EXERCISE 9.1

Determine the order and degree (if defined) of differential equations given in Exercises 1 to 10.

1. $\frac{d^4y}{dx^4} + \sin(y''') = 0$

Solution:

The given differential equation is,

1. $\frac{d^4y}{dx^4} + \sin(y''') = 0$

⇒ y'''' + sin (y''') = 0

The highest order derivative present in the differential equation is y'''', so its order is three. Hence, the given differential equation is not a polynomial equation in its derivatives, so its degree is not defined.

2. y' + 5y = 0

Solution:

The given differential equation is y' + 5y = 0

The highest order derivative present in the differential equation is y', so its order is one.

Therefore, the given differential equation is a polynomial equation in its derivatives.

So, its degree is one.

3. $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

Solution:-

The given differential equation is,

$$\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$$

The highest order derivative present in the differential equation is $\frac{d^2s}{dt^2}$.

The order is two. Therefore, the given differential equation is a polynomial equation in $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt}$.

So, its degree is one.

4. $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

Solution:-

The given differential equation is,

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$.

The order is two. Therefore, the given differential equation is not a polynomial.

So, its degree is not defined.

5. $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$

Solution:-

The given differential equation is,

$$\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$$

$$\Rightarrow \frac{d^2y}{dx^2} - \cos 3x - \sin 3x = 0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$.

The order is two. Therefore, the given differential equation is a polynomial equation in $\frac{d^2y}{dx^2}$ and the power is 1.

Therefore, its degree is one.

6. $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

Solution:

The given differential equation is, $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

The highest order derivative present in the differential equation is y'''.

The order is three. Therefore, the given differential equation is a polynomial equation in y''', y'' and y'.

Then, the power raised to y''' is 2.

Therefore, its degree is two.

7. $y''' + 2y'' + y' = 0$

Solution:

The given differential equation is, $y''' + 2y'' + y' = 0$

The highest order derivative present in the differential equation is y'''.

The order is three. Therefore, the given differential equation is a polynomial equation in y''', y'' and y'.

Then, the power raised to y''' is 1.

Therefore, its degree is one.

8. y' + y = e^x

Solution:

The given differential equation is y' + y = e^x

= y' + y – e^x = 0

The highest order derivative present in the differential equation is y'.

The order is one. Therefore, the given differential equation is a polynomial equation in y'.

Then, the power raised to y' is 1.

Therefore, its degree is one.

9. y''' + $(y')^2$ + 2y = 0

Solution:

The given differential equation is, y''' + $(y')^2$ + 2y = 0

The highest order derivative present in the differential equation is y''.

The order is two. Therefore, the given differential equation is a polynomial equation in y'' and y'.

Then, the power raised to y'' is 1.

Therefore, its degree is one.

10. y''' + 2y' + sin y = 0

Solution:-

The given differential equation is, y''' + 2y' + sin y = 0

The highest order derivative present in the differential equation is y''.

The order is two. Therefore, the given differential equation is a polynomial equation in y'' and y'.

Then the power raised to y'' is 1.

Therefore, its degree is one.

11. The degree of the differential equation.

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0 \text{ is}$$

(A) 3 (B) 2 (C) 1 (D) not defined

Solution:-

(D) not defined

The given differential equation is,

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0 \text{ is}$$

The highest order derivative present in the differential equation is

$$\frac{d^2y}{dx^2}$$

The order is three. Therefore, the given differential equation is not a polynomial.

Therefore, its degree is not defined.

12. The order of the differential equation

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0 \text{ is}$$

(A) 2 (B) 1 (C) 0 (D) not defined

Solution:-

(A) 2

The given differential equation is,

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$$

The highest order derivative present in the differential equation is

$$\frac{d^2y}{dx^2}$$

.

Therefore, its order is two.

EXERCISE 9.2

In each of the Exercises 1 to 10, verify that the given functions (explicit or implicit) are a solution of the corresponding differential equation:

1. y = ex + 1 : y″ – y′ = 0

Solution:-

From the question, it is given that $y = e^x + 1$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) \qquad \text{... [Equation (i)]}$$

Now, differentiating equation (i) both sides with respect to x, we have,

$$\frac{d}{dx}(y') = \frac{d}{dx}(e^x)$$

⇒ $y'' = e^x$

Then,

Substituting the values of y’ and y” in the given differential equations, we get,

$y'' - y' = e^x - e^x$ = RHS.

Therefore, the given function is a solution of the given differential equation.

2. $y = x^2 + 2x + C$: $y' - 2x - 2 = 0$

Solution:-

From the question, it is given that $y = x^2 + 2x + C$

Differentiating both sides with respect to x, we get

$$y' = \frac{d}{dx}(x^2 + 2x + C)$$

$y' = 2x + 2$

Then,

Substituting the values of y’ in the given differential equations, we get

$= y' - 2x - 2$

= 2x + 2 – 2x – 2

= 0

= RHS

Therefore, the given function is a solution of the given differential equation.

3. y = cos x + C : y′ + sin x = 0

Solution:-

From the question, it is given that y = cos x + C

Differentiating both sides with respect to x, we get

$$y' = \frac{d}{dx}(\cos x + C)$$

y' = -sinx

Then,

Substituting the values of y' in the given differential equations, we get

= y' + sinx

= – sinx + sinx

= 0

= RHS

Therefore, the given function is a solution of the given differential equation.

4. $y = \sqrt{(1 + x^2)}$: $y' = ((xy)/(1 + x^2))$

Solution:-

From the question it is given that y = $\sqrt{1+x^2}$

Differentiating both sides with respect to x, we get,

$$y' = \frac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$\Rightarrow y' = \frac{1}{2\sqrt{1+x^2}}.\frac{d}{dx}(1+x^2)$$

By differentiating $(1 + x^2)$ we get,

$$\Rightarrow y' = \frac{2x}{2\sqrt{1+x^2}}$$

On simplifying we get,

$$\Rightarrow y' = \frac{x}{\sqrt{1+x^2}}$$

By multiplying and dividing $\sqrt{(1 + x^2)}$

$$\Rightarrow y' = \frac{x}{1+x^2} \times \sqrt{1+x^2}$$

Substituting the value of $\sqrt{(1 + x^2)}$

$$\Rightarrow y' = \frac{x}{1+x^2}.y$$

Substituting the value of $\sqrt{(1 + x^2)}$

$$\Rightarrow y' = \frac{x}{1+x^2}.y$$

$$\Rightarrow y' = \frac{xy}{1+x^2}$$

Therefore, LHS = RHS

Therefore, the given function is a solution of the given differential equation.

5. y = Ax : xy′ = y (x ≠ 0)

Solution:-

From the question, it is given that y = Ax

Differentiating both sides with respect to x, we get

$$y' = \frac{d}{dx}(Ax)$$

y' = A

Then,

Substituting the values of y' in the given differential equations, we get

= xy'

= x × A

= Ax

= Y … [from the question]

= RHS

Therefore, the given function is a solution of the given differential equation.

6. y = x sinx: xy' = y + x ($\sqrt{(x^2 – y^2)}$) (x ≠ 0 and x>y or x< – y)

Solution:-

From the question it is given that y = xsinx

Differentiating both sides with respect to x, we get,

$$y' = \frac{d}{dx}(x\sin x)$$

$$\Rightarrow y' = \sin x\frac{d}{dx}(x) + x.\frac{d}{dx}(\sin x)$$

$$\Rightarrow y' = \sin x + x\cos x$$

Then,

Substituting the values of y' in the given differential equations, we get, Then,

Substituting the values of y′ in the given differential equations, we get,

LHS = xy′ = x(sinx + xcosx)

= xsinx + x^2cosx

From the question substitute y instead of xsinx, we get,

= y + $x^2 . \sqrt{1 - \sin^2 x}$

$= y + x^2 \sqrt{1 - \left(\frac{y}{x}\right)^2}$

$= y + x\sqrt{(y)^2 - (x)^2}$

= RHS

Therefore, the given function is a solution of the given differential equation

7. xy = logy + C : $y' = \dfrac{y^2}{1 - xy} \quad (xy \neq 1)$

Solution:-

From the question it is given that xy = logy + C

Differentiating both sides with respect to x, we get,

$$\frac{d}{dx}(xy) = \frac{d}{dx}(\log y)$$

$$\Rightarrow y.\frac{d}{dx}(x) + x.\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

On simplifying, we get.

$$\Rightarrow y + xy' = \frac{1}{y}\frac{dy}{dx}$$

By cross multiplication,

$$\Rightarrow y^2 + xyy' = y'$$

$$\Rightarrow (xy - 1)y' = -y^2$$

$$\Rightarrow y' = \frac{y^2}{1-xy}$$

$$\Rightarrow y' = \frac{y^2}{1-xy}$$

By comparing LHS and RHS

LHS = RHS

Therefore, the given function is the solution of the corresponding differential equation.

8. y – cos y = x : (y sin y + cos y + x) y′ = y

Solution:-

From the question it is given that y – cosy = x

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} - \frac{d}{dx}\cos y = \frac{d}{dx}(x)$$

$$\Rightarrow y' + \sin y . y' = 1$$

$\Rightarrow$ y' (1 + siny) = 1

$\Rightarrow y' = \frac{1}{1+\sin y}$

Then,

Substituting the values of y' in the given differential equations, we get,

Consider LHS = (ysiny + cosy + x)y'

$$= (y\sin y + \cos y + y - \cos y) \times \frac{1}{1 + \sin y}$$

$$= y(1 + \sin y) \times \frac{1}{1 + \sin y}$$

On simplifying we get,

= y

= RHS

Therefore, the given function is the solution of the corresponding differential equation.

9. $x + y = \tan^{-1}y$: $y^2 y' + y^2 + 1 = 0$

Solution:-

From the question it is given that $x + y = \tan^{-1}y$

Differentiating both sides with respect to x, we get,

$$\frac{d}{dx}(x+y) = \frac{d}{dx}(\tan^{-1}y)$$

$$\Rightarrow 1 + y' = \left[\frac{1}{1+y^2}\right]y'$$

By transposing y' to RHS and it becomes – y' and take out y' as common for both, we get,

$$\Rightarrow y'\left[\frac{1}{1+y^2} - 1\right] = 1$$

On simplifying,

$$\Rightarrow y'\left[\frac{1-(1+y^2)}{1+y^2}\right] = 1$$

$$\Rightarrow y'\left[\frac{-y^2}{1+y^2}\right] = 1$$

$$\Rightarrow y' = \frac{-(1+y^2)}{y^2}$$

Then,

Substituting the values of y' in the given differential equations, we get,

Consider, LHS = $y^2y' + y^2 + 1$

$= y^2\left[\frac{-(1+y^2)}{y^2}\right] + y^2 + 1$

$= -1 - y^2 + y^2 + 1$

= 0

= RHS

Therefore, the given function is the solution of the corresponding differential equation.

10. $y = \sqrt{a^2 - x^2}\ x \in (-a, a): \quad x + y\dfrac{dy}{dx} = 0\ (y \neq 0)$

Solution:-

From the question it is given that $y = \sqrt{a^2 - x^2}$

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{a^2 - x^2})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}}.\frac{d}{dx}(a^2 - x^2)$$

$$= \frac{1}{2\sqrt{a^2 - x^2}}(-2x)$$

$$= \frac{-x}{\sqrt{a^2 - x^2}}$$

Then,

Substituting the values of y' in the given differential equations, we get,

Consider LHS = $x + y\frac{dy}{dx}$

$$= x + \sqrt{a^2 - x^2} \times \frac{-x}{\sqrt{a^2 - x^2}}$$

On simplifying, we get,

= x −x

= 0

By comparing LHS and RHS

LHS = RHS.

Therefore, the given function is the solution of the corresponding differential equation.

11. The number of arbitrary constants in the general solution of a differential equation of fourth order is:

(A) 0 (B) 2 (C) 3 (D) 4

Solution:-

(D) 4

The solution which contains arbitrary constants is called the general solution (primitive) of the differential equation.

12. The number of arbitrary constants in the particular solution of a differential equation of third order is:

(A) 3 (B) 2 (C) 1 (D) 0

Solution:-

(D) 0

The solution free from arbitrary constants, i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants, is called a particular solution of the differential equation.

EXERCISE 9.3 PAGE NO: 391

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

1. $\frac{x}{a}+\frac{y}{b}=1$

Solution:-

From the question it is given that $\frac{x}{a}+\frac{y}{b}=1$

Differentiating both sides with respect to x, we get,

$$\frac{1}{a}+\frac{1}{b}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}y'=0 \quad \text{... [Equation (i)]}$$

Now, differentiating equation (i) both sides with respect to x, we get,

$$0+\frac{1}{b}y''=0$$

$$\Rightarrow \frac{1}{b}y''=0$$

By cross multiplication, we get,

$$\Rightarrow y''=0$$

∴the required differential equation is y'' = 0.

2. $y^2 = a(b^2 - x^2)$

Solution:-

From the question it is given that $y^2 = a(b^2 - x^2)$

Differentiating both sides with respect to x, we get,

$$2y\frac{dy}{dx} = a(-2x)$$

$\Rightarrow 2yy' = -2ax$

$\Rightarrow yy' = (-2/2)ax$

$\Rightarrow yy' = -ax$ ... [we call it as equation (i)]

Now, differentiating equation (i) both sides, we get,

$\Rightarrow y' \times y' + yy'' = -a$

$\Rightarrow (y')^2 + yy'' = -a$ ...[we call it as equation (ii)]

Then,

Dividing equation (2) by (1), we get,

$$\frac{(y')^2 + yy''}{yy'} = \frac{-a}{-ax}$$

$\Rightarrow x(y')^2 + xyy'' = yy'$

Now, differentiating equation (i) both sides, we get,

$\Rightarrow y' \times y' + yy'' = -a$

$\Rightarrow (y')^2 + yy'' = -a$ ...[we call it as equation (ii)]

Then,

Dividing equation (ii) by (i), we get,

$$\frac{(y')^2 + yy''}{yy'} = \frac{-a}{-ax}$$

$\Rightarrow x(y')^2 + xyy'' = yy'$

Transposing yy' to LHS it becomes $- yy'$

$\Rightarrow xyy'' + x(y')^2 - yy' = 0$

∴ the required differential equation is $xyy'' + x(y')^2 - yy' = 0$.

3. $y = ae^{3x} + be^{-2x}$

Solution:-

From the question it is given that $y = ae^{3x} + be^{-2x}$ ... [we call it as equation (i)]

Differentiating both sides with respect to x, we get,

$y' = 3ae^{3x} - 2be^{-2x}$ ... [equation (ii)]

Now, differentiating equation (ii) both sides, we get,

$y'' = 9ae^{3x} + 4be^{-2x}$ ... [equation (iii)]

Then, multiply equation (i) by 2 and afterwards add it to equation (ii),

We have,

$\Rightarrow (2ae^{3x} + 2be^{-2x}) + (3ae^{3x} - 2be^{-2x}) = 2y + y'$

$\Rightarrow 5ae^{3x} = 2y + y'$

$$\Rightarrow ae^{3x} = \frac{2y + y'}{5}$$

So now, let us multiply equation (ii) by 3 and subtracting equation (ii),

We have

$\Rightarrow (3ae^{3x} + 3be^{-2x}) - (3ae^{3x} - 2be^{-2x}) = 3y - y'$

$\Rightarrow 5be^{-2x} = 3y - y'$

$$\Rightarrow be^{-2x} = \frac{3y - y'}{5}$$

Substitute the value of ae^{3x} and be^{-2x} in y'',

$$y'' = 9 \times \frac{2y+y'}{5} + 4 \times \frac{2y+y'}{5}$$

$$\Rightarrow y'' = \frac{18y + 9y'}{5} + \frac{12y - 4y'}{5}$$

On simplifying we get,

$$\Rightarrow y'' = \frac{30y + 5y'}{5}$$

$\Rightarrow y'' = 6y + y'$

$\Rightarrow y'' - y' - 6y = 0$

$\therefore$the required differential equation is $y'' - y' - 6y = 0$.

4. $y = e^{2x}(a + bx)$

Solution:-

From the question it is given that $y = e^{2x}(a + b x)$ … [we call it as equation (i)]

Differentiating both sides with respect to x, we get,

$y' = 2e^{2x}(a + b x) + e^{2x} \times b$ … [equation (ii)]

Then, multiply equation (i) by 2 and afterwards subtract it from equation (ii),

We have,

$y' - 2y = e^{2x}(2a + 2bx + b) - e^{2x}(2a + 2bx)$

$y' - 2y = 2ae^{2x} + 2e^{2x}bx + e^{2x}b - 2ae^{2x} - 2bxe^{2x}$

$y' - 2y = be^{2x}$ … [equation (iii)]

Now, differentiating equation (iii) both sides, We have,

$\Rightarrow$ y'' – 2y = 2be^{2x} … [equation (iv)]

Then,

Dividing equation (iv) by (iii), we get,

$$\frac{y'' - 2y'}{y' - 2y} = 2$$

By cross multiplication,

$$\Rightarrow y'' - 2y' = 2y' - 4y$$

Transposing 2y' and -4y to LHS it becomes – 2y' and 4y

$$\Rightarrow y'' - 4y' - 4y = 0$$

∴ the required differential equation is y'' – 4y' – 4y = 0.

5. y = e^{x} (a cos x + b sin x)

Solution:

From the question, it is given that y = e^{x}(a cos x + b sin x)

… [we call it as equation (i)]

Differentiating both sides with respect to x, we get,

$\Rightarrow$y' = e^{x}(a cos x + b sin x) + e^{x}(-a sin x + b cos x)

$\Rightarrow$ y' = e^{x}[(a + b)cos x – (a – b) sin x)] … [equation (ii)]

Now, differentiating equation (ii) both sides,

We have,

y'' = e^{x}[(a + b) cos x – (a – b)sin x)] + e^{x}[-(a + b)sin x – (a – b) cos x)]

On simplifying, we get,

$\Rightarrow$ y'' = e^{x}[2bcosx – 2asinx]

$\Rightarrow$ y'' = 2e^{x}(b cos x – a sin x) … [equation (iii)]

Now, adding equations (i) and (iii), we get,

$$y + \frac{y''}{2} = e^x[(a+b)\cos x - (a-b)\sin x]$$

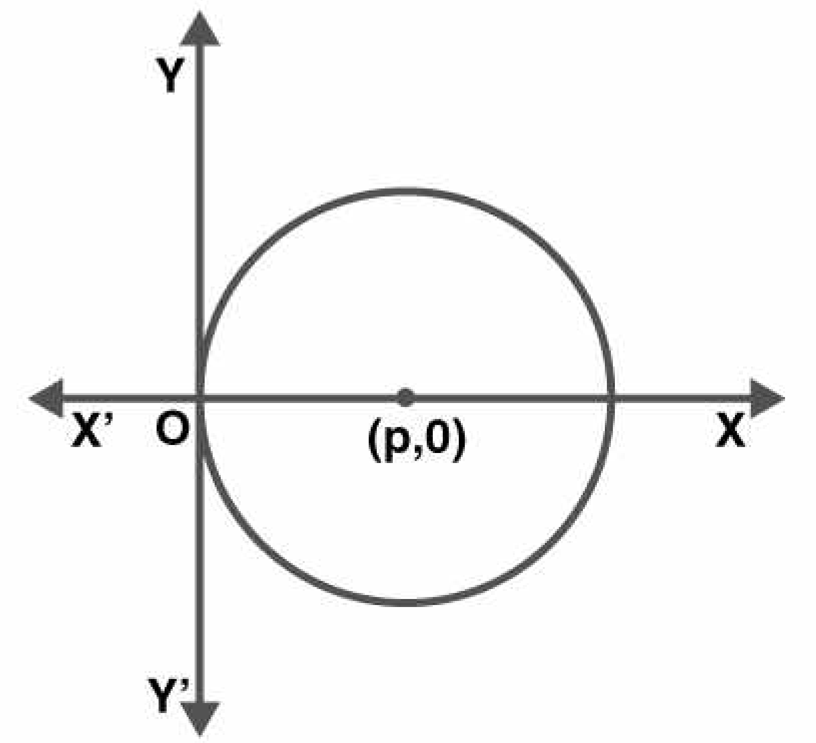
$$y + \frac{y''}{2} = y'$$

$$\Rightarrow 2y + y'' = 2y'$$

Therefore, the required differential equation is $2y + y'' = 2y' = 0$.

6. Form the differential equation of the family of circles touching the y-axis at the origin.

Solution:



By looking at the figure, we can say that the centre of the circle touching the y- axis at the origin lies on the x-axis.

Let us assume (p, 0) is the centre of the circle.

Hence, it touches the y-axis at the origin, and its radius is p.

Now, the equation of the circle with centre (p, 0) and radius (p) is

$\Rightarrow (x - p)^2 + y^2 = p^2$

$\Rightarrow x^2 + p^2 - 2xp + y^2 = p^2$

Transposing p^2 and – 2xp to RHS then it becomes – p^2 and 2xp

$\Rightarrow x^2 + y^2 = p^2 - p^2 + 2px$

$\Rightarrow x^2 + y^2 = 2px$ … [equation (i)]

Now, differentiating equation (i) both sides,

We have,

$\Rightarrow 2x + 2yy' = 2p$

$\Rightarrow x + yy' = p$

Now, on substituting the value of 'p' in the equation, we get,

$\Rightarrow x^2 + y^2 = 2(x + yy')x$

$\Rightarrow 2xyy' + x^2 = y^2$

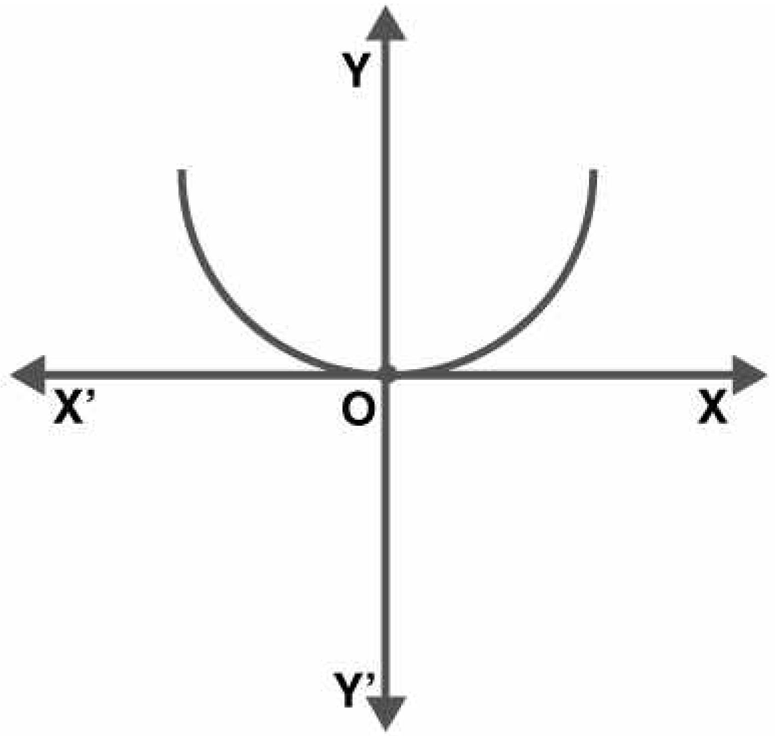
Hence, $2xyy' + x^2 = y^2$ is the required differential equation.

7. Form the differential equation of the family of parabolas having a vertex at origin and axis along positive y-axis.

Solution:

The parabola having the vertex at the origin and the axis along the positive y-axis is

$x^2 = 4ay$ … [equation (i)



Now, differentiating equation (i) both sides with respect to x,

$2x = 4ay'$ ... [equation (ii)]

Dividing equation (ii) by equation (i), we get,

$$\Rightarrow \frac{2x}{x^2} = \frac{4ay'}{4ay}$$

On simplifying, we get,

$$\Rightarrow \frac{2}{x} = \frac{y'}{y}$$

By cross multiplication,

$\Rightarrow xy' = 2y$

Transposing 2y to LHS it becomes -2y.

$\Rightarrow xy' - 2y = 0$

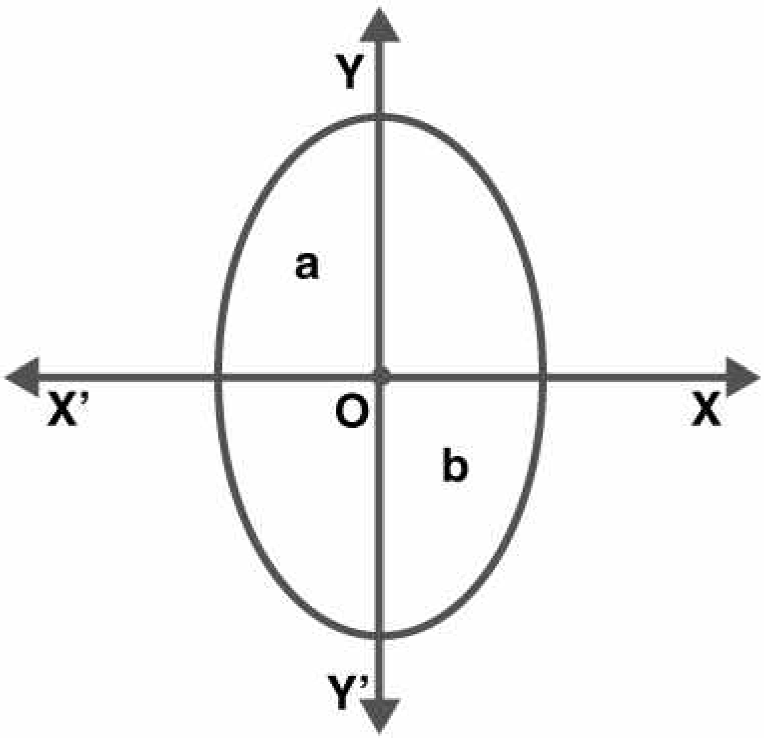
Therefore, the required differential equation is $xy' - 2y = 0$.

8. Form the differential equation of the family of ellipses having foci on the y-axis and centre at the origin.

Solution:

By observing the figure we can say that, the equation of the family of ellipses having foci on y – axis and the centre at origin.

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ ... [equation (i)]



Now, differentiating equation (i) both sides with respect to x,

$$\frac{2x}{b^2}+\frac{2yy'}{a^2}=0$$

$$\Rightarrow \frac{x}{b^2}+\frac{yy'}{a^2}=0 \qquad \text{... [equation (ii)]}$$

Now, again differentiating equation (ii) both sides with respect to x,

$$\frac{1}{b^2}+\frac{y'y'+yy"}{a^2}=0$$

On simplifying,

$$\Rightarrow \frac{1}{b^2}+\frac{1}{a^2}(y'^2+yy")=0$$

$$\Rightarrow \frac{1}{b^2}=-\frac{1}{a^2}(y'^2+yy")$$

Now substitute the Let us substitute the value in equation (ii), we get,

$$x\left[-\frac{1}{a^2}(y'^2+yy")\right]+\frac{yy'}{a^2}=0$$

Now, again differentiating equation (ii) both sides with respect to x,

$$\frac{1}{b^2}+\frac{y'y'+yy''}{a^2}=0$$

On simplifying,

$$\Rightarrow \frac{1}{b^2}+\frac{1}{a^2}(y'^2+yy'')=0$$

$$\Rightarrow \frac{1}{b^2}=-\frac{1}{a^2}(y'^2+yy'')$$

Now substitute the value in equation (ii), we get,

$$x\left[-\frac{1}{a^2}(y'^2+yy'')\right]+\frac{yy'}{a^2}=0$$

On simplifying,

$\Rightarrow -x(y')^2 - xyy'' + yy' = 0$

$\Rightarrow xyy'' + x(y')^2 - yy' = 0$

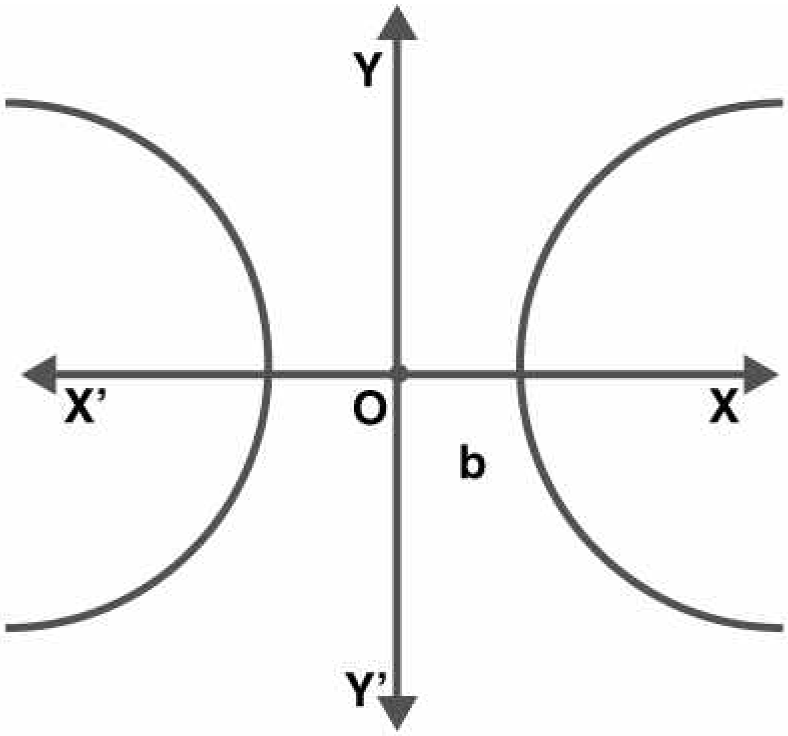
Hence, $xyy'' + x(y')^2 - yy' = 0$ is the required differential equation.

9. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at the origin.

Solution:

By observing the figure we can say that, the equation of the family of hyperbolas having foci on x – axis and the centre at origin is

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... [equation (i)]


Y
X'
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Y'

Now, differentiating equation (i) both sides with respect to x,

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$$

$$\Rightarrow \frac{x}{a^2} - \frac{yy'}{b^2} = 0 \qquad \text{... [equation (ii)]}$$

Now, again differentiating equation (ii) both sides with respect to x,

$$\frac{1}{a^2} - \frac{y'y' + yy''}{b^2} = 0$$

On simplifying,

$$\Rightarrow \frac{1}{a^2} - \frac{1}{b^2}(y'^2 + yy'') = 0$$

$$\Rightarrow \frac{1}{a^2} = \frac{1}{b^2}(y'^2 + yy'')$$

Now substitute the value in equation (ii), we get,

$$\frac{x}{b^2}((y'^2 + yy'') - \frac{yy'}{b^2} = 0$$

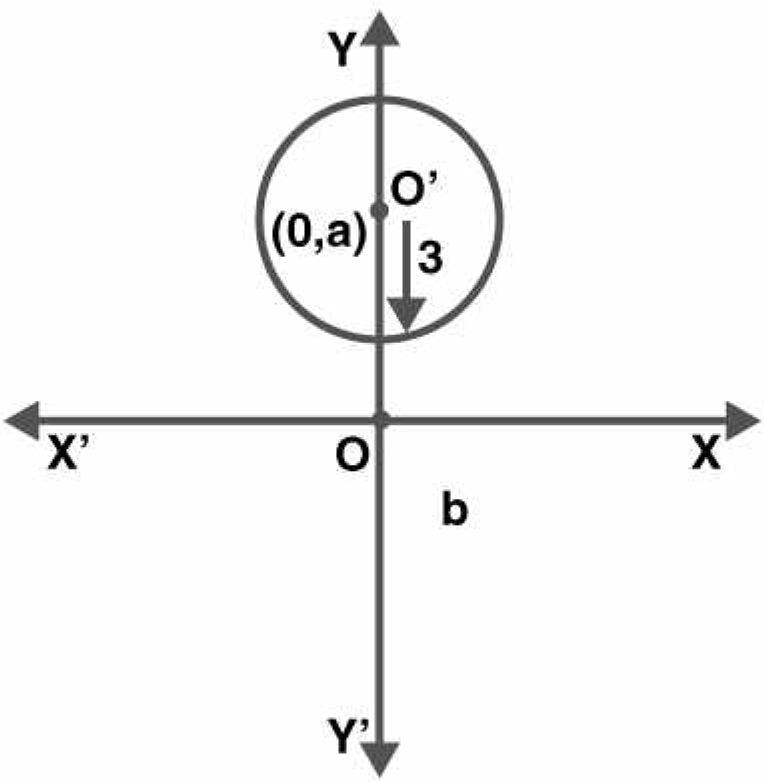
$\Rightarrow x (y')^2 + xyy'' – yy' = 0$

$\Rightarrow xyy'' + x(y')^2 – yy' = 0$

Hence, $xyy'' + x (y')^2 – yy' = 0$is the required differential equation.

10. Form the differential equation of the family of circles having a centre on the y-axis and a radius of 3 units.

Solution:



Let us assume the centre of the circle on the y-axis be (0, a).

We know that the differential equation of the family of circles with centre at (0, a) and radius 3 is: $x^2 + (y- a)^2 = 3^2$

$\Rightarrow x^2 + (y- a)^2 = 9$ … [equation (i)]

Now, differentiating equation (i) both sides with respect to x,

$\Rightarrow 2x + 2(y – a) \times y' = 0$ … [dividing both side by 2]

$\Rightarrow x + (y – a) \times y' = 0$

Transposing x to the RHS, it becomes – x.

$\Rightarrow (y – a) \times y' = x$

$$\Rightarrow y - a = \frac{-x}{y'}$$

Now, substitute the value of (y - a) in equation (i), we get,

$$x^2 + \left(\frac{-x}{y'}\right)^2 = 9$$

Take out the x^2 as common,

$$\Rightarrow x^2\left[1 + \frac{1}{(y')^2}\right] = 9$$

On simplifying,

$$\Rightarrow x^2((y')^2 + 1) = 9(y')^2$$

$$\Rightarrow (x^2 - 9)(y')^2 + x^2 = 0$$

Hence, $(x^2 - 9)\,(y')^2 + x^2 = 0$ is the required differential equation.

11. Which of the following differential equations has $y = c_1 e^x + c^2 e^{-x}$ as the general solution?

(A) $\frac{d^2y}{dx^2} + y = 0$ (B) $\frac{d^2y}{dx^2} - y = 0$ (C) $\frac{d^2y}{dx^2} + 1 = 0$ (D) $\frac{d^2y}{dx^2} - 1 = 0$

Solution:

(B) $\frac{d^2y}{dx^2} - y = 0$

Explanation:

From the question it is given that $y = c_1e^x + c_2e^{-x}$

Now, differentiating given equation both sides with respect to x,

$$\frac{dy}{dx} = c_1e^x - c_2e^{-x} \quad \text{... [equation (i)]}$$

Now, again differentiating equation (i) both sides with respect to x,

$$\frac{d^2y}{dx^2} = c_1e^x + c_2e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y$$

$$\Rightarrow \frac{d^2y}{dx^2} - y = 0$$

Hence, $\frac{d^2y}{dx^2} - y = 0$ is the required differential equation.

From the question it is given that $y = c_1e^x + c_2e^{-x}$

Now, differentiating given equation both sides with respect to x,

$$\frac{dy}{dx} = c_1e^x - c_2e^{-x} \quad \text{... [equation (i)]}$$

Now, again differentiating equation (i) both sides with respect to x,

$$\frac{d^2y}{dx^2} = c_1e^x + c_2e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y$$

$$\Rightarrow \frac{d^2y}{dx^2} - y = 0$$

Hence, $\frac{d^2y}{dx^2} - y = 0$ is the required differential equation.

12. Which of the following differential equations has y = x as one of its particular solution?

(A) $\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = x$ (B) $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + xy = x$

(C) $\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 0$ (D) $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + xy = 0$

Solution:

(C) $\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 0$

Explanation:

From the question it is given that y = x

Now, differentiating given equation both sides with respect to x,

$$\frac{dy}{dx} = 1 \quad \text{... [equation (i)]}$$

Now, again differentiating equation (i) both sides with respect to x,

$$\frac{d^2y}{dx^2} = 0$$

Then,

Substitute the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given options,

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy$$

$= 0 - (x^2 \times 1) + (x \times x)$

$= -x^2 + x^2$

$= 0$

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For each of the differential equations in Exercises 1 to 10, find the general solution:

1. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

Solution:

Given

$\Rightarrow \frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$

We know that $1 - \cos x = 2\sin^2(x/2)$ and $1 + \cos x = 2\cos^2(x/2)$

Using this formula in above function we get

$\Rightarrow \frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$

We have sin x/cos x = tan x using this we get

$\Rightarrow \frac{dy}{dx} = \tan^2\frac{x}{2}$

From the identity $\tan^2 x = \sec^2 x - 1$, the above equation can be written as

$\Rightarrow \frac{dy}{dx} = (\sec^2\frac{x}{2} - 1)$

Now by rearranging and taking integrals on both sides we get

$\Rightarrow \int dy = \int \sec^2\frac{x}{2}\, dx - \int dx$

On integrating we get

$\Rightarrow y = 2\tan\frac{x}{2} - x + c$

2. $\frac{dy}{dx} = \sqrt{4-y^2}\ (-2 < y < 2)$

Solution:

Given

$$\Rightarrow \frac{dy}{dx} = \sqrt{4 - y^2}$$

On rearranging we get

On rearranging we get

$$\Rightarrow \frac{dy}{\sqrt{4 - y^2}} = dx$$

Now taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{\sqrt{4 - y^2}} = \int dx$$

We know that,

$$\Rightarrow \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$$

Then above equation becomes

$$\Rightarrow \sin^{-1} \frac{y}{2} = x + c$$

3. $\dfrac{dy}{dx} + y = 1 \; (y \neq 1)$

Solution:

$\Rightarrow \frac{dy}{dx} + y = 1$

On rearranging we get

$\Rightarrow$ dy = (1 - y) dx

Separating variables by variable separable method we get

$\Rightarrow \frac{dy}{1-y} = dx$

Now by taking integrals on both sides we get

$\Rightarrow \int \frac{dy}{1-y} = \int dx$

On integrating

$\Rightarrow$ - log (1 - y) = x + log c

$\Rightarrow$ - log (1 - y) – log c = x

$\Rightarrow$ log (1 - y) c = -x

$\Rightarrow (1-y)c = e^{-x}$

Above equation can be written as

$\Rightarrow (1-y) = \frac{1}{c}e^{-x}$

$y = 1 + \frac{1}{c}e^{-x}$

Y = 1 + A e^{-x}

$\Rightarrow (1-y) = \frac{1}{c}e^{-x}$

$y = 1 + \frac{1}{c}e^{-x}$

Y = 1 + A e^{-x}

4. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Solution:

Given

$\Rightarrow\ \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$

Dividing both sides by (tan x) (tan y) we get

$$\therefore \frac{\sec^2 x\tan y\, dx}{\tan x\tan y} + \frac{\sec^2 y\tan x\, dy}{\tan x\tan y} = 0$$

On simplification we get

$$\Rightarrow \frac{\sec^2 x\, dx}{\tan x} + \frac{\sec^2 y\, dy}{\tan y} = 0$$

Integrating both sides,

$$\Rightarrow \int \frac{\sec^2 x\ dx}{\tan x} = \int \frac{\sec^2 y\, dy}{\tan y}$$

$\Rightarrow$ let tan x = t &tan y = u

Then

$\sec^2 x\ dx = dt \& \sec^2 y\, dy = du$

By substituting these in above equation we get

$$\therefore \int \frac{dt}{t} = -\int \frac{du}{u}$$

On integrating

$\Rightarrow$ log t = -log u + log c

Or,

$\Rightarrow$ log (tan x) = -log (tan y) + log c

$$\Rightarrow\ \log \tan x = \log \frac{c}{\tan y}$$

$\Rightarrow$ (tan x) (tan y) = c

5. $(e^x + e^{-x})\, dy - (e^x - e^{-x})\, dx = 0$

Solution:

Given

$\Rightarrow (e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$

On rearranging the above equation we get

$$\Rightarrow dy = \frac{(e^x - e^{-x})dx}{e^x + e^{-x}}$$

Taking Integrals both sides,

$$\Rightarrow \int dy = \int \frac{(e^x - e^{-x})dx}{e^x + e^{-x}}$$

Now let $(e^x + e^{-x}) = t$

Then, $(e^x - e^{-x})dx = dt$

$$\therefore y = \int \frac{dt}{t}$$

On integrating

$$\because \int \frac{dx}{x} = \log x$$

So,

$\Rightarrow y = \log t$

Now by substituting the value of t we get

$\Rightarrow y = \log(e^x + e^{-x}) + C$

6. $\frac{dy}{dx} = (1+x^2)(1+y^2)$

Solution:

$$\Rightarrow \frac{dy}{dx} = (1+x^2)(1+y^2)$$

Separating variables by variable separable method,

$$\Rightarrow \frac{dy}{1 + y^2} = dx(1 + x^2)$$

Now taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{1 + y^2} = \int dx + \int x^2 dx$$

On integrating we get

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + c$$

7. $y \log y \, dx - x \, dy = 0$

Solution:

Given

y log y dx – x dy = 0

On rearranging we get

⇒ (y log y) dx = x dy

Separating variables by using variable separable method we get

$$\Rightarrow \frac{dx}{x} = \frac{dy}{ylogy}$$

Now integrals on both sides,

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{ylogy}$$

⇒ let logy = t

Then

$$\Rightarrow \frac{1}{y} dy = dt$$

$$\Rightarrow \log x = \int \frac{dt}{t}$$

⇒ Log x + log c = log t

Now by substituting the value of t

⇒ Log x + log c = log (log y)

Now by using logarithmic formulae we get

⇒ Log c x = log y

⇒ Log y = cx

$$\Rightarrow y = e^{cx}$$

8. $x^5 \frac{dy}{dx} = -y^5$

Solution:

Given

$\Rightarrow x^5 \frac{dy}{dx} = -y^5$

Separating variables by using variable separable method we get

$\Rightarrow \frac{dy}{y^5} = \frac{-dx}{x^5}$

On rearranging

$\Rightarrow \frac{dy}{y^5} + \frac{dx}{x^5} = 0$

Integrating both sides,

$\Rightarrow \int \frac{dy}{y^5} + \int \frac{dx}{x^5} = a$

Let a be a constant,

$\Rightarrow \int y^{-5} dy + \int x^{-5} dx = a$

On integrating we get

$\Rightarrow -4y^{-4} - 4x^{-4} + c = a$

On simplification we get

$\Rightarrow -x^{-4} - y^{-4} = c$

The above equation can be written as

$\Rightarrow \frac{1}{x^4} + \frac{1}{y^4} = c$

9. $\frac{dy}{dx} = \sin^{-1} x$

Solution:

Given

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} x$$

Separating variables by using variable separable method we get

$$\Rightarrow dy = \sin^{-1} x\, dx$$

Taking integrals on both sides,

Separating variables by using variable separable method we get

$$\Rightarrow dy = \sin^{-1} x\, dx$$

Taking integrals on both sides,

$$\Rightarrow \int dy = \int \sin^{-1} x \ dx$$

Now to integrate $\sin^{-1}x$ we have to multiply it by 1

to use product rule

$$\int u.v\, dx = u \int v \ dx - \int \left(\frac{d}{dx}u\right)(\int vdx)dx\}$$

Then we get

$$\Rightarrow y = \int 1. \sin^{-1} x\, dx$$

According to product rule and ILATE rule, the above equation can be written as

$$\therefore y = \{\sin^{-1} x \int 1. dx - \int (\frac{d}{dx}\sin^{-1} x)(\int 1. dx)dx\}$$

On integrating we get

$$\Rightarrow y = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Now

$$\Rightarrow \text{let } 1 - x^2 = t$$

Then

$$\Rightarrow -2x\, dx = dt$$

$$\Rightarrow xdx = -\frac{dt}{2}$$

Substituting these in above equation we get

$$\Rightarrow y = x \sin^{-1} x + \int \frac{1}{2\sqrt{t}} dt$$

On simplification above equation can be written as

$$\Rightarrow y = x\sin^{-1}x + \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$\Rightarrow y = x\sin^{-1}x + \frac{1}{2}\sqrt{t} + c$$

Substituting the value of t, we get

$$\Rightarrow y = x\sin^{-1}x + \sqrt{1-x^2} + c$$

10. $e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$

Solution:

Given

$$\Rightarrow e^x \tan y\, dx + 1(1-e^x)\sec^2 y\, dy = 0$$

On rearranging above equation can be written as

$$\Rightarrow (1-e^x)\sec^2 y\, dy = -e^x \tan y\, dy = 0$$

Separating the variables by using variable separable method,

$$\Rightarrow \frac{\sec^2 y}{\tan y}dy = -\frac{e^x}{1-e^x}dx$$

Now by taking integrals on both sides, we get

$$\Rightarrow \int \frac{\sec^2 y}{\tan y}dy = \int \frac{e^{-x}}{1-e^x}dx$$

Let tan y = t and 1 – e^x = u

Then on differentiating

$$(\sec^2 y\, dy = dt) \,\&\, (e^x dx = du)$$

Substituting these in above equation we get

$$\therefore \int \frac{dt}{t} = \int \frac{du}{u}$$

On integrating we get

$\Rightarrow$ Log t = log u + log c

Substituting the values of t and u on above equation.

$$\Rightarrow \log(\tan y) = \log(1-e^x) + \log c$$

$$\Rightarrow \log \tan y = \log c(1-e^x)$$

By using logarithmic formula above equation can be written as

$$\Rightarrow \tan y = c(1-e^x)$$

For each of the differential equations in Exercises 11 to 14, find a particular solution

Satisfying the given condition:

11. $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x;\ y = 1$ when $x = 0$

Solution:

Given

$\Rightarrow (x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$

Separating variables by using variable separable method,

$$\Rightarrow dy = \frac{2x^2 + x}{(x + 1)(x^2 + 1)}dx$$

Taking integrals on both sides, we get

$$\Rightarrow \int dy = \int \frac{2x^2 + x}{(x + 1)(x^2 + 1)}dx \quad \text{...... 1}$$

Integrating it partially using partial fraction method,

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{Ax^2 + A(Bx + C)(x + 1)}{(x + 1)(x^2 + 1)}$$

$\Rightarrow 2x^2 + x = Ax^2 + A + Bx + Cx + C$

$\Rightarrow 2x^2 + x = (A + B)x^2 + (B + C)x + A + C$

Now comparing the coefficients of x^2 and x

$\Rightarrow A + B = 2$

$\Rightarrow B + C = 1$

$\Rightarrow A + C = 0$

Solving them we will get the values of A, B, C

$$A = \frac{1}{2}, B = \frac{3}{2}, C = -\frac{1}{2}$$

Putting the values of A, B, C in 1 we get

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{1}{2}\frac{1}{(x + 1)} + \frac{1}{2}\frac{3x - 1}{x^2 + 1}$$

Now taking integrals on both sides

$$\Rightarrow \int dy = \frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{3x-1}{x^2+1}dx$$

On integrating

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{2}\int \frac{x}{x^2+1}dx - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\int \frac{2x}{x^2+1}dx - \frac{1}{2}\tan^{-1}x \quad ....2$$

For second term

let $x^2 + 1 = t$

Then, 2x dx = dt

$$\therefore \frac{3}{4}\int \frac{2x}{x^2+1}dx = \frac{3}{4}\int \frac{dt}{t}$$

$$\text{so, } I = \frac{3}{4}\log t$$

Substituting the value of t we get

$$I = \frac{3}{4}\log(x^2+1)$$

Then 2 becomes

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c$$

Taking 4 common

$$\Rightarrow y = \frac{1}{4}[2\log(x+1) + 3\log(x^2+1)] - \frac{1}{2}\tan^{-1}x + c$$

$$\Rightarrow y = \frac{1}{4}[\log(x+1)^2 + \log(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + c$$

$$\Rightarrow y = \frac{1}{4}[\log\{(x+1)^2(x^2+1)^3\}] - \frac{1}{2}\tan^{-1}x + c \quad ....3$$

Now, we are given that y = 1 when x = 0

$$\therefore 1 = \frac{1}{4}[\log\{(0+1)^2(0^2+1)\}] - \frac{1}{2}\tan^{-1}0 + c$$

$$1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + c$$

Therefore,

C = 1

Putting the value of c in 3 we get

$$y = \frac{1}{4}[\log\{(x + 1)^2(x^2 + 1)^3\}] - \frac{1}{2}\tan^{-1}x + 1$$

12. $x(x^2 - 1)\frac{dy}{dx} = 1$; $y = 0$ when $x = 2$

Solution:

Given

$$x(x^2 + 1)\frac{dy}{dx} = 1$$

Separating variables by variable separable method,

$$\Rightarrow dy = \frac{dx}{x(x^2 + 1)}$$

$X^2 + 1$ can be written as $(x + 1)(x – 1)$ we get

$$\Rightarrow dy = \frac{dx}{x(x + 1)(x - 1)}$$

Taking integrals on both sides,

$$\Rightarrow \int dy = \int \frac{dx}{x(x + 1)(x - 1)} \text{.....1}$$

Now by using partial fraction method,

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{c}{x - 1} \text{.....2}$$

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{A(x - 1)(x + 1) + B(x)(x - 1) + C(x)(x + 1)}{x(x + 1)(x - 1)}$$

Or

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{(A + B + C)x^2 + (B - C)x - A}{x(x + 1)(x - 1)}$$

Now comparing the values of A, B, C

A + B + C = 0

B-C = 0

A = -1

Solving these we will get that B = ½ and C = ½

Now putting the values of A, B, C in 2

$$\Rightarrow \frac{1}{x(x+1)(x-1)} = -\frac{1}{x} + \frac{1}{2}\left(\frac{1}{x+1}\right) + \frac{1}{2}\left(\frac{1}{x-1}\right)$$

Now taking integrals we get

$$\Rightarrow \int dy = -\int \frac{1}{x}dx + \frac{1}{2}\int \left(\frac{1}{x+1}\right)dx + \frac{1}{2}\int \left(\frac{1}{x-1}\right)dx$$

On integrating

$$\Rightarrow y = -\log x + \frac{1}{2}\log(x+1) + \frac{1}{2}\log(x-1) + \log c$$

$$\Rightarrow y = \frac{1}{2}\log\left[\frac{c^2(x-1)(x+1)}{x^2}\right\} \quad .....3$$

Now we are given that y = 0 when x = 2

$$0 = \frac{1}{2}\log\left[\frac{c^2(2-1)(2+1)}{4}\right\}$$

$$\Rightarrow \log\frac{3c^2}{4} = 0$$

We know $e^0 = 1$ by substituting we get

$$\Rightarrow \frac{3c^2}{4} = 1$$

$$\Rightarrow 3c^2 = 4$$

$$\Rightarrow c^2 = 4/3$$

Now putting the value of c^2 in 3

Then,

$$y = \frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

13. $\cos\left(\frac{dy}{dx}\right) = a \quad (a \in \mathbf{R});\ y = 1$ when $x = 0$

Solution:

Given

$$\cos\left(\frac{dy}{dx}\right) = a$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$$

dy = $\cos^{-1}$ a dx

Integrating both sides, we get

$$\int dy = \cos^{-1} a \int dx$$

y = x $\cos^{-1}$ a + C ….. 1

Now y = 1 when x = 0

Then

1 = 0 $\cos^{-1}$ a + C

y = x $\cos^{-1}$ a + C ….. 1

Now y = 1 when x = 0

Then

1 = 0 $\cos^{-1}$ a + C

Hence C = 1

Substituting C = 1 in equation (1), we get:

y = x $\cos^{-1}$ a + 1

(y – 1)/x = $\cos^{-1}$ a

$$\Rightarrow \cos\left(\frac{y-1}{x}\right) = a$$

14. $\frac{dy}{dx} = y\tan x\,;\ y = 1$ when $x = 0$

Solution:

Given

$$\frac{dy}{dx} = y \tan x$$

Separating variables by variable separable method,

$$\Rightarrow \frac{dy}{y} = \tan x \, dx$$

Taking Integrals both sides, we get

$$\Rightarrow \int \frac{dy}{y} = \int \tan x \, dx$$

On integrating

$\Rightarrow$ Log y = -log (cos x) + log c

Using standard trigonometric identity we get

$\Rightarrow$ Log y = log (sec x) + log c

Using logarithmic formula in above equation we get

$\Rightarrow$ Log y = log c (sec x)

$\Rightarrow$ y = c (sec x) 1

Now we are given that y = 1 when x = 0

$\Rightarrow$ 1 = c (sec 0)

$\Rightarrow$ 1 = c × 1

$\Rightarrow$ c = 1

Putting the value of c in 1

$\Rightarrow$ y = sec x

$\Rightarrow$ c = 1

Putting the value of c in 1

$\Rightarrow$ y = sec x

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$

Solution:

To find the equation of a curve that passes through point (0, 0) and has differential equation $y' = e^x \sin x$

So, we need to find the general solution of the given differential equation and the put the given point in to find the value of constant.

So, $\Rightarrow \frac{dy}{dx} = e^x \sin x$

Separating variables by variable separable method, we get

$\Rightarrow dy = e^x \sin x \, dx$

Integrating both sides,

$\Rightarrow \int dy = \int e^x \sin x \, dx$ 1

Now by using product rule we get

$\int u.v \, dx = u \int v \, dx - \int \{\frac{d}{dx} u \int v \, dx\} dx$

Now let

$I = \int e^x \sin x \, dx$

$\Rightarrow I = \sin x \int e^x dx - \int (\frac{d}{dx} \sin x \int e^x dx) dx$

$\Rightarrow I = e^x \sin x - \int \cos x \, e^x dx$

Now by integrating we get

$\Rightarrow I = e^x \sin x - [\cos x \int e^x dx + \int \sin x \, e^x dx]$

From 1 we have

$\Rightarrow I = e^x \sin x - e^x \cos x - I$

Now on simplifying

$\Rightarrow 2I = e^x \sin x - e^x \cos x$

$\Rightarrow 2I = e^x(\sin x - \cos x)$

$\Rightarrow I = e^x \dfrac{(\sin x - \cos x)}{2}$

Substituting I in 1 we get

$\Rightarrow y = e^x \dfrac{(\sin x - \cos x)}{2} + c$ 2

Now we are given that the curve passes through point (0, 0)

$\therefore 0 = e^0 \dfrac{(\sin 0 - \cos 0)}{2} + c$

$\Rightarrow 0 = \dfrac{1(0-1)}{2} + c$

$\Rightarrow c = \dfrac{1}{2}$

Substituting the value of C in 2

$\Rightarrow y = e^x \dfrac{(\sin x - \cos x)}{2} + \dfrac{1}{2}$

On rearranging

$\Rightarrow 2y = e^x(\sin x - \cos x) + 1$

Hence

$\Rightarrow 2y - 1 = e^x(\sin x - \cos x)$

16. For the differential equation $xy\dfrac{dy}{dx} = (x+2)(y+2)$

Find the solution curve passing through the point (1, –1).

Solution:

For this question, we need to find the particular solution at point (1,-1) for the given differential equation.
Given differential equation is

$$\Rightarrow\ xy\frac{dy}{dx} = (x + 2)(y + 2)$$

Separating variables by variable separable method, we get

$$\Rightarrow\ \frac{y}{y + 2}dy = \frac{(x + 2)dx}{x}$$

This can also be written as

Taking Integrals both sides, we get

$$\Rightarrow\ \int\left(1 - \frac{2}{y + 2}\right)dy = \int\left(1 + \frac{2}{x}\right)dx$$

Splitting the integrals

$$\Rightarrow\ \int dy - 2\int\frac{1}{y + 2}dy = \int dx + 2\int\frac{1}{x}dx$$

$$\Rightarrow\ y - 2\log(y + 2) = x + 2\log x + c_{...1}$$

Now separating like terms on each side,

$$\Rightarrow\ y - x - c = 2\log x + 2\log(y + 2)$$

$$\Rightarrow\ y - x - c = \log x^2 + \log(y + 2)^2$$

Using logarithmic formula we get

$$\Rightarrow\ y - x - c = \log\{x^2(y + 2)^2\} - i)$$

Now we are given that, the curve passes through (1, -1)
Substituting the values of x and y, to find the value of c

$$\Rightarrow\ -1 - 1 - c = \log\{1(-1 + 2)^2\}$$

⇒ -2-c = log (1)
We know that log 1 0
⇒ c = -2 + 0
So c = -2
Substituting the value of c in 1

$$y - x - c = \log\{x^2(y + 2)^2\}$$

$$y - x + 2 = \log\{x^2(y + 2)^2\}$$

17. Find the equation of a curve passing through the point (0, –2) given that at any point (*x*, *y*) on the curve, the product of the slope of its tangent and *y* coordinate

of the point is equal to the *x* coordinate of the point.

Solution:

We know that slope of a tangent is = $\frac{dy}{dx}$.

So we are given that the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

$$y\frac{dy}{dx} = x$$

Now separating variables by variable separable method,

⇒ y dy = x dx

⇒ y dy = x dx

Taking integrals both sides,

$$\Rightarrow \int ydy = \int xdx$$

On integrating we get

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\Rightarrow y^2 - x^2 = 2c \quad ...1$$

Now the curve passes through (0, -2).

∴ 4-0 = 2c

⇒ c = 2

Putting the value of c in 1 we get

$$\Rightarrow y^2 - x^2 = 4$$

18. At any point (*x*, *y*) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, –3). Find the equation of the curve given that it passes through (–2, 1).

Solution:

We know that (x, y) is the point of contact of curve and its tangent.

Slope (m1) for line joining (x, y) and (-4, -3) is $\frac{y+3}{x+4}$......1

Also we know that slope of tangent of a curve is $\frac{dy}{dx}$.

$\therefore$ slope (m2) of tangent $= \frac{dy}{dx}$ 2

Now, according to the question, we can write as

(m2) = 2(m1)

$\Rightarrow \frac{dy}{dx} = \frac{2(y + 3)}{x + 4}$

Separating variables by variable separable method, we get

$\Rightarrow \frac{dy}{y + 3} = \frac{2dx}{x + 4}$

Taking integrals on both sides,

$\Rightarrow \int \frac{dy}{y + 3} = 2\int \frac{dx}{x + 4}$

On integrating we get

$\Rightarrow \log(y + 3) = 2\log(x + 4) + \log c$

Using logarithmic formula above equation can be written as

$\Rightarrow \int \frac{dy}{y + 3} = 2\int \frac{dx}{x + 4}$

On integrating we get

$\Rightarrow \log(y + 3) = 2\log(x + 4) + \log c$

Using logarithmic formula above equation can be written as

$\Rightarrow \log(y + 3) = \log c(x + 4)^2$

$\Rightarrow y + 3 = c(x + 4)^2$ 3

Now, this equation passes through the point (-2, 1).

$\Rightarrow 1 + 3 = c(-2 + 4)^2$

$\Rightarrow 4 = 4c$

$\Rightarrow c = 1$

Substitute the value of c in 3

$\Rightarrow y + 3 = (x + 4)^2$

19. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after *t* seconds.

Solution:

Let the rate of change of the volume of the balloon be k where k is a constant

$$\therefore \frac{dy}{dt} = k$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \{\text{volume of sphere} = \frac{4}{3}\pi r^3\}$$

On differentiating with respect to r we get

$$\Rightarrow \frac{4}{3}\pi 3r^2\frac{dr}{dt} = k$$

On rearranging

$$\Rightarrow 4\pi r^2 dr = kdt$$

Taking integrals on both sides,

$$\Rightarrow 4\pi\int r^2 dr = k\int dt$$

On integrating we get

$$\Rightarrow \frac{4\pi r^3}{3} = kt + c \quad ....1$$

Now, from the question we have

At t = 0, r = 3:

⇒ 4π × 33 = 3(k × 0 + c)

⇒ 108 π = 3c

⇒ c = 36π

At t = 3, r = 6:

Now, from the question we have

At t = 0, r = 3:

$\Rightarrow 4\pi \times 33 = 3(k \times 0 + c)$

$\Rightarrow 108\pi = 3c$

$\Rightarrow c = 36\pi$

At t = 3, r = 6:

$\Rightarrow 4\pi \times 6^3 = 3(k \times 3 + c)$

$\Rightarrow k = 84\pi$

Substituting the values of k and c in 1

$\Rightarrow 4\pi r^3 = 3(84\pi t + 36\pi)$

$\Rightarrow 4\pi r^3 = 4\pi(63t + 27)$

$\Rightarrow r^3 = 63t + 27$

$\Rightarrow r = \sqrt[3]{63t + 27}$

So the radius of balloon after t seconds is $\sqrt[3]{63t + 27}$

20. In a bank, the principal increases continuously at the rate of *r*% per year. Find the value of *r* if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$).

Solution:

Let t be time, p be principal and r be rate of interest

According the information principal increases at the rate of r% per year.

$$\therefore \frac{dp}{dt} = \left(\frac{r}{100}\right)p$$

Separating variables by variable separable method, we get

$$\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$$

Taking integrals on both sides,

$$\Rightarrow \int \frac{dp}{p} = \frac{r}{100}\int dt$$

On integrating we get

$$\Rightarrow \log p = \frac{rt}{100} + k$$

$$\Rightarrow p = e^{\frac{rt}{100}+k} \quad ....1$$

Given that t = 0, p = 100.

$$\Rightarrow 100 = e^{k} \quad ....2$$

$$\Rightarrow 100 = e^{k} \quad ....2$$

Now, if t = 10, then p = 2 × 100 = 200

So,

$$\Rightarrow 200 = e^{\frac{rt}{10}+k}$$

$$\Rightarrow 200 = e^{\frac{rt}{10}}.e^{k}$$

From 2

$$\Rightarrow 200 = e^{\frac{rt}{10}} \times 100$$

$$\Rightarrow e^{\frac{r}{10}} = 2$$

$$\Rightarrow \frac{r}{10} = \log 2$$

$$\Rightarrow r = 6.93$$

So r is 6.93%.

21. In a bank, the principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank. How much will it be worth after 10 years?

($e^{0.5}$ = 1.648).

Solution:

Let p and t be principal and time respectively.

Given that principal increases continuously at rate of 5% per year.

$$\therefore \frac{dp}{dt} = \left(\frac{5}{100}\right)p$$

Separating variables by variable separable method,

$$\Rightarrow \frac{dp}{p} = \frac{p}{25}$$

Taking integrals on both sides,

$$\Rightarrow \int \frac{dp}{p} = \frac{1}{20}\int dt$$

$$\Rightarrow \log p = e^{\frac{t}{20}+c} \quad ...1$$

When t = 0, p = 1000

$$\Rightarrow 1000 = e^{c}$$

At t = 10

$$\Rightarrow p = e^{\frac{1}{2}+c}$$

The above equation can be written as

$$\Rightarrow p = e^{0.5} \times e^{c}$$

$$\Rightarrow p = 1.648 \times 1000\ (e^{0.5} = 1.648)$$

$$\Rightarrow p = 1648$$

So after 10 years the total amount would be Rs.1648

22. In a culture, the bacteria count is 1,00,000. The number increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

Solution:

Let y be the number of bacteria at any instant t.
Given that the rate of growth of bacteria is proportional to the number present

$$\therefore \frac{dy}{dt} \propto y$$

$$\Rightarrow \frac{dy}{dt} = ky \text{ (k is a constant)}$$

Separating variables by variable separable method we get,

$$\Rightarrow \frac{dy}{dt} = kdt$$

Taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{y} = k\int dt$$

On integrating we get

⇒ Log y = k t + c…1

Let y' be the number of bacteria at t = 0.

⇒ Log y' = c

Substituting the value of c in 1

⇒ log y = k t + log y'

⇒ Log y- log y' = k t

Using logarithmic formula we get

$$\Rightarrow \log\frac{y}{y'} = kt \quad ....2$$

Also, given that number of bacteria increases by 10% in 2 hours.
Therefore,

$$\Rightarrow y = \frac{110}{100}y'$$

$\Rightarrow \frac{y}{y'} = \frac{11}{10}$ 3

Substituting this value in 2, we get

$\Rightarrow k \times 2 = \log\frac{11}{10}$

$\Rightarrow k = \frac{1}{2}\log\frac{11}{10}$

So, 2 becomes

$\Rightarrow \frac{1}{2}\log\frac{11}{10} \times t = \log\frac{y}{y'}$

$\Rightarrow t = \frac{2\log\frac{y}{y'}}{\log\frac{11}{10}}$ ...4

Now, let the time when number of bacteria increase from 100000 to 200000 be t'.

$\Rightarrow y = 2y'$ at $t = t'$

So from 4, we have

$\Rightarrow t' = \frac{2\log\frac{y}{y'}}{\log\frac{11}{10}} = \frac{2\log 2}{\log\frac{11}{10}}$

So bacteria increases from 100000 to 200000 in $\frac{2\log 2}{\log\frac{11}{10}}$ hours.

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

Solution:

(A) $e^x + e^{-y} = C$

Explanation:

We have

$$\Rightarrow \frac{dy}{dx} = e^{x+y}$$

Using laws of exponents we get

$$\Rightarrow \frac{dy}{dx} = e^{x} \times e^{y}$$

Separating variables by variable separable method we get

$$\Rightarrow e^{-y}dy = e^{x}dx$$

Now taking integrals on both sides

$$\Rightarrow \int e^{-y}dy = \int e^{x}dx$$

On integrating

$$\Rightarrow -e^{-y} = e^{x} + c$$

$$\Rightarrow e^{x} + e^{-y} = -c$$

Or,

$$e^{x} + e^{-y} = c$$

So the correct option is A.

EXERCISE 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1. $(x^2 + x y)\, dy = (x^2 + y^2)\, dx$

Solution:

On rearranging the given equation we get

$$\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$$

$$\text{Let } f(x,y) = \frac{x^2+y^2}{x^2+xy}$$

Here, substituting x = k x and y = k y

$$f(kx, ky) = \frac{(kx)^2 + (ky)^2}{(kx)^2 + kx.ky}$$

Taking k^2 common

$$= \frac{k^2}{k^2} \cdot \frac{x^2 + y^2}{x^2 + xy}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$(x^2 + x y)\, dy = (x^2 + y^2)\, dx$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}$$

To solve it we make the substitution.

$y = v x$

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

We have dy/dx, substituting this in above equation

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{x^2 + x.vx}$$

Taking x^2 common

$$v + x\frac{dv}{dx} = \frac{x^2(1 + v^2)}{x^2(1 + v)}$$

On simplification we get

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{1 + v}$$

On rearranging the above equation we get

$$x\frac{dv}{dx} = \frac{1 + v^2}{1 + v} - v = \frac{1 + v^2 - v - v^2}{1 + v}$$

$$x\frac{dv}{dx} = \frac{1 - v}{1 + v}$$

$$\frac{1 + v}{1 - v}dv = \frac{1}{x}dx$$

Taking integrals on both side,

$$\int \frac{1+1-1+v}{1-v}\, dv = \int \frac{1}{x}\, dx$$

$$\int \frac{2-(1-v)}{1-v}\, dv = \log x + c$$

$$\int \left(\frac{2}{1-v} - 1\right) dv = \log x + c$$

$$\frac{2\log(1-v)}{-1} - v = \log x + c$$

$$-2\log(1-v) - v = \log x + c$$

Putting $v = \frac{y}{x}$ $\quad -2\log\left(1-\frac{y}{x}\right) - \frac{y}{x} = \log x + c$

$$2\log\left(1-\frac{y}{x}\right) + \frac{y}{x} = -\log x - c$$

$$\log\left(\frac{x-y}{x}\right)^2 + \log x = -\frac{y}{x} - c$$

$$\frac{(x-y)^2}{x} = e^{\frac{-y}{x} - c}$$

$$\frac{(x-y)^2}{x} = e^{\frac{-y}{x}} . e^{-c}$$

2. $y' = \frac{x+y}{x}$

Solution:

Given

$y' = \frac{x+y}{x}$

The above equation can be written as

$$\frac{dy}{dx} = \frac{x + y}{x}$$

Let $f(x,y) = \frac{x + y}{x}$

Here, putting x = k x and y = k y

$$f(kx, ky) = \frac{kx + ky}{kx}$$

$$= \frac{k}{k} \cdot \frac{x + y}{x}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$y' = \frac{x + y}{x}$$

Then the above equation can be written as

$$\frac{dy}{dx} = \frac{x + y}{x}$$

To solve it we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting the value of v we get

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

On simplification we get

$$v + x\frac{dv}{dx} = 1 + v$$

On rearranging we get

$$x\frac{dv}{dx} = 1$$

$$dv = \frac{1}{x}dx$$

Now taking integrals on both side we get

$$\int dv = \int \frac{1}{x}dx$$

On integrating we get

v = log x + C

Now by substituting the value of v

$$\frac{y}{x} = \log x + C$$

y = x log x + C x

3. (x – y) dy – (x + y) dx = 0

Solution:

Given (x - y) dy = (x + y) dx

On rearranging above equation we can write as

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

Let $f(x, y) = \frac{x + y}{x - y}$

Now by substituting x = k x and y = k y

$$f(kx, ky) = \frac{kx + ky}{kx - ky}$$

On simplification we get

$$f(kx, ky) = \frac{x + y}{x - y}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

(x - y) dy – (x + y) dx = 0

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

For further simplification we make the substitution.

y = v x

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting the value of dv/dx we get

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx}$$

Taking x as common we get

$$v + x\frac{dv}{dx} = \frac{1 + v}{1 - v}$$

On rearranging

$$x\frac{dv}{dx} = \frac{1 + v}{1 - v} - v$$

Now taking LCM and computing we get

$$x\frac{dv}{dx} = \frac{1 + v - v + v^2}{1 - v}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{1 - v}$$

$$\frac{1 - v}{1 + v^2}dv = \frac{1}{x}dx$$

Taking integrals on both sides we get,

$$\int \frac{1 - v}{1 + v^2}dv = \int \frac{1}{x}dx$$

Now by splitting the integrals we get

$$\int \frac{1}{1 + v^2}dv - \int \frac{v}{1 + v^2}dv = \int \frac{1}{x}dx \quad .....1$$

$$\text{Let, } I_1 = \int \frac{v}{1 + v^2}dv$$

Put $1 + v^2 = t$

2v dv = dt

$$vdv = \frac{1}{2}dt$$

Now by applying integral we get

$$\frac{1}{2}\int \frac{1}{t}dt$$

$\frac{1}{2}\log t$

Now by substituting the value of t we get

$\frac{1}{2}\log(1 + v^2)$

From equation 1 we have

$\therefore \tan^{-1}v - \frac{1}{2}\log(1 + v^2) = \log x + C$

Now by substituting the value of v we get

$\tan^{-1}\frac{y}{x} - \frac{1}{2}\log(1 + \left(\frac{y}{x}\right)^2) = \log x + C$

On rearranging we get

$\tan^{-1}\frac{y}{x} = \log x + \frac{1}{2}\log\left(\frac{x^2 + y^2}{x^2}\right) + C$

$\tan^{-1}\frac{y}{x} = \frac{1}{2}\left(2\log x + \log\left(\frac{x^2 + y^2}{x^2}\right)\right) + C$

Using logarithmic formula we get

$\tan^{-1}\frac{y}{x} = \frac{1}{2}\left(\log\left(\frac{x^2 + y^2}{x^2} \times x^2\right)\right) + C$

$\tan^{-1}\frac{y}{x} = \frac{1}{2}(\log x^2 + y^2) + C$

4. $(x^2 - y^2)dx + 2xy\,dy = 0$

Solution:

The given equation can be written as

$$2xy\,dy = -(x^2 - y^2)dx$$

On rearranging we get

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$$

$$\text{Let } f(x, y) = -\frac{x^2 - y^2}{2xy}$$

Here, substituting x = k x and y = k y

$$f(kx, ky) = -\frac{k^2x^2 - k^2y^2}{2k^2xy}$$

Now by taking k^2 common

$$f(kx, ky) = -\frac{k^2}{k^2}.\frac{x^2 - y^2}{2xy}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$(x^2 - y^2)dx + 2xy\,dy = 0$$

Again on rearranging

$$2xy\,dy = -(x^2 - y^2)dx$$

The above equation can be written as

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$$

To solve above equation and for further simplification we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = -\frac{x^2 - v^2x^2}{2x.\,vx}$$

Now taking x^2 as common

$$v + x\frac{dv}{dx} = -\frac{x^2(1 - v^2)}{2vx^2}$$

On rearranging

$$x\frac{dv}{dx} = -\frac{1 - v^2}{2v} - v$$

Now taking LCM and computing

$$x\frac{dv}{dx} = \frac{-1 + v^2 - 2v^2}{2v}$$

On simplification

$$x\frac{dv}{dx} = \frac{-1 - v^2}{2v}$$

Rearranging the above equation we get

$$-\frac{2v}{1 + v^2}dv = \frac{1}{x}dx$$

Now by multiplying the above equation by negative sign we get

$$\frac{2v}{1 + v^2}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{2v}{1 + v^2}dv = -\int \frac{1}{x}dx \quad .....1$$

$$\text{Let}, I_1 = \int \frac{2v}{1 + v^2}dv$$

Put $1 + v^2 = t$

$2v\ dv = dt$

$$vdv = \frac{1}{2}dt$$

Taking integral we get

$$\int \frac{1}{t}dt$$

Log t

From 1 we have

$\therefore \log(1 + v^2) = -\log x + \log C$

Now by substituting the value of v we get

$$\log\left(1 + \left(\frac{y}{x}\right)^2\right) = -\log x + \log C$$

By using logarithmic formula we get

$$\log\left(\frac{x^2 + y^2}{x^2}\right) = \log\frac{C}{x}$$

On simplification

$x^2 + y^2 = Cx$

5. $x^2\dfrac{dy}{dx} = x^2 - 2y^2 + xy$

Solution:

The given question can be written as

$$\frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$$

Let $f(x,y) = \dfrac{x^2 - 2y^2 + xy}{x^2}$

Now by substituting x = k x and y = k y

$$f(kx, ky) = \frac{k^2x^2 - 2k^2y^2 + kxky}{k^2x^2}$$

Now by taking k^2 common we get

$$f(kx, ky) = \frac{k^2}{k^2} \cdot \frac{x^2 - 2y^2 + xy}{x^2}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$$

On rearranging we get

$$\frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$$

To solve above equation and to make simplification easier we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{x^2 - 2v^2x^2 + x.vx}{x^2}$$

On rearranging we get

$$v + x\frac{dv}{dx} = \frac{1 - 2v^2 + v}{1}$$

$$v + x\frac{dv}{dx} = 1 - 2v^2 + v$$

On simplification

$$x\frac{dv}{dx} = 1 - 2v^2$$

By separating the variables using variable separable method,

By separating the variables using variable separable method,

$$\frac{1}{1 - 2v^2}dv = \frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1}{1 - 2v^2}dv = \int \frac{1}{x}dx$$

The above equation can be written as

$$\int \frac{1}{1 - (\sqrt{2}v)^2}dv = \int \frac{1}{x}dx$$

$$\int \frac{1}{1^2 - (\sqrt{2}v)^2}dv = \int \frac{1}{x}dx$$

On integrating using standard trigonometric identity we get

$$\frac{1}{\sqrt{2}}.\frac{1}{2.1}.\log\left|\frac{1 + \sqrt{2}v}{1 - \sqrt{2}v}\right| = \log|x| + C$$

Now by substituting the value of v we get

$$\frac{1}{2\sqrt{2}}\log\left|\frac{1 + \sqrt{2}\frac{y}{x}}{1 - \sqrt{2}\frac{y}{x}}\right| = \log|x| + C$$

On simplification

$$\frac{1}{2\sqrt{2}}\log\left|\frac{x + \sqrt{2}y}{x - \sqrt{2}y}\right| = \log|x| + C$$

6. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$

Solution:

The given question can be written as

$xdy = (\sqrt{x^2+y^2} + y)dx$

On rearranging the above equation we get

$$\frac{dy}{dx} = \frac{(\sqrt{x^2+y^2} + y)}{x}$$

Let $f(x,y) = \frac{(\sqrt{x^2+y^2} + y)}{x}$

Here, putting x = k x and y = k y

$$f(kx, ky) = \frac{(\sqrt{k^2x^2+k^2y^2} + ky)}{kx}$$

Now taking k as common

$$f(kx, ky) = \frac{(\sqrt{k^2x^2 + k^2y^2} + ky)}{kx}$$

Now taking k as common

$$f(kx, ky) = \frac{k}{k} \cdot \frac{(\sqrt{x^2 + y^2} + y)}{x}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$xdy - ydx = \sqrt{x^2 + y^2}dx$$

By separating the variables using variable separable method we get

$$xdy = (\sqrt{x^2 + y^2} + y)dx$$

On rearranging we get

$$\frac{dy}{dx} = \frac{(\sqrt{x^2 + y^2} + y)}{x}$$

To solve above equation we make the substitution.

$y = v x$

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2 + x^2v^2} + vx}{x}$$

Taking x as common and computing we get

$$v + x\frac{dv}{dx} = \frac{x\sqrt{1 + v^2} + vx}{x}$$

On simplification

$$v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

Again separating variables we get

$$\frac{1}{\sqrt{1+v^2}}dv = \frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1}{\sqrt{1+v^2}}dv = \int \frac{1}{x}dx$$

Using $\int \frac{1}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2+a^2}$, the above equation can be written as

$$\int \frac{1}{\sqrt{1+v^2}}dv = \int \frac{1}{x}dx$$

Using $\int \frac{1}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2+a^2}$, the above equation can be written as

$$\log(v + \sqrt{1 + v^2}) = \log x + \log C$$

Now by using logarithmic formula we get

$$\log\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) = \log Cx$$

On simplifying we get

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = Cx$$

Taking LCM

$$\frac{y}{x} + \sqrt{\frac{x^2 + y^2}{x^2}} = Cx$$

$$\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = Cx$$

On rearranging

$$y + \sqrt{x^2+y^2} = Cx^2$$

7. $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y\,dx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x\,dy$

Solution:

The given question can be written as

$$\frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\text{Let } f(x,y)=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

Now by substituting x = kx and y = ky

Now by substituting x = kx and y = ky

$$f(kx, ky) = \frac{\left\{kx\cos\left(\frac{ky}{kx}\right) + ky\sin\left(\frac{ky}{kx}\right)\right\} ky}{\left\{ky\sin\left(\frac{ky}{kx}\right) - kx\cos\left(\frac{ky}{kx}\right)\right\} kx}$$

Now by taking k^2 as common we get

$$f(kx, ky) = \frac{k^2}{k^2} \cdot \frac{\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y}{\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x}$$

$= k^0.f(x,y)$

Therefore, the given differential equation is homogeneous.

$$\frac{dy}{dx} = \frac{\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\} y}{\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x}$$

To solve above equation we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now by substituting dy/dx value and on rearranging we get

$$v + x\frac{dv}{dx} = \frac{\{x\cos(v) + vx\sin(v)\}vx}{\{vx\sin(v) - x\cos(v)\}x}$$

Taking x as common and simplifying we get

$$v + x\frac{dv}{dx} = \frac{\{\cos(v) + v\sin(v)\}v}{\{v\sin(v) - \cos(v)\}}$$

On rearranging and computing we get

$$x\frac{dv}{dx} = \frac{\{\cos(v) + v\sin(v)\}v}{\{v\sin(v) - \cos(v)\}} - v$$

Taking LCM and simplifying we get

$$x\frac{dv}{dx} = \frac{v\cos(v) + v^2\sin(v) - v^2\sin(v) + v\cos(v)}{v\sin(v) - \cos(v)}$$

$$x\frac{dv}{dx} = \frac{2v\cos(v)}{v\sin(v) - \cos(v)}$$

Separating the variables by using variable separable method we get

$$\frac{v\sin(v) - \cos v}{2v\cos v}dv = \frac{1}{x}dx$$

Now by splitting the numerator we get

$$\frac{v\sin v}{2v\cos v}dv - \frac{\cos v}{2v\cos v}dv = \frac{1}{x}dx$$

$$\frac{v\sin(v) - \cos v}{2v\cos v}dv = \frac{1}{x}dx$$

Now by splitting the numerator we get

$$\frac{v\sin v}{2v\cos v}dv - \frac{\cos v}{2v\cos v}dv = \frac{1}{x}dx$$

On simplification we get

$$\frac{1}{2}\tan v dv - \frac{1}{2}.\frac{1}{v}dv = \frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\frac{1}{2}\int \tan v dv - \frac{1}{2}.\int \frac{1}{v}dv = \int \frac{1}{x}dx$$

On integrating we get

$$\frac{1}{2}\log \sec v - \frac{1}{2}\log v = \log x + \log k$$

Using logarithmic formula we get

log secv – logv = 2logkx

Now by substituting the value of v we get

$$\log \sec\left(\frac{y}{x}\right) - \log\left(\frac{y}{x}\right) = 2\log kx$$

Ahain using logarithmic formula we gte

$$\log\left(\frac{x}{y}\sec\left(\frac{y}{x}\right)\right) = \log(kx)^2$$

On simplification

$$\frac{x}{y}\sec\left(\frac{y}{x}\right) = k^2x^2$$

We know that sec x = 1/cos x, by using this in above equation we get

$$\frac{1}{xy\cos\left(\frac{y}{x}\right)} = k^2$$

On rearranging

$$xy\cos\left(\frac{y}{x}\right) = \frac{1}{k^2}$$

Where C is integral constant

$$C = \frac{1}{k^2}$$

$$xy\cos\left(\frac{y}{x}\right) = C$$

8. $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

Solution:

The given question can be written as

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

On rearranging we get

$$\frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

Let $f(x,y) = \dfrac{y - x\sin\left(\frac{y}{x}\right)}{x}$

Now put x= k x and y = k y

$$f(kx, ky) = \frac{ky - kx\sin\left(\frac{ky}{kx}\right)}{kx}$$

By taking k as common we get

$$f(kx, ky) = \frac{k}{k}.\frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

On rearranging the above equation

$$\frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

To solve above equation we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{vx - x\sin\left(\frac{vx}{x}\right)}{x}$$

On simplification we get

$$v + x\frac{dv}{dx} = v - \sin v$$

$$x\frac{dv}{dx} = -\sin v$$

Now separating variables by variable separable method we get

$$\frac{1}{\sin v}dv = -\frac{1}{x}dx$$

We know that 1/sin x = cosec x then above equation becomes

$$\text{cosec}\,v\,dv = -\frac{1}{x}dx$$

Taking integration on both side, we get

$$\int \text{cosec}\,v\,dv = -\int \frac{1}{x}dx$$

On integrating we get

Log (cosec v – cot v) = -log x + log C

Now by substituting the value of v we get

$$\log(\text{cosec}\frac{y}{x} - \cot\frac{y}{x}) = \log\frac{C}{x}$$

On simplifying we get

$$\text{cosec}\frac{y}{x} - \cot\frac{y}{x} = \frac{C}{x}$$

We know that 1/sin x = cosec x and cot x = cos x/ sin x then above equation becomes

$$\frac{1}{\sin\frac{y}{x}} - \frac{\cos\frac{y}{x}}{\sin\frac{y}{x}} = \frac{C}{x}$$

On rearranging we get

$$1 - \cos\frac{y}{x} = \frac{C}{x}.\sin\frac{y}{x}$$

$$x(1 - \cos\frac{y}{x}) = C\sin\frac{y}{x}$$

9. $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$

Solution:

Given

$$y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$$

The given equation can be written as

$$x\log\left(\frac{y}{x}\right)dy - 2x\,dy = -y\,dx$$

Taking dy common

$$\left(x\log\left(\frac{y}{x}\right)dy - 2x\right)dy = -y\,dx$$

On rearranging we get

$$\frac{dy}{dx} = \frac{-y}{x\log\left(\frac{y}{x}\right)dy - 2x}$$

$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

$$\text{Let } f(x,y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

Now put x = k x and y = k y

$$f(kx, ky) = \frac{ky}{2kx - kx\log\left(\frac{ky}{kx}\right)}$$

Taking k as common

$$f(kx, ky) = \frac{k}{k}.\frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$\text{ydx} + \text{xlog}\left(\frac{y}{x}\right)\text{dy} - 2\text{xdy} = 0$$

$$\text{xlog}\left(\frac{y}{x}\right)\text{dy} - 2\text{xdy} = -\text{ydx}$$

On rearranging

$$\frac{dy}{dx} = \frac{-y}{\text{xlog}\left(\frac{y}{x}\right)\text{dy} - 2x}$$

Simplifying we get

$$\frac{dy}{dx} = \frac{y}{2x - \text{xlog}\left(\frac{y}{x}\right)}$$

To solve it we make the substitution.

y = v x

$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

To solve it we make the substitution.

y = v x

Differentiating equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting dy/dx value we get

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)}$$

On simplification

$$v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$$

$$x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

Taking LCM and simplifying we get

$$x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$

$$x\frac{dv}{dx} = \frac{-v + v\log v}{2 - \log v}$$

By separating the variables using variable separable method we get

$$\frac{2 - \log v}{-v + v\log v}dv = \frac{1}{x}dx$$

$$\frac{2 - \log v}{v(\log v - 1)}dv = \frac{1}{x}dx$$

On simplifying we get

$$\frac{1-(\log v-1)}{v(\log v-1)}dv = \frac{1}{x}dx$$

$$\frac{1}{v(\log v-1)}dv - \frac{1}{v}dv = \frac{1}{x}dx$$

Integrating both sides, we get $\int \frac{1}{v(\log v-1)}dv - \int \frac{1}{v}dv = \int \frac{1}{x}dx$ ...1

Let, $I_1 = \int \frac{1}{v(\log v-1)}dv$

Put, log v – 1 = t

$$\frac{1}{v}dv = dt$$

Let, $I_1 = \int \frac{1}{v(\log v - 1)} dv$

Put, log v – 1 = t

$\frac{1}{v} dv = dt$

On integrating

$\int \frac{1}{t} dt$

Log t

Substituting the value of t

Log (log v - 1)

From equation 1 we have

∴ Log (log v - 1) – log (v) = log (x) + log (c)

By using logarithmic formula we get

$\log\left(\frac{\log v - 1}{v}\right) = \log(Cx)$

$\frac{\log v - 1}{v} = Cx$

On simplification we get

$\frac{\log\left(\frac{y}{x}\right) - 1}{\frac{y}{x}} = Cx$

$\frac{x}{y}\left(\log\left(\frac{y}{x}\right) - 1\right) = Cx$

$\log\left(\frac{y}{x}\right) - 1 = Cy$

10. $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Solution:

Given question can be written as

$\frac{dy}{dx} = \frac{-e^{x/y}\left(1 - \frac{x}{y}\right)}{(1 + e^{x/y})}$

Let $f(x,y) = \dfrac{-e^{x/y}\left(1-\frac{x}{y}\right)}{(1+e^{x/y})}$

Now put x = k x and y = k y

$$f(kx,ky) = \frac{-e^{kx/ky}\left(1-\frac{kx}{ky}\right)}{(1+e^{kx/ky})}$$

$$= \frac{-e^{x/y}\left(1-\frac{x}{y}\right)}{(1+e^{x/y})}$$

$= k^0 f(x, y)$

Therefore, the given differential equation is homogeneous.

$$\left(1+e^{x/y}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$$

On rearranging

$$\left(1+e^{x/y}\right)dx = -e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy$$

$$\frac{dx}{dy} = \frac{-e^{x/y}\left(1-\frac{x}{y}\right)}{(1+e^{x/y})}$$

To solve above equation we make the substitution.

x = v y

Differentiation above equation with respect to x, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

On rearranging and substituting for dy/dx value we get

$$v + y\frac{dv}{dy} = \frac{-e^{vy/y}\left(1 - \frac{vy}{y}\right)}{(1 + e^{vy/y})}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^{v} + ve^{v}}{1+e^{v}} - v$$

Now taking LCM and simplifying we get

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^{v} + ve^{v} - v - ve^{v}}{1+e^{v}}$$

The above equation can be written as

$$\Rightarrow y\frac{dv}{dy} = -\left[\frac{v+e^{v}}{1+e^{v}}\right]$$

$$\Rightarrow \left[\frac{1+e^{v}}{v+e^{v}}\right]dv = -\frac{dy}{y}$$

Integrating both sides we get

$$\Rightarrow \log\left(v+e^{v}\right) = -\log y + \log C = \log\left(\frac{C}{y}\right)$$

Using logarithmic formula the above equation can be written as

$$\Rightarrow \left[\frac{x}{y} + e^{\frac{x}{y}}\right] = \frac{C}{y}$$

$$\Rightarrow x + ye^{\frac{x}{y}} = C$$

$$\Rightarrow y\frac{dv}{dy} = -\left[\frac{v+e^v}{1+e^v}\right]$$

$$\Rightarrow \left[\frac{1+e^v}{v+e^v}\right]dv = -\frac{dy}{y}$$

Integrating both sides we get

$$\Rightarrow \log\left(v+e^v\right) = -\log y + \log C = \log\left(\frac{C}{y}\right)$$

Using logarithmic formula the above equation can be written as

$$\Rightarrow \left[\frac{x}{y}+e^{\frac{x}{y}}\right] = \frac{C}{y}$$

$$\Rightarrow x + ye^{\frac{x}{y}} = C$$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. (x + y) dy + (x – y) dx = 0; y = 1 when x = 1

Solution:

Given

(x + y) dy + (x - y) dx = 0

The above equation can be written as

$$\frac{dy}{dx} = -\frac{(x-y)}{(x+y)}$$

$$\text{Let } f(x,y) = -\frac{(x-y)}{(x+y)}$$

Now put x= k x and y = k y

$$f(kx, ky) = -\frac{(kx-ky)}{(kx+ky)}$$

By taking k common from both numerator and denominator we get

$$= \frac{k}{k}.-\frac{(x-y)}{(x+y)}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

(x + y) dy + (x - y) dx = 0

Again above equation can be written as

$$\frac{dy}{dx} = -\frac{(x-y)}{(x+y)}$$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = -\frac{(x-vx)}{(x+vx)}$$

Taking x common and simplifying we get

$$v + x\frac{dv}{dx} = -\frac{(1-v)}{(1+v)}$$

On rearranging

$$x\frac{dv}{dx} = -\frac{(1-v)}{(1+v)} - v$$

Taking LCM and simplifying

$$x\frac{dv}{dx} = \frac{-1+v-v-v^2}{(1+v)}$$

$$x\frac{dv}{dx} = \frac{-1-v^2}{(1+v)}$$

$$x\frac{dv}{dx} = \frac{-(1+v^2)}{(1+v)}$$

Then above equation can be written as

$$\frac{1+v}{1+v^2}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1+v}{1+v^2}dv = -\int \frac{1}{x}dx$$

Splitting the denominator,

$$\int \frac{1}{1+v^2}dv + \int \frac{v}{1+v^2}dv = -\int \frac{1}{x}dx$$

On integrating we get

$$\tan^{-1}v + \frac{1}{2}\log(1+v^2) = -\log x + C$$

Now by substituting the value of v we get

On integrating we get

$$\tan^{-1}v + \frac{1}{2}\log(1 + v^2) = -\log x + C$$

Now by substituting the value of v we get

$$\tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left(1 + \left(\frac{y}{x}\right)^2\right) = -\log x + C$$

y = 1 when x = 1

$$\tan^{-1}\frac{1}{1} + \frac{1}{2}\log\left(1 + \left(\frac{1}{1}\right)^2\right) = -\log 1 + C$$

The above equation becomes,

$$\frac{\pi}{4} + \frac{1}{2}\log 2 = 0 + C$$

$$C = \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$\therefore \tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left(1 + \left(\frac{y}{x}\right)^2\right) = -\log x + C$$

where, $C = \frac{\pi}{4} + \frac{1}{2}\log 2$

$$\therefore \tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left(1 + \left(\frac{y}{x}\right)^2\right)$$

$$= -\log x + \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$2\tan^{-1}\frac{y}{x} + \log\left(\frac{x^2+y^2}{x^2}\right)$$

$$= -2\log x + \frac{\pi}{2} + \log 2$$

On simplifying we get

$$2\tan^{-1}\frac{y}{x} + \log\left(\frac{x^2+y^2}{x^2}\right) + \log x^2 = \frac{\pi}{2} + \log 2$$

$$2\tan^{-1}\frac{y}{x} + \log(x^2+y^2) = \frac{\pi}{2} + \log 2$$

The required solution of the differential equation.

12. $x^2dy + (xy + y^2)dx = 0$; y = 1 when x = 1

Solution:

Given

$x^2dy + (x\,y + y^2)\,dx = 0$

On rearranging we get

$$\frac{dy}{dx} = -\frac{(xy + y^2)}{x^2}$$

Let $f(x,y) = -\frac{(xy + y^2)}{x^2}$

Now put x = k x and y = k y

$$f(kx, ky) = -\frac{(kxky + k^2y^2)}{k^2x^2}$$

Taking k^2 common we get

$$= \frac{k^2}{k^2} . -\frac{(xy + y^2)}{x^2}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$x^2dy + (x\,y + y^2)\,dx = 0$

Above equation can be written as

$$\frac{dy}{dx} = -\frac{(xy + y^2)}{x^2}$$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting dy/dx value we get

$$v + x\frac{dv}{dx} = -\frac{(x.vx + v^2x^2)}{x^2}$$

$$v + x\frac{dv}{dx} = -\frac{(vx^2 + v^2x^2)}{x^2}$$

On computing and simplifying

$$v + x\frac{dv}{dx} = -v - v^2$$

$$x\frac{dv}{dx} = -v - v^2 - v$$

$$x\frac{dv}{dx} = -v(v + 2)$$

$$\frac{1}{v(v+2)}dv = -\frac{1}{x}dx$$

$$x\frac{dv}{dx} = -v(v + 2)$$

$$\frac{1}{v(v+2)}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1}{v(v+2)}dv = -\int \frac{1}{x}dx$$

Dividing and multiplying above equation by 2 we get

$$\frac{1}{2}\int \frac{2}{v(v+2)}dv = -\int \frac{1}{x}dx$$

Adding and subtracting v to the numerator we get

$$\frac{1}{2}\int \frac{2+v-v}{v(v+2)}dv = -\int \frac{1}{x}dx$$

Now splitting the denominator we get

$$\frac{1}{2}\int \left(\frac{2+v}{v(v+2)} - \frac{v}{v(v+2)}\right)dv = -\int \frac{1}{x}dx$$

$$\frac{1}{2}\int \left(\frac{1}{v} - \frac{1}{v+2}\right)dv = -\int \frac{1}{x}dx$$

On integrating we get

$$\frac{1}{2}(\log v - \log(v + 2)) = -\log x + \log C$$

Using logarithmic formula,

$$\frac{1}{2}\left(\log\frac{v}{v+2}\right) = \log\frac{C}{x}$$

$$\log\left(\frac{\frac{y}{x}}{\frac{y}{x}+2}\right) = 2\log\frac{C}{x}$$

$$\log\left(\frac{y}{y+2x}\right) = \log\left(\frac{c}{x}\right)^2$$

On simplification we get

$$\frac{y}{y+2x} = \left(\frac{c}{x}\right)^2$$

$$\frac{x^2y}{y+2x} = c^2$$

y = 1 when x = 1

$$c^2 = \frac{1}{1+2} = \frac{1}{3}$$

$$\therefore \frac{x^2y}{y+2x} = \frac{1}{3}$$

$$c^2 = \frac{1}{1+2} = \frac{1}{3}$$

$$\therefore \frac{x^2y}{y+2x} = \frac{1}{3}$$

$3x^2y = y + 2x$

$y + 2x = 3x^2y$

The required solution of the differential equation.

13. $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0;\ y = \frac{\pi}{4}$ when $x = 1$

Solution:

Given

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx = -xdy$$

The above equation can be written as

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right] = -x\frac{dy}{dx}$$

On rearranging

$$\frac{dy}{dx} = -\frac{\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}$$

We know f(x, y) = dy/dx using this in above equation we get

$$f(x,y) = -\frac{\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}$$

Now put x = k x and y = k y

$$f(kx, ky) = -\frac{\left[kx\sin^2\left(\frac{ky}{kx}\right) - ky\right]}{kx}$$

Taking k as common

$$= \frac{k}{k}.-\frac{\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$$

On rearranging

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx = -xdy$$

On rearranging

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx = -xdy$$

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right] = -x\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}$$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = -\frac{\left[x\sin^2\left(\frac{vx}{x}\right) - vx\right]}{x}$$

$$v + x\frac{dv}{dx} = -\left[\frac{x\sin^2 v - vx}{x}\right]$$

$$v + x\frac{dv}{dx} = -\sin^2 v - v$$

On computing and simplifying we get

$$x\frac{dv}{dx} = -[\sin^2 v - v] - v$$

$$x\frac{dv}{dx} = -\sin^2 v + v - v$$

$$x\frac{dv}{dx} = -\sin^2 v$$

$$\frac{1}{\sin^2 v}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \frac{1}{\sin^2 v}dv = -\int \frac{1}{x}dx$$

$\int \text{cosec}^2 v dv$ = - log x – log C

On integrating we get

-cot v = - log x – log C

Cot v = log x + log C

Substituting the value of v we get

$$\cot\frac{y}{x} = \log(Cx)$$

Substituting the value of v we get

$$\cot\frac{y}{x} = \log(Cx)$$

$$y = \frac{\pi}{4} \text{ when } x = 1$$

$$\cot\frac{\pi/4}{1} = \log(C.1)$$

$$\cot\frac{\pi}{4} = \log C$$

1 = C

e^1 = C

$$\therefore \cot\frac{y}{x} = \log(ex)$$

The required solution of the differential equation.

14. $\dfrac{dy}{dx} - \dfrac{y}{x} + \text{cosec}\left(\dfrac{y}{x}\right) = 0$; $y = 0$ when $x = 1$

Solution:

Given

$\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0$

On rearranging we get

$\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\left(\frac{y}{x}\right)$

Let $f(x, y) = \frac{y}{x} - \text{cosec}\left(\frac{y}{x}\right)$

Now put x= k x and y = k y

$f(kx, ky) = \frac{ky}{kx} - \text{cosec}\left(\frac{ky}{kx}\right)$

$= \frac{y}{x} - \text{cosec}\left(\frac{y}{x}\right)$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0$

$\frac{dy}{dx} = \frac{y}{x} - \text{cosec}\left(\frac{y}{x}\right)$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \operatorname{cosec}\left(\frac{vx}{x}\right)$$

On simplification

$$v + x\frac{dv}{dx} = v - \operatorname{cosec}v$$

$$x\frac{dv}{dx} = -\operatorname{cosec}v$$

$$\frac{1}{\operatorname{cosec}v}dv = -\frac{1}{x}dx$$

Taking integrals on both sides, we get

$$\int \sin v\, dv = -\int \frac{1}{x}dx$$

On integrating we get

- Cos v = - log x + C

Substituting the value of v

$$-\cos\frac{y}{x} = -\log x + C$$

y = 0 when x = 1

$$-\cos\frac{0}{1} = -\log 1 + C$$

- 1 = C

$$\therefore -\cos\frac{y}{x} = -\log x - 1$$

$$\cos\frac{y}{x} = \log x + \log e$$

$$\cos\frac{y}{x} = \log|ex|$$

The required solution of the differential equation.

15. $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$; $y = 2$ when $x = 1$

Solution:

Given

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

The above equation can be written as

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

Let $f(x, y) = \frac{2xy + y^2}{2x^2}$

Now put x = k x and y = k y

$$f(kx, ky) = \frac{2kxky + (ky)^2}{2(kx)^2}$$

Taking k^2 common

$$= \frac{k^2}{k^2}.\frac{2xy + y^2}{2x^2}$$

$= k^0.f(x, y)$

Therefore, the given differential equation is homogeneous.

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

On rearranging

$$\frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

To solve it we make the substitution.

y = v x

Differentiating above equation with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

On rearranging and substituting the value of dy/dx we get

$$v + x\frac{dv}{dx} = \frac{2x.vx + (vx)^2}{2x^2}$$

$$v + x\frac{dv}{dx} = \frac{2vx^2 + v^2x^2}{2x^2}$$

On computing and simplification we get

$$v + x\frac{dv}{dx} = \frac{2v + v^2}{2}$$

$$v + x\frac{dv}{dx} = v + \frac{1}{2}v^2$$

$$x\frac{dv}{dx} = \frac{1}{2}v^2$$

$$2\frac{1}{v^2}dv = \frac{1}{x}dx$$

Taking integration on both sides, we get

$$\int 2\frac{1}{v^2}dv = \int \frac{1}{x}dx$$

On integrating we get

$$-\frac{2}{v} = \log x + C$$

Substituting the value of v we get

$$-\frac{2}{y/x} = \log x + C$$

$$-\frac{2x}{y} = \log x + C$$

y = 2 when x = 1

$$-\frac{2.1}{2} = \log 1 + C$$

- 1 = C

$$\therefore -\frac{2x}{y} = \log x - 1$$

$$\frac{2x}{y} = 1 - \log x$$

$$y = \frac{2x}{1 - \log|x|} : x \neq e, x \neq 0$$

The required solution of the differential equation.

16. A homogeneous differential equation of the from $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution.

(A) y = v x (B) v = y x (C) x = v y (D) x = v

Solution:

(C) x = v y

Explanation:

Since, $\frac{dx}{dy}$ is given equal to $h\left(\frac{x}{y}\right)$.

Therefore,

$h\left(\frac{x}{y}\right)$ is a function of $\frac{x}{y}$.

Therefore, we shall substitute, x = v y is the answer

17. Which of the following is a homogeneous differential equation?

A. (4x + 6y + 5) dy – (3y + 2x + 4) dx = 0
B. (x y) dx – ($x^3 + y^3$) dy = 0
C. ($x^3 + 2y^2$) dx + 2xy dy = 0
D. y^2dx + (x^2 – x y – y^2) dy = 0

Solution:

D. y^2dx + (x^2 – x y – y^2) dy = 0

Explanation:

We have

$y^2dx + (x^2 - x\,y - y^2)\,dy = 0$

On rearranging

$$\frac{dy}{dx} = -\frac{x^2 - xy - y^2}{y^2}$$

$$\text{Let } f(x,y) = -\frac{x^2 - xy - y^2}{y^2}$$

Now put x = k x and y = k y

$$f(kx, ky) = -\frac{(kx)^2 - kxky - (ky)^2}{(ky)^2}$$

$$= \frac{k^2}{k^2}.-\frac{x^2 - xy - y^2}{y^2}$$

$= k^0.f(x, y)$

Therefore, the given differential equations is homogeneous.

EXERCISE 9.6

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For each of the differential equations given in question, find the general solution:

1. $\frac{dy}{dx} + 2y = \sin x$

Solution:

Given

$\frac{dy}{dx} + 2y = \sin x$

Given equation in the form of $\frac{dy}{dx} + py = Q$ where, p = 2 and Q = sin x

Now, I.F. = $e^{\int p dx} = e^{\int 2dx} = e^{2x}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times \text{I.F.})dx + C$

$\Rightarrow ye^{2x} = \int \sin x . e^{2x}dx + C$........1

Let I = $\int \sin x . e^{2x}dx$

Integrating using chain rule we get

$$\Rightarrow I = \sin x \int e^{2x}dx - \int \left(\frac{d}{dx}(\sin x).e^{\int 2dx}\right)dx$$

$$= \sin x.\frac{e^{2x}}{2} - \int \left(\cos x.\frac{e^{2x}}{2}\right)dx$$

On integrating and computing we get

$$= \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x \int e^{2x} - \int \left(\frac{d}{dx}(\cos x).\int e^{2x}dx\right)dx\right]$$

$$= \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x\frac{e^{2x}}{2} - \int \left[(-\sin x).\frac{e^{2x}}{2}\right]dx\right]$$

$$= \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}\int (\sin x.e^{2x})dx$$

Above equation can be written as

$$= \frac{e^{2x}}{4}(2\sin x - \cos x) - \frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I = \frac{e^{2x}}{4}(2\sin x - \cos x)$$

$$\Rightarrow I = \frac{e^{2x}}{5}(2\sin x - \cos x)$$

Now, putting the value of I in 1, we get,

$$\Rightarrow ye^{2x} = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

$$\Rightarrow y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$$

Therefore, the required general solution of the given differential equation is

$$y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$$

2. $\frac{dy}{dx} + 3y = e^{-2x}$

Solution:

Given

$$\frac{dy}{dx} + 3y = e^{-2x}$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = 3 and Q = e^{-2x}

Now, I.F. = $e^{\int p dx} = e^{\int 3 dx} = e^{3x}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times I.F.)dx + C$

$$\Rightarrow ye^{3x} = \int (e^{-2x} \times e^{2x})dx + C$$

$$\Rightarrow ye^{3x} = \int e^{x} dx + C$$

On integrating we get

$\Rightarrow ye^{3x} = e^{x} + C$

$\Rightarrow y = e^{-2x} + Ce^{-3x}$

Therefore, the required general solution of the given differential equation is $y = e^{-2x} + Ce^{-3x}$

3. $\frac{dy}{dx}+\frac{y}{x}=x^2$

Solution:

Given

$\frac{dy}{dx}+\frac{y}{x}=x^2$

This is equation in the form of $\frac{dy}{dx}+py=Q$

Where, p = $\frac{1}{x}$ and Q = x^2

Now, I.F. = $e^{\int pdx}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int(Q\times I.F.)dx+C$

$$\Rightarrow y(x)=\int(x^2.x)dx+C$$

$$\Rightarrow xy=\int(x^3)dx+C$$

On integrating we get

$$\Rightarrow xy=\frac{x^4}{4}+C$$

Therefore, the required general solution of the given differential equation is

$xy=\frac{x^4}{4}+C$

4. $\frac{dy}{dx}+(\sec x)y=\tan x\left(0\le x<\frac{\pi}{2}\right)$

Solution:

Given

$\frac{dy}{dx} + (\sec x)y = \tan x$

Given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, p = sec x and Q = tan x)

Now, I.F. = $e^{\int p dx} = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$

Thus, the solution of the given differential equation is given by the relation:

y (I.F.) = $\int (Q \times \text{I.F.})dx + C$

$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + C$

$\Rightarrow y(\sec x + \tan x) = \int \sec x \tan x dx + \int \tan^2 x dx + C$

$\Rightarrow y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1)dx + C$

⇒ y (sec x + tan x) = sec x + tan x – x+ C

Therefore, the required general solution of the given differential equation is

y (sec x + tan x) = sec x + tan x – x+ C.

y (I.F.) = $\int (Q \times \text{I.F.})dx + C$

$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + C$

$\Rightarrow y(\sec x + \tan x) = \int \sec x \tan x dx + \int \tan^2 x dx + C$

$\Rightarrow y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1)dx + C$

⇒ y (sec x + tan x) = sec x + tan x – x+ C

Therefore, the required general solution of the given differential equation is

y (sec x + tan x) = sec x + tan x – x+ C.

5. $\cos^2 x \frac{dy}{dx} + y = \tan x \quad \left(0 \le x < \frac{\pi}{2}\right)$

Solution:

Given

$\cos^2 \frac{dy}{dx} + y = \tan x$

The above equation can be written as

$\Rightarrow \frac{dy}{dx} + \sec^2 x.y = \sec^2 x \tan x$

Given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, $p = \sec^2 x$ and $Q = \sec^2 x \tan x$

Now, I.F. = $e^{\int p dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times \mathrm{I.F.}) dx + C$

$\Rightarrow y.\, e^{\tan x} = \int e^{\tan x} dx + C$ 1

Now, Let t = tanx

$\Rightarrow \frac{d}{dx}(\tan x) = \frac{dt}{dx}$

$\Rightarrow \sec^2 x = \frac{dt}{dx}$

$\Rightarrow \sec^2 x dx = dt$

Thus, the equation 1 becomes,

$\Rightarrow y.\, e^{\tan x} = \int (e^t.t) dt + C$

$$\Rightarrow y.e^{\tan x} = \int (e^t.t)dt + C$$

$$\Rightarrow y.e^{\tan x} = \int (t.e^t)dt + C$$

Using chain rule for integration we get

$$\Rightarrow y.e^{\tan x} = t.\int e^t dt - \int \left(\frac{d}{dt}(t).\int e^t dt\right)dt + C$$

$$\Rightarrow y.e^{\tan x} = t.e^t - \int e^t dt + C$$

On integrating we get

$\Rightarrow te^{\tan x} = (t - 1)e^t + C$

$\Rightarrow te^{\tan x} = (\tan x - 1)e^{\tan x} + C$

$\Rightarrow y = (\tan x - 1) + C\,e^{-\tan x}$

Therefore, the required general solution of the given differential equation is
$y = (\tan x - 1) + C\,e^{-\tan x}$.

6. $x\dfrac{dy}{dx} + 2y = x^2 \log x$

Solution:

Given

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

The above equation can be written as

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x\log x$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, $p = \frac{2}{x}$ and $Q = x \log x$

Now, I.F. $= e^{\int p dx} = e^{\int \frac{2}{x}dx} = e^{2(\log x)} = e^{\log x^2} = x^2$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) $= \int (Q \times \text{I.F.})dx + C$

$$\Rightarrow y.x^2 = \int (x\log x.x^2)dx + C$$

The above equation becomes

$$\Rightarrow x^2y = \int (x^3 \log x)dx + C$$

On integrating using chain rule we get

$$\Rightarrow x^2y = \log x. \int x^3 dx - \int \left[\frac{d}{dx}(\log x). \int x^3 dx\right] dx + C$$

$$\Rightarrow x^2y = \log x. \frac{x^4}{4} - \int \left(\frac{1}{x}.\frac{x^4}{4}\right) dx + C$$

$$\Rightarrow x^2y = \frac{x^4 \log x}{4} - \frac{1}{4}\int x^3 dx + C$$

Integrating and simplifying we get

$$\Rightarrow x^2y = \frac{x^4 \log x}{4} - \frac{1}{4}.\frac{x^4}{4} + C$$

$$\Rightarrow x^2y = \frac{1}{16}x^4(4\log x - 1) + C$$

$$\Rightarrow y = \frac{1}{16}x^2(4\log x - 1) + Cx^{-2}$$

Therefore, the required general solution of the given differential equation

$$y = \frac{1}{16}x^2(4\log x - 1) + Cx^{-2}$$

7. $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x}\log x$

Solution:

Given

$$x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$$

The above equation can be written as

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2}$$

The given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, p = $\frac{1}{x\log x}$ and Q = $\frac{2}{x^2}$

Now, I.F. = $e^{\int p dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

Thus, the solution of the given differential equation is given by the relation:

y (I.F.) = $\int (Q \times \text{I.F.})dx + C$

$\Rightarrow y.\log x = \int \left[\frac{2}{x^2}.\log x\right]dx + C$ 1

Now, $\int \left[\frac{2}{x^2}.\log x\right]dx = 2\int \left(\log x.\frac{1}{x^2}\right)dx$

On integrating using chain rule we get

$= 2\left[\log x.\int \frac{1}{x^2}dx - \int \left\{\frac{d}{dx}(\log x).\int \frac{1}{x^2}dx\right\}dx\right]$

$= 2\left[\log x\left(-\frac{1}{x}\right) - \int \left(\frac{1}{x}.\left(-\frac{1}{x}\right)\right)dx\right]$

$= 2\left[-\frac{\log x}{x} + \int \frac{1}{x^2}dx\right]$

$= 2\left[-\frac{\log x}{x} - \frac{1}{x}\right]$

$= -\frac{2}{x}(1 + \log x)$

Now, substituting the value in 1, we get,

$\Rightarrow y.\log x = -\frac{2}{x}(1 + \log x) + C$

Therefore, the required general solution of the given differential equation is

$y.\log x = -\frac{2}{x}(1 + \log x) + C$

8. $(1 + x^2)$ dy + 2xy dx = cot x dx ($x \neq 0$)

Solution:

Given

$(1 + x^2)dy + 2xydx = \cot x dx$

The above equation can be written as

$\Rightarrow \frac{dy}{dx} + \frac{2xy}{(1 + x^2)} = \frac{\cot x}{1 + x^2}$

The given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, p = $\frac{2x}{(1+x^2)}$ and Q =$\frac{\cot x}{1+x^2}$)

Now, I.F. = $e^{\int pdx} = e^{\int \frac{2x}{(1+x^2)}dx} = e^{\log(1+x^2)} = 1 + x^2$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times \mathrm{I.F.})dx + C$

$$\Rightarrow y.(1+x^2) = \int \left[\frac{\cot x}{1+x^2}.(1+x^2)\right]dx + C$$

$$\Rightarrow y.(1+x^2) = \int \cot x dx + C$$

On integrating we get

$\Rightarrow y(1+x^2) = \log|\sin x| + C$

Therefore, the required general solution of the given differential equation is

$y(1+x^2) = \log|\sin x| + C$

9. $x\frac{dy}{dx} + y - x + xy\cot x = 0 \quad (x \neq 0)$

Solution:

Given

$x\frac{dy}{dx} + y - x + xy\cot x = 0$

The above equation can be written as

$$\Rightarrow x\frac{dy}{dx} + y(1 + x\cot x) = x$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right)y = 1$$

The given equation is in the form of $\frac{dy}{dx} + px = Q$

Where, p = $\frac{1}{x} + \cot x$ and Q = 1

Now, I.F. = $e^{\int pdx} = e^{\int\left(\frac{1}{x}+\cot x\right)dy} = e^{\log x + \log(\sin x)} = e^{\log(x\sin x)} = x\sin x$

Thus, the solution of the given differential equation is given by the relation

x (I.F.) = $\int (Q \times \mathrm{I.F.})dy + C$

$$\Rightarrow y(x\sin x) = \int [1 \times x\sin x]dx + C$$

$$\Rightarrow y(x\sin x) = \int [x\sin x]dx + C$$

By splitting the integrals we get

$$\Rightarrow y(x\sin x) = x\int \sin x dx - \int \left[\frac{d}{dx}(x).\int \sin x dx\right] + C$$

$$\Rightarrow y(x\sin x) = x(-\cos x) - \int 1.(-\cos x)dx + C$$

On integrating we get

$\Rightarrow$ y (x sin x) = -x cos x + sin x + C

$$\Rightarrow y = \frac{-x\cos x}{x\sin x} + \frac{\sin x}{x\sin x} + \frac{C}{x\sin x}$$

$$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}$$

Therefore, the required general solution of the given differential equation is

$$y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}$$

10. $(x+y)\frac{dy}{dx} = 1$

Solution:

Given

$(x+y)\frac{dy}{dx} = 1$

The above equation can be written as

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+y}$$

$$\Rightarrow \frac{dx}{dy} = x + y$$

$$\Rightarrow \frac{dx}{dy} - x = y$$

The given equation is in the form of $\frac{dy}{dx} + px = Q$

Where, p = -1 and Q = y

Now, I.F. = $e^{\int p dy} = e^{\int -dy} = e^{-y}$

Thus, the solution of the given differential equation is given by the relation:

x (I.F.) = $\int (Q \times \text{I.F.})dy + C$

$$\Rightarrow xe^{-y} = \int [y.e^{-y}]dy + C$$

$$\Rightarrow xe^{-y} = y\int e^{-dy} - \int \left[\frac{d}{dy}(y)\int e^{-y}dy\right]dy + C$$

$$\Rightarrow xe^{-y} = y\int e^{-dy} - \int \left[\frac{d}{dy}(y)\int e^{-y}dy\right]dy + C$$

$$\Rightarrow xe^{-y} = y(-e^{-y}) - \int (-e^{-y})dy + C$$

On integrating and computing we get

$$\Rightarrow xe^{-y} = -ye^{-y} + \int e^{-y}dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$$

=> $x = -y - 1 + C e^{y}$

=> $x + y + 1 = C e^{y}$

Therefore, the required general solution of the given differential equation is $x + y + 1 = C e^{y}$.

11. $y\,dx + (x - y^2)dy = 0$

Solution:

Given

$ydx + (x - y^2)dy = 0$

The above equation can be written as

$\Rightarrow ydx = (y^2 - x)dy$

$\Rightarrow \frac{dx}{dy} = \frac{(y^2 - x)}{y} = y - \frac{x}{y}$

On simplifying we get

$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = y$

The above equation is in the form of $\frac{dy}{dx} + px = Q$

Where, p = $\frac{1}{y}$ and Q = y

Now, I.F. = $e^{\int pdy} = e^{\int \frac{dy}{y}} = e^{\log y} = y$

Thus, the solution of the given differential equation is given by the relation

x (I.F.) = $\int (Q \times I.F.)dy + C$

$\Rightarrow x.y = \int [y.y]dy + C$

$\Rightarrow x.y = \int y^2 dy + C$

On integrating we get

$\Rightarrow x.y = \frac{y^3}{3} + C$

On integrating we get

$$\Rightarrow x.y = \frac{y^3}{3} + C$$

$$\Rightarrow x = \frac{y^3}{3} + \frac{C}{y}$$

Therefore, the required general solution of the given differential equation is

$x = \frac{y^3}{3} + \frac{C}{y}$.

12. $(x+3y^2)\frac{dy}{dx} = y \quad (y > 0)$

Solution:

Given

$(x + 3y^2)\frac{dy}{dx} = y$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x + 3y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x + 3y^2}{y} = \frac{x}{y} + 3y$$

On simplification

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = -1/y and Q = 3y

Now, I.F. $= e^{\int p dy} = e^{-\int \frac{dy}{y}} = e^{-\log y} = e^{\log(\frac{1}{y})} = \frac{1}{y}$

Thus, the solution of the given differential equation is given by the relation:

x (I.F.) = $\int (Q \times \text{I.F.}) dy + C$

$$\Rightarrow x.\frac{1}{y} = \int \left[3y.\frac{1}{y}\right] dy + C$$

On integrating we get

$$\Rightarrow \frac{x}{y} = 3y + C$$

$\Rightarrow x = 3y^2 + Cy$

Therefore, the required general solution of the given differential equation is $x = 3y^2 + Cy$.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\frac{dy}{dx} + 2y \tan x = \sin x;\ y = 0 \text{ when } x = \frac{\pi}{3}$

Solution:

Given

$\frac{dy}{dx} + 2y\tan x = \sin x$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = 2 tanx and Q =sin x

Now, I.F. = $e^{\int p dx} = e^{\int 2\tan x dx} = e^{2\log(\sec x)} = e^{\log(\sec^2 x)} = \sec^2 x$

Thus, the solution of the given differential equation is given by the relation:

y (I.F.) = $\int (Q \times \text{I.F.})dx + C$

$$\Rightarrow y.(\sec^2 x) = \int [\sin x.\sec^2 x]dx + C$$

$$\Rightarrow y.(\sec^2 x) = \int [\sec x.\tan x]dx + C$$

On integrating we get

$\Rightarrow y.(\sec^2 x) = \sec x + C$..........1

Now, it is given that y = 0 at x = $\frac{\pi}{3}$

$0 \times \sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + C$

⇒ 0 = 2 + C

⇒ C = -2

Now, Substituting the value of C = -2 in 1, we get,

$\Rightarrow y.(\sec^2 x) = \sec x - 2$

⇒ y = cos x – 2$\cos^2$x

Therefore, the required general solution of the given differential equation is y = cos x – 2$\cos^2$x.

14. $(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$; $y = 0$ when $x = 1$

Solution:

Given

$(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} + \frac{2xy}{(1+x^2)} = \frac{1}{(1+x^2)^2}$

The given equation is in the form of $\frac{dy}{dx} + py = Q$

Where, p = $\frac{2x}{(1+x^2)}$ and Q = $\frac{1}{(1+x^2)^2}$

Now, I.F. = $e^{\int p dx} = e^{\int \frac{2x}{(1+x^2)}dx} = e^{\log(1+x^2)} = 1 + x^2$

Thus, the solution of the given differential equation is given by the relation

y(I.F.) = $\int (Q \times I.F.)dx + C$

$\Rightarrow y.(1+x^2) = \int \left[\frac{1}{(1+x^2)^2}.(1+x^2)\right]dx + C$

$\Rightarrow y.(1+x^2) = \int \frac{1}{(1+x^2)}dx + C$

On integrating we get

$\Rightarrow y.(1+x^2) = \tan^{-1} x + C$.....1

Now, it is given that y = 0 at x = 1

0 = $\tan^{-1}$ 1+ C

$\Rightarrow$ C = $-\frac{\pi}{4}$

Now, Substituting the value of C = $-\frac{\pi}{4}$ in (1), we get,

$\Rightarrow y.(1+x^2) = \tan^{-1} x - \frac{\pi}{4}$

Therefore, the required general solution of the given differential equation is

$y.(1+x^2) = \tan^{-1} x - \frac{\pi}{4}$

15. $\frac{dy}{dx} - 3y\cot x = \sin 2x;\ y = 2$ when $x = \frac{\pi}{2}$

Solution:

Given

$\frac{dy}{dx} - 3y\cot x = \sin 2x$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = -3cot x and Q = sin 2x

Now, I.F. = $e^{\int p dx} = e^{-3\int \cot x dx} = e^{-3\log|\sin x|} = e^{\log\left|\frac{1}{\sin^3 x}\right|} = \frac{1}{\sin^3 x}$

Thus, the solution of the given differential equation is given by the relation

y (I.F.) = $\int (Q \times I.F.)dx + C$

$$\Rightarrow y.\frac{1}{\sin^3 x} = \int \left[\sin 2x.\frac{1}{\sin^3 x}\right] dx + C$$

$$\Rightarrow y\operatorname{cosec}^3 x = 2\int (\cot x \operatorname{cosec} x)dx + C$$

On integrating we get

$\Rightarrow$ y cosec^{3}x = 2cosecx + C

$\Rightarrow$ y = $-\frac{2}{\operatorname{cosec}^2 x} + \frac{3}{\operatorname{cosec}^3 x}$

$\Rightarrow$ y = -2sin^{2}x + Csin^{3}x.............1

Now, it is given that y = 2 when x = $\frac{\pi}{2}$

Thus, we get,

2 = -2 + C

$\Rightarrow$ C = 4

Now, Substituting the value of C = 4 in 1, we get,

y = -2sin^{2}x + 4sin^{3}x

$\Rightarrow$ y = 4sin^{3}x - 2sin^{2}x

Therefore, the required general solution of the given differential equation is

y = 4sin^{3}x - 2sin^{2}x.

16. Find the equation of a curve passing through the origin, given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Solution:

Let F(x, y) be the curve passing through origin and let (x, y) be a point on the curve.

We know the slope of the tangent to the curve at (x, y) is $\frac{dy}{dx}$

According to the given conditions, we get,

$$\frac{dy}{dx} = x + y$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} - y = x$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = -1 and Q = x

Now, I.F. = $e^{\int p dx} = e^{\int(-1)dx} = e^{-x}$

Thus, the solution of the given differential equation is given by the relation:

y(I.F.) = $\int (Q \times I.F.)dx + C$

$\Rightarrow ye^{-x} = \int xe^{-x}dx + C$ 1

Now, $\int xe^{-x}dx = x\int e^{-x}dx - \int \left[\frac{d}{dx}(x).\int e^{-x}\,dx\right]dx$

On integrating

$$= x(e^{-x}) - \int(-e^{-x})dx$$

$$= x(e^{-x}) + (-e^{-x})$$

$$= -e^{-x}(x + 1)$$

Thus, from equation 1, we get,

$\Rightarrow ye^{-x} = -e^{-x}(x+1) + C$

$\Rightarrow$ y = -(x+1) + C e^x

$\Rightarrow$ x + y + 1 = C e^x 2

Now, it is given that the curve passes through the origin.

Thus, equation 2 becomes

1 = C

⇒ C = 1

Substituting C = 1 in equation 2, we get,

x + y + 1 = e^x

Therefore, the required general solution of the given differential equation is

$x + y + 1 = e^x$

17. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Solution:

Let F(x, y) be the curve and let (x, y) be a point on the curve.
We know the slope of the tangent to the curve at (x, y) is $\frac{dy}{dx}$
According to the given conditions, we get,

$$\frac{dy}{dx} + 5 = x + y$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} - y = x - 5$$

This is equation in the form of $\frac{dy}{dx} + py = Q$
Where, p = -1 and Q = x - 5
Now, I.F. = $e^{\int p dx} = e^{\int(-1)dx} = e^{-x}$
Thus, the solution of the given differential equation is given by the relation:

y(I.F.) = $\int (Q \times \text{I.F.})dx + C$

$$\Rightarrow ye^{-x} = \int (x-5)e^{-x}dx + C \text{............1}$$

Now, $\int (x-5)e^{-x}dx = (x-5)\int e^{-x}dx - \int\left[\frac{d}{dx}(x-5).\int e^{-x}\,dx\right]dx$

$$= (x-5)(e^{-x}) - \int(-e^{-x})dx$$

On integrating we get

$$= (x-5)(e^{-x}) + (-e^{-x})$$

$$= (4-x)e^{-x}$$

Thus, from equation 1, we get,

$$\Rightarrow ye^{-x} = (4-x)e^{-x} + C$$

$$\Rightarrow y = 4 - x + Ce^x$$

$$\Rightarrow x + y - 4 = Ce^x$$

Thus, equation (2) becomes:

$0 + 2 – 4 = C e^0$

⇒ – 2 = C

⇒ C = -2

Substituting C = -2 in equation (2), we get,

x + y – 4 =-2e^x

⇒ y = 4 – x – 2e^x

Therefore, the required general solution of the given differential equation is

y = 4 – x – 2e^x

$$x\frac{dy}{dx} - y = 2x^2$$

18. The integrating factor of the differential equation is

A. e^{-x}, B. e^{-y}, C. 1/x, D. x

Solution:

C. 1/x

Explanation:

Given

$$x\frac{dy}{dx} - y = 2x^2$$

On simplification we get

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x$$

This is equation in the form of $\frac{dy}{dx} + py = Q$

Where, p = -1/x and Q =2x

Now, I.F. = $e^{\int p dx} = e^{\int -\frac{1}{x}dx} = e^{\log(x^{-1})} = x^{-1} = \frac{1}{x}$

Hence the answer is 1/x

19. The integrating factor of the differential equation

$$(1-y^2)\frac{dx}{dy}+yx = ay(-1<y<1) \text{ is}$$

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Solution:

(D) $\frac{1}{\sqrt{1-y^2}}$

Explanation:

Given

$(1-y^2)\frac{dy}{dx}+yx = ay$

On rearranging we get

$\Rightarrow \frac{dy}{dx}+\frac{yx}{1-y^2} = \frac{ay}{1-y^2}$

This is equation in the form of $\frac{dy}{dx}+py = Q$

Where, p = $\frac{y}{1-y^2}$ and Q = $\frac{a}{1-y^2}$

Now, I.F. =

$$e^{\int pdy} = e^{\int \frac{y}{1-y^2}dy} = e^{\frac{1}{2}\log(1-y^2)} = e^{\log\left[\frac{1}{\sqrt{(1-y^2)}}\right]}$$

$$= \frac{1}{\sqrt{(1-y^2)}}$$

MISCELLANEOUS EXERCISE

1. For each of the differential equations given below, indicate its order and degree (if defined)

(i) $(d^2y/dx^2) + 5x(dy/dx)^2 – 6y = \log x$

(ii) $(dy/dx)^3 – 4 (dy/dx)^2 + 7y = \sin x$

(iii) $(d^4y/dx^4) – \sin (d^3y/dx^3) = 0$

Solution:

(i) $(d^2y/dx^2) + 5x(dy/dx)^2 – 6y = \log x$

Rearranging the given equation, we get

$(d^2y/dx^2) + 5x(dy/dx)^2 – 6y – \log x = 0$

Hence, the highest order derivative present in the given differential equation is d^2y/dx^2.

Therefore, the order is 2.

Also, the highest power raised to d^2y/dx^2 is 1.

Hence, the degree is 1.

(ii) $(dy/dx)^3 – 4 (dy/dx)^2 + 7y = \sin x$

Rearranging the given equation, we get

$(dy/dx)^3 – 4 (dy/dx)^2 + 7y – \sin x = 0$

Hence, the highest order derivative present in the given differential equation is dy/dx.

Therefore, the order is 1.

And the highest power raised to dy/dx is 3.

Hence, the degree is 3.

(iii) $(d^4y/dx^4) – \sin (d^3y/dx^3) = 0$

The highest order derivative present in the given differential equation is d^4y/dx^4. Hence, the order is 4.

Since the given differential equation is not a polynomial equation, the degree of the equation is not defined.

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution to the corresponding differential equation.

(i) $y = ae^x + be^{-x} + x^2$: $x (d^2y/dx^2) + 2(dy/dx) – xy + x^2 – 2 = 0$

(ii) $y = e^x (a \cos x + b \sin x)$: $(d^2y/dx^2) – 2(dy/dx) + 2y = 0$

(iii) $y = x \sin 3x$: $(d^2y/dx^2) + 9y - 6 \cos 3x = 0$

(iv) $x^2 = 2y^2 \log y$: $(x^2 + y^2)(dy/dx) - xy = 0$

Solution:

(i) $y = ae^x + be^{-x} + x^2$: $x (d^2y/dx^2) + 2(dy/dx) - xy + x^2 - 2 = 0$

Given: $y = ae^x + be^{-x} + x^2$

Differentiate the function with respect to x, and we get

$dy/dx = ae^x - be^{-x} + 2x$ … (1)

Now, again differentiate with respect to x, and we get

$d^2y/dx^2 = ae^x + be^{-x} + 2$ …(2)

To check whether the given function is the solution of the given differential equation, substitute (1) and (2) in the given differential equation.

L.H.S of the given differential equation

$= x (d^2y/dx^2) + 2(dy/dx) - xy + x^2$

Now, substituting the values, we get

$= x (ae^x + be^{-x} + 2) + 2 (ae^x - be^{-x} + 2x) - x (ae^x + be^{-x} + x^2) + x^2 - 2$

$= (xae^x + x\, be^{-x} + 2x) + (2ae^x - 2be^{-x} + 4x) - (xae^x + xbe^{-x} + x^3) + x^2 - 2$

On simplifying the above equation, we get

$= 2ae^x - 2be^{-x} - x^3 + x^2 + 6x - 2 \neq 0$

Hence, L.H.S $\neq$ R.H.S.

Therefore, the given function is not a solution to the corresponding differential equation.

(ii) $y = e^x(a \cos x + b \sin x)$: $(d^2y/dx^2) - 2(dy/dx) + 2y = 0$

Given: $y = e^x(a \cos x + b \sin x)$

The given function can be written as follows:

$y = e^x a \cos x + e^x b \sin x$

Differentiating the function on both sides, we get

$dy/dx = (a + b)e^x \cos x + (b - a)e^x \sin x$ …(1)

Again, differentiate the above equation on both sides with respect to x, and we get

$d^2y/dx^2 = [(a + b) (d/dx) (e^x \cos x] + [(b-a) (d/dx) (e^x \sin x]$

$d^2y/dx^2 = [(a+ b) (e^x \cos x – e^x \sin x)] + [(b- a)(e^x \sin x + e^x \cos x)]$

$d^2y/dx^2 = e^x [(a+ b) (\cos x – \sin x) + (b- a)(\sin x + \cos x)]$

On simplifying the above equation, we get

$d^2y/dx^2 = 2e^x(b \cos x – a \sin x)$ …(2)

Now, substitute (1) and (2) in the given differential equation.

L.H.S = $(d^2y/dx^2) – 2(dy/dx) + 2y$

= $[2e^x(b \cos x – a \sin x)] – 2[(a + b)e^x \cos x + (b – a)e^x \sin x] + 2 e^x(a \cos x + b \sin x)$

= $e^x [(2b \cos x – 2a \sin x)- (2a \cos x + 2b \cos x) – (2b \sin x – 2a \sin x) + (2a \cos x + 2b \sin x)]$

= $e^x [2b \cos x – 2a \sin x- 2a \cos x – 2b \cos x – 2b \sin x + 2a \sin x + 2a \cos x + 2b \sin x]$

= $e^x[0]$

= 0 = R.H.S

As L.H.S = R.H.S, the given function is the solution of the corresponding differential equation.

(iii) $y = x \sin 3x$: $(d^2y/dx^2) + 9y – 6 \cos 3x = 0$

Given: $y = x \sin 3x$

Now, differentiating the given function with respect to x, and we get

$dy/dx = \sin 3x + x. \cos 3x. 3$

$dy/dx = \sin 3x + 3x \cos 3x$ …(1)

Again differentiate (1) with respect to x, we get

$d^2y/dx^2 = (d/dx) (\sin 3x) + 3 (d/dx) (x \cos 3x)$

$d^2y/dx^2 = 3 \cos 3x + 3 [\cos 3x + x (- \sin 3x). 3]$

On simplifying the above equation, we get

$d^2y/dx^2 = 6 \cos 3x – 9x \sin 3x$ …(2)

Now, substitute (1) and (2) in the given differential equation, and we get the following:

L.H.S = $(d^2y/dx^2) + 9y – 6 \cos 3x$

= $(6 \cos 3x – 9x \sin 3x) + 9(x \sin 3x) – 6 \cos 3x$

= $6 \cos 3x – 9x \sin 3x + 9x \sin 3x – 6 \cos 3x$

= 0 = R.H.S

As L.H.S = R.H.S, the given function is the solution of the corresponding differential equation.

(iv) $x^2 = 2y^2 \log y$: $(x^2 + y^2)(dy/dx) – xy = 0$

Given: $x^2 = 2y^2 \log y$

Now, differentiate the function with respect to x, and we get

$2x = 2 (d/dx) (y^2 \log y)$

On simplifying the above equation, we get

$x = (d/dx) (y^2 \log y)$

$x = [2y \log y .(dy/dx) + y^2. (1/y). (dy/dx)]$

$x = (dy/dx)[2y \log y + y]$

Hence, we get

$dy/dx = x / [y(1 + 2 \log y)]$ …(1)

Now, substitute (1) in the given differential equation.

L.H.S = $(x^2 + y^2)(dy/dx) – xy$

$= [2y^2 \log y + y^2] . [x / [y(1 + 2 \log y)]] – xy$

$= [y^2(2 \log y + 1)] . [x / [y(1 + 2 \log y)]] – xy$

$= xy – xy$

= 0 = R.H.S

As L.H.S = R.H.S, the given function is the solution of the corresponding differential equation.

3. Form the differential equation representing the family of curves given by $(x- a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

Solution:

Given equation: $(x- a)^2 + 2y^2 = a^2$

The given equation can be written as:

$\Rightarrow x^2 + a^2 – 2ax + 2y^2 = a^2$

On rearranging the above equation, we get

$\Rightarrow 2y^2 = 2ax – x^2$ …(1)

Now, differentiate equation (1) with respect to x,

⇒ 2 . 2y (dy/dx) = 2a – 2x

⇒ 2y(dy/dx) = (2a – 2x) /2

⇒ dy/dx = (a-x)/2y

⇒ dy/dx = $(2ax – 2x^2)$ / 4xy … (2)

From equation (1), we get

$2ax = 2y^2 + x^2$

Substitute the value in equation (2), and we get

dy/dx = $[2y^2 + x^2 – 2x^2]$/4xy

dy/dx = $(2y^2 – x^2)$ / 4xy

Therefore, the differential equation representing the family of curves given by $(x- a)^2 + 2y^2 = a^2$ is $(2y^2 – x^2)$ / 4xy.

4. Prove that $x^2 – y^2 = C (x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 – 3xy^2)$ dx = $(y^3 – 3x^2y)$ dy, where C is a parameter.

Solution:

Given differential equation: $(x^3 – 3xy^2)$ dx = $(y^3 – 3x^2y)$ dy

The equation can be rewritten as:

dy/dx = $(x^3 – 3xy^2)$ / $(y^3 – 3x^2y)$ …(1)

The above equation is a homogeneous equation.

To simplify the equation, let us assume y = vx.

⇒ (d/dx) y = (d/dx) (vx)

⇒ dy/dx = v + x(dv/dx) …(2)

Using equations (1) and (2), we get

⇒v + x(dv/dx) = $(x^3 – 3xy^2)$ / $(y^3 – 3x^2y)$

⇒v + x(dv/dx) = $(x^3 – 3x(vx)^2)$ / $((vx)^3 – 3x^2(vx))$

⇒v + x(dv/dx) = $[(1-3v^2)/(v^3-3v)]$

⇒x(dv/dx) = $[(1-3v^2)/(v^3-3v)]$ – v

On simplifying the above equation, we get

$\Rightarrow x(dv/dx) = (1-v^4)/(v^3 - 3v)$

Rearranging the above equation,

$\Rightarrow [(v^3 - 3v)/(1-v^4)]dv = (dx/x)$.

Integrate both sides, and we get

$$\Rightarrow \int \left(\frac{v^3-3v}{1-v^4}\right)dv = logx + logC' \ldots (3)$$

$$\Rightarrow \int \left(\frac{v^3-3v}{1-v^4}\right)dv = \int \frac{v^3dv}{1-v^4} - 3\int \frac{vdv}{1-v^4}$$

$$\Rightarrow \int \left(\frac{v^3-3v}{1-v^4}\right)dv = I_1 - 3I_2 \ldots (4)$$

Where $I_1 = \int [(v^3dv)/(1-v^4)]$ and $I_2 = \int [(vdv)/(1-v^4)]$

Now, let us assume $1- v^4 = t$

Hence, we get

$\Rightarrow (d/dv)(1-v^4) = (dt/dv)$

$\Rightarrow -4v^3 = dt/dv$

$\Rightarrow v^3\, dv = -dt/4$

Now,

$I_1 = \int -(dt/4t) = (-¼) \log t = -(¼) \log(1-v^4) \ldots(5)$

Similarly,

$I_2 = \int [(vdv)/(1-v^4)] = \int [(vdv)/(1-(v^2)^2)]$

Assume that $v^2 = p$

Hence, we get

$\Rightarrow (d/dv)v^2 = dp/dv$

$\Rightarrow 2v = dp/dv$

$\Rightarrow vdv = dp/2$

Now,

$I_2 = (½) \int [dp/(1-p^2)]$

$$I_2 = \frac{1}{2\times 2} log\left|\frac{1+p}{1-p}\right|$$

$$I_2 = \frac{1}{4} log\left|\frac{1+v^2}{1-v^2}\right| \ldots (6)$$

Using equations (4), (5) and (6), we get

$$\int \left(\frac{v^3-3v}{1-v^4}\right) dv = -\frac{1}{4} log(1-v^4) - \frac{3}{4} log\left|\frac{1+v^2}{1-v^2}\right| \ldots (7)$$

Now, using equations (2) and (7)

$$-\frac{1}{4} log(1-v^4) - \frac{3}{4} log\left|\frac{1+v^2}{1-v^2}\right| = logx + logc'$$

$$-\frac{1}{4} log\left[(1-v^4)\left|\frac{1+v^2}{1-v^2}\right|^3\right] = logC'x$$

$$-\frac{1}{4} log\left[(1-v^2)(1+v^2)\left|\frac{1+v^2}{1-v^2}\right|^3\right] = logC'x$$

$$\Rightarrow \frac{(1+v^2)^4}{(1-v^2)^2} = (C'x)^{-4}$$

Now, replace v with y/x in the above equation and simplify it.

Hence, we get

$\Rightarrow (x^2 – y^2)^2 = C'^4 (x^2 + y^2)^4$

Now, take the square root on both sides of the above equation, and we get

$\Rightarrow (x^2 – y^2) = C'^2 (x^2 + y^2)^2$

$\Rightarrow (x^2 – y^2) = C' (x^2 + y^2)^2$, where $C = C'^2$

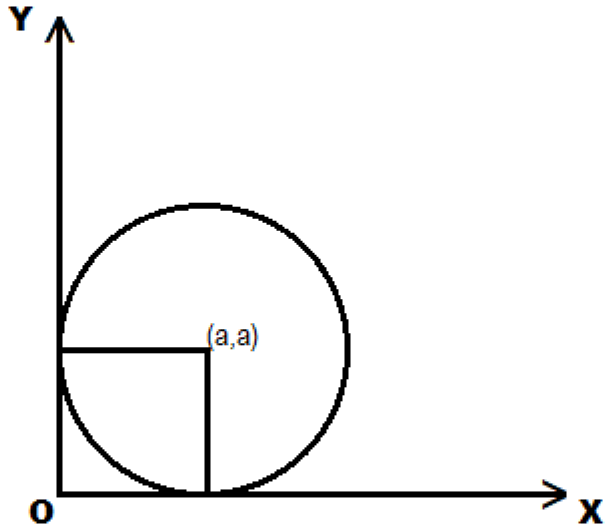
Hence, $x^2 – y^2 = C (x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 – 3xy^2) dx = (y^3 – 3x^2y) dy$, it is proved.

5. Form the differential equation of the family of circles in the first quadrant, which touches the coordinate axes.

Solution:

We know that the equation of a circle with centre (a, a) and radius "a" in the first quadrant, which touches the coordinate axes, is:

$(x-a)^2 + (y-a)^2 = a^2$ …(1)



Differentiating the above equation of both sides with respect to x, we get

$\Rightarrow 2(x-a) + 2(y-a)(dy/dx) = 0$

Now, the equation can be written as

$\Rightarrow (x-a) + (y-a)y' = 0$

$\Rightarrow (x-a) + yy' - ay' = 0$

$\Rightarrow x + yy' - a(1+ y') = 0$

$\Rightarrow x + yy' = a (1 + y')$

Rearranging the above equation, we get

$\Rightarrow a = (x + yy') / (1+y')$

Now, substitute the value of "a" in equation (1), and we get

$$\Rightarrow \left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left[\frac{x+yy'}{1+y'}\right]^2$$

$$\Rightarrow \left[\frac{(x-y)y'}{(1+y')}\right]^2+\left[\frac{y-x}{1+y'}\right]^2=\left[\frac{x+yy'}{1+y'}\right]^2$$

On simplifying the above equation, we get

$\Rightarrow (x-y)^2.y'^2 + (y-x)^2 = (x+ yy')^2$

Therefore, the differential equation of the family of circles in the first quadrant, which touches the coordinate axes, is $(x-y)^2.y'^2 + (y-x)^2 = (x+ yy')^2$.

6. Find the general solution of the differential equation $(dy/dx) + \sqrt{[(1-y^2)/(1-x^2)]} = 0$.

Solution:

Given: $(dy/dx) + \sqrt{[(1-y^2)/(1-x^2)]} = 0$.

The given differential equation can be written as $dy/dx = -\sqrt{[(1-y^2)/(1-x^2)]}$

$\Rightarrow dy /\sqrt{(1-y^2)} = -dx/\sqrt{(1-x^2)}$

Now, integrate both sides, and we get

$\Rightarrow \sin^{-1} y = -\sin^{-1} x + C$

Now, rearrange the equation, and we get

$\Rightarrow \sin^{-1} x + \sin^{-1} y = C$.

7. Show that the general solution of the differential equation $(dy/dx) + [(y^2 + y + 1) / (x^2 + x + 1)] = 0$ is given by $(x + y + 1) = A (1 - x - y - 2xy)$, where A is the parameter.

Solution:

Given differential equation: $(dy/dx) + [(y^2 + y + 1) / (x^2 + x + 1)] = 0$

Rearranging the given equation, we get

$dy/dx = -[(y^2 + y + 1) / (x^2 + x + 1)]$

$\Rightarrow dy/(y^2 + y + 1) = -dx/ (x^2 + x + 1)$

Now, integrate both sides, and we get

$\Rightarrow \int[dy/(y^2 + y + 1)] = -\int [dx/ (x^2 + x + 1)]$

$$\Rightarrow \int \frac{dy}{(y+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} = -\int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \frac{2}{\sqrt{3}} tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] = -\frac{2}{\sqrt{3}} tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + C$$

$$\Rightarrow tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right] + tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] = \frac{\sqrt{3}C}{2}$$

On simplifying the above equation, we get

$$\Rightarrow \left(\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\frac{4xy+2x+2y+1}{3}}\right) = tan\left[\frac{\sqrt{3}C}{2}\right]$$

$$\Rightarrow \left(\frac{\frac{2x+2y+2}{\sqrt{3}}}{\frac{3-(4xy+2x+2y+1)}{3}}\right) = C_1 \text{ Where, } C_1 = tan\left[\frac{\sqrt{3}C}{2}\right]$$

[la

$$\Rightarrow \frac{\sqrt{3}(2x+2y+2)}{3-(4xy+2x+2y+1)} = C_1$$

$\Rightarrow 2\sqrt{3}\ (x + y + 1) = C_1\ (3 – 4xy + 2x + 2y + 1)$

$\Rightarrow 2\sqrt{3}\ (x + y + 1) = C_1\ (2 – 4xy – 2x – 2y)$

$\Rightarrow 2\sqrt{3}\ (x + y + 1) = 2C_1\ (1 – 2xy – x – y)$

$\Rightarrow \sqrt{3}\ (x + y + 1) = C_1\ (1 – 2xy – x – y)$

$\Rightarrow \sqrt{3}\ (x + y + 1) = C_1\ (1 – x – y – 2xy)$

$\Rightarrow (x + y + 1) = (C_1/\sqrt{3})(1 – x – y – 2xy)$

$\Rightarrow (x + y + 1) = A\ (1 – x – y – 2xy)$, where $A = (C_1/\sqrt{3})$

Hence, proved.

8. Find the equation of the curve passing through the point (0, π/4) whose differential equation is sin x cos y dx + cos x sin y dy = 0.

Solution:

The given differential equation is sin x cos y dx + cos x sin y dy = 0.

It can also be written as:

⇒ (sin x cos y dx + cos x sin y dy) / cos x cos y = 0.

We know that sin x / cos = tan x,

And simplify the above equation

⇒ tan x dx + tan y dy = 0

⇒ log (sec x) + log (sec y) = log C

⇒ log (sec x. sec y) = log C

On simplification, we get

sec x sec y = C

It is given that the curve passes through the point (0, π/4).

⇒ 1 × √2 = C

⇒ C = √2

Hence,

sec x × sec y = √2

⇒ sec x × (1/cos y) = √2

⇒ cos y = sec x / √2

Hence, the equation of the curve passing through the point (0, π/4) whose differential

equation is sin x cos y dx + cos x sin y dy = 0 is cos y = sec x /√2.

9. Find the particular solution of the differential equation $(1 + e^{2x})$ dy + $(1 + y^2)e^x$ dx = 0, given that y = 1 when x = 0.

Solution:

Given differential equation: $(1 + e^{2x})$ dy + $(1 + y^2)e^x$ dx = 0

Rearranging the equation, we get

⇒ $[dy/(1+y^2)] + [(e^x\, dx)/(1 + e^{2x})] = 0$

Integrating both sides of the equation, we get

$\tan^{-1} y + \int [(e^x\, dx) / (1+e^{2x})] = C$ …(1)

Let $e^x = t$, and hence, $e^{2x} = t^2$

$(d/dx)(e^x) = (dt/dx)$

$\Rightarrow e^x = dt/dx$

$\Rightarrow e^x dx = dt$

Substituting the value in equation (1), we get

$\tan^{-1} y + \int [(dt) / (1+t^2)] = C$

$\Rightarrow \tan^{-1} y + \tan^{-1} t = C$

$\Rightarrow \tan^{-1} y + \tan^{-1}(e^x) = C$

If x = 0 and y = 1, we get

$\Rightarrow \tan^{-1} 1 + \tan^{-1}(e^0) = C$

$\Rightarrow \tan^{-1} 1 + \tan^{-1} 1 = C$

$\Rightarrow (\pi/4) + (\pi/4) = C$

$\Rightarrow C = \pi/2$

Hence, $\tan^{-1} y + \tan^{-1}(e^x) = \pi/2$, which is the particular solution of the given differential equation.

10. Solve the differential equation $y\, e^{x/y}\, dx = (xe^{x/y} + y^2)dy$ ($y \neq 0$).

Solution:

Given: $y\, e^{x/y}\, dx = (xe^{x/y} + y^2)dy$

Rearranging the given equation, we get

$y\, e^{x/y}(dx/dx) = xe^{x/y} + y^2$

$\Rightarrow e^{x/y}[y(dx/dy) - x] = y^2$

$\Rightarrow [e^{x/y}[y(dx/dy) - x]]/y^2 = 1$ ….(1)

Assume that $e^{x/y} = z$

Differentiate with respect to y, and we get

$(d/dy)(e^{x/y}) = dz/dy$

$\Rightarrow e^{x/y}[(y(dx/dy) - x)/y^2] = dz/dy$ …(2)

Comparing equations (1) and (2), we get

⇒dz/dy = 1

⇒ dz = dy

Now, integrating both sides, we get

⇒ z = y + C

⇒ $e^{x/y}$ = y + C, which is the solution of the given differential equation.

11. Find a particular solution of the differential equation (x – y) (dx + dy) = dx – dy,

given that y = –1, when x = 0. (Hint: put x – y = t)

Solution:

Given differential equation: (x – y) (dx + dy) = dx – dy

On simplifying the above equation, we get

⇒ (dy/dx) = (1-x+y)/ (x -y +1)

⇒ (dy/dx) = [1 – (x-y)] / [1 + (x-y)] …(1)

Given: x – y = t …(2)

(d/dx) (x – y) = dt/dx

⇒ 1 – (dy/dx) = dt/dx

⇒ 1 – (dt/dx) = dy/dx …(3)

Using the equations (1), (2) and (3), we get

⇒ 1 – (dt/dx) =(1-t)/(1+t)

⇒ dt/dx = 1 – [(1-t)/(1+t)]

On simplification, we get

⇒dt/dx = 2t / (1+t)

⇒ [(1+t)/t]dt = 2 dx

⇒ [1 + (1/t)]dt = 2dx

Now, integrating both sides, we get

⇒ t + log |t| = 2x + C

⇒ (x-y) + log |x-y| = 2x + C

⇒ log |x-y| = x + y + C

When x = 0 and y = -1, we get

⇒ log 1 = 0 – 1 + C

⇒ C = 1

Hence, log |x-y| = x + y + 1.

Therefore, log |x-y| = x + y +1 is a particular solution of the differential equation (x – y) (dx + dy) = dx – dy.

12. Solve the differential equation [($e^{-2\sqrt{x}}$ /√x) – (y/√x)](dx/dy) = 1 (x ≠ 0).

Solution:

Given: [($e^{-2\sqrt{x}}$ /√x) – (y/√x)](dx/dy) = 1

Rearranging the given equation, we get

⇒ dy/dx = ($e^{-2\sqrt{x}}$ /√x) – (y/√x)

⇒ (dy/dx) + (y/√x) = ($e^{-2\sqrt{x}}$ /√x)

The above equation is a linear equation of the form (dy/dx) + Py = Q

Where, P = 1/√x and Q = $e^{-2\sqrt{x}}$/√x

Now, I.F = $e^{\int P\,dx} = e^{\int 1/\sqrt{x}\,dx} = e^{2\sqrt{x}}$

Hence, the general solution of the given differential equation is:

y. (I.F) = ∫ (Q × I.F) dx + C

Now, substituting the values, we get

⇒ $ye^{2\sqrt{x}}$ = ∫ [($e^{-2\sqrt{x}}$/√x) ×$e^{2\sqrt{x}}$]dx + C

⇒$ye^{2\sqrt{x}}$ = ∫ (1/√x) dx + C

⇒$ye^{2\sqrt{x}}$ = 2√x +C, which is the solution of the given differential equation.

13. Find a particular solution of the differential equation (dy/dx) + y cot x = 4x cosec x, (x≠ 0), given that y = 0 when x = π/2.

Solution:

Given: (dy/dx) + y cot x = 4x cosec x

The given equation is a linear differential equation of the form (dy/dx) + Py = Q

Where

P = cot x and Q = 4x cosec x

Now, I.F = $e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log|\sin x|} = \sin x$

Hence, the general solution of the given differential equation is:

y. (I.F) = ∫ (Q × I.F) dx + C

⇒ y sin x = ∫ (4x cosec x × sin x) dx + C

⇒ y sin x = 4 ∫x dx + C

⇒ y sin x = 4 (x^2/2) + C

⇒ y sin x =2x^2 + C

when x = π/2 and y = 0,

Substituting the values in the above equation, we get

⇒ 0 =2$(\pi/2)^2$ + C

⇒ 0 = 2(π^2/4) + C

⇒ 0 = π^2/2 + C

⇒ C = – π^2/2

Hence, y sin x =2x^2 – (π^2/2)

Therefore, the particular solution of the differential equation (dy/dx) + y cot x = 4x cosec x is y sin x =2x^2 – (π^2/2).

14. Find a particular solution of the differential equation (x + 1) (dy/dx) = 2e^{-y} – 1. Given that y = 0 when x = 0.

Solution:

Given differential equation: (x + 1) (dy/dx) = 2e^{-y} – 1

Rearranging the equation, we get

⇒ dy/2e^{-y} – 1 = dx/(x + 1)

⇒ (e^y dy)/(2-e^y) = dx/(x+1)

Integrate on both sides, we get

∫ [(e^y dy)/(2-e^y)] = log |x+1| + log C …(1)

Assume that 2-e^y = t

$\Rightarrow$ (d/dy) (2-e^y) = dt/dy

$\Rightarrow$ -e^y = dt/dy

$\Rightarrow$ -e^y dy = dt

Substituting the value in equation (1), we get

$\Rightarrow\int$ [(dt)/(t)] = log |x+1| + log C

$\Rightarrow$ – log |t| = log |C (x+1)|

$\Rightarrow$ – log |2-e^y| = log |C (x+1)|

$\Rightarrow$ 1/ (2-e^y) = C (x+1)

$\Rightarrow$2-e^y = 1/[C(x+1)]

When x = 0 and y = 0, we get

$\Rightarrow$ 2 – 1 = 1/C

We get C = 1.

Therefore, 2-e^y = 1/[1(x+1)]

$\Rightarrow$2-e^y = 1/(x+1)

$\Rightarrow$ e^y = 2 – [1/(x+1)]

On simplification, we get

e^y = (2x +1) / (x+1)

$\Rightarrow$ y = log |(2x +1) / (x+1)|, where x $\neq$ -1.

Therefore, the particular solution of the differential equation (x + 1) (dy/dx) = 2e^{-y}-1 is y = log |(2x +1) / (x+1)|, where x $\neq$ -1.

15. The population of a village increases continuously at a rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Solution:

Let us assume that the population at any instant (t) be y.

Also, given that the rate of Increase in population is proportional to the number of inhabitants at any instant.

(dy/dx) $\propto$ y

$\Rightarrow$ (dy/dx) = ky

⇒ dy/y = kdt (Where k is a constant)

Now, integrating both sides of the above equation, we get

log y = kt + C … (1)

In 1999, t = 0 and y = 20000, we get log 20000 = C … (2)

In 2004, t = 5 and y = 25000, we get 1og 25000=k.5 + C

⇒ log 25000 = 5k + Iog 20000

⇒ 5k = log(25000/20000) = log(5/4)

⇒ k = (⅕) log(5/4) …(3)

In 2009, t = 10 years.

Substitute the values of k, t and C in (1), and we get

log y = 10 [(⅕) log(5/4)] + log 20000

On simplification, we get

y = (20000)(5/4)(5/4)

y = 31250

Therefore, the population of the village in 2009 was 31250.

16. The general solution of the differential equation [(y dx – x dy)/y] = 0 is:

1. **xy = C (B) x = Cy^2 (C) y = Cx (D) y = Cx^2**

Solution:

The differential equation is [(y dx – x dy)/y] = 0.

The given equation can be written as:

(ydx / y) – (xdy/y) = 0

Thus, we get

dx = xdy /y

dx/x = dy/y

(1/x)dx – (1/y)dy = 0

Now, integrating the above equation on both sides, we get

log |x| – log |y| = log k

⇒ log |x/y| = log k

⇒ x/y = k

⇒ y = (1/k) x

⇒ y = Cx [Where C = 1/k]

Hence, the correct answer is option (C) y = Cx.

17. The general solution of a differential equation of the type (dx/dy) + P_1x = Q_1 is:

A. $ye^{\int P_1 dy} = \int (Q_1 e^{\int P_1 dy}) dy + C$
B. $ye^{\int P_1 dx} = \int (Q_1 e^{\int P_1 dx}) dx + C$
C. $xe^{\int P_1 dy} = \int (Q_1 e^{\int P_1 dy}) dy + C$
D. $xe^{\int P_1 dx} = \int (Q_1 e^{\int P_1 dx}) dx + C$

Solution:

As we know, the integrating factor of the differential equation (dx/dy) + P_1x = Q_1 is $e^{\int P1\, dy}$.

Hence,

⇒ x. (I.F) = ∫ (Q_1 × I.F) dy + C

⇒ x. $e^{\int P1\, dy}$ = ∫ (Q_1 × $e^{\int P1\, dy}$) dy + C

Hence, the correct answer is option (C).

$$xe^{\int P_1 dy} = \int (Q_1 e^{\int P_1 dy}) dy + C$$

18. The general solution of the differential equation e^x dy + (y e^x + 2x) dx = 0 is:

A. **x e^y + x^2 = C**
B. **x e^y + y^2 = C**
C. **y e^x + x^2 = C**
D. **y e^y + x^2 = C**

Solution:

The correct answer is option (C) y e^x + x^2 = C

Explanation:

The given differential equation is $e^x dy + (y e^x + 2x) dx = 0$

$\Rightarrow e^x (dy/dx) + ye^x + 2x = 0$

Hence, we get

$\Rightarrow (dy/dx) + y = -2xe^{-x}$.

The above equation is the linear differential equation of the form $(dy/dx) + Py = Q$, where $P = 1$ and $Q = -2xe^{-x}$.

Now,

$I.F = e^{\int P\,dx} = e^{\int dx} = e^x$.

$\Rightarrow y. (I.F) = \int (Q \times I.F)\, dx + C$

$\Rightarrow y. e^x = \int (-2xe^{-x} \times e^x)\, dx + C$

$\Rightarrow ye^x = \int -2x\, dx + C$

$\Rightarrow ye^x = -x^2 + C$

On rearranging the above equation, we get

$\Rightarrow ye^x + x^2 = C$

Hence, option (C) is the general solution of the given differential equation.