Learning Integration!

$$\int_0^1 x^5 e^{x^2}\,dx$$

$f(x)\,g'(x)$ $\qquad \int f(x)g'(x) = f(x)\,g(x) - \int f'(x)g(x)\,dx$

$$\int_0^1 x^5 e^{x^2}\,dx$$

$u = x^2 \quad du = 2x\,dx \quad dx = \frac{du}{2x}$

$x = 0 \quad u = 0 \qquad x^5 = x\cdot x^4 = x\cdot u^2$

$x = 1 \quad u = 1$

$$= \frac{1}{2}\int_0^1 \cancel{x}\cdot u^2 e^u \cdot \frac{du}{\cancel{x}} = \frac{1}{2}\int_0^1 u^2 e^u\,du = \frac{1}{2}\left(u^2 e^u\Big|_0^1 - 2\int_0^1 \underbrace{u e^u\,du}\right)$$

$$= \frac{1}{2}\left(u^2 e^u\Big|_0^1 - 2\left(u e^u\Big|_0^1 - \int e^u\right)\right) = \frac{1}{2}\left(u e^u\Big|_0^1 - 2\,u e^u\Big|_0^1 + 2\,e^u\Big|_0^1\right)$$

$$= \frac{1}{2}\left(e - \cancel{2e} + \cancel{2e} - 2\right) = \boxed{\frac{e}{2} - 1}$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \quad . \quad e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

$$I = \int_0^1 x^5 e^{x^2}\,dx = \int_0^1 x^5 \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}\,dx = \sum_{k=0}^{\infty} \frac{1}{k!}\int_0^1 x^5\cdot x^{2k}\,dx$$

$$= I = \sum_{k=0}^{\infty} \frac{1}{k!}\int_0^1 x^{5+2k}\,dx = \sum_{k=0}^{\infty} \frac{1}{k!}\,\frac{x^{6+2k}}{6+2k}\Bigg|_0^1$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\,\frac{1}{6+2k} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{k!(3+k)}$$

$$(k+3)! = (k+3)(k+2)(k+1)k! \qquad k! = \frac{(k+3)!}{(k+1)(k+2)\cancel{(k+3)}}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{(k+3)!} \cdot (k+1)(k+2)$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{(k+3)!} = \frac{k^2+3k+2}{(k+3)!}$$

$$= \frac{(k+3)^2-3(k+3)+2}{(k+3)!} = \frac{(k+3)^2}{(k+3)!} - \frac{3(k+3)}{(k+3)!} + \frac{2}{(k+3)!}$$

① $$\frac{\cancel{(k+3)}(k+3)}{\cancel{(k+3)}(k+2)!} = \frac{(k+3)+1}{(k+2)!\;(k+2)!} = \frac{\cancel{k+2}}{\cancel{(k+2)}(k+1)!}\frac{1}{} + \frac{1}{(k+2)!}$$

② $$\frac{-3\cancel{(k+3)}}{\cancel{(k+3)}(k+2)!} = \frac{-3}{(k+2)!}$$ ③ $$\frac{2}{(k+3)!}$$

$$\frac{1}{(k+1)!} - \frac{2}{(k+2)!} + \frac{2}{(k+3)!}$$

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)!} - \sum_{k=0}^{\infty}\frac{2}{(k+2)!} + 2\sum_{k=0}^{\infty}\frac{1}{(k+3)!}$$

$$\sum_{k=1}^{\infty}\frac{1}{k!} = e-1 \qquad 2\sum_{k=2}^{\infty}\frac{1}{k!} = -2(e-1-1) = -2e+4$$

$$2\sum_{k=3}^{\infty}\frac{1}{k!} = e-1-1-\frac{1}{2} = 2e - \cancel{2}\cdot\frac{5}{\cancel{2}} = 2e-5$$

$$\frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{1}{k!(3+k)}\right) = \frac{1}{2}\left(e-1-\cancel{2e}+4+\cancel{2e}-5\right)$$

$$\frac{1}{2}(e-2) = \boxed{\frac{e}{2}-1}$$

$$a\,I(a) = \int_0^\infty \frac{2x\sin(ax)}{(x^2+b^2)^2}\,dx$$

taking

$$I(a) + a\,I'(a) = \int_0^\infty \frac{2x^2\cos(ax)}{(x^2+b^2)^2}\,dx$$

$$I(a) + a\,I'(a) = 2\int_0^\infty \frac{x^2\cos(ax)}{(x^2+b^2)^2}\,dx$$

$$\frac{x^2\cos(ax)}{(x^2+b^2)^2} \quad \frac{\cos(ax)}{(x^2+b^2)} - \frac{b^2\cos(ax}{(x^2+b^2)^2}$$

$$I(a) + a\,I'(a) = 2\left(\underbrace{\int_0^\infty \frac{\cos(ax)}{x^2+b^2}} - b^2\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,dx\right.$$

$$\cancel{I(a)} + a\,I'(a) = \cancel{2}I(a) - 2b^2\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,dx$$

$$a\,I'(a) - I(a) = -2b^2\int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,dx$$

$$\cancel{I'(a)} + a\,I''(a) - \cancel{I'(a)} = +2b^2\int_0^\infty \frac{+\sin(ax)\times}{(x^2+b^2)^2}\,dx$$

$$aI''(a) = 2b^2 \int_0^{\infty} \frac{\sin(ax)}{(x^2+b^2)^2} x \, dx$$

$$\cancel{a} I''(a) = \cancel{2}b^2 \frac{\cancel{a} I(a)}{\cancel{2}}$$

$$\boxed{I''(a) = b^2 I(a)}$$

$$z^2 - b^2 = 0 \qquad (z-b)(z+b) = 0$$

$$z = b \quad z = -b \qquad y_{omo} = c_1 e^{ba} + c_2 e^{-ba}$$

$$I(0) = \int_0^{\infty} \frac{1}{x^2+b^2} dx = \frac{1}{b} \arctan\left(\frac{x}{b}\right)\Big|_0^{\infty}$$

$$= \frac{\pi}{2b}$$

$$y(0) = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 = \frac{\pi}{2b}$$

$$y(\alpha) = \lim_{a \to +\infty} \frac{1}{a} \int_0^{\infty} \frac{x \sin(ax)}{(x^2+b^2)} dx$$

$$y(\alpha) = 0 \qquad c_1 e^{ba} + c_2 e^{-ba} = 0$$

$$\lim_{a \to +\infty} \qquad \to 0$$

$\Rightarrow \quad c_1 = 0 \quad \Rightarrow \quad c_2 + c_1 = \frac{\pi}{2b}$

$$c_2 = \frac{\pi}{2b} \quad \Rightarrow \quad \boxed{y(a) = \frac{\pi}{2b} e^{-ba}}$$

$I(1)$

$b = 1$

$$y(1) = \frac{\pi}{2} e^{-1} = \boxed{\frac{\pi}{2e}}$$
