})/d\alpha}{\sqrt{\alpha} + \sqrt{\beta}} d\alpha = \pi \log(\sqrt{\alpha} + \sqrt{\beta}) + C, \quad \dots(5)$$

where C is independent of α . Using the well known property $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ of definite integrals, (1) yields

$$\begin{aligned} G(\alpha, \beta) &= \int_0^{\pi/2} \log(\alpha \cos^2 \theta + \beta \sin^2 \theta) d\theta = \int_0^{\pi/2} \log\{\alpha \cos^2(\pi/2 - \theta) + \beta \sin^2(\pi/2 - \theta)\} d\theta \\ &= \int_0^{\pi/2} \log(\beta \cos^2 \theta + \alpha \sin^2 \theta) d\theta = G(\beta, \alpha), \text{ using (1)} \end{aligned}$$

In view of the relation $G(\alpha, \beta) = G(\beta, \alpha)$, it follows that C occurring in (5) is also independent of β . Hence, C is an absolute constant. Now, putting $\alpha = \beta = 1$ in (1), we get

$$G(1, 1) = \int_0^{\pi/2} \log(\cos^2 \theta + \sin^2 \theta) d\theta = \int_0^{\pi/2} (\log 1) d\theta = 0, \text{ as } \log 1 = 0$$

Putting $\alpha = \beta = 1$ in (5) and using $G(1, 1) = 0$, we get $0 = \pi \log 2 + C$ so that $C = -\pi \log 2$. Hence, from (1) and (5), we get

$$\int_0^{\pi/2} \log(\alpha \cos^2 \theta + \beta \sin^2 \theta) d\theta = \pi \{\log(\sqrt{\alpha} + \sqrt{\beta}) - \log 2\} = \pi \log\{(\sqrt{\alpha} + \sqrt{\beta})/2\} \dots (6)$$

(ii). Replacing α and β by a^2 and b^2 respectively in equation (6) of part (i), we get

$$\int_0^{\pi/2} \log(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \pi \log\{(a+b)/2\}$$

Ex. 4. Show that $\int_0^\infty \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2} dx = \frac{\pi}{2} \log \left\{ \frac{(\alpha + \beta)^{\alpha + \beta}}{\alpha^\alpha \beta^\beta} \right\}, \alpha > 0, \beta > 0$

$$\text{Sol. Let } G(\alpha, \beta) = \int_0^\infty \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2} dx \dots (1)$$

Differentiating both sides of (1) w.r.t. ' α ' and using the Leibnitz's rule of differentiation under the integral sign, we have

$$\frac{\partial G}{\partial \alpha} = \int_0^\infty \frac{\partial}{\partial \alpha} \left\{ \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2} \right\} dx = \int_0^\infty \frac{\tan^{-1} \beta x}{x(1 + \alpha^2 x^2)} dx \dots (2)$$

Again, differentiating both sides of (2) w.r.t. ' β ' and using the Leibnitz's rule of differentiation under the integral sign, we have

$$\begin{aligned} \frac{\partial^2 G}{\partial \beta \partial \alpha} &= \int_0^\infty \frac{\partial}{\partial \beta} \left\{ \frac{\tan^{-1} \beta x}{x(1 + \alpha^2 x^2)} \right\} dx = \int_0^\infty \frac{dx}{(1 + \alpha^2 x^2)(1 + \beta^2 x^2)} \\ &= \int_0^\infty \frac{1}{\alpha^2 - \beta^2} \left(\frac{\alpha^2}{1 + \alpha^2 x^2} - \frac{\beta^2}{1 + \beta^2 x^2} \right) dx, \text{ on resolving into partial fractions} \\ &= \frac{1}{\alpha^2 - \beta^2} \int_0^\infty \left\{ \frac{1}{(1/\alpha)^2 + x^2} - \frac{1}{(1/\beta)^2 + x^2} \right\} dx = \frac{1}{\alpha^2 - \beta^2} [\alpha \tan^{-1} \alpha x - \beta \tan^{-1} \beta x]_0^\infty, \alpha \neq \beta \end{aligned}$$

$$\text{or } \frac{\partial^2 G}{\partial \alpha \partial \beta} = \frac{1}{\alpha^2 - \beta^2} \left(\alpha \times \frac{\pi}{2} - \beta \times \frac{\pi}{2} \right) = \frac{1}{(\alpha - \beta)(\alpha + \beta)} \cdot \frac{\pi(\alpha - \beta)}{2} = \frac{\pi}{2(\alpha + \beta)}, \alpha > 0, \beta > 0 \dots (3)$$

It is easy to show that (3) remains valid even for $\alpha = \beta$.

Now, integrating (3) w.r.t. ' β ', we have

$$\partial G / \partial \alpha = (\pi/2) \times \log(\alpha + \beta) + f(\alpha), f(\alpha), \text{ being an arbitrary function of } \alpha \dots (4)$$

Putting $\beta = 0$ in (2) yields $\partial G / \partial \alpha = 0$. Next, putting $\beta = 0$ and $\partial G / \partial \alpha = 0$ in (4) yields $0 = (\pi/2) \times \log \alpha + f(\alpha)$ so that $f(\alpha) = -(\pi/2) \times \log \alpha$. Hence, (4) reduces to

$$\partial G / \partial \alpha = (\pi/2) \times \log(\alpha + \beta) - (\pi/2) \times \log \alpha \dots (5)$$

Integrating (5) w.r.t. ' α ', we get

$$G(\alpha, \beta) = \frac{\pi}{2} \int \log(\alpha + \beta) \cdot 1 d\alpha - \frac{\pi}{2} \int \log \alpha \cdot 1 d\alpha = \frac{\pi}{2} \left\{ \log(\alpha + \beta) \cdot \alpha - \int \frac{\alpha d\alpha}{\alpha + \beta} \right\} - \frac{\pi}{2} \left\{ \log \alpha \cdot \alpha - \int \frac{1}{\alpha} \cdot \alpha d\alpha \right\}$$

$$\text{or } G(\alpha, \beta) = \frac{\pi}{2} \left\{ \alpha \log(\alpha + \beta) - \int \left(1 - \frac{\beta}{\alpha + \beta} \right) d\alpha \right\} - \frac{\pi}{2} (\alpha \log \alpha - \alpha)$$

$$\therefore G(\alpha, \beta) = (\pi/2) \{ \alpha \log(\alpha + \beta) - \alpha + \beta \log(\alpha + \beta) \} - (\pi/2) \times (\alpha \log \alpha - \alpha) + g(\beta),$$

where $g(\beta)$ is an arbitrary function of β .

$$\text{Thus, } G(\alpha, \beta) = (\pi/2) \times \{ (\alpha + \beta) \log(\alpha + \beta) - \alpha \log \alpha \} + g(\beta) \dots (6)$$

Putting $\alpha = 0$ in (1) yields $G(0, \beta) = 0$. Next, putting $\alpha = 0$ and $G(0, \beta) = 0$ in (6) gives

$$0 = (\pi/2) \times \{ \beta \log \beta - \lim_{\alpha \rightarrow 0} \alpha \log \alpha \} + g(\beta) \dots (7)$$

$$\text{But, } \lim_{\alpha \rightarrow 0} \alpha \log \alpha = \lim_{\alpha \rightarrow 0} \frac{\log \alpha}{(1/\alpha)} = \lim_{\alpha \rightarrow 0} \frac{(1/\alpha)}{(-1/\alpha)} = 0, \text{ using L' Hospital's rule} \dots (8)$$

Using (8), (7) reduces to

$$0 = (\pi/2) \times \beta \log \beta + g(\beta) \quad \text{so that} \quad g(\beta) = -(\pi/2) \times \beta \log \beta \dots (9)$$

$$\text{Using (9), (6) reduces to } G(\alpha, \beta) = (\pi/2) \times \{ (\alpha + \beta) \log(\alpha + \beta) - \alpha \log \alpha - \beta \log \beta \}$$

$$\text{or } G(\alpha, \beta) = (\pi/2) \times \{ \log(\alpha \times \beta)^{\alpha + \beta} - \log \alpha^\alpha - \log \beta^\beta \}$$

$$\text{or } \int_0^\infty \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2} dx = \frac{\pi}{2} \log \left\{ \frac{(\alpha + \beta)^{\alpha + \beta}}{\alpha^\alpha \beta^\beta} \right\}, \quad \text{using (1)}$$

EXERCISE 21(B)

Assuming the validity of differentiation under the integral sign, show that

$$1. \int_0^\infty \frac{\log(1 + a^2 x^2)}{1 + b^2 x^2} dx = \frac{\pi}{b} \log \left(1 + \frac{a}{b} \right)$$

$$2. \int_0^\infty e^{-a^2 x^2 - b^2/x^2} dx = (\sqrt{\pi}/2a) \times e^{-2ab}, a > 0, b \geq 0$$

Deduce that $\int_0^{\infty} e^{-(x^2+a^2/x^2)} dx = (\sqrt{\pi}/2) \times e^{-2a}, a > 0$

$$3. \int_0^1 \frac{x^a - x^b}{\log x} dx = \log \frac{a+1}{b+1}$$

$$4. \int_0^{\infty} \frac{\cos \lambda x}{x} (e^{-ax} - e^{-bx}) dx = \frac{1}{2} \log \frac{b^2 + \lambda^2}{a^2 + \lambda^2}, a > 0, b > 0$$

$$5. \int_0^{\pi/2} \frac{d\theta}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^2} = \frac{\pi(a^2 + b^2)}{4a^3 b^3}$$

21.6A. Evaluation of integral $\int_{g(\alpha)}^{h(\alpha)} f(x, \alpha) dx$, where $g(\alpha)$ and $h(\alpha)$ are functions of parameter α . **Working rule.**

Step 1. Let $F(\alpha) = \int_{g(\alpha)}^{h(\alpha)} f(x, \alpha) dx \quad \dots(1)$

Step 2. Differentiating both sides of (1) w.r.t. ' α ' and using general Leibnitz's rule of differentiation under integral sign (see Art 21.3), we get

$$\frac{dF}{d\alpha} = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial F}{\partial \alpha} d\alpha + \frac{dh}{d\alpha} f(h(\alpha), \alpha) - \frac{dg}{d\alpha} f(g(\alpha), \alpha) \quad \dots(2)$$

While writing (2), write $dh/d\alpha = 0$ if h is independent of α (or write $dg/d\alpha = 0$ if g is independent of α)

Steps 3 to 5. Read Art 21.4 A for complete discussion.

21.6B. Solved examples of type 3 base on Art.21.6A

Ex.1. Assuming the validity of differentiation under the integral sign, show that

$$\int_{\pi/2-\alpha}^{\pi/2} \sin \theta \cos^{-1}(\cos \alpha \operatorname{cosec} \theta) d\theta = \frac{\pi}{2}(1 - \cos \theta) \quad [\text{Delhi Maths (H) 2001, 03, 04, 09}]$$

Sol. Let $F(\alpha) = \int_{\pi/2-\alpha}^{\pi/2} \sin \theta \cos^{-1}(\cos \alpha \operatorname{cosec} \theta) d\theta \quad \dots(1)$

Here the lower limit of integral is function of the parameter α while the upper limit is independent of α . Hence, differentiating both sides of (1) w.r.t. ' α ' and using the general form of Leibnitz's rule of differentiation under the integral sign, we have

$$\begin{aligned} \frac{dF}{d\alpha} &= \int_{\pi/2-\alpha}^{\pi/2} \frac{\partial}{\partial \alpha} \{ \sin \theta \cos^{-1}(\cos \alpha \operatorname{cosec} \theta) \} d\theta + \frac{d(\pi/2)}{d\alpha} \sin \frac{\pi}{2} \cos^{-1} \left(\cos \alpha \operatorname{cosec} \frac{\pi}{2} \right) \\ &\quad - \frac{d(\pi/2 - \alpha)}{d\alpha} \sin(\pi/2 - \alpha) \cos^{-1} \{ \cos \alpha \operatorname{cosec} (\pi/2 - \alpha) \} \\ &= \int_{\pi/2-\alpha}^{\pi/2} \frac{\sin \theta \operatorname{cosec} \theta \sin \alpha}{\sqrt{1 - \cos^2 \alpha \operatorname{cosec}^2 \theta}} d\theta - (-1) \times \cos \alpha \times \cos^{-1}(1) \\ &= \sin \alpha \int_{\pi/2-\alpha}^{\pi/2} \frac{\sin \theta d\theta}{\sqrt{\sin^2 \theta - \cos^2 \alpha}} = \sin \alpha \int_{\pi/2-\alpha}^{\pi/2} \frac{\sin \theta d\theta}{\sqrt{(1 - \cos^2 \theta) - (1 - \sin^2 \alpha)}} \text{, as } \cos^{-1}(1) = 0 \\ &= -\sin \alpha \int_{\sin \alpha}^0 \frac{dt}{\sqrt{\sin^2 \alpha - t^2}}, \text{ putting } \cos \theta = t \text{ and } -\sin \theta d\theta = dt \end{aligned}$$

$$= -\sin \alpha \left[\sin^{-1} \frac{t}{\sin \alpha} \right]_{-\sin \alpha}^0 = -(\sin \alpha) \times \left(0 - \frac{\pi}{2} \right) = \frac{\pi}{2} \sin \alpha$$

Thus, $dF = (\pi/2) \times \sin \alpha d\alpha$

Integrating, $F(\alpha) = -(\pi/2) \times \cos \alpha + C$, C being an arbitrary constant ... (2)

Putting $\alpha = 0$ in (1) yields $F(0) = 0$. Next, putting $\alpha = 0$ and $F(0) = 0$ in (2) yields $C = -\pi/2$. Hence, (2) reduces to

$$F(\alpha) = -(\pi/2) \times \cos \alpha + \pi/2 \quad \text{or} \quad \int_{\pi/2-\alpha}^{\alpha} \sin \theta \cos^{-1}(\cos \alpha \operatorname{cosec} \theta) d\theta = \frac{\pi}{2}(1 - \cos \alpha), \text{ by (1)}$$

Ex. 2. What are the points of the extrema of the function $y = \int_0^x \frac{\sin t}{t} dt$, $x > 0$?

(a) $0, \pm n\pi$ (b) $\pm n\pi$ only (c) $n\pi$ only (d) $0, n\pi$ only, where $n = 1, 2, 3, \dots$ [I.A.S. (Prel) 2009]

Sol. Ans. (c). Given $y(x) = \int_0^x \frac{\sin t}{t} dt$, $x > 0$... (1)

Differentiating both sides of (1) w.r.t. 'x' and using the general form of Leibnitz's rule of differentiation under the integral sign, we obtain

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x} \left(\frac{\sin t}{t} \right) dt + \frac{dx}{dx} \times \frac{\sin x}{x} - \frac{d0}{dx} \times \lim_{t \rightarrow 0} \frac{\sin t}{t}$$

or $\frac{dy}{dx} = 0 + \frac{\sin x}{x} - (0 \times 1) \quad \text{or} \quad \frac{dy}{dx} = \frac{\sin x}{x}, x > 0$... (2)

For extremen values of y , we have $dy/dx = 0$, i.e., $\sin x/x = 0$ or $\sin x = 0$, since $x > 0$.

Thus, $x = n\pi$, where $n = 1, 2, 3, \dots$. Hence the choice (c) is correct.

Ex. 3 The function $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$ has (a) two maxima and two minima points

(b) two maxima and three minima points (c) Three maxima and two minima points

(d) One maximum point and one minimum point [I.A.S. (Prel.) 2009]

Sol. Ans. (b). Given $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$... (1)

Differentiating both sides of (1) w.r.t. 'x' and using the general form of Leibnitz's rule of differentiation under the integral sign, we obtain

$$f'(x) = \int_0^{x^2} \frac{\partial}{\partial x} \left(\frac{t^2 - 5t + 4}{2 + e^t} \right) dt + \frac{dx^2}{dx} \times \frac{x^4 - 5x^2 + 4}{2 + e^{x^2}} - \frac{d0}{dx} \times \frac{4}{3}$$

or $f'(x) = \frac{2x(x^4 - 5x^2 + 4)}{2 + e^{x^2}} = \frac{2(x^5 - 5x^3 + 4x)}{2 + e^{x^2}}$... (2)

For maximum and minimum values of $f(x)$, we have

$$f'(x) = 0 \Rightarrow x(x^4 - 5x^2 + 4) = 0 \quad \text{or} \quad x(x^2 - 1)(x^2 - 4) = 0, \text{ giving } x = 0, 1, -1, 2, -2.$$

Differentiating both sides of (2) w.r.t. 'x', we get

$$f''(x) = 2 \times \frac{(5x^4 - 15x^2 + 4)(2 + e^{x^2}) - (x^5 - 5x^3 + 4x) \times e^{x^2} \times 2x}{(2 + e^{x^2})^2}$$

$$\text{or } f''(x) = \frac{2(5x^4 - 15x^2 + 4)(2 + e^{x^2}) - 4x^2 e^{x^2} (x-1)(x+1)(x-2)(x+2)}{(2 + e^{x^2})^2} \quad \dots(3)$$

Now, (3) $\Rightarrow f''(0) > 0$, $f''(1) < 0$, $f''(-1) < 0$, $f''(2) > 0$ and $f''(-2) > 0$, showing that the given function $f(x)$ has maxima at two points (namely, at $x = 1$ and $x = -1$) and it has minima at three points (namely, at $x = 0$, $x = 2$ and $x = -2$).

Ex. 4. If $F(x) = (1/x^2) \times \int_4^x \{4t^2 - 3F'(t)\} dt$, then what is $F'(4)$?

(a) 32/19 (b) 64/3 (c) 64/19 (d) 16/3. [I.A.S. (Prel.) 2009]

Sol. Ans. (c). Given $F(x) = (1/x^2) \times \int_4^x \{4t^2 - 3F'(t)\} dt \quad \dots(1)$

Differentiating both sides of (1) w.r.t. 'x', we obtain

$$F'(x) = -\frac{2}{x^3} \int_4^x \{4t^2 - 3F'(t)\} dt + \frac{1}{x^2} \frac{d}{dx} \int_4^x \{4t^2 - 3F'(t)\} dt \quad \dots(2)$$

Using the general form of Leibnitz's rule of differentiating under the integral sign in the second term on R.H.S of (2), (2) reduces to

$$F'(x) = -\frac{2}{x^3} \int_4^x \{4t^2 - 3F'(t)\} dt + \frac{1}{x^2} \left[\int_4^x \frac{\partial}{\partial x} \{4t^2 - 3F'(t)\} dt + (dx/dx) \times \{4x^2 - 3F'(x)\} - (d4/dx) \times \{64 - 3F'(4)\} \right]$$

$$\text{Hence, } F'(4) = 0 + 0 + \{64 - 3F'(4)\} / 16 - 0 \quad \Rightarrow \quad F'(4) = 64/19$$

Exercise 21(C)

1. Show that $\int_0^a \frac{\log(1+ax)}{1+x^2} dx = \frac{1}{2} \log(1+a^2) \tan^{-1} a$. Deduce that $\int_0^a \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 8$

2. Show that $y = \frac{1}{k} \int_0^x \sin k(x-t) dt$ satisfies the equation $d^2y/dx^2 + k^2y = f(x)$, where k is a constant.

3. Show that $\int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x} dx = \text{constant}$.

21.7A Determination of the value of an integral when certain standard known integral is given with its value. Working rule.

$$\text{Let } \int_a^b f(x, \alpha) dx = F(\alpha) \quad \dots(1)$$

be given, a and b being independent of the parameter α . Then, differentiating both sides of (1) and using Leibnitz's rule of differentiating under the integral sign, we get a new integral on L.H.S of (1) and its value on the R.H.S. of (1)

Remark. Sometimes one or more than one successive differentiation w.r.t. ' α ' may be required to get the required results.

21.7B. Solved examples of type 4 based on Art. 21.7 A

Ex.1. Given $\int_0^x \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$. Using the rule of differentiation under the integral

sign, show that
$$\int_0^x \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2a^3} \tan^{-1} \frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)}$$

Sol. Given
$$\int_0^x \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. 'x' and using Leibnitz's rule of differentiation under the integral sign, we obtain

$$\int_0^x \left\{ -\frac{1}{(x^2 + a^2)^2} \times 2a \right\} dx = \frac{1}{a} \times \frac{1}{1 + (x/a)^2} \times \left(-\frac{x}{a^2} \right) - \frac{1}{a^2} \tan^{-1} \frac{x}{a}$$

Hence,
$$\int_0^x \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2a^3} \tan^{-1} \frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)}$$

Ex. 2. From the value of $\int_0^1 x^m dx$, deduce the value of $\int_0^1 x^m (\log x)^n dx$, $m \geq 0$ and n is a positive integer

Sol. We have,
$$\int_0^1 x^m dx = \left[x^{m+1} / (m+1) \right]_0^1 = 1/(m+1) = (m+1)^{-1} \quad \dots(1)$$

Differentiating both sides of (1) w.r.t. 'm' and using the Leibnitz's rule of differentiation under the integral sign, we get

$$\int_0^1 x^m \log x dx = (-1) (m+1)^{-2} \quad \dots(2)$$

Again, differentiations both sides of (2) w.r.t. 'm', as before, we get

$$\int_0^1 x^m (\log x)^2 dx = (-1) (-2) (m+1)^{-3} \quad \dots(3)$$

Continuing likewise till (1) is differentiated n times w.r.t. 'm', we finally obtain

$$\int_0^1 x^m (\log x)^n dx = (-1) (-2) (-3) \dots (-n) (m+1)^{-(n+1)} = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

Ex.3. From the value of $\int_0^\infty e^{-ax^2} dx$ deduce the value of $\int_0^\infty e^{-ax^2} x^{2n} dx$.

Sol. Let $ax^2 = t$, i.e., $x = (t/a)^{1/2}$ so that $dx = (1/2\sqrt{a}) \times t^{-1/2} dt$. Then, we have

$$\int_0^\infty e^{-ax^2} dx = \int_0^\infty e^{-t} \times \frac{1}{2\sqrt{a}} \times t^{-1/2} dt = \frac{1}{2\sqrt{a}} \int_0^\infty e^{-t} t^{(1/2)-1} dt$$

or
$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2\sqrt{a}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \times a^{-1/2} \quad \dots (1)$$

[\therefore By definition of Gamma function, (refer Art 20.2) $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma(n)$]

Differentiating both sides of (1) w.r.t. 'a' and using Leibnitz's rule of differentiation under the integral sign, we obtain

$$\int_0^\infty e^{-ax^2} (-x^2) dx = \frac{\sqrt{\pi}}{2} \times \left(-\frac{1}{2}\right) \times a^{-3/2} \quad \text{or} \quad \int_0^\infty e^{-ax^2} x^2 dx = \frac{\sqrt{\pi}}{2} \times \frac{1}{2} \times a^{-3/2} \quad \dots(2)$$

Differentiating both sides of (2) w.r.t. 'a' and proceeding as before, we get

$$\int_0^{\infty} e^{-ax^2} \times (-1) \times (x^2)^2 dx = \frac{\sqrt{\pi}}{2} \times \frac{1}{2} \times \left(-\frac{3}{2}\right) a^{-5/2} \quad \text{or} \quad \int_0^{\infty} e^{-ax^2} (x^2)^2 dx = \frac{\sqrt{\pi}}{2} \times \frac{1}{2} \times \frac{3}{2} \times a^{-5/2}$$

Continuing likewise till (1) is differentiated n times w.r.t. ' a ', we get

$$\begin{aligned} \int_0^{\infty} e^{-ax^2} (x^2)^n dx &= \frac{\sqrt{\pi}}{2} \times \frac{1}{2} \times \frac{3}{2} \times \dots \times \frac{2n-1}{2} \times a^{-(1/2)-n} \\ &= \frac{\sqrt{\pi}}{2} \times \frac{1}{a^{n+1/2}} \times \frac{2n-1}{2} \times \frac{2n-3}{2} \times \dots \times \frac{3}{2} \times \frac{1}{2} \\ &= \frac{1}{2a^{n+1/2}} \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \dots \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right), \text{ as } \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) \\ &= [\Gamma(n+1/2)] / 2a^{n+1/2}, \quad \text{as} \quad (n-1)\Gamma(n-1) = \Gamma(n) \end{aligned}$$

EXERCISE 21 (D)

1. Starting with $\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$, $a > 0$, $|b| < a$, deduce that

$$(i) \int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^2} \qquad (ii) \int_0^{\pi} \frac{\cos x dx}{(a+b\cos x)^2} = -\frac{\pi b}{(a^2-b^2)^{3/2}}$$

2. Starting from $\int_0^{\infty} e^{-ax} dx = \frac{1}{a}$ for $a > 0$, deduce that $\int_0^{\infty} x^m e^{-ax} dx = (m!) / a^{m+1}$

3. Using the value of the integral $\int_0^{\infty} \frac{dx}{x^2+ax}$, show that

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^{n+1}} = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \cdot \frac{1}{a^{n+1/2}}$$

MISCELLANEOUS PROBLEMS ON CHAPTER 21

Ex.1, If $f(x) = \int_0^x \sqrt{1+t^6} dt$, $x > 0$, Then find $f'(2)$. (Pune 2010)

Sol. Differentiating both sides of the given equation and using Leibnitz's rule of differentiation under integral sign, we have

$$f'(x) = \int_0^x \frac{\partial}{\partial x} \sqrt{1+t^6} dt + \sqrt{1+x^6} \times \frac{dx}{dx} - \sqrt{1+0} \times \frac{d0}{dx}$$

$$\text{Thus, } f'(x) = \sqrt{1+x^6} \text{ and so } f'(2) = \sqrt{1+2^6} = \sqrt{65}$$

Ex.2. The value of $\lim_{x \rightarrow 0} \frac{x e^{x^2}}{\int_0^x e^{t^2} dt}$ is

(a) 0 (b) 1 (c) does not exist (d) -1 (I.A.S. 2004)

Sol. Ans. (b). We have,

$$\lim_{x \rightarrow 0} \frac{x e^{x^2}}{\int_0^x e^{t^2} dt} \quad \text{Form } \left[\frac{\infty}{\infty} \right]$$

$$= \lim_{x \rightarrow 0} \frac{d(x e^{x^2}) / dx}{\frac{d}{dx} \int_0^x e^{t^2} dt}, \text{ by L' Hospital's rule}$$

$$= \lim_{x \rightarrow 0} \frac{(1 \times e^{x^2}) + (x \times e^{x^2} \times 2x)}{\int_0^x \frac{\partial}{\partial x} (e^{t^2}) dt + e^{x^2} \times \frac{dx}{dx} - e^0 \times \frac{d0}{dx}}$$

[Using the Leibnitz' rule of differentiation under the integral sign]

$$= \lim_{x \rightarrow 0} \frac{(1+x^2)e^{x^2}}{0 + e^{x^2} - 0} = \lim_{x \rightarrow 0} (1+x^2) = 1$$