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A family of log-gamma integrals and associated results

Junesang Choi [a], H.M. Srivastava [b,*]

[a] *Department of Mathematics, College of Natural Sciences, Dongguk University, Kyongju 780-714, South Korea*
[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, V8W 3P4, Canada*

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Abstract

The authors present a systematic investigation of the following log-gamma integral:

$$\int_0^z \log \Gamma(t+1)\,dt$$

and of its several related integral formulas. Relevant connections among the various mathematical constants involved naturally in the evaluation of the proposed integral are pointed out. Some approximate numerical values of the derivative $\zeta'(-1,a)$ of the Hurwitz zeta function are also considered. Importance of such derivatives as $\zeta'(-1,a)$ lies in their usefulness in the effective Lagrangian theory of quark confinement.

Keywords: Double gamma functions; Riemann and Hurwitz (or generalized) zeta functions; Determinants of the Laplacians; Glaisher–Kinkelin constant; Catalan constant; Euler–Maclaurin summation formula

* Corresponding author.
E-mail addresses: junesang@mail.dongguk.ac.kr (J. Choi), hmsri@uvvm.uvic.ca (H.M. Srivastava).

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1. Introduction and preliminaries

The following integral formula is well known (cf., e.g., [16, p. 24]; see also [19, Entry 6.441.3, p. 661]):

$$\int_0^1 \log \Gamma(t+1)\,dt = \frac{1}{2}\log(2\pi) - 1, \tag{1.1}$$

where Γ denotes the familiar gamma function whose Weierstrass canonical product form is given (among several equivalent expressions) by

$$\big\{\Gamma(z)\big\}^{-1} = z e^{\gamma z} \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right) e^{-z/k} \right\}, \tag{1.2}$$

γ being the Euler–Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577215664901532860606512\ldots. \tag{1.3}$$

Furthermore, a generalization of the integral formula (1.1) in the following form:

$$\int_z^{z+1} \log \Gamma(t+1)\,dt = (z+1)\log(z+1) - z - 1 + \frac{1}{2}\log(2\pi), \tag{1.4}$$

can be proved by employing Gauss's multiplication formula for the gamma function Γ (cf. [16, Eq. (11), p. 4], [19, Entry 6.441.1, p. 661], and [22, Eq. (17), p. 89]; see also [23, Eq. 1.2 (40), p. 18]). More generally, it is easy to see from (1.4) that (cf. [16, Eq. (20), p. 24]; see also [19, Entry 6.441.5, p. 662])

$$\int_z^{z+n} \log \Gamma(t+1)\,dt = \sum_{k=1}^{n} (z+k)\log(z+k) - nz - \frac{1}{2}n(n+1) + \frac{1}{2}n\log(2\pi)$$
$$\big(n \in \mathbb{N} := \{1,2,3,\ldots\}\big). \tag{1.5}$$

The Barnes G-function ($1/G =: \Gamma_2$ being the so-called double gamma function) was defined and studied by Barnes [2] and others in about 1900. In fact, it was recently revived in the study of the determinants of the Laplacians (cf. [3,24–27]). Among several equivalent forms, we recall here the following definition:

$$\begin{aligned} \big\{\Gamma_2(z+1)\big\}^{-1} &= G(z+1) \\ &:= (2\pi)^{z/2} \exp\left(-\frac{1}{2}\big[(1+\gamma)z^2 + z\big]\right) \\ &\quad \cdot \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^k \exp\left(-z + \frac{z^2}{2k}\right) \right\}. \end{aligned} \tag{1.6}$$

For sufficiently large $x \in \mathbb{R}^+$ and for $a \in \mathbb{C}$, we have the following Stirling formula for the G-function:

$$\log G(x+a+1) = \frac{x+a}{2}\log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax + \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right)\log x + O(x^{-1}) \quad (x \to \infty), \tag{1.7}$$

where A is the Glaisher–Kinkelin constant given by

$$\log A = \frac{1}{12} - \zeta'(-1) = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k\log k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n + \frac{n^2}{4}\right] \cong 1.282427130\ldots, \tag{1.8}$$

where $\zeta(z)$ denotes the Riemann zeta function defined by Eq. (1.14) below.

The G-function satisfies the following fundamental functional relationships:

$$G(1) = 1 \quad \text{and} \quad G(z+1) = \Gamma(z)G(z), \tag{1.9}$$

which may be compared with the following basic relationships for the familiar gamma function:

$$\Gamma(1) = 1 \quad \text{and} \quad \Gamma(z+1) = z\Gamma(z). \tag{1.10}$$

The Hurwitz (or generalized) zeta function $\zeta(z,a)$ defined by

$$\zeta(z,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^z} \quad \left(\Re(z) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\right) \tag{1.11}$$

can be continued meromorphically to the whole complex z-plane by means of some familiar contour integral representations (cf. [28, p. 266]) except for a simple pole at $z = 1$ (with its residue 1).

By differentiating both sides of (1.11) partially with respect to a, we obtain

$$\frac{\partial}{\partial a}\zeta(z,a) = -z\zeta(z+1,a). \tag{1.12}$$

Another familiar derivative formula for $\zeta(z,a)$ is derivable from Hermite's formula for $\zeta(z,a)$, and we have [23, Eq. 2.2(17), p. 92]]

$$\left.\frac{d}{dz}\zeta(z,a)\right|_{z=0} = \log\Gamma(a) - \frac{1}{2}\log(2\pi). \tag{1.13}$$

The Riemann zeta function $\zeta(z)$ defined by

$$\zeta(z) := \begin{cases} \displaystyle\sum_{k=1}^{\infty}\frac{1}{k^z} = \frac{1}{1-2^{-z}}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^z} & \left(\Re(z) > 1\right), \\ \displaystyle\frac{1}{1-2^{1-z}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^z} & \left(\Re(z) > 0;\ z \neq 1\right), \end{cases} \tag{1.14}$$

can, just as the Hurwitz (or generalized) zeta function $\zeta(z,a)$, be continued meromorphically to the whole complex z-plane except for a simple pole at $z=1$ (with its residue 1). Clearly, we have

$$\zeta(z,1)=\zeta(z)=\frac{1}{2^z-1}\zeta\left(z,\frac{1}{2}\right). \tag{1.15}$$

Finally, we recall the following known relationship between the Hurwitz zeta function $\zeta(z,a)$ and the Bernoulli polynomials $B_n(a)$ of degree n in a (cf. [1, Theorem 12.13, p. 264]; see also [23, Eq. 2.1(17), p. 85]):

$$\zeta(-n,a)=-\frac{B_{n+1}(a)}{n+1}\quad \bigl(n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\bigr), \tag{1.16}$$

which, for $a=1$, yields

$$\zeta(0)=B_1=-\frac{1}{2}\quad\text{and}\quad \zeta(-n)=-\frac{B_{n+1}}{n+1}\quad (n\in\mathbb{N}), \tag{1.17}$$

where B_n denotes the Bernoulli numbers given by

$$B_n:=B_n(0)=(-1)^nB_n(1)=\frac{1}{2^{1-n}-1}B_n\left(\frac{1}{2}\right)\quad (n\in\mathbb{N}_0). \tag{1.18}$$

The object of this paper is to present a systematic investigation of the following log-gamma integral:

$$\int_0^z \log\Gamma(t+1)\,dt$$

and of its several related integral formulas. Relevant connections among the various mathematical constants involved naturally in the evaluation of the proposed integral are pointed out. Some approximate numerical values of the derivative $\zeta'(-1,a)$, which is very important in the effective Lagrangian theory of quark confinement (cf. [11, p. 1638]; see also [9, p. 323]), are also considered.

2. A set of mathematical constants

Choi et al. [8, Eq. (36), p. 112] introduced a class of mathematical constants A_p $(p>0)$ defined by

$$\begin{aligned}\log A_p:=\lim_{n\to\infty}\Biggl[&\sum_{k=1}^n\left(k+\frac{1}{p}\right)\log\left(k+\frac{1}{p}\right)\\&-\left\{\frac{n^2}{2}+\left(\frac{1}{2}+\frac{1}{p}\right)n+\frac{1}{2p^2}+\frac{1}{2p}+\frac{1}{12}\right\}\log\left(n+\frac{1}{p}\right)+\frac{n^2}{4}+\frac{n}{2p}\Biggr],\end{aligned} \tag{2.1}$$

by making use of the Euler–Maclaurin summation formula (cf. [20, p. 318] and [10, p. 117]):

$$\sum_{k=1}^{n} f(k) \sim C_0 + \int_a^n f(x)\,dx + \frac{1}{2} f(n) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n), \tag{2.2}$$

where C_0 is an arbitrary constant to be determined in each special case and B_r are the Bernoulli numbers occurring in (1.17) and (1.18). It is easy to see that (cf. [8, p. 114])

$$\log A_1 = \log A - \frac{1}{4}, \tag{2.3}$$

where A denotes the Glaisher–Kinkelin constant defined by (1.8). Choi et al. [8, Eq. (37), p. 112] also presented another class of mathematical constants C_p $(p > 1)$:

$$\begin{aligned}\log C_p := \lim_{n\to\infty}\Bigg[&\sum_{k=1}^{n}\left(k - \frac{1}{p}\right)\log\left(k - \frac{1}{p}\right) \\ &- \left\{\frac{n^2}{2} + \left(\frac{1}{2} - \frac{1}{p}\right)n + \frac{1}{2p^2} - \frac{1}{2p} + \frac{1}{12}\right\}\log\left(n - \frac{1}{p}\right) + \frac{n^2}{4} - \frac{n}{2p}\Bigg],\end{aligned} \tag{2.4}$$

where (obviously)

$$C_p = A_{-p}. \tag{2.5}$$

In what follows, we shall also make use of the Catalan constant $\mathcal{G}$ defined by

$$\mathcal{G} := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cong 0.915965594177219015\ldots. \tag{2.6}$$

By using the theory of the double gamma function Γ_2, the following log-gamma integral was evaluated by Choi and Srivastava [5, Eq. (2.29), p. 100]:

$$\int_0^{1/2} \log\Gamma(t+1)\,dt = -\frac{1}{2} - \frac{7}{24}\log 2 + \frac{1}{4}\log\pi + \frac{3}{2}\log A \tag{2.7}$$

or, equivalently,

$$A = 2^{7/36} \cdot \pi^{-1/6} \cdot \exp\left(\frac{1}{3} + \frac{2}{3}\int_0^{1/2} \log\Gamma(t+1)\,dt\right), \tag{2.8}$$

which was derived earlier by Glaisher [17, p. 47] by using an elementary technique. Recently, Gosper [18, p. 71] made use of the polylogarithm function (cf. [21]) in order to evaluate the log-gamma integral (2.7) in the form:

$$\int_0^{1/2} \log\Gamma(t+1)\,dt = -\frac{1}{2} + \frac{\gamma}{8} - \frac{1}{6}\log 2 + \frac{3}{8}\log\pi - \frac{3\zeta'(2)}{4\pi^2}, \tag{2.9}$$

so that, by comparing (2.7) and (2.9), we obtain

$$\zeta'(2) = \pi^2\left(\frac{\gamma}{6} + \frac{1}{6}\log(2\pi) - 2\log A\right). \tag{2.10}$$

On the other hand, if we apply the known relationship [27, Eq. (A.11), p. 462]:

$$\log A = \frac{1}{12} - \zeta'(-1) \tag{2.11}$$

in (2.10), we find that [6, Eq. (18), p. 240]

$$\zeta'(-1) = \frac{1}{12}\left[1 - \gamma - \log(2\pi)\right] + \frac{\zeta'(2)}{2\pi^2}, \tag{2.12}$$

which can also be obtained by appealing appropriately to Riemann's functional equation for $\zeta(z)$ (cf. [28, p. 269]; see also [16, Eq. 1.12(23), p. 35]). The relationship (2.12) is recorded *erroneously* in the aforecited work of Voros [27, Eq. (6.25), p. 453]].

In order to get a more general class of mathematical constants than those given by (2.1), we begin by differentiating the function

$$f(x) := \left(x + \frac{1}{p}\right)^q \log\left(x + \frac{1}{p}\right) \quad (p > 0;\ q \in \mathbb{N}) \tag{2.13}$$

l times ($l \in \mathbb{N}$). We thus obtain

$$f^{(l)}(x) = \left(x + \frac{1}{p}\right)^{q-l}\left\{\left(\prod_{j=1}^{l}(q - j + 1)\right)\log\left(x + \frac{1}{p}\right) + P_l(q)\right\} \\ (l \in \mathbb{N}), \tag{2.14}$$

where $P_l(q)$ is a polynomial of degree $l - 1$ in q satisfying the following recurrence relation:

$$P_l(q) := \begin{cases} (q - l + 1)P_{l-1}(q) + \prod_{j=1}^{l-1}(q - j + 1) & (l \in \mathbb{N} \setminus \{1\}), \\ 1 & (l = 1). \end{cases} \tag{2.15}$$

In fact, by mathematical induction on $l \in \mathbb{N}$, we can give an explicit expression for $P_l(q)$ as follows:

$$P_l(q) = \left(\prod_{j=1}^{l}(q - j + 1)\right)\left(\sum_{j=1}^{l}\frac{1}{q - j + 1}\right) \quad (l \in \mathbb{N}). \tag{2.16}$$

By substituting from (2.13) and (2.14) into the Euler–Maclaurin summation formula (2.2) with $a = 0$, we get a class of mathematical constants $C_{p,q}$ defined by

$$\log C_{p,q} := \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(k + \frac{1}{p}\right)^q \log\left(k + \frac{1}{p}\right) + \frac{1}{(q+1)^2}\left(n + \frac{1}{p}\right)^{q+1} \right. \\ \left. - \frac{1}{q+1}\left(n + \frac{1}{p}\right)^{q+1}\log\left(n + \frac{1}{p}\right) - \frac{1}{2}\left(n + \frac{1}{p}\right)^q \log\left(n + \frac{1}{p}\right)\right.$$

$$
\begin{aligned}
&- \sum_{r=1}^{[(q+1)/2]} \frac{B_{2r}}{(2r)!}\left(n+\frac{1}{p}\right)^{q-2r+1} \\
&\cdot \left\{\left(\prod_{j=1}^{2r-1}(q-j+1)\right)\log\left(n+\frac{1}{p}\right)+P_{2r-1}(q)\right\}\Bigg] \\
&(p>0;\ q\in\mathbb{N}),
\end{aligned}
\tag{2.17}
$$

where $[x]$ denotes (as usual) the greatest integer $\leqq x$.

Setting $q=1$ in (2.17) and comparing the resulting equation with (2.1), it is easy to get a relationship between A_p and $C_{p,1}$ as follows:

$$
C_{p,1}=\exp\left(\frac{1}{4p^2}-\frac{1}{12}\right)A_p \quad (p>0). \tag{2.18}
$$

3. Evaluation of $\int_0^z \log\Gamma(t+1)\,dt$ and its related integrals

First of all, we recall that the following integral formula of Barnes [2, p. 283] was derived recently by Choi et al. [7, Eq. (2.4), p. 385]:

$$
\begin{aligned}
\int_0^z \log\Gamma(t+a)\,dt &= \frac{1}{2}\big[\log(2\pi)+1-2a\big]z-\frac{z^2}{2}+(z+a-1)\log\Gamma(z+a) \\
&\quad -\log G(z+a)+(1-a)\log\Gamma(a)+\log G(a),
\end{aligned}
\tag{3.1}
$$

which, for $a=1$, reduces at once to Alexeweisky's theorem (cf., e.g., [19, Entry 6.441.4, p. 661]):

$$
\int_0^z \log\Gamma(t+1)\,dt=\frac{1}{2}\big[\log(2\pi)-1\big]z-\frac{z^2}{2}+z\log\Gamma(z+1)-\log G(z+1). \tag{3.2}
$$

Next, in view of (1.6) and (1.13), the expression for Γ_2 (given by Choi and Seo [4, Eq. (2.8), p. 164]) can be rewritten in its equivalent form:

$$
G(z+1)=\frac{1}{A}\big\{\Gamma(z+1)\big\}^z\exp\left(\frac{1}{12}-\zeta'(-1,z+1)\right), \tag{3.3}
$$

where the prime in $\zeta'(s,z)$ denotes differentiation with respect to the first argument s. Thus, upon substituting for $G(z+1)$ from (3.3) into (3.2), we obtain another evaluation for the log-gamma integral in (3.2) as follows:

$$
\int_0^z \log\Gamma(t+1)\,dt=\zeta'(-1,z+1)+\log A-\frac{1}{12}+\frac{z}{2}\big[\log(2\pi)-1\big]-\frac{z^2}{2}. \tag{3.4}
$$

It is easy to deduce from (1.11) that

$$
\zeta(s,z+n)=\zeta(s,z)+\sum_{k=0}^{n-1}\frac{1}{(k+z)^s} \quad (n\in\mathbb{N}_0), \tag{3.5}
$$

which, upon differentiating with respect to s and setting $s = -1$, yields

$$\zeta'(-1, z+n) = \zeta'(-1, z) + \sum_{k=0}^{n-1} (k+z)\log(k+z) \quad (n \in \mathbb{N}_0). \tag{3.6}$$

In view of (3.6), the log-gamma integral (3.4) can be rewritten in the following equivalent form:

$$\int_0^z \log\Gamma(t+1)\,dt = \zeta'(-1, z) + \log A - \frac{1}{12} + z\log z + \frac{z}{2}\big[\log(2\pi) - 1\big] - \frac{z^2}{2}, \tag{3.7}$$

which was also proved earlier by Gosper [18, p. 75] who made use of polylogarithms and the reflection formula of $\zeta(s, a)$ (cf. [28, p. 269]; see also [16, Eq. 1.10(6), p. 26]).

Here we choose to recall the process of evaluation of the integral formula (3.4) presented by Elizalde and Romeo [15, p. 456] (see also [13] and [14, p. 16]). Indeed, by differentiating (1.12) with respect to z, we get

$$\frac{\partial^2}{\partial z \partial t}\zeta(z, t+1) = -\zeta(z+1, t+1) - z\frac{\partial}{\partial z}\zeta(z+1, t+1), \tag{3.8}$$

which, by virtue of (1.12) itself, yields

$$\frac{\partial^2}{\partial z \partial t}\zeta(z, t+1) = \frac{1}{z}\frac{\partial}{\partial t}\zeta(z, t+1) - z\frac{\partial}{\partial z}\zeta(z+1, t+1) \quad \big(z \in \mathbb{C}\setminus\{0, 1\}\big). \tag{3.9}$$

Now, for $z \neq 1$ and $\Re(t) > -1$, $\zeta(z, t+1)$ is an analytic function of z and t, and therefore, the partial derivatives with respect to z and t commute in this region. Furthermore, by taking $z = -1$ in (3.9) and making use of (1.13), we obtain

$$\frac{\partial^2}{\partial t \partial z}\zeta(z, t+1)\bigg|_{z=-1} = -\frac{\partial}{\partial t}\zeta(z, t+1)\bigg|_{z=-1} + \log\Gamma(t+1) - \frac{1}{2}\log(2\pi). \tag{3.10}$$

By integrating both sides of (3.10) with respect to t over the interval (a, b) $(b > a > -1)$, we find that

$$\int_a^b \log\Gamma(t+1)\,dt = \left[\zeta(-1, t+1) + \zeta'(-1, t+1) + \frac{t}{2}\log(2\pi)\right]_a^b, \tag{3.11}$$

where, as before,

$$\zeta'(-1, t+1) = \frac{\partial}{\partial z}\zeta(z, t+1)\bigg|_{z=-1}. \tag{3.12}$$

In particular, upon setting $a = 0$ and $b = z > -1$ in (3.11), and then applying (1.16), (1.17), and (2.11), we arrive at the integral formula (3.4).

The special case of (3.7) when $z = 1/2$, with application of (1.15), (1.17), and (2.11), is seen to correspond to (2.7).

By setting $z = 1/3$ in (3.7), we obtain

$$\int_0^{1/3} \log \Gamma(t+1)\,dt = \zeta'\left(-1, \frac{1}{3}\right) + \log A - \frac{11}{36} - \frac{1}{3}\log 3 + \frac{1}{6}\log(2\pi), \tag{3.13}$$

which, when compared with the known result [6, Eq. (19), p. 240]:

$$\int_0^{1/3} \log \Gamma(t+1)\,dt = -\frac{1}{3} - \frac{\pi\sqrt{3}}{54} - \frac{25}{72}\log 3 + \frac{1}{6}\log(2\pi) + \frac{\sqrt{3}}{36\pi}\zeta\left(2, \frac{1}{3}\right) + \frac{4}{3}\log A, \tag{3.14}$$

yields the following relationship between $\zeta'(-1, 1/3)$ and $\zeta(2, 1/3)$:

$$\zeta'\left(-1, \frac{1}{3}\right) = -\frac{1}{36} - \frac{\pi\sqrt{3}}{54} - \frac{1}{72}\log 3 + \frac{1}{3}\log A + \frac{\sqrt{3}}{36\pi}\zeta\left(2, \frac{1}{3}\right). \tag{3.15}$$

In its special case when $z = 1/4$, we find from the integral formula (3.7) that

$$\int_0^{1/4} \log \Gamma(t+1)\,dt = \zeta'\left(-1, \frac{1}{4}\right) + \log A - \frac{23}{96} - \frac{3}{8}\log 2 + \frac{1}{8}\log \pi, \tag{3.16}$$

which, when compared with the known result [5, Eq. (2.30), p. 100]:

$$\int_0^{1/4} \log \Gamma(t+1)\,dt = -\frac{1}{4} - \frac{3}{8}\log 2 + \frac{1}{8}\log \pi + \frac{9}{8}\log A + \frac{\mathcal{G}}{4\pi}, \tag{3.17}$$

also gives us an interesting identity in the form:

$$\zeta'\left(-1, \frac{1}{4}\right) = -\frac{1}{96} + \frac{1}{8}\log A + \frac{\mathcal{G}}{4\pi}, \tag{3.18}$$

$\mathcal{G}$ being the Catalan constant defined by (2.6).

It is fairly straightforward to derive the integral formula (1.5) by combining (3.7) and (3.8).

In terms of the mathematical constants A_p and C_p introduced in Section 2, the following log-gamma integrals were evaluated earlier by Choi et al. [8, Eqs. (39) and (40), p. 114]:

$$\int_0^{1/p} \log \Gamma(1+t)\,dt = \log A - \log A_p + \frac{1}{2p}\log(2\pi) - \frac{1}{2p} - \frac{3}{4p^2} \quad (p > 0) \tag{3.19}$$

and

$$\int_0^{1/p} \log \Gamma(1-t)\,dt = \log C_p - \log A + \frac{1}{2p}\log(2\pi) - \frac{1}{2p} + \frac{3}{4p^2} \quad (p > 1). \tag{3.20}$$

With a view to deriving a relationship between

$$A_p \quad \text{and} \quad \zeta'\left(-1, \frac{1}{p}\right) \quad (p > 0)$$

analogous to that in (2.11), we set

$$f(x) := \left(x + \frac{1}{p}\right)^{-s} \quad (p > 0)$$

in the Euler–Maclaurin summation formula (2.2) with $a = 0$. We then obtain

$$\begin{aligned}\sum_{k=1}^{n}\left(k+\frac{1}{p}\right)^{-s} \sim C(s,p) + \frac{(n+1/p)^{1-s} - p^{s-1}}{1-s} + \frac{1}{2}\left(n+\frac{1}{p}\right)^{-s} \\ - \sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}(s)_{2r-1}\left(n+\frac{1}{p}\right)^{-s-2r+1} + \mathcal{R}(s,p;n;m) \\ \big(\Re(s) > -2m-1;\ m \in \mathbb{N}_0\big),\end{aligned} \tag{3.21}$$

where $C(s, p)$ is a constant depending on s and p, $(s)_n$ denotes the Pochhammer symbol defined by

$$(s)_n := \frac{\Gamma(s+n)}{\Gamma(s)} \quad (n \in \mathbb{N}_0),$$

an empty sum is interpreted (as usual) to be nil, and the remainder part $\mathcal{R}(s, p; n; m)$ satisfies the following limit relationship:

$$\lim_{n\to\infty} \mathcal{R}(s,p;n;m) = 0 \quad \big(\Re(s) > -2m-1;\ m \in \mathbb{N}_0\big).$$

In fact, $C(s, p)$ can be expressed *explicitly* as follows by applying the same $f(x)$ to (2.2) and reducing the domain to $\Re(s) > 1$:

$$\begin{aligned}C(s,p) &= \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(k+\frac{1}{p}\right)^{-s} - \frac{(n+1/p)^{1-s} - p^{s-1}}{1-s}\right] \\ &= \zeta\left(s, \frac{1}{p}\right) - p^s - \frac{p^{s-1}}{s-1} \quad \big(\Re(s) > 1\big).\end{aligned} \tag{3.22}$$

Now it follows from (3.21) and (3.22) that

$$\begin{aligned}\zeta\left(s,\frac{1}{p}\right) = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(k+\frac{1}{p}\right)^{-s} - \frac{(n+1/p)^{1-s}}{1-s} - \frac{1}{2}\left(n+\frac{1}{p}\right)^{-s} \right. \\ \left. + \sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}(s)_{2r-1}\left(n+\frac{1}{p}\right)^{-s-2r+1}\right] + p^s \\ \big(m \in \mathbb{N},\ \Re(s) > -2m-1, \text{ and } s \neq 1;\ m = 0 \text{ and } \Re(s) > 1\big).\end{aligned} \tag{3.23}$$

We note that the right-hand side of (3.23) is analytic for

$$\Re(s) > -2m-1 \quad (m \in \mathbb{N}_0) \text{ and } s \neq 1.$$

Thus we can differentiate (3.23) with respect to s under the limit sign and set $s=-1$ in the resulting equation. We, therefore, obtain

$$\begin{aligned}\zeta'\left(s,\frac{1}{p}\right)=\lim_{n\to\infty}\Bigg[&-\sum_{k=1}^{n}\left(k+\frac{1}{p}\right)^{-s}\log\left(k+\frac{1}{p}\right)-\frac{(n+1/p)^{1-s}}{(s-1)^2}\\&+\frac{(n+1/p)^{1-s}}{1-s}\log\left(n+\frac{1}{p}\right)+\frac{1}{2}\left(n+\frac{1}{p}\right)^{-s}\log\left(n+\frac{1}{p}\right)\\&+\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}(s)_{2r-1}\left(n+\frac{1}{p}\right)^{-s-2r+1}\\&\cdot\left\{\sum_{j=1}^{2r-1}\frac{1}{s+j-1}-\log\left(n+\frac{1}{p}\right)\right\}\Bigg]+p^s\log p\\&\big(m\in\mathbb{N},\ \Re(s)>-2m-1,\ \text{and } s\neq 1;\ m=0 \text{ and } \Re(s)>1\big),\end{aligned}\tag{3.24}$$

where we have also used the following elementary identity:

$$\frac{d}{ds}(s)_n=(s)_n\left(\sum_{k=1}^{n}\frac{1}{s+k-1}\right)\quad(n\in\mathbb{N}_0),\tag{3.25}$$

an empty sum being understood (as usual) to be nil.

By setting $s=-1$ and $s=-2$ in (3.24), we find that

$$\begin{aligned}\zeta'\left(-1,\frac{1}{p}\right)=-\lim_{n\to\infty}\Bigg[&\sum_{k=1}^{n}\left(k+\frac{1}{p}\right)\log\left(k+\frac{1}{p}\right)\\&-\left\{\frac{n^2}{2}+\left(\frac{1}{p}+\frac{1}{2}\right)n+\frac{1}{2p^2}+\frac{1}{2p}+\frac{1}{12}\right\}\log\left(n+\frac{1}{p}\right)\\&+\frac{n^2}{4}+\frac{n}{2p}\Bigg]-\frac{1}{4p^2}+\frac{1}{12}+\frac{1}{p}\log p\quad(p>0)\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}\zeta'\left(-2,\frac{1}{p}\right)=-\lim_{n\to\infty}\Bigg[&\sum_{k=1}^{n}\left(k+\frac{1}{p}\right)^2\log\left(k+\frac{1}{p}\right)-\left\{\frac{n^3}{3}+\left(\frac{1}{p}+\frac{1}{2}\right)n^2\right.\\&\left.+\left(\frac{1}{p^2}+\frac{1}{p}+\frac{1}{6}\right)n+\frac{1}{3p^3}+\frac{1}{2p^2}+\frac{1}{6p}\right\}\log\left(n+\frac{1}{p}\right)+\frac{n^3}{9}\\&+\frac{n^2}{3p}+\left(\frac{1}{3p^2}-\frac{1}{12}\right)n\Bigg]-\frac{1}{9p^3}+\frac{1}{12p}+\frac{\log p}{p^2}\quad(p>0).\end{aligned}\tag{3.27}$$

It should be remarked in passing that the *main* result in an interesting paper by Elizalde [12, Eq. (17), p. 349], which incidentally was proven by appropriately applying Watson's lemma and Laplace's method to Hermite's integral representation for $\zeta(s,a)$ (cf., e.g., [16, Eq. 1.10(7), p. 26]), actually provides a rather useful *asymptotic* expansion for $\zeta'(-n,a)$ when

$$|a| \to \infty \quad (n \in \mathbb{N}).$$

Comparing (2.1) and (3.26), we finally obtain the desired relationship:

$$\log A_p = -\zeta'\left(-1, \frac{1}{p}\right) + \frac{1}{12} - \frac{1}{4p^2} + \frac{1}{p}\log p \quad (p > 0), \tag{3.28}$$

which, in view of (2.3), immediately yields (2.11) when $p = 1$.

Next, in view of the following well-known reflection formula:

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)},$$

we can evaluate the following integral (cf. [8, Eq. (43), p. 114]):

$$\int_0^{\pi/p} \log\sin t \, dt = \pi\left(\log A_p - \log C_p - \frac{1}{p}\log(2p)\right) \quad (p > 1), \tag{3.29}$$

which, by virtue of another known result [5, Eq. (2.2), p. 95]:

$$\int_0^{\pi/p} \log\sin t \, dt = \frac{\pi}{p}\log\left(\frac{\sin(\pi/p)}{2\pi}\right) + \pi\log\left[\frac{G(1+1/p)}{G(1-1/p)}\right] \tag{3.30}$$

yields the following relationship between the mathematical constants A_p and C_p:

$$A_p = \left[\frac{p}{\pi}\sin\left(\frac{\pi}{p}\right)\right]^{1/p} \frac{G(1+1/p)}{G(1-1/p)} C_p \quad (p > 1). \tag{3.31}$$

Making use of (3.3), (3.6), (3.28), and (3.31), it is not difficult to obtain the following relationship analogous to that in (3.28):

$$\log C_p = -\zeta'\left(-1, 1 - \frac{1}{p}\right) + \frac{1}{12} - \frac{1}{4p^2} \quad (p > 1). \tag{3.32}$$

The relationship (3.28) can also be deduced *directly* by setting $z = 1/p$ in (3.7) and comparing the resulting equation with (3.19).

Gosper [18, p. 74] evaluated the following integral:

$$\int_0^{1/5} \log\sin(\pi t)\, dt = \frac{1}{25\pi}\left[2\left(1 + \frac{\sqrt{5}}{5}\right)\pi^2 - \frac{1+\sqrt{5}}{2}\zeta\left(2, \frac{1}{5}\right) - \zeta\left(2, \frac{2}{5}\right)\right]\sin\frac{\pi}{5} - \frac{\log 2}{5}, \tag{3.33}$$

which, upon comparing with (3.29), yields

$$\log\left[\frac{G(1+1/5)}{G(4/5)}\right] = \frac{1}{25\pi}\left[2\left(1 + \frac{\sqrt{5}}{5}\right)\pi^2 - \frac{1+\sqrt{5}}{2}\zeta\left(2, \frac{1}{5}\right) - \zeta\left(2, \frac{2}{5}\right)\right]\sin\frac{\pi}{5} - \frac{1}{5}\log\left(\frac{\sin(\pi/5)}{5}\right). \tag{3.34}$$

If we set $s=-q$ $(q\in\mathbb{N})$ in (3.24), we obtain

$$\begin{aligned}
-\zeta'\left(-q,\frac{1}{p}\right)=\lim_{n\to\infty}\Bigg[&\sum_{k=1}^{n}\left(k+\frac{1}{p}\right)^{q}\log\left(k+\frac{1}{p}\right)+\frac{1}{(q+1)^2}\left(n+\frac{1}{p}\right)^{q+1}\\
&-\frac{1}{q+1}\left(n+\frac{1}{p}\right)^{q+1}\log\left(n+\frac{1}{p}\right)-\frac{1}{2}\left(n+\frac{1}{p}\right)^{q}\log\left(n+\frac{1}{p}\right)\\
&-\sum_{r=1}^{m}\frac{B_{2r}}{(2r)!}\left(n+\frac{1}{p}\right)^{q-2r+1}\left\{\left(\prod_{j=1}^{2r-1}(q-j+1)\right)\log\left(n+\frac{1}{p}\right)\right.\\
&\left.+\left(\prod_{j=1}^{2r-1}(q-j+1)\right)\left(\sum_{j=1}^{2r-1}\frac{1}{q-j+1}\right)\right\}\Bigg]-p^{-q}\log p\\
&(m,q\in\mathbb{N};\ q<2m+1).
\end{aligned}\tag{3.35}$$

Thus, by comparing (2.17) and (3.35), and then applying (2.16), we get the following relationship between $C_{p,q}$ and $\zeta'(-q,1/p)$:

$$\log C_{p,q}=-\zeta'\left(-q,\frac{1}{p}\right)+p^{-q}\log p\quad(p>0;\ q\in\mathbb{N}).\tag{3.36}$$

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