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1.1 Implicit Form

$x = 2\sin(t)$; $y = \frac{5}{2}\sin(2t) = 5\sin(t)\sqrt{1-\sin^2(t)}$.

$\Rightarrow y(x) = 5\left(\frac{x}{2}\right)\sqrt{1-\frac{x^2}{4}}$ $\qquad y^2 = \frac{25}{4}x^2\left(1-\frac{x^2}{4}\right)$

1.2 Tangent

$\langle x(t), y(t)\rangle \to \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle = \langle 2\cos(t), 5\cos(2t)\rangle$

1.3 Normal

$f(x,y) = \frac{25}{4}x^2\left(1-\frac{x^2}{4}\right) - y^2 = 0$

$\nabla f(x,y) = \left\langle \frac{25}{4}(2x - x^3), -2y\right\rangle$ $\quad \vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\left\langle \frac{25}{4}(2x-x^3), -2\ldots\right.}{\sqrt{4y^2 + \left(\frac{25}{4}\right)^2(2x\ldots}}$

Alternatively

$\left\langle -\frac{dy(t)}{dt}, \frac{dx(t)}{dt}\right\rangle$ for zero dot product

$= \langle -5\cos(2t), 2\cos(t)\rangle$.

1.4 X axis sym $\quad \boxed{\text{Yes}}$

$\forall \langle x(t_0), y(t_0)\rangle \; \exists t_1$ s.t. $\langle x(t_1), y(t_1)\rangle$

where $x(t_1) = x(t_0) \wedge y(t_1) = -y(t_0)$

$\Rightarrow \quad 2\sin(t_1) = 2\sin t_0 \qquad \Rightarrow \sin(2t_1) = -\sin(2t_0)$

$\Rightarrow t_1 = t_0 \qquad \#2) \Rightarrow t_1 = -t_0$

or $t_1 = \pi - t_0 \qquad$ or $t_1 = \pi - t_0$

$\therefore \forall t_0, \exists t_1 = \pi - t_0$ S.t. $y(t_0) = -y(t_1)$ & $x(t_0) = x(t_1)$

1.5 Y axis sym

$\boxed{\text{Yes}}$

$\forall \langle x(t_0), y(t_0)\rangle \ \exists \ t_1$ s.t. $\langle x(t_1), y(t_1)\rangle$

where $x(t_1) = -x(t_0) \cap y(t_1) = y(t_0)$

$\Rightarrow -\sin(t_1) = \sin(t_0)$ $\qquad \Rightarrow \sin(2t_1) = \sin(2t_0)$

$\Rightarrow t_1 = -t_0$ $\qquad \Rightarrow t_1 = t_0$

or $t_1 = t_0 + \pi$ $\qquad$ or $t_1 = t_0 + \pi$

$\therefore \forall t_0, \exists t_1 = t_0 + \pi$ s.t. $y(t_0) = y(t_1)$ & $x(t_1) = -x(t_0)$

1.6 Area

$$A = 4 \cdot \int_0^2 \frac{5}{2} x\sqrt{1 - \frac{x^2}{4}}\, dx$$

$$u = 1 - \frac{x^2}{4} \Rightarrow du = -\frac{1}{2}x\,dx \Rightarrow -2du = x\,dx$$

$$A = 4\int_1^0 (-5)\sqrt{u}\,du = 20\int_0^1 \sqrt{u}\,du$$

$$= \frac{40}{3}\left(u^{3/2}\right)\Big|_0^1 = \frac{40}{3}$$

1.7 Approximate perimeter

for some timestep $\Delta = \frac{2\pi}{n}$:

$$dS = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \Rightarrow S \approx \sum_{i=0}^{n} \sqrt{\left(\frac{dx}{dt}(i\Delta)\right)^2 + \left(\frac{dy}{dt}(i\Delta)\right)^2}\ \frac{2\pi}{n}$$

We incrementally sum over timesteps of tangent to approximate perimeter

2.1 translation & translation $\boxed{\text{True}}$

$$S_1 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \quad S_1 S_2 = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix}$$

Since $a+c = c+a$, we can arbitrarily reassign $S1$ & $S2$ $\Rightarrow$ Commutative ✓.

2.2 translation & rotation $\boxed{\text{False}}$

$$S_1 = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_1 S_2 = \begin{bmatrix} \cos t & -\sin t & tx \\ \sin t & \cos t & ty \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_1 S_2 \neq S_2 S_1$$

$$S_2 S_1 = \begin{bmatrix} \cos t & -\sin t & tx\cos t - ty\sin t \\ \sin t & \cos t & tx\sin t + ty\cos t \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Not Commutative. X

2.3 Scaling & Rotation of diff fixed points $\boxed{\text{False}}$

first to origin, second to α, β: $\beta = 0$ for convenience
$\alpha = 1$, $t = \frac{\pi}{2}$.

$$\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = S_1 \qquad \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} = S_2$$

$s_1 = 1$
$s_2 = 2$

$$T = \begin{bmatrix} 1 & 0 & +\alpha \\ 0 & 1 & +\beta \\ 0 & 0 & 1 \end{bmatrix} \qquad T' = \begin{bmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix}$$

Must Show:

$$\underset{①}{S_1 T S_2 T'} \overset{?}{=} \underset{②}{T S_2 T' S_1}$$

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_1 T S_2 T' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$TS_2T'S_1 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Not Comm

2.4 Scaling & Scaling Same point $\boxed{\text{True}}$

Point $\langle \alpha, \beta \rangle$.

Must show

$$T S_1 \underbrace{\cancel{T'T}}_{I} S_2 T' = T S_2 \underbrace{\cancel{T'T}}_{I} S_1 T'$$

$\Rightarrow$ Need to show $S_1 S_2 = S_2 S_1$

as $T'T = I$

$$S_1 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_1 S_2 = \begin{bmatrix} \alpha\gamma & 0 & 0 \\ 0 & \beta\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{since } \begin{matrix} \alpha\gamma = \gamma\alpha \\ \beta\varphi = \varphi\beta \end{matrix}$$

We can arbitrarily re assign

$\Rightarrow$ Commutative.

3.1 Homography

$$\begin{bmatrix} a_1 & a_2 & t_x \\ a_3 & a_4 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

1/ $(1,0) \to (6,2)$

2/ $(0,1) \to (7,3)$

3/ $(1,1) \to (6,3)$

4/ $(0,0) \to (7,2)$

By 4: $t_x = 7 \quad t_y = 2$

By 1: $a_1 + 7 = 6 \quad a_1 = -1$

$a_3 + 2 = 2 \quad a_3 = 0$

By 2: $a_2 + 7 = 7 \quad a_2 = 0$

$a_4 + 2 = 3 \quad a_4 = 1$

By 3: $a_1 + a_2 + 7 = 6 \Rightarrow -1 + 7 = 6$ ✓

$a_3 + a_4 + 2 = 3 \quad 1 + 2 = 3$ ✓

3.2 mapping (2,5)

$$\begin{bmatrix} -1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix} \Rightarrow (2,5) \to (5,7)$$

4.1 on edge

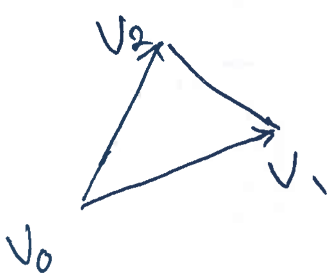
Note: $V_0 = (V_{0x}, V_{0y})$
$V_1 = (V_{1x}, V_{1y})$
$V_2 = (V_{2x}, V_{2y})$

define
$$\left.\begin{aligned} e_0 &= V_0(t) + V_1(1-t) \\ e_1 &= V_1(t) + V_2(1-t) \\ e_2 &= V_2(t) + V_0(1-t) \end{aligned}\right\} \quad \boxed{0 \le t \le 1}$$

Given point (x, y), check if point (x, y) lies on any edge ① & $0 \le t \le 1$ ②

Ex
$$\left.\begin{aligned} e_{0x} &= x = V_{0x}(t) + V_{1x}(1-t) \\ e_{0y} &= y = V_{0y}(t) + V_{1y}(1-t) \end{aligned}\right\}$$
2 eqn, 1 unknown
solve for t,
check if equal
for condition ①

4.2 Area of triangle



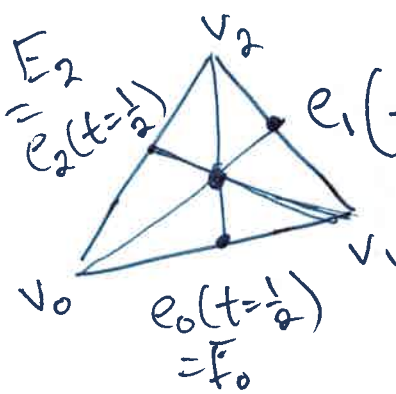
define $\overrightarrow{V_0V_2}$ & $\overrightarrow{V_0V_1}$

Area $= \left\|\overrightarrow{V_0V_2} \times \overrightarrow{V_0V_1}\right\| \left(\frac{1}{2}\right)$

Cross product.

4.3 Centroid

$(C_x, C_y) =$ intersection of midpoint.

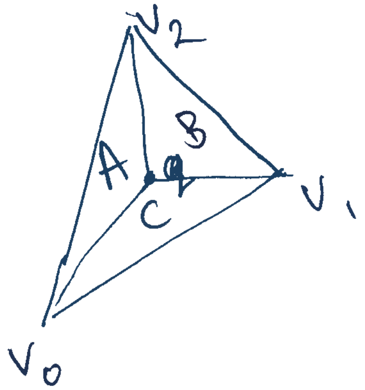


$e_1(t=\frac{1}{2}) = E_1$

define $l_1 = V_0(t) + E_1(1-t)$
$l_2 = V_1(t) + E_2(1-t)$

Find point of intersection between l_1 & l_2 ⇒ centroid.

4.4 Inside triangle



Using part 4.2:

$$|A|+|B|+|C| = \|\vec{v_0v_1} \times \vec{v_0v_2}\| \left(\tfrac{1}{2}\right)$$

If q inside triangle!

$$|A| = \|\vec{v_0q} \times \vec{v_0v_2}\| (1/2)$$

$$|B| = \|\vec{v_2q} \times \vec{v_2v_1}\| (1/2)$$

$$|C| = \|\vec{v_0q} \times \vec{v_0v_1}\| (1/2)$$

If $|A|+|B|+|C| \neq \|\vec{v_0v_1} \times \vec{v_0v_2}\| (1/2)$

$\Rightarrow$ Not in triangle.