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Integral representations for the Gamma function, the Beta function, and the Double Gamma function

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A variety of integral representations for some special functions have been developed. Here we aim at presenting certain (new or known) integral representations for $\log \Gamma(a)$, $B(\alpha, \beta)$, and $\log \Gamma_2(a)$ by using some of the known integral representations of the Hurwitz (or generalized) Zeta function $\zeta(s, a)$. As a by-product of our main formulas, several integral representations for the Glaisher–Kinkelin constant A and the Psi (or Digamma) function $\psi(a)$ are also given. Relevant connections of some of the results presented here with those obtained in earlier works are indicated. We also indicate the potential for the usefulness of these results.

Keywords: Euler–Mascheroni constant; Glaisher–Kinkelin constant; Gamma function; Double Gamma function; Weierstrass factorization theorem; Beta function; Psi (or Digamma) function; Riemann Zeta function; Hurwitz (or generalized) Zeta function; Hurwitz–Lerch Zeta function; Weierstrass canonical product; determinants of the Laplacians

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1. Introduction, definitions, and preliminaries

We begin by recalling the definitions and useful properties of each of the following special functions: the Euler Gamma function $\Gamma(z)$ extends the familiar factorials

$$n! \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\})$$

to a function defined on the set of real or complex numbers and is usually defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad \big(\Re(z) > 0\big). \tag{1.1}$$

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Among various equivalent forms of the Gamma function $\Gamma(z)$, we choose to recall here the one given by Weierstrass:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \right\} \quad (z \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}), \tag{1.2}$$

where γ denotes the Euler–Mascheroni constant (see [11,16]) defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.5772156649015325 \cdots , \tag{1.3}$$

and the other by Gauss:

$$\Gamma(z) = \lim_{n\to\infty} \left\{ \frac{(n-1)! n^z}{z(z+1)\cdots(z+n-1)} \right\}. \tag{1.4}$$

For explicit evaluations of the derivatives of the Gamma function, see the recent work by Choi and Srivastava [7].

The Psi (or Digamma) function $\psi(z)$ is defined by

$$\psi(z) := \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(t)\, dt. \tag{1.5}$$

The Polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) = \frac{d^n}{dz^n} \psi(z) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \notin \mathbb{Z}_0^-). \tag{1.6}$$

The Beta function $B(\alpha, \beta)$ is a function of two complex variables α and β defined by

$$B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt = B(\beta, \alpha) \quad \big(\Re(\alpha) > 0; \Re(\beta) > 0\big), \tag{1.7}$$

which is closely related to the Gamma function as follows:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\alpha, \beta \notin \mathbb{Z}_0^-). \tag{1.8}$$

Formula (1.8) continues the Beta function $B(\alpha, \beta)$ analytically for all complex values of its arguments α and β, except when

$$\alpha, \beta \in \mathbb{Z}_0^-.$$

The Hurwitz (or generalized) Zeta function $\zeta(s, a)$ is defined by

$$\zeta(s, a) := \sum_{k=0}^{\infty} (k+a)^{-s} \quad \big(\Re(s) > 1; a \notin \mathbb{Z}_0^-\big). \tag{1.9}$$

The special case $a = 1$ of the Hurwitz Zeta function $\zeta(s, a)$ [20, Section 2.3] is the Riemann Zeta function $\zeta(s)$ [19] defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\[2ex] \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0; s \neq 1\big). \end{cases} \tag{1.10}$$

It is noted that both the Riemann Zeta function $\zeta(s)$ and the Hurwitz Zeta function $\zeta(s,a)$ can be continued meromorphically to the whole complex s-plane, except for a simple pole only at $s=1$ with their respective residue 1, in various ways. Several recent investigations dealing extensively with the Riemann Zeta function $\zeta(s)$, the Hurwitz Zeta function $\zeta(s,a)$, the Hurwitz–Lerch Zeta function $\Phi(z,s.a)$, and their various generalizations include those by Garg *et al.* [10], Lin *et al.* [15], and Choi *et al.* [9] (see also [8] and the recent monograph by Srivastava and Choi [20]).

Barnes [1] defined the double Gamma function $\Gamma_2 = 1/G$ satisfying each of the following properties:

(a) $G(z+1)=\Gamma(z)G(z) \quad (z \in \mathbb{C})$;
(b) $G(1)=1$;
(c) Asymptotically,

$$\log G(z+n+2) = \left(\frac{n+1+z}{2}\right)\log(2\pi) + \left(\frac{n^2}{2}+n+\frac{5}{12}+\frac{z^2}{2}+(n+1)z\right)\log n \\ -\frac{3n^2}{4} - n - nz - \log A + \frac{1}{12} + O(n^{-1}) \quad (n\to\infty),$$

where Γ is the Gamma function and A is called the Glaisher–Kinkelin constant defined by

$$\log A = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k\log k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\log n + \frac{n^2}{4}\right\}, \tag{1.11}$$

the numerical value of A being given by

$$A \cong 1.282427130\cdots.$$

From this definition, Barnes [1] deduced several explicit Weierstrass canonical product forms of the double Gamma function Γ_2:

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1) = (2\pi)^{(1/2)z}\exp\left(-\frac{1}{2}z - \frac{1}{2}(\gamma+1)z^2\right) \\ \cdot\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k \exp\left(-z+\frac{z^2}{2k}\right)\right\}; \tag{1.12}$$

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1) = (2\pi)^{(1/2)z}\exp\left(-\frac{1}{2}z(z+1) - \frac{1}{2}\gamma z^2\right) \\ \cdot\prod_{k=1}^{\infty}\frac{\Gamma(k)}{\Gamma(z+k)}\exp\left(z\psi(k)+\frac{1}{2}z^2\psi'(k)\right); \tag{1.13}$$

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1) = (2\pi)^{(1/2)z}\exp\left[\left(\gamma-\frac{1}{2}\right)z - \left(\frac{\pi^2}{6}+1+\gamma\right)\frac{z^2}{2}\right]\Gamma(z+1) \\ \cdot\prod_{m=0}^{\infty}{\prod_{n=0}^{\infty}}'\left(1+\frac{z}{m+n}\right)\exp\left(-\frac{z}{m+n}+\frac{z^2}{2(m+n)^2}\right), \tag{1.14}$$

where the prime denotes the exclusion of the case $n=m=0$, the Psi function ψ is given by Equation (1.5), and γ denotes the Euler–Mascheroni constant given by Equation (1.3). Each form of these products is convergent for all finite values of $|z|$, by the Weierstrass factorization theorem.

The double Gamma function satisfies the following relations:

$$G(1) = 1 \quad \text{and} \quad G(z+1) = \Gamma(z)G(z) \quad (z \in \mathbb{C}). \tag{1.15}$$

For sufficiently large real x and $a \in \mathbb{C}$, we have the Stirling formula for the G-function:

$$\begin{aligned}\log G(x+a+1) = \left(\frac{x+a}{2}\right)\log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax \\ + \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right)\log x + O(x^{-1}) \quad (x \to \infty).\end{aligned} \tag{1.16}$$

The following special values of G [1] may be recalled here:

$$\left\{\Gamma_2\left(\frac{1}{2}\right)\right\}^{-1} = G\left(\frac{1}{2}\right) = 2^{1/24} \cdot \pi^{-1/4} \cdot \mathrm{e}^{1/8} \cdot A^{-3/2}; \tag{1.17}$$

$$G(n+2) = 1!\,2! \cdots n! \quad \text{and} \quad G(n+1) = \frac{(n!)^n}{1 \cdot 2 \cdot 3^2 \cdot 4^3 \cdots n^{n-1}} \quad (n \in \mathbb{N}). \tag{1.18}$$

It is remarked in passing that, in about 1900, the double Gamma function Γ_2 and the multiple Gamma functions Γ_n were defined and studied systematically by Barnes [1–4]. Nonetheless, except possibly for the citations of Γ_2 in the exercises by Whittaker and Watson [24, p. 264] and also by Gradshteyn and Ryzhik [13, p. 661, Entry 6.441 (4); p. 937, Entry 8.333], these functions were revived only in about the middle of the 1980s in the study of the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ (see, e.g., [5,14,17,18,21,23]).

Sometimes a problem that began as one part of mathematics could ultimately be resolved by techniques from another seemingly very different part of mathematics. It can also be said that, if there exists a relationship between two meaningful mathematical functions, some properties of one function can be deduced from the corresponding properties of the other function. Fortunately, there are certain useful relationships among some of the above functions as follows (see, e.g., [20, Chapter 2]):

$$\log\Gamma(a) = \zeta'(0,a) + \frac{1}{2}\log(2\pi), \tag{1.19}$$

where (and *throughout this paper*)

$$\zeta'(s,a) = \frac{\partial}{\partial s}\{\zeta(s,a)\},$$

unless other statement is provided;

$$B(\alpha,\beta) = (2\pi)^{1/2}\exp\zeta'(0,\alpha) + \zeta'(0,\beta) - \zeta'(0,\alpha+\beta)); \tag{1.20}$$

$$\begin{aligned}\Gamma_2(a) &= A\{\Gamma(a)\}^{1-a}\exp\left(-\frac{1}{12} + \zeta'(-1,a)\right) \\ &= A(2\pi)^{(1/2)(1-a)}\exp\left(-\frac{1}{12} + (1-a)\zeta'(0,a) + \zeta'(-1,a)\right) \quad (a > 0),\end{aligned} \tag{1.21}$$

where A is the Glaisher–Kinkelin constant given in Equation (1.11).

Here we aim mainly at demonstrating how easily certain (new or known) integral representations of $\log\Gamma(a)$, $B(\alpha,\beta)$, and $\log\Gamma_2(a)$ can be deduced from some known integral representations of the generalized Zeta function $\zeta(s,a)$. As a by-product of our main formulas, we give some integral representations of A and $\psi(a)$. Relevant connections of some of our results presented here with those obtained in earlier works are also indicated together with the potential for their usefulness.

2. Integral representations of $\log\Gamma(a)$, $B(\alpha,\beta)$, and $\log\Gamma_2(a)$

We begin by recalling some of the known integral representations of the generalized Zeta function $\zeta(s,a)$ as Lemma 1 (see, e.g., [20, Section 2.2]).

LEMMA 1 *Each of the following relationships holds true*:

$$\zeta(s,a)=\frac{1}{2}\,a^{-s}+\frac{a^{1-s}}{s-1}+2\int_0^\infty (a^2+y^2)^{-(1/2)s}$$

$$\cdot\left[\sin\left(s\arctan\frac{y}{a}\right)\right]\frac{\mathrm{d}y}{\mathrm{e}^{2\pi y}-1}\quad (s\in\mathbb{C}\setminus\{1\}); \tag{2.1}$$

$$\zeta(s,a)=\frac{a^{-s+1}}{s-1}+\frac{a^{-s}}{2}+\sum_{k=4}^{n}\frac{\Gamma(k+s-1)}{\Gamma(s)}\frac{B_k}{k!}a^{-k-s+1}$$

$$+\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{1}{\mathrm{e}^t-1}-\sum_{k=0}^{n}\frac{B_k}{k!}t^{k-1}\right)\mathrm{e}^{-at}t^{s-1}\,\mathrm{d}t$$

$$\big(\Re(s)>-(2n-1);\ \Re(a)>0;\ n\in\mathbb{N}_0\big), \tag{2.2}$$

where B_n $(n\in\mathbb{N}_0)$ are the Bernoulli numbers [20, *Section* 1.6]);

$$\zeta(s,a)=\cos\left(\frac{1}{2}\pi s\right)\sin(2\pi a)\int_0^\infty\frac{t^{-s}}{\cosh(2\pi t)-\cosh(2\pi a)}\,\mathrm{d}t$$

$$+\sin\left(\frac{1}{2}\pi s\right)\int_0^\infty\frac{t^{-s}\left[\cosh(2\pi a)-\mathrm{e}^{-2\pi t}\right]}{\cosh(2\pi t)-\cosh(2\pi a)}\,\mathrm{d}t$$

$$\big(\Re(s)<1\quad when\quad 0<\Re(a)<1;\ \Re(s)<0\quad when\quad a=1\big); \tag{2.3}$$

$$\zeta(s,a)=\frac{\pi 2^{s-2}}{s-1}\int_0^\infty\left[t^2+(2a-1)^2\right]^{(1/2)(1-s)}\frac{\cos[(s-1)\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,\mathrm{d}t$$

$$\left(s\in\mathbb{C}\setminus\{1\};\ \Re(a)>\frac{1}{2}\right). \tag{2.4}$$

Differentiating both sides of Equations (2.1) to (2.4) with respect to s under the sign of integration (for validity of this process, see [24, Corollary, p. 74]) and setting $s=0$ and $s=-1$ in the resulting identities, we obtain Lemma 2.

LEMMA 2 *Each of the following results holds true:*

$$\zeta'(0,a) = \left(a - \frac{1}{2}\right)\log a - a + 2\int_0^\infty \frac{\arctan(y/a)}{e^{2\pi y}-1}\,dy \tag{2.5}$$

and

$$\zeta'(-1,a) = -\frac{a^2}{4} + \left(\frac{a(a-1)}{2}\right)\log a + \int_0^\infty \left[2\arctan\left(\frac{y}{a}\right)\cos\left(\arctan\frac{y}{a}\right) + \log\left(a^2+y^2\right)\cdot\sin\left(\arctan\frac{y}{a}\right)\right]\frac{\sqrt{a^2+y^2}}{e^{2\pi y}-1}\,dy; \tag{2.6}$$

$$\zeta'(0,a) = \left(a - \frac{1}{2}\right)\log a - a + \int_0^\infty \left(\frac{1}{e^t-1} - \sum_{k=0}^{n}\frac{B_k}{k!}t^{k-1}\right)\frac{e^{-at}}{t}\,dt \quad \left(\Re(a) > 0;\, n \in \mathbb{N}\right) \tag{2.7}$$

and

$$\zeta'(-1,a) = \left(\frac{a(a-1)}{2}\right)\log a - \frac{a^2}{4} - \int_0^\infty \left(\frac{1}{e^t-1} - \sum_{k=0}^{n}\frac{B_k}{k!}t^{k-1}\right)\frac{e^{-at}}{t^2}\,dt \quad \left(\Re(a) > 0;\, n \in \mathbb{N}\setminus\{1\}\right); \tag{2.8}$$

$$\zeta'(0,a) = -\sin(2\pi a)\int_0^\infty \frac{\log t}{\cosh(2\pi t)-\cosh(2\pi a)}\,dt + \frac{\pi}{2}\int_0^\infty \frac{\cosh(2\pi a)-e^{-2\pi t}}{\cosh(2\pi t)-\cosh(2\pi a)}\,dt \quad \left(0 < \Re(a) < 1\right) \tag{2.9}$$

and

$$\zeta'(-1,a) = \frac{\pi}{2}\sin(2\pi a)\int_0^\infty \frac{t}{\cosh(2\pi t)-\cosh(2\pi a)}\,dt + \int_0^\infty \frac{[\cosh(2\pi a)-e^{-2\pi t}]t\log t}{\cosh(2\pi t)-\cosh(2\pi a)}\,dt \quad \left(0 < \Re(a) < 1\right); \tag{2.10}$$

$$\begin{aligned}\zeta'(0,a) = &-\frac{\pi}{4}(1+\log 2)\int_0^\infty \sqrt{t^2+(2a-1)^2}\,\frac{\cos[\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,dt \\ &- \frac{\pi}{4}\int_0^\infty \sqrt{t^2+(2a-1)^2}\,\frac{\sin[\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\arctan\left(\frac{t}{2a-1}\right)dt \\ &+ \frac{\pi}{8}\int_0^\infty \sqrt{t^2+(2a-1)^2}\,\log[t^2+(2a-1)^2]\,\frac{\cos[\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,dt \\ &\left(\Re(a) > \frac{1}{2}\right)\end{aligned} \tag{2.11}$$

and

$$\zeta'(-1,a) = -\frac{\pi}{32}(1+2\log 2)\int_0^\infty [t^2+(2a-1)^2]\frac{\cos[2\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,\mathrm{d}t$$
$$-\frac{\pi}{16}\int_0^\infty [t^2+(2a-1)^2]\frac{\sin[2\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\arctan\left(\frac{t}{2a-1}\right)\mathrm{d}t$$
$$+\frac{\pi}{32}\int_0^\infty [t^2+(2a-1)^2]\log[t^2+(2a-1)^2]\frac{\cos[2\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,\mathrm{d}t$$
$$\left(\Re(a) > \frac{1}{2}\right). \tag{2.12}$$

Proof We need to prove only Equations (2.7) and (2.8). Differentiating (2.2) with respect to s, we get

$$\zeta'(s,a) = -\frac{a^{1-s}\log a}{s-1} - \frac{a^{1-s}}{(s-1)^2} - \frac{1}{2}a^{-s}\log a + \sum_{k=4}^{n}\frac{\Gamma(s+k-1)}{\Gamma(s)}\frac{B_k}{k!}a^{-k-s+1}$$
$$\cdot\left(\sum_{j=1}^{k-1}\frac{1}{s+j-1} - \log a\right) - \frac{\psi(s)}{\Gamma(s)}\int_0^\infty\left(\frac{1}{\mathrm{e}^t-1} - \sum_{k=0}^{n}\frac{B_k}{k!}t^{k-1}\right)\mathrm{e}^{-at}t^{s-1}\,\mathrm{d}t$$
$$+\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{1}{\mathrm{e}^t-1} - \sum_{k=0}^{n}\frac{B_k}{k!}t^{k-1}\right)\mathrm{e}^{-at}t^{s-1}\log t\,\mathrm{d}t$$
$$\big(\Re(s) > -(2n-1);\ \Re(a) > 0;\ n\in\mathbb{N}_0\big). \tag{2.13}$$

It is observed that

$$\left.\frac{\psi(s)}{\Gamma(s)}\right|_{s=-k} = -\frac{1}{\Gamma(\ell-k+1)}\sum_{j=0}^{\ell}\prod_{\substack{p=0\\(p\neq j)}}^{\ell}(p-k) \quad (k\in\mathbb{N}_0;\ \ 0\leqq k\leqq \ell) \tag{2.14}$$

and

$$\left.\frac{1}{\Gamma(s)}\right|_{s=-k} = 0 \quad (k\in\mathbb{N}_0). \tag{2.15}$$

Some special cases of Equation (2.14) are

$$\left.\frac{\psi(s)}{\Gamma(s)}\right|_{s=-1} = 1 \quad \text{and} \quad \left.\frac{\psi(s)}{\Gamma(s)}\right|_{s=0} = -1. \tag{2.16}$$

By setting $s=0$ and $s=-1$ in Equation (2.13) and with the aid of Equations (2.15) and (2.16), we arrive at Equations (2.7) and (2.8).

Now, by applying the identities in Lemma 2 to Equations (1.19), (1.20), and (1.21), we obtain our desired integral expressions as in the following theorem.

THEOREM *Each of the following relationships holds true*:

$$\log\Gamma(a)=\left(a-\frac{1}{2}\right)\log a-a+\frac{1}{2}\log(2\pi)+2\int_0^\infty\frac{\arctan y/a}{\mathrm{e}^{2\pi y}-1}\,\mathrm{d}y\quad\big(\Re(a)>0\big);\tag{2.17}$$

$$\log\Gamma(a)=\left(a-\frac{1}{2}\right)\log a-a+\frac{1}{2}\log(2\pi)+\int_0^\infty\left(\frac{1}{e^t-1}-\sum_{k=0}^{n}\frac{B_k}{k!}t^{k-1}\right)\frac{\mathrm{e}^{-at}}{t}\,\mathrm{d}t\quad\big(\Re(a)>0;\,n\in\mathbb{N}\big);\tag{2.18}$$

$$\log\Gamma(a)=\frac{1}{2}\log(2\pi)-\sin(2\pi a)\int_0^\infty\frac{\log t}{\cosh(2\pi t)-\cosh(2\pi a)}\,\mathrm{d}t+\frac{\pi}{2}\int_0^\infty\frac{\cosh(2\pi a)-\mathrm{e}^{-2\pi t}}{\cosh(2\pi t)-\cosh(2\pi a)}\,\mathrm{d}t\quad\big(0<\Re(a)<1\big);\tag{2.19}$$

$$\begin{aligned}\log\Gamma(a)={}&\frac{1}{2}\log(2\pi)-\frac{\pi}{4}(1+\log 2)\int_0^\infty\sqrt{t^2+(2a-1)^2}\,\frac{\cos[\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,\mathrm{d}t\\&-\frac{\pi}{4}\int_0^\infty\sqrt{t^2+(2a-1)^2}\,\frac{\sin[\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\arctan\left(\frac{t}{2a-1}\right)\mathrm{d}t\\&+\frac{\pi}{8}\int_0^\infty\sqrt{t^2+(2a-1)^2}\log[t^2+(2a-1)^2]\frac{\cos[\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,\mathrm{d}t\\&\left(\Re(a)>\frac{1}{2}\right);\end{aligned}\tag{2.20}$$

$$\begin{aligned}\log\Gamma_2(a)={}&\log A-\frac{1}{12}+(1-a)\log\Gamma(a)-\frac{a^2}{4}+\left(\frac{a(a-1)}{2}\right)\log a\\&+\int_0^\infty\left[2\arctan\left(\frac{y}{a}\right)\cos\left(\arctan\frac{y}{a}\right)+\log\left(a^2+y^2\right)\cdot\sin\left(\arctan\frac{y}{a}\right)\right]\\&\cdot\frac{\sqrt{a^2+y^2}}{\mathrm{e}^{2\pi y}-1}\,\mathrm{d}y\quad\big(\Re(a)>0\big);\end{aligned}\tag{2.21}$$

$$\begin{aligned}\log\Gamma_2(a)={}&\log A-\frac{1}{12}+(1-a)\log\Gamma(a)+\left(\frac{a(a-1)}{2}\right)\log a-\frac{a^2}{4}\\&-\int_0^\infty\left(\frac{1}{\mathrm{e}^t-1}-\sum_{k=0}^{n}\frac{B_k}{k!}t^{k-1}\right)\frac{\mathrm{e}^{-at}}{t^2}\,\mathrm{d}t\quad\big(\Re(a)>0;\,n\in\mathbb{N}\setminus\{1\}\big);\end{aligned}\tag{2.22}$$

$$\begin{aligned}\log\Gamma_2(a)={}&\log A-\frac{1}{12}+(1-a)\log\Gamma(a)+\frac{\pi}{2}\sin(2\pi a)\int_0^\infty\frac{t}{\cosh(2\pi t)-\cosh(2\pi a)}\,\mathrm{d}t\\&+\int_0^\infty\frac{[\cosh(2\pi a)-\mathrm{e}^{-2\pi t}]t\log t}{\cosh(2\pi t)-\cosh(2\pi a)}\,\mathrm{d}t\quad\big(0<\Re(a)<1\big);\end{aligned}\tag{2.23}$$

$$\log \Gamma_2(a) = \log A - \frac{1}{12} + (1-a)\log\Gamma(a) - \frac{\pi}{32}(1+2\log 2)\int_0^\infty [t^2+(2a-1)^2] \cdot \frac{\cos[2\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,\mathrm{d}t - \frac{\pi}{16}\int_0^\infty [t^2+(2a-1)^2] \cdot \frac{\sin[2\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\arctan\left(\frac{t}{2a-1}\right)\mathrm{d}t + \frac{\pi}{32}\int_0^\infty [t^2+(2a-1)^2]\log[t^2+(2a-1)^2]\frac{\cos[2\arctan(t/(2a-1))]}{\cosh^2((1/2)\pi t)}\,\mathrm{d}t \quad \left(\Re(a) > \frac{1}{2}\right); \tag{2.24}$$

$$B(\alpha,\beta) = \frac{\alpha^{\alpha-1/2}\beta^{\beta-1/2}}{(\alpha+\beta)^{\alpha+\beta-1/2}}(2\pi)^{1/2}\mathrm{e}^{I_1(\alpha,\beta)} \quad (\alpha>0;\ \beta>0), \tag{2.25}$$

where, for convenience,

$$I_1(\alpha,\beta) := 2\rho\int_0^\infty \arctan\left(\frac{(t^3+t)\rho^3}{\alpha\beta(\alpha+\beta)}\right)\frac{\mathrm{d}t}{\mathrm{e}^{2\pi t\rho}-1} \quad (\rho^2 := \alpha^2+\alpha\beta+\beta^2);$$

$$B(\alpha,\beta) = \frac{\alpha^{\alpha-1/2}\beta^{\beta-1/2}}{(\alpha+\beta)^{\alpha+\beta-1/2}}(2\pi)^{1/2}\mathrm{e}^{I_2(\alpha,\beta;n)} \quad (\alpha>0;\ \beta>0), \tag{2.26}$$

where, for convenience,

$$I_2(\alpha,\beta;n) := \int_0^\infty \left(\frac{1}{\mathrm{e}^t-1} - \sum_{k=0}^n \frac{B_k}{k!}t^{k-1}\right)(\mathrm{e}^{-\alpha t}+\mathrm{e}^{-\beta t}-\mathrm{e}^{-(\alpha+\beta)t})\frac{\mathrm{d}t}{t} \quad (n\in\mathbb{N}). \tag{2.27}$$

3. Further observations and special cases

The special case $n=1$ of Equation (2.18) becomes *Binet's first expression* for $\log\Gamma(a)$ as an infinite integral (see, e.g., [20, Chapter 12]):

$$\log\Gamma(a) = \left(a-\frac{1}{2}\right)\log a - a + \frac{1}{2}\log(2\pi) + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{\mathrm{e}^t-1}\right)\frac{\mathrm{e}^{-at}}{t}\,\mathrm{d}t \quad (\Re(a)>0). \tag{3.1}$$

It is noted that Equation (2.1) is Hermite's formula for $\zeta(s,a)$ and Equation (2.17) is called Binet's second expression for $\log\Gamma(a)$. Binet's two expressions for $\log\Gamma(a)$ are of great importance in showing the way in which $\log\Gamma(a)$ behaves as $|a|\to\infty$.

Vignéras [22] gave the following integral representation for the double Gamma function Γ_2:

$$\log\Gamma_2(a+1) = -\int_0^\infty \frac{\mathrm{e}^{-t}}{t(1-\mathrm{e}^{-t})^2}\left(1-at-\frac{a^2t^2}{2}-\mathrm{e}^{-at}\right)\mathrm{d}t + (1+\gamma)\frac{a^2}{2} - \frac{3}{2}\log\pi \quad (\Re(a)>-1). \tag{3.2}$$

It is interesting to see that Equation (3.2) is comparable with Equations (2.21) to (2.24).

Glaisher [12, p. 47] expressed the Glaisher–Kinkelin constant A given in Equation (1.11) as an integral:

$$A = 2^{7/36}\pi^{-1/6}\exp\left(\frac{1}{3}+\frac{2}{3}\int_0^{1/2}\log\Gamma(t+1)\,\mathrm{d}t\right). \tag{3.3}$$

By setting $a=1$ in Equations (2.21), (2.22), and (2.24), and with the aid of Equation (1.15), we can also obtain integral representations of $\log A$ as follows:

$$\log A = \frac{1}{3} - 2\int_0^\infty \left(\frac{1}{2}\sin(\arctan t)\log(1+t^2)+\cos(\arctan t)\arctan t\right)\frac{\sqrt{1+t^2}}{\mathrm{e}^{2\pi t}-1}\,\mathrm{d}t; \tag{3.4}$$

$$\log A = \frac{1}{3} + \int_0^\infty \left(\frac{1}{\mathrm{e}^t-1} - \sum_{k=0}^{n}\frac{B_k}{k!}t^{k-1}\right)\frac{\mathrm{e}^{-t}}{t^2}\,\mathrm{d}t \quad (n\in\mathbb{N}\setminus\{1\}); \tag{3.5}$$

$$\begin{aligned}\log A &= \frac{1}{12} + \frac{\pi}{16}\int_0^\infty (t^2+1)\frac{\sin(2\arctan t)}{\cosh^2\left(\frac{1}{2}\pi t\right)}\arctan t\,\mathrm{d}t + \frac{\pi}{32}\int_0^\infty [1+\log 4 \\ &\quad - \log(t^2+1)](t^2+1)\frac{\cos(2\arctan t)}{\cosh^2((1/2)\pi t)}\,\mathrm{d}t.\end{aligned} \tag{3.6}$$

Setting $a=1/2$ in (2.23) with the aid of (1.17) yields

$$\log A = \frac{1}{12}(1+\log 2) - \int_0^\infty \frac{\left(\mathrm{e}^{-2\pi t}-\cosh\pi\right)t\log t}{\cosh(2\pi t)-\cosh\pi}\,\mathrm{d}t. \tag{3.7}$$

It is noted that Equation (2.25) is a known formula [6] where the following trigonometric identities:

$$\begin{aligned}\arctan a + \arctan b &= \arctan\left(\frac{a+b}{1-ab}\right) \quad (ab<1), \\ \arctan a - \arctan b &= \arctan\left(\frac{a-b}{1+ab}\right) \quad (ab>-1)\end{aligned} \tag{3.8}$$

are used. The special case $n=1$ of Equation (2.26) was also recorded earlier [6].

The formula (2.22) (with $n=2$) can be used to obtain an asymptotic formula for $\log\Gamma_2(a)$ by first observing that, for some $M>0$ and for all $a>0$,

$$\left|\int_0^\infty\left(\frac{1}{\mathrm{e}^t-1}-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}\right)\frac{\mathrm{e}^{-at}}{t^2}\,\mathrm{d}t\right| < \frac{M}{a}.$$

Thus, by employing the following well-known asymptotic formula $\log\Gamma(a)$:

$$\begin{aligned}\log\Gamma(a) &= \left(a-\frac{1}{2}\right)\log a - a + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{n}\frac{B_{2k}}{2k(2k-1)a^{2k-1}} + O(a^{-2n-1}) \\ &\qquad \left(|a|\longrightarrow\infty;\ |\arg(a)|\leqq\pi-\epsilon\ (0<\epsilon<\pi);\ n\in\mathbb{N}_0\right),\end{aligned} \tag{3.9}$$

we have

$$\begin{aligned}\log\Gamma_2(a) &= \log A + \frac{3a^2}{4} - a - \frac{1}{12} - \left(\frac{a^2}{2}-a+\frac{5}{12}\right)\log a \\ &\quad + \frac{1}{2}(1-a)\log(2\pi) + O(a^{-1}) \quad (a\longrightarrow\infty;\ a>0),\end{aligned} \tag{3.10}$$

which may be compared with Equation (1.16).

Differentiating (2.17) and (2.18) under the sign of integration with respect to a, we obtain integral representations of the ψ-function as follows:

$$\psi(a) = \log a - \frac{1}{2a} - 2\int_0^\infty \frac{y\,\mathrm{d}y}{(y^2+a^2)(\mathrm{e}^{2\pi y}-1)} \quad \big(\Re(a) > 0\big), \tag{3.11}$$

which is a known formula (see [24, p. 251, Example]);

$$\psi(a) = \log a - \frac{1}{2a} - \int_0^\infty \left(\frac{1}{\mathrm{e}^t-1} - \sum_{k=0}^{n} \frac{B_k}{k!} t^{k-1}\right) \mathrm{e}^{-at}\,\mathrm{d}t \quad \big(\Re(a) > 0;\, n \in \mathbb{N}\big), \tag{3.12}$$

which, upon setting $n = 1$, yields a known result (see [20, p. 15, Eq. (21)]):

$$\psi(a) = \log a - \frac{1}{2a} - \int_0^\infty \left((1-\mathrm{e}^{-t})^{-1} - \frac{1}{t} - \frac{1}{2}\right) \mathrm{e}^{-at}\,\mathrm{d}t \quad \big(\Re(a) > 0\big). \tag{3.13}$$

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