EXERCISE 7.1 PAGE NO: 299

Find an anti-derivative (or integral) of the following functions by the method of inspection.

1. sin 2x

2. cos 3x

3. e^{2x}

4. $(ax + b)^2$

5. sin 2x – 4 e^{3x}

Solution:

1. sin 2x

The anti-derivative of sin 2x is a function of x whose derivative is sin 2x

We know that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

We get,

$$\sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

On further calculation, we get

$$\sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Hence, the anti derivative of sin 2x is – 1 / 2 cos 2x

2. cos 3x

The anti-derivative of cos 3x is a function of x whose derivative is cos 3x

We know that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$

We get,

$$\cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

On further calculation, we get

$$\cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Hence, the anti derivative of cos 3x is 1 / 3 sin 3x

3. e^{2x}

The anti-derivative of e^{2x} is the function of x whose derivative is e^{2x}

We know that,

$$\frac{d}{dx}\left(e^{2x}\right)=2e^{2x}$$

We get,

$$e^{2x}=\frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$

On further calculation, we get

$$e^{2x}=\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Hence, the anti derivative of e^{2x} is $1 / 2\ e^{2x}$

4. $(ax + b)^2$

The anti-derivative of $(ax + b)^2$ is the function of x whose derivative is $(ax + b)^2$

We know that,

$$\frac{d}{dx}(ax+b)^3=3a(ax+b)^2$$

On further multiplication, we get

$$(ax+b)^2=\frac{1}{3a}\frac{d}{dx}(ax+b)^3$$

Hence,

$$(ax+b)^2=\frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Thus, the anti derivative of $(ax + b)^2$ is $1 / 3a\ (ax + b)^3$

5. $\sin 2x - 4\ e^{3x}$

The anti-derivative of $(\sin 2x - 4\ e^{3x})$ is the function of x whose derivative of $(\sin 2x - 4e^{3x})$

We know that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)=\sin 2x-4e^{3x}$$

Hence, the anti derivative of $(\sin 2x - 43^{3x})$ is $(-1 / 2 \cos 2x - 4 / 3\ e^{3x})$

Find the following integrals in Exercises 6 to 20:

6. $\int\left(4e^{3x}+1\right)dx$

Solution:

We get,

$$= 4\int e^{3x}dx + \int 1dx$$

On further calculation, we obtain,

$$= 4\left(\frac{e^{3x}}{3}\right) + x + C$$

Therefore,

$$= \frac{4}{3}e^{3x} + x + C$$

7. $\int x^2\left(1 - \frac{1}{x^2}\right)dx$

Solution:

We get,

$$= \int (x^2 - 1)dx$$

On further calculation, we obtain,

$$= \int x^2 dx - \int 1dx$$

Hence,

$$= \frac{x^3}{3} - x + C$$

8. $\int (ax^2 + bx + c)dx$

Solution:

By taking the terms separately, we get,

$$= a\int x^2 dx + b\int x dx + c\int 1.dx$$

On further calculation, we obtain,

$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$

So, we get,

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

9. $\int (2x^2 + e^x)dx$

Solution:

By taking the terms separately, we get,

$$= 2\int x^2 dx + \int e^x dx$$

On further calculation, we get,

$$= 2\left(\frac{x^3}{3}\right) + e^x + C$$

Therefore,

$$= \frac{2}{3}x^3 + e^x + C$$

10. $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

Solution:

We get,

$$= \int\left(x + \frac{1}{x} - 2\right)dx$$

By taking the terms separately, we get,

$$= \int x dx + \int\frac{1}{x}dx - 2\int 1.dx$$

Hence, we get,

$$= \frac{x^2}{2} + \log|x| - 2x + C$$

11. $\int\frac{x^3 + 5x^2 - 4}{x^2}dx$

Solution:

We get,

$$= \int\left(x + 5 - 4x^{-2}\right)dx$$

By taking the terms separately, we get,

$$= \int x dx + 5\int 1.dx - 4\int x^{-2}dx$$

On further calculation, we obtain,

$$= \frac{x^2}{2} + 5x - 4\left(\frac{x^{-1}}{-1}\right) + C$$

Hence, we get,

$$= \frac{x^2}{2} + 5x + \frac{4}{x} + C$$

12. $\int \frac{x^3+3x+4}{\sqrt{x}}dx$

Solution:

We get,

$$= \int\left(x^{\frac{5}{2}}+3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\right)dx$$

On further calculation, we get,

$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+\frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}}+C$$

So,

$$= \frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$$

Hence,

$$= \frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8\sqrt{x}+C$$

13. $\int \frac{x^3-x^2+x-1}{x-1}dx$

Solution:

By dividing, we get,

$$= \int\left(x^2+1\right)dx$$

By taking the terms separately, we get,

$$= \int x^2dx+\int 1dx$$

Therefore, we obtain,

$$= \frac{x^3}{3}+x+C$$

14. $\int(1-x)\sqrt{x}dx$

Solution:

We get,

$$= \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx$$

On further calculation, we get,

$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$

So,

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$

Hence, we get,

$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$$

15. $\int \sqrt{x}\left(3x^2 + 2x + 3\right) dx$

Solution:

We get,

$$= \int \left(3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}} \right) dx$$

By taking the terms separately, we get,

$$= 3\int x^{\frac{5}{2}} dx + 2\int x^{\frac{3}{2}} dx + 3\int x^{\frac{1}{2}} dx$$

On further calculation, we get

$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}} \right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right) + 3\frac{\left(x^{\frac{3}{2}} \right)}{\frac{3}{2}} + C$$

Therefore, we get,

$$= \frac{6}{7} x^{\frac{7}{2}} + \frac{4}{5} x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

16. $\int \left(2x - 3\cos x + e^x\right) dx$

Solution:

By taking the terms separately, we get,

$$= 2\int x dx - 3\int \cos x dx + \int e^x dx$$

On further calculation, we get,

$$= \frac{2x^2}{2} - 3(\sin x) + e^x + C$$

Hence, we get,

$$= x^2 - 3\sin x + e^x + C$$

17. $\int \left(2x^2 - 3\sin x + 5\sqrt{x}\right) dx$

Solution:

By taking the terms separately, we get,

$$= 2\int x^2 dx - 3\int \sin x dx + 5\int x^{\frac{1}{2}} dx$$

On further calcualtion, we get,

$$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

Therefore, we get,

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$$

18. $\int \sec x(\sec x + \tan x) dx$

Solution:

On multiplication, we get,

$$= \int \left(\sec^2 x + \sec x \tan x\right) dx$$

By taking separately, we get,

$$= \int \sec^2 x dx + \int \sec x \tan x dx$$

We get,

$$= \tan x + \sec x + C$$

19. $\int \frac{\sec^2 x}{\cos ec^2 x} dx$

Solution:

We get,

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx$$

So,

$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$

We get,

$$= \int \tan^2 x dx$$

On further calculation, we get,

$$= \int (\sec^2 x - 1) dx$$

By taking separately, we get,

$$= \int \sec^2 x dx - \int 1 dx$$

Therefore, we get,

$$= \tan x - x + C$$

20. $\int \frac{2 - 3\sin x}{\cos^2 x} dx$

Solution:

By separating the terms, we get,

$$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$$

On further calculation, we get,

$$= \int 2\sec^2 x dx - 3\int \tan x \sec x dx$$

Hence, we obtain,

$$= 2\tan x - 3\sec x + C$$

Choose the correct answer in Exercises 21 and 22

21. The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ **equals**

(A) $(1 / 3)\ x^{1/3} + (2)\ x^{1/2} + C$
(B) $(2 / 3)\ x^{2/3} + (1 / 2)\ x^2 + C$
(C) $(2 / 3)\ x^{3/2} + (2)\ x^{1/2} + C$
(D) $(3 / 2)\ x^{3/2} + (1 / 2)\ x^{1/2} + C$

Solution:

Given

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$

We get,

$$=\int x^{\frac{3}{2}}dx+\int x^{-\frac{1}{2}}dx$$

On further calcualtion, we get,

$$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+C$$

Therefore, we get,

$$=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$$

Here, the correct answer is option (C)

22. If d / dx f (x) = $4x^3 - 3 / x^4$ such that f (2) = 0. Then f (x) is
(A) $x^4 + 1 / x^3 - 129 / 8$
(B) $x^3 + 1 / x^4 + 129 / 8$
(C) $x^4 + 1 / x^3 + 129 / 8$
(D) $x^3 + 1 / x^4 - 129 / 8$
Solution:

Given
d / dx f (x) = $4x^3 - 3 / x^4$
The anti derivative of $4x^3 - 3 / x^4$ = f (x)
Hence,

$$f(x)=\int 4x^3-\frac{3}{x^4}dx$$

By taking separately, we get,

$$f(x)=4\int x^3dx-3\int\left(x^{-4}\right)dx$$

We get,

$$f(x)=4\left(\frac{x^4}{4}\right)-3\left(\frac{x^{-3}}{-3}\right)+C$$

Now, we get,

$$f(x) = x^4 + \frac{1}{x^3} + C$$

Also, f (2) = 0

By substituting x = 2, we get,

$$f(2) = (2)^4 + \frac{1}{(2)^3} + C = 0$$

$$16 + \frac{1}{8} + C = 0$$

On further calculation, we get,

$$C = -\left(16 + \frac{1}{8}\right)$$

By taking L.C.M, we get,

$$C = \frac{-129}{8}$$

Hence, f (x) = $x^4 + 1 / x^3 - 129 / 8$

Therefore, the correct answer is option (A).

EXERCISE 7.2

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Integrate the functions in Exercises 1 to 37:

1. $2x / 1 + x^2$

Solution:

Let us take $1 + x^2 = t$

So, we get,

$2x\,dx = dt$

$$\int \frac{2x}{1+x^2}dx$$

We get,

$$= \int \frac{1}{t}dt$$

On further calculation, we get,

$$= \log|t| + C$$

Now, substituting $t = 1 + x^2$ we get,

$$= \log|1+x^2| + C$$

$$= \log(1 + x^2) + C$$

2. $(\log x)^2 / x$

Solution:

Let us take,

$\log |x| = t$

On differentiating, we get,

$$\frac{1}{x}dx = dt$$

$$\int \frac{(\log|x|)^2}{x}dx$$

We get,

$$= \int t^2 dt$$

On further calcualtion, we get,

$$= \frac{t^3}{3} + C$$

By substituting $t = \log |x|$ we get,

$$= \frac{(\log|x|)^3}{3} + C$$

3. 1 / (x + x log x)
Solution:

Given

$$\frac{1}{x + x\log x}$$

This can be written as

$$= \frac{1}{x(1+\log x)}$$

Let us take,

$1 + \log x = t$

We get,

1 / x dx = dt

So,

$$\int \frac{1}{x(1+\log x)}dx$$

We get,

$$= \int \frac{1}{t}dt$$

On calcualting further, we get

$$= \log|t| + C$$

Hence, we get,

$$= \log|1+\log x| + C$$

4. sin x sin (cos x)
Solution:

Let us take cos x = t

By differentiating, we get

- sin x dx = dt

Now,

$$\int \sin x \cdot \sin(\cos x)\,dx$$

We obtain,

$= -\int \sin t\, dt$

On further calculation, we get

$= -[-\cos t] + C$

$= \cos t + C$

By substituting t = cos x, we get

$= \cos(\cos x) + C$

5. Sin (ax + b) cos (ax + b)
Solution:

Given

$\sin(ax+b)\cos(ax+b)$

On integrating the above function, we get

$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2}$

We obtain,

$= \frac{\sin 2(ax+b)}{2}$

Let 2 (ax + b) = t

We get,

2a dx = dt

We get,

$\int \frac{\sin 2(ax+b)}{2} dx = \frac{1}{2}\int \frac{\sin t\; dt}{2a}$

On further calculation, we get,

$= \frac{1}{4a}[-\cos t] + C$

By putting t = 2 (ax + b), we get

$= \frac{-1}{4a}\cos 2(ax+b) + C$

6. √ax + b
Solution:

Let us take,
ax + b = t
We get,
a dx = dt
Hence,
dx = 1 / a dt
Now,

$$\int (ax+b)^{\frac{1}{2}}\,dx$$

We get,

$$=\frac{1}{a}\int t^{\frac{1}{2}}dt$$

On further calculation, we get

$$=\frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

Hence, we get,

$$=\frac{2}{3a}(ax+b)^{\frac{3}{2}}+C$$

7. x √x + 2
Solution:

Let us take,
(x + 2) = t
We get,
dx = dt
Now,

$$\int x\sqrt{x+2}\,dx$$

We get,

$$=\int (t-2)\sqrt{t}\,dt$$

On further calculating, we get

$$=\int\left(t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)dt$$

By taking separately, we get

$$=\int t^{\frac{3}{2}}dt-2\int t^{\frac{1}{2}}dt$$

So,

$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

By further calculation, we get

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

8. x √1 + 2x²

Solution:

Let us take,

$1 + 2x^2 = t$

We get,

$4x\,dx = dt$

$$\int x\sqrt{1+2x^2}\,dx$$

We obtain,

$$= \int \frac{\sqrt{t}\,dt}{4}$$

So,

$$= \frac{1}{4}\int t^{\frac{1}{2}}dt$$

On further calculation, we get

$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6}(1+2x^2)^{\frac{3}{2}} + C$$

9. (4x + 2) √x² + x + 1

Solution:

Let us take,

$x^2 + x + 1 = t$

We get,

$(2x + 1)\, dx = dt$

$$\int (4x+2)\sqrt{x^2+x+1}\; dx$$

We obtain,

$$= \int 2\sqrt{t}\, dt$$

$$= 2\int \sqrt{t}\, dt$$

On further calculation, we get

$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

10. 1 / (x - √x)

Solution:

Given

$$\frac{1}{x-\sqrt{x}}$$

This can be written as

$$= \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let us take,

$$\left(\sqrt{x}-1\right) = t$$

We get,

$$\frac{1}{2\sqrt{x}}\, dx = dt$$

$$\int \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)} dx = \int \frac{2}{t}\, dt$$

On further calculation, we get

$$= 2\log|t| + C$$

Hence, we obtain,

$$= 2\log|\sqrt{x} - 1| + C$$

11. x / (√x + 4), x > 0

Solution:

Let us take,

x + 4 = t

We get,

dx = dt

$$\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(t-4)}{\sqrt{t}}dt$$

So,

$$= \int\left(\sqrt{t} - \frac{4}{\sqrt{t}}\right)dt$$

On further calculation, we get

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t^{\frac{1}{2}}(t - 12) + C$$

By substituting t = x + 4, we obtain

$$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x + 4 - 12) + C$$

$$= \frac{2}{3}\sqrt{x+4}(x - 8) + C$$

12. $(x^3 - 1)^{1/3} x^5$

Solution:

Let us take,
$x^3 - 1 = t$
We get,
$3x^2 dx = dt$

$\int (x^3 - 1)^{\frac{1}{3}} x^5 dx$

We get,

$= \int (x^3 - 1)^{\frac{1}{3}} x^3 \cdot x^2 dx$

By putting $x^3 - 1 = t$, we obtain

$= \int t^{\frac{1}{3}} (t+1) \frac{dt}{3}$

$= \frac{1}{3} \int \left(t^{\frac{4}{3}} + t^{\frac{1}{3}} \right) dt$

On further calculation, we get

$= \frac{1}{3} \left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}} \right] + C$

$= \frac{1}{3} \left[\frac{3}{7} t^{\frac{7}{3}} + \frac{3}{4} t^{\frac{4}{3}} \right] + C$

$= \frac{1}{7} (x^3 - 1)^{\frac{7}{3}} + \frac{1}{4} (x^3 - 1)^{\frac{4}{3}} + C$

13. $x^2 / (2 + 3x^3)^3$
Solution:

Let us take,
$2 + 3x^3 = t$
We get,
$9x^2 dx = dt$

$\int \frac{x^2}{(2 + 3x^3)^3} dx$

So,

$= \frac{1}{9} \int \frac{dt}{(t)^3}$

On further calculation, we get

$$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right] + C$$

$$= \frac{-1}{18}\left(\frac{1}{t^2}\right) + C$$

$$= \frac{-1}{18\left(2+3x^3\right)^2} + C$$

14. 1 / x (log x) ^{m}, x > 0, m ≠ 1
Solution:

Let us take,
log x = t
We get,

$$\frac{1}{x}dx = dt$$

$$\int \frac{1}{x(\log x)^m} dx$$

We obtain,

$$= \int \frac{dt}{(t)^m}$$

On further calculation, we get

$$= \left(\frac{t^{-m+1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

15. x / (9 – 4x^2)
Solution:

Let us take,
$9 - 4x^2 = t$
We get,
$-8x\, dx = dt$
Now take,

$$\int \frac{x}{9-4x^2} dx$$

So,

$$= \frac{-1}{8} \int \frac{1}{t} dt$$

By further calculating, we obtain

$$= \frac{-1}{8} \log|t| + C$$

$$= \frac{-1}{8} \log|9 - 4x^2| + C$$

16. e^{2x+3}

Solution:

Let us take,

2x + 3 = t

We get,

2dx = dt

Now

$$\int e^{2x+3} dx$$

We obtain,

$$= \frac{1}{2} \int e^t dt$$

On further calculation, we get

$$= \frac{1}{2}(e^t) + C$$

$$= \frac{1}{2} e^{(2x+3)} + C$$

17. $\frac{x}{e^{x^2}}$

Solution:

Let us take,

$x^2 = t$

We get,

2x dx = dt

$$\int \frac{x}{e^{x^2}} dx$$

So,

$$= \frac{1}{2}\int \frac{1}{e^t}dt$$

$$= \frac{1}{2}\int e^{-t}dt$$

On further calculation, we get

$$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

$$= \frac{-1}{2e^{x^2}} + C$$

18. $\dfrac{e^{tan^{-1}x}}{1+x^2}$

Solution:

Let us take,

$\tan^{-1} x = t$

We get,

$$\frac{1}{1+x^2}dx = dt$$

$$\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$$

We obtain,

$$= \int e^t dt$$

By further calculation, we get

$$= e^t + C$$

$$= e^{\tan^{-1}x} + C$$

19. $\dfrac{e^{2x}-1}{e^{2x}+1}$

Solution:

By dividing numerator and denominator by e^x, we find

$$\frac{\frac{\left(e^{2x}-1\right)}{e^x}}{\frac{\left(e^{2x}+1\right)}{e^x}}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Let us assume,

$e^x+e^{-x}=t$

So,

$\left(e^x-e^{-x}\right)dx=dt$

$$\int\frac{e^{2x}-1}{e^{2x}+1}dx=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

We get,

$$=\int\frac{dt}{t}$$

By calculating further, we get

$=\log|t|+C$

$=\log\left|e^x+e^{-x}\right|+C$

20. $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Solution:

Let us assume,

$e^{2x}+e^{-2x}=t$

We get,

$\left(2e^{2x}-2e^{-2x}\right)dx=dt$

$2\left(e^{2x}-e^{-2x}\right)dx=dt$

Now

$$\int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx$$

We get,

$$=\int\frac{dt}{2t}$$

$= \frac{1}{2}\int \frac{1}{t}dt$

On calculating further, we get

$= \frac{1}{2}\log|t| + C$

$= \frac{1}{2}\log\left|e^{2x} + e^{-2x}\right| + C$

21.

$\tan^2(2x - 3)$

Solution:

$\tan^2(2x-3) = \sec^2(2x-3) - 1$

Let us take,

$2x - 3 = t$

We get,

$2dx = dt$

Now,

$\int \tan^2(2x-3)\,dx = \int\left[\left(\sec^2(2x-3)\right) - 1\right]dx$

By separating, we obtain

$= \frac{1}{2}\int\left(\sec^2 t\right)dt - \int 1dx$

$= \frac{1}{2}\int \sec^2 t\,dt - \int 1dx$

On further calculation, we get

$= \frac{1}{2}\tan t - x + C$

$= \frac{1}{2}\tan(2x-3) - x + C$

22.

$\sec^2(7 - 4x)$

Solution:

Let us take,

7 – 4x = t

We get,

$$-4dx = dt$$

Hence,

$$\int \sec^2(7-4x)\,dx = \frac{-1}{4}\int \sec^2 t\,dt$$

On calculating further, we get

$$= \frac{-1}{4}(\tan t) + C$$

$$= \frac{-1}{4}\tan(7-4x) + C$$

23.

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

Solution:

Let us take,

$\sin^{-1} x = t$

$$\frac{1}{\sqrt{1-x^2}}dx = dt$$

$$\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int t\,dt$$

We get,

$$= \frac{t^2}{2} + C$$

By substituting $t = \sin^{-1} x$, we get

$$= \frac{(\sin^{-1}x)^2}{2} + C$$

24.

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$$

Solution:

$$\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$$

This can be written as

$$=\frac{2\cos x-3\sin x}{2(3\cos x+2\sin x)}$$

Let us assume,

$3\cos x+2\sin x=t$

$(-3\sin x+2\cos x)dx=dt$

$$\int\frac{2\cos x-3\sin x}{6\cos x+4\sin x}\,dx=\int\frac{dt}{2t}$$

On further calculation, we get

$$=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

Therefore, we get

$$=\frac{1}{2}\log|2\sin x+3\cos x|+C$$

25. $\dfrac{1}{\cos^2 x\,(1-\tan x)^2}$

Solution:

$$\frac{1}{\cos^2 x(1-\tan x)^2}=\frac{\sec^2 x}{(1-\tan x)^2}$$

Let us assume,

$(1-\tan x)=t$

$-\sec^2 x dx=dt$

$$\int\frac{\sec^2 x}{(1-\tan x)^2}dx=\int\frac{-dt}{t^2}$$

We get,

$$=-\int t^{-2}dt$$

$$=+\frac{1}{t}+C$$

Therefore, we get

$$= \frac{1}{(1-\tan x)} + C$$

26. $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

Solution:

Let us take,

$$\sqrt{x} = t$$

$$\frac{1}{2\sqrt{x}}dx = dt$$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos t\, dt$$

By further calculation, we get

$$= 2\sin t + C$$

$$= 2\sin\sqrt{x} + C$$

27. $\sqrt{\sin 2x}\cos 2x$

Solution:

Let us take,

$$\sin 2x = t$$

$$2\cos 2x\, dx = dt$$

$$\Rightarrow \int \sqrt{\sin 2x}\,\cos 2x\, dx = \frac{1}{2}\int \sqrt{t}\, dt$$

On further calculation, we get

$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{3}t^{\frac{3}{2}} + C$$

By substituting t = sin 2x, we get

$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

28. $\frac{\cos x}{\sqrt{1+\sin x}}$

Solution:

Let us take,

$1+\sin x = t$

$\cos x \, dx = dt$

$$\int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$$

By further calculation, we get

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{1+\sin x} + C$$

29. $\cot x \log \sin x$

Solution:

Take

$\log \sin x = t$

By differentiation we get

$\frac{1}{\sin x} \cdot \cos x \, dx = dt$

So we get

$\cot x \, dx = dt$

Integrating both sides

$$\int \cot x \log \sin x \, dx = \int t \, dt$$

We get

$$= \frac{t^2}{2} + C$$

Substituting the value of t

$$= \frac{1}{2}(\log \sin x)^2 + C$$

30.

$\frac{\sin x}{1+\cos x}$

Solution:

Take 1 + cos x = t
By differentiation
- sin x dx = dt
By integrating both sides

$$\int \frac{\sin x}{1+\cos x}\, dx = \int -\frac{dt}{t}$$

So we get
= - log |t| + C
Substituting the value of t
= - log |1 + cos x| + C

31.

$$\frac{\sin x}{(1+\cos x)^2}$$

Solution:

Take 1 + cos x = t
By differentiation
- sin x dx = dt
Integrating both sides

$$\int \frac{\sin x}{(1+\cos x)^2}\, dx = \int -\frac{dt}{t^2}$$

We get

$$= -\int t^{-2} dt$$

It can be written as

$$= \frac{1}{t} + C$$

Substituting the value of t

$$= \frac{1}{1+\cos x} + C$$

32.

$$\frac{1}{1+\cot x}$$

Solution:

It is given that

$$I = \int \frac{1}{1+\cot x} dx$$

We can write it as

$$= \int \frac{1}{1+\frac{\cos x}{\sin x}} dx$$

By taking LCM

$$= \int \frac{\sin x}{\sin x + \cos x} dx$$

Multiply and divide by 2 in numerator and denominator

$$= \frac{1}{2} \int \frac{2\sin x}{\sin x + \cos x} dx$$

It can be written as

$$= \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx$$

On further calculation

$$= \frac{1}{2} \int 1\, dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

We get

$$= \frac{1}{2}(x) + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

Take sin x + cos x = t

By differentiation

(cos x – sin x) dx = dt

We get

$$I = \frac{x}{2} + \frac{1}{2} \int \frac{-(dt)}{t}$$

By integration

$$= \frac{x}{2} - \frac{1}{2} \log|t| + \mathrm{C}$$

Substituting the value of t

$$= \frac{x}{2} - \frac{1}{2} \log|\sin x + \cos x| + \mathrm{C}$$

33.

$$\frac{1}{1-\tan x}$$

Solution:

It is given that

$$I=\int\frac{1}{1-\tan x}dx$$

We can write it as

$$=\int\frac{1}{1-\frac{\sin x}{\cos x}}dx$$

By taking LCM

$$=\int\frac{\cos x}{\cos x-\sin x}dx$$

Multiply and divide by 2 in numerator and denominator

$$=\frac{1}{2}\int\frac{2\cos x}{\cos x-\sin x}dx$$

It can be written as

$$=\frac{1}{2}\int\frac{(\cos x-\sin x)+(\cos x+\sin x)}{(\cos x-\sin x)}dx$$

On further calculation

$$=\frac{1}{2}\int 1\,dx+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

We get

$$=\frac{x}{2}+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

Take cos x - sin x = t

By differentiation

(-sin x – cos x) dx = dt

We get

$$I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$

By integration

$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

Substituting the value of t

$$= \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$$

34.
$$\frac{\sqrt{\tan x}}{\sin x \cos x}$$
Solution:

It is given that

$$I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$$

By multiplying cos x to both numerator and denominator

$$= \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$$

On further calculation

$$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$$

So we get

$$= \int \frac{\sec^2 x \, dx}{\sqrt{\tan x}}$$

Take tan x = t

We get $\sec^2 x$ dx = dt

$$I = \int \frac{dt}{\sqrt{t}}$$

By integration we get

$$= 2\sqrt{t} + C$$

Substituting the value of t

$$= 2\sqrt{\tan x} + C$$

35.
$$\frac{(1 + \log x)^2}{x}$$
Solution:

Consider

1 + log x = t

So we get

$$\frac{1}{x}dx = dt$$

Integrating both sides

$$\int\frac{(1+\log x)^2}{x}\,dx = \int t^2 dt$$

We get

$$=\frac{t^3}{3}+C$$

Substituting the value of t

$$=\frac{(1+\log x)^3}{3}+C$$

36.
$$\frac{(x+1)(x+\log x)^2}{x}$$
Solution:

It is given that

$$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2$$

We can write it as

$$=\left(1+\frac{1}{x}\right)(x+\log x)^2$$

Consider x + log x = t

By differentiation

$$\left(1+\frac{1}{x}\right)dx = dt$$

Integrating both sides

$$\int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx = \int t^2 dt$$

So we get

$$= \frac{t^3}{3} + C$$

Substituting the value of t

$$= \frac{1}{3}(x + \log x)^3 + C$$

37.
$\frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8}$
Solution:

It is given that

$$\frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8}$$

Consider $x^4 = t$

We get $4x^3\, dx = dt$

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8} dx = \frac{1}{4} \int \frac{\sin\left(\tan^{-1} t\right)}{1+t^2} dt \quad \ldots(1)$$

Similarly take $\tan^{-1} t = u$

By differentiation we get

$$\frac{1}{1+t^2} dt = du$$

Using equation (1) we get

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right) dx}{1+x^8} = \frac{1}{4} \int \sin u \, du$$

By integration

$$= \frac{1}{4}(-\cos u) + C$$

Substituting the value of u

$$= \frac{-1}{4} \cos\left(\tan^{-1} t\right) + C$$

Now substituting the value of t

$$= \frac{-1}{4} \cos\left(\tan^{-1} x^4\right) + C$$

Choose the correct answer in Exercises 38 and 39.

38. $\int \frac{10x^9 + 10^x log_e 10dx}{x^{10} + 10^x}$ $equals$

$(A) 10^x - x^{10} + C$
$(B) 10^x + x^{10} + C$
$(C) (10^x - x^{10})^{-1} + C$
$(D) log(10^x + x^{10}) + C$

Solution:

Take $x^{10} + 10^x = t$

Differentiating both sides

$$(10x^9 + 10^x \log_e 10)dx = dt$$

Integrating both sides we get

$$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t}$$

So we get

$= \log t + C$

Substituting the value of t

$= \log (10^x + x^{10}) + C$

Therefore, D is the correct answer.

39. $\int \frac{dx}{sin^2 x cos^2 x}$ $equals$

(A) tan x + cot x + C
(B) tan x – cot x + C
(C) tan x cot x + C
(D) tan x – cot 2x + C

Solution:

It is given that

$$I = \int \frac{dx}{\sin^2 x \cos^2 x}$$

We can write it as

$$= \int \frac{1}{\sin^2 x \cos^2 x} dx$$

Here we get

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

By separating the terms

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

We get

$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$

By integration

= tan x – cot x + C

Therefore, B is the correct answer.

EXERCISE 7.3

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1. $\sin^2(2x + 5)$

Solution:-

We have,

By standard trigonometric identity, $\sin^2x = (1 - \cos4x)/2$

$$\sin^2(2x+5) = \frac{1-\cos2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2}$$

Taking integrals on both sides, we get,

$$= \int \sin^2(2x+5)dx = \int \frac{1-\cos(4x+10)}{2}dx$$

Splitting the integrals,

$$= \frac{1}{2}\int 1.dx - \frac{1}{2}\int \cos(4x+10)dx$$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos(4x+10)dx$$

On integrating, we get,

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$$

$$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$$

2. sin 3x cos 4x

Solution:-

By standard trigonometric identity sinA cosB = ½ {sin(A + B) + cos(A - B)}

$$\int \sin3x\cos4x dx = \frac{1}{2}\int\{\sin(3x+4x)+\sin(3x-4x)\}dx$$

On simplifying,

$$= \frac{1}{2}\int\left\{\sin 7x + \sin(-x)\right\}dx$$

$$= \frac{1}{2}\int\left\{\sin 7x - \sin x\right\}dx$$

Splitting the integrals, we have,

$$= \frac{1}{2}\int \sin 7x dx - \frac{1}{2}\int \sin x dx$$

On integrating, we get,

$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + C$$

$$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$$

3. cos 2x cos 4x cos 6x

Solution:-

By standard trigonometric identity cosA cosB = ½ {cos(A + B) + cos(A - B)}

$$\int \cos 2x \cos 4x \cos 6x dx = \int \cos 2x\left[\frac{1}{2}\left\{\cos(4x+6x) + \cos(4x-6x)\right\}\right]dx$$

$$= \frac{1}{2}\int\left\{\cos 2x \cos 10x + \cos 2x \cos(-2x)\right\}dx$$

We know that, cos(-x) = cos x,

$$= \frac{1}{2}\int\left\{\cos 2x \cos 10x + \cos^2 2x\right\}dx$$

Again by, standard trigonometric identity cosA cosB = ½ {cos(A + B) + cos(A - B)} and $\cos^2 2x = (1 + \cos 4x)/2$

$$= \frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x) + \cos(2x-10x)\right\} + \left(\frac{1+\cos 4x}{2}\right)\right]dx$$

On simplifying, we get,

$$= \frac{1}{4}\int\left(\cos 12x + \cos 8x + 1 + \cos 4x\right)dx$$

By integrating,

$$= \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + x + \frac{\sin 4x}{4}\right] + C$$

4. $\sin^3 (2x + 1)$

Solution:-

Given, $\sin^3(2x+1)$

By splitting,

$$= \int \sin^3(2x+1)dx = \int \sin^2(2x+1).\sin(2x+1)dx$$

We know that, $\sin^2 x = 1 - \cos^2 x$

$$= \int\left(1 - \cos^2(2x+1)\right)\sin(2x+1)dx$$

Let us assume cos (2x+1) = t

Then,

=> $-2\sin(2x+1)dx = dt$

=> $\sin(2x+1)dx = \frac{-dt}{2}$

$\sin^3(2x+1) = \frac{-1}{2}\int\left(1 - t^2\right)dt$

$$= \frac{-1}{2}\left\{t - \frac{t^3}{3}\right\}$$

Now substitute the value 't' in equation,

$$= \frac{-1}{2}\left\{\cos(2x+1) - \frac{\cos^3(2x+1)}{3}\right\}$$

$$= \frac{-\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C$$

5. $\sin^3 x \cos^3 x$
Solution:-

Given, $\int \sin^3 x \cos^3 x.dx$

By splitting the given function,

$$= \int \cos^3 x.\sin^2 x.\sin x.dx$$

We know that, $\sin^2 x = 1 - \cos^2 x$

$$= \int \cos^3 x\left(1-\cos^2 x\right)\sin x.dx$$

So, let us assume cosx = t

Then,

$\Rightarrow$ -sinx × dx = dt

$\sin^3 x \cos^3 x \quad = -\int t^3\left(1-t^2\right)dt$

$$= -\int\left(t^3 - t^5\right)dt$$

On integrating, we get,

$$= -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$$

Now substitute the value 't' in equation,

$$= -\left\{\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right\} + C$$

$$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$$

6. sin x sin 2x sin 3x
Solution:-

By standard trigonometric identity sinA sinB = -½ {cos (A + B) - cos (A - B)}

$$\int \sin x \sin 2x \sin 3x\, dx = \int \sin x . \frac{1}{2}\left[\left\{\cos(2x-3x) - \cos(2x+3x)\right\}\right] dx$$

On simplifying, we get,

$$= \frac{1}{2}\int \left\{\sin x \cos(-x) - \sin x \cos 5x\right\} dx$$

We know that, cos(-x) = cos x,

$$= \frac{1}{2}\int \left\{\sin x \cos x - \sin x \cos 5x\right\} dx$$

Splitting the integrals, by using sin 2x = 2sinx cosx, we get,

$$= \frac{1}{2}\int \frac{\sin 2x}{2} dx - \frac{1}{2}\int \sin x \cos 5x\, dx$$

On integrating the first term, and substituting sinA cosB = ½ {sin(A + B) + sin (A - B)}

$$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int \left\{\frac{1}{2}\sin(x+5x) + \sin(x-5x)\right\} dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int (\sin 6x + \sin(-4x)) dx$$

Computing and simplifying, we get,

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{4}\right] + C$$

$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + C$$

$$= \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$$

7. sin 4x sin 8x
Solution:-

By standard trigonometric identity sinA sinB = -½ {cos (A + B) - cos (A - B)}

Then,

$$\int \sin 4x \sin 8x dx = \int \left\{ \frac{1}{2} \cos(4x - 8x) - \cos(4x + 8x) \right\} dx$$

$$= \frac{1}{2} \int (\cos(-4x) - \cos 12x dx$$

We know that, cos (-x) = cos x,

$$= \frac{1}{2} \int \{ \cos 4x - \cos 12x \} dx$$

On integrating we get,

$$= \frac{1}{2} \left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right] + C$$

8. $\frac{1 - \cos x}{1 + \cos x}$

Solution:-

By standard trigonometric identity, we have,

$$\frac{1 - \cos x}{1 + \cos x} = \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}}$$

We know that, (Sin x/cos x) = tan x

$$= 2 \tan^2 \frac{x}{2}$$

Also, we know that, $\tan^{-1}$ x = sec x

$$= \left(\sec^2 \frac{x}{2} - 1 \right)$$

Integrating both the sides, we get,

$$\therefore \int \frac{1-\cos x}{1+\cos x}dx = \int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$= \left[\frac{\tan\frac{x}{2}}{\frac{1}{2}} - x\right] + C$$

$$= 2\tan\frac{x}{2} - x + C$$

9. $\dfrac{\cos x}{1+\cos x}$

Solution:-

By standard trigonometric identity, we have,

$$\frac{\cos x}{1+\cos x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

We know that, (Sin x/cos x) = tan x and takeout ½ as common, we get

$$= \frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]$$

Integrating both the sides, we get,

$$\int \frac{\cos x}{1+\cos x}dx = \int \frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]dx$$

Using standard trigonometric identity $\tan^2 x + 1 = \sec^2 (x)$

$$= \frac{1}{2}\int\left[2-\sec^2\frac{x}{2}\right]dx$$

On integrating, we get,

$$= \frac{1}{2}\left[2x - \frac{\tan\frac{x}{2}}{\frac{1}{2}}\right] + C$$

$$= x - \tan\frac{x}{2} + C$$

10. $\sin^4 x$
Solution:-
By splitting the given function, we get,

$\sin^4 x = \sin^2 x \sin^2 x$

By standard trigonometric identity, we have, $\sin^2 x = (1 - \cos 2x)/2$

$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{4}(1-\cos 2x)^2$$

By using the formula $(a - b)^2 = a^2 - 2ab + b^2$, we get,

$$= \frac{1}{4}\left[1+\cos^2 2x - 2\cos 2x\right]$$

From the standard trigonometric identity, $\cos^2 2x = (1 + \cos 4x)/2$

$$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$$

On simplifying, we get,

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right]$$

Integrating on both the sides,

$$\int \sin^4 x dx = \frac{1}{4}\int\left[\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right]dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \frac{1}{2}\left(\frac{\sin 4x}{4}\right) - \frac{2\sin 2x}{2}\right] + C$$

By simplifying,

$$= \frac{1}{8}\left[3x + \left(\frac{\sin 4x}{4}\right) - 2\sin 2x\right] + C$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

11. $\cos^4 2x$

Solution:-

By splitting the given function,

$\cos^4 2x = (\cos^2 2x)^2$

By standard trigonometric identity, we have, $\cos^2 2x = (1 + \cos 4x)/2$

$$= \left(\frac{1 + \cos 4x}{2}\right)^2$$

On simplifying, we get,

$$= \frac{1}{4}\left[1 + \cos^2 4x - 2\cos 4x\right]$$

By standard trigonometric identity, we have, $\cos^2 2x = (1 + \cos 4x)/2$

$$= \frac{1}{4}\left[1 + \left(\frac{1 + \cos 8x}{2}\right) + 2\cos 4x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{1}{2}\cos 8x + 2\cos 4x\right]$$

By simplifying,

$$= \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\cos 8x + 2\cos 4x\right]$$

Integrating both side,

$$\int \cos^4 2x dx = \int\left[\frac{3}{8} + \frac{1}{8}\cos 8x + \frac{1}{2}\cos 4x\right]dx$$

$$= \frac{3x}{8} + \frac{1}{64}\sin 8x + \frac{1}{8}\sin 4x + C$$

12. $\frac{\sin^2 x}{1+\cos x}$

Solution:-

By standard trigonometric identity, we have,

$$\frac{\sin^2 x}{1+\cos x} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$= \frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

On simplifying, we get,

$$= 2\sin^2\frac{x}{2}$$

From the standard trigonometric identity, we have, $1 - \cos x = 2\sin^2\frac{x}{2}$

= 1- cosx

On integrating both the sides, we get,

$$\int \frac{\sin^2 x}{1+\cos x} dx = \int (1-\cos x) dx$$

$$= x - \sin x + C$$

13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

Solution:-

By using the trigonometry identity i.e.,

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

So, we have,

$$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{-2\sin \frac{2x+2\alpha}{2} \sin \frac{2x-2\alpha}{2}}{-2\sin\sin \frac{x+\alpha}{2} \sin\sin \frac{x-\alpha}{2}}$$

By simplifying, we get,

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

Then,

From the identity sin 2x = 2 sin x cos x, we have

$$= \frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

On simplifying, we get,

$$= 4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

By using the trigonometry identity 2 cos A cos B = cos (A + B) + cos (A - B), we have

$$= 2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

$= 2[\cos(x) + \cos\alpha]$

$= 2\cos x + 2\cos\alpha$

Then,

Integrating on both the sides,

$$\int \therefore \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx = \int(2\cos x + 2\cos\alpha)dx$$

We have,

$= 2[\sin x + x\cos\alpha] + C$

14. $\dfrac{\cos x - \sin x}{1+\sin 2x}$

Solution:-

Given = $\dfrac{\cos x - \sin x}{1+\sin 2x}$

By using the standard trigonometric identity, $(1 + \sin 2x) = \sin^2 x + \cos^2 x + 2\sin x\cos x$.

Then,

$$= \frac{\cos x - \sin x}{(\sin^2 x + \cos^2 x) + 2\sin x\cos x}$$

$$= \frac{\cos x - \sin x}{(\sin x + \cos x)^2}$$

Now,

Let us assume that, sinx + cosx = t

And also, (cosx-sinx)dx = dt

Integrating on both the sides and substitute the value of (cosx – sinx) dx and (sinx + cosx) we get,

$$= \int \frac{\cos x - \sin x}{1+\sin 2x} dx = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx$$

$$= \int \frac{dt}{t^2}$$

$$= -t^{-1} + C$$

$$= -\frac{1}{t} + C$$

$$= \frac{-1}{\sin x + \cos x} + C$$

15. $\tan^3 2x \sec 2x$

Solution:-

By splitting the given function, we have,

$\tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$

From the standard trigonometric identity, $\tan^2 2x = \sec^2 2x – 1$,

$= (\sec^2 2x - 1) \tan 2x \sec 2x$

By multiplying, we get,

$= (\sec^2 2x \times \tan 2x \sec 2x) – (\tan 2x \sec 2x)$

Integrating both sides,

$$\int \tan^3 2x \sec 2x dx = \int \sec^2 2x \tan 2x \sec 2x dx - \int \tan 2x \sec 2x dx$$

$$= \int \sec^2 2x \tan 2x \sec 2x dx - \frac{\sec 2x}{2} + C$$

Then,

Let us assume sec2x = t

And also assume 2sec2x tan2x dx = dt

$$\int \tan^3 2x \sec 2x dx = \frac{1}{2}\int t^2 dt - \frac{\sec 2x}{2} + C$$

On simplifying, we get,

$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$

$$= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$$

16. $\tan^4 x$

Solution:-

By splitting the given function, we have,

$\tan^4 x = \tan^2 x \times \tan^2 x$

Then,

From trigonometric identity, $\tan^2 x = \sec^2 x - 1$

$= (\sec^2 x - 1) \tan^2 x$

By multiplying, we get,

$= \sec^2 x \tan^2 x - \tan^2 x$

Again by using trigonometric identity, $\tan^2 x = \sec^2 x - 1$

$= \sec^2 x \tan^2 x - (\sec^2 x - 1)$

$= \sec^2 x \tan^2 x - \sec^2 x + 1$

Now, integrating on both sides we get,

$$\int \tan^4 x dx = \int \sec^2 x \tan^2 x dx - \int \sec^2 x dx - \int 1 . dx$$

$$= \int \sec^2 x \tan^2 x dx - \tan x + x + C$$

Then, let us assume tanx = t

And also assume $\sec^2 x$ dx =dt

$$\int \sec^2 x \tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

$$\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Solution:-

By splitting up the given function,

$$\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}=\frac{\sin^3 x}{\sin^2 x\cos^2 x}+\frac{\cos^3 x}{\sin^2 x\cos^2 x}$$

By simplifying, we get,

$$=\frac{\sin x}{\cos^2 x}+\frac{\cos x}{\sin^2 x}$$

We know that, (sinx/cosx) = tanx and (1/cosx) = secx.

Again, we have (cosx/sinx) = cotx and (1/sinx) = cosecx

= tanx secx + cotx cosecx

Integrating on both the sides, we get

$$\int\frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}dx=\int(\tan x\sec x+\cot x\operatorname{cosec}x)dx$$

= secx – cosecx + C

18. $\dfrac{\cos 2x+2\sin^2 x}{\cos^2 x}$

Solution:-

By using the standard trigonometric identity, $2\sin^2 x = (1 - \cos 2x)$

$$\frac{\cos 2x+2\sin^2 x}{\cos^2 x}=\frac{\cos 2x+(1-\cos 2x)}{\cos^2 x}$$

By simplification, we get,

$$=\frac{1}{\cos^2 x}$$

We know that, $(1/\cos^2 x) = \sec^2 x$

$= \sec^2 x$

Integrating on both sides, we get,

$$\int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx=\int\sec^2 x\,dx$$

= tanx + C

19. $\dfrac{1}{\sin x \cos^3 x}$

Solution:-

For further simplification, the given function can be written as,

$$\frac{1}{\sin x \cos^3 x} = \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x}$$

Divide both numerator and denominator by $\cos^2 x$

$$= \tan x \sec^2 x + \frac{\frac{1}{\cos^2 x}}{\frac{\sin x \cos x}{\cos^2 x}}$$

On simplification, we get,

$$= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

By applying the integrals, we get,

$$\int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x} dx$$

Let us assume that, $\tan x = t$

Then, $\sec^2 x \, dx = dt$

By substituting above values, we get,

$$\int \frac{1}{\sin x \cos^3 x} dx = \int t dt + \int \frac{1}{t} dt$$

On integrating,

$$= \frac{t^2}{2} + \log|t| + C$$

Now, by substituting the value of 't' we get,

$$= \frac{1}{2} \tan^2 x + \log|\tan x| + C$$

20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$

Solution:-

We know that, $(\cos x + \sin x)^2 = \cos^2 x + \sin^2 x + 2\sin x\cos x$

$$\frac{\cos 2x}{(\cos x+\sin x)^2} = \frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x}$$

And also we know that, $\cos^2 x + \sin^2 x = 1$ and $2\sin x \cos x = \sin 2x$,

Then,

$$= \frac{\cos 2x}{1+\sin 2x}$$

By applying the integrals, we get,

$$\int \frac{\cos 2x}{(\cos x+\sin x)^2}dx = \int \frac{\cos 2x}{1+\sin 2x}dx$$

Let us assume that, $1 + \sin 2x = t$

So, $2\cos 2x\, dx = dt$

By substituting above values, we get,

$$\int \frac{\cos 2x}{(\cos x+\sin x)^2}dx = \frac{1}{2}\int \frac{1}{t}dt$$

On integrating,

$$= \frac{1}{2}\log|t| + C$$

Now, by substituting the value of 't' we get,

$$= \frac{1}{2}\log|1+\sin 2x| + C$$

$$= \frac{1}{2}\log\left|(\cos x+\sin x)^2\right| + C$$

$= \log|\sin x + \cos x| + C$

21. $\sin^{-1}(\cos x)$

Solution:-

Given, $\sin^{-1}(\cos x)$

Let us assume $\cos x = t$ … [equation (i)]

Then, substitute 't' in place of $\cos x$

= $\sin^{-1}(t)$

Sinx = $\sqrt{1-t^2}$

So, now differentiating both sides of (i), we get,

(-sinx)dx = dt

dx = $\frac{-dt}{\sin x}$

dx = $\frac{-dt}{\sqrt{1-t^2}}$

By applying the integrals, we get,

$$\int \sin^{-1}(\cos x)\,dx = \int \sin^{-1}t\left(\frac{-dt}{\sqrt{1-t^2}}\right)$$

$$= \int \frac{\sin^{-1}t}{\sqrt{1-t^2}}\,dt$$

Let us assume that, $\sin^{-1} t = v$

$$\frac{dt}{\sqrt{1-t^2}} = dv$$

$\int \sin^{-1}(\cos x)\, dx = - \int v\,dv$

On integrating,

$$= -\frac{v^2}{2} + C$$

Now, by substituting the value of 'V' and 't', we get,

$$= -\frac{\left(\sin^{-1}t\right)^2}{2} + C$$

$$= -\frac{\left(\sin^{-1}(\cos x)\right)^2}{2} + C \quad \text{... [equation (ii)]}$$

As we know that,

sin-1x + cos-1 x = $\frac{\pi}{2}$

22. $\dfrac{1}{\cos(x-a)\cos(x-b)}$

Solution:-

Multiplying and dividing by sin (a - b) to given function, we get,

$$\frac{1}{\cos(x-a)\cos(x-b)}=\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

For further simplification, the given function can be written as,

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$

Using sin (A - B) = sin A cos B – cos A sin B formula, we get,

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right]$$

We know that, sinx/cosx = tan x by applying this formula we get,

$$=\frac{1}{\sin(a-b)}\left[\tan(x-b)-\tan(x-a)\right]$$

Taking integrals,

$$\int\frac{1}{\cos(x-a)\cos(x-b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x-b)-\tan(x-a)\right]dx$$

On integrating,

$$=\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x-b)\right|+\log\left|\cos(x-a)\right|\right]$$

We know that, log (a/b) = log a – log b, using in above equation, we get,

$$=\frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Choose the correct answer in Exercises 23 and 24.

23. $\int\dfrac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$ is equal to

(A) $\tan x+\cot x+C$ (B) $\tan x+\operatorname{cosec} x+C$

(C) $-\tan x+\cot x+C$ (D) $\tan x+\sec x+C$

Solution:-

(A) $\tan x + \cot x + C$

By splitting the denominators of given equation,

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx$$

On simplifying, we get,

$$= \int \left(\sec^2 x - \operatorname{cosec}^2 x \right) dx$$

As we know that,

$\int \sec^2 x \, dx = \tan x + c$

$\int \operatorname{cosec}^2 x \, dx = - \cot x + c$

$= \tan x + \cot x + C$

24. $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$ equals

(A) $-\cot(ex^x) + C$

(B) $\tan(xe^x) + C$

(C) $\tan(e^x) + C$

(D) $\cot(e^x) + C$

Solution:-

(B) $\tan(xe^x) + C$

Let us assume that, $(xe^x) = t$

Differentiating both sides we get,

$((e^x \times x) + (e^x \times 1))\, dx = dt$

$e^x (x + 1) = dt$

Applying integrals,

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx = \int \frac{dt}{\cos^2 t}$$

We know that, $(1/\cos^2 t) = \sec^2 t$

$$= \int \sec^2 t . dt$$

$= \tan t + C$

Substituting the value of 't',

$= \tan(e^x x) + C$

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Integrate the functions in Exercises 1 to 23.

1. $\dfrac{3x^2}{x^6+1}$

Solution:-

Let us assume that $x^3 = t$

Then, $3x^2\,dx = dt$

By applying integrals, we get,

$$\int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$

On integrating,

$= \tan^{-1}t + C$

No, Substitute the value of t,

$= \tan^{-1}(x^3) + C$

2.

$\dfrac{1}{\sqrt{1+4x^2}}$

Solution:

Take $2x = t$

We get $2x\,dx = dt$

Integrating both sides

$$\int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

Using the formula

$$\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|$$

We get

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right]+C$$

Substituting the value of t

$$= \frac{1}{2}\log\left|2x + \sqrt{4x^2 + 1}\right| + C$$

3.

$$\frac{1}{\sqrt{(2-x)^2 + 1}}$$

Solution:

Take 2 - x = t

We get - dx = dt

Integrating both sides

$$\int \frac{1}{\sqrt{(2-x)^2 + 1}}dx = -\int \frac{1}{\sqrt{t^2 + 1}}dt$$

Using the formula

$$\int \frac{1}{\sqrt{x^2 + a^2}}dt = \log\left|x + \sqrt{x^2 + a^2}\right|$$

We get

$$= -\log\left|t + \sqrt{t^2 + 1}\right| + C$$

Substituting the value of t

$$= -\log\left|2 - x + \sqrt{(2-x)^2 + 1}\right| + C$$

$$= \log\left|\frac{1}{(2-x) + \sqrt{x^2 - 4x + 5}}\right| + C$$

4.

$$\frac{1}{\sqrt{9 - 25x^2}}$$

Solution:

Take 5x = t

We get 5dx = dt

Integrating both sides

$$\int \frac{1}{\sqrt{9 - 25x^2}}\, dx = \frac{1}{5}\int \frac{1}{\sqrt{9 - t^2}}dt$$

We get

$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2 - t^2}}\, dt$$

On further calculation

$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$

Substituting the value of t

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

5.

$$\frac{3x}{1+2x^4}$$

Solution:

Take $\sqrt{2}\, x^2 = t$

We get $2\sqrt{2}\, x\, dx = dt$

Integrating both sides

$$\int \frac{3x}{1+2x^4} dx = \frac{3}{2\sqrt{2}} \int \frac{dt}{1+t^2}$$

On further calculation

$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$

Substituting the value of t

$$= \frac{3}{2\sqrt{2}} \tan^{-1}\left(\sqrt{2}x^2\right) + C$$

6.

$$\frac{x^2}{1-x^6}$$

Solution:

Take $x^3 = t$

We get $3x^2\, dx = dt$

Integrating both sides

$$\int \frac{x^2}{1-x^6} dx = \frac{1}{3} \int \frac{dt}{1-t^2}$$

On further calculation

$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$

Substituting the value of t

$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

7.
$$\frac{x-1}{\sqrt{x^2-1}}$$
Solution:

By separating the terms

$$\int \frac{x-1}{\sqrt{x^2-1}}\,dx = \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx \qquad \ldots(1)$$

Take

$$\int \frac{x}{\sqrt{x^2-1}}dx$$

If $x^2 - 1 = t$ we get $2x\,dx = dt$

$$\int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

It can be written as

$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

By integration

$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$

$$= \sqrt{t}$$

Substituting the value of t

$$= \sqrt{x^2-1}$$

Using equation (1) we get

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx$$

From formula

$$\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|$$

We get

$$= \sqrt{x^2 - 1} - \log\left|x + \sqrt{x^2 - 1}\right| + C$$

8.

$$\frac{x^2}{\sqrt{x^6 + a^6}}$$

Solution:

Take $x^3 = t$

We get $3x^2\,dx = dt$

Integrating both sides

$$\int \frac{x^2}{\sqrt{x^6 + a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 + \left(a^3\right)^2}}$$

On further calculation

$$= \frac{1}{3}\log\left|t + \sqrt{t^2 + a^6}\right| + C$$

Substituting the value of t

$$= \frac{1}{3}\log\left|x^3 + \sqrt{x^6 + a^6}\right| + C$$

9.

$$\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$$

Solution:

Take $\tan x = t$

We get $\sec^2 x\,dx = dt$

Integrating both sides

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

On further calculation

$$= \log\left|t + \sqrt{t^2 + 4}\right| + C$$

Substituting the value of t

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

10.
$\frac{1}{\sqrt{x^2+2x+2}}$
Solution:

It is given that

$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{(x+1)^2+(1)^2}}dx$$

Take x + 1 = t

We get dx = dt

Integrating both sides

$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{t^2+1}}dt$$

On further calculation

$$= \log\left|t+\sqrt{t^2+1}\right| + C$$

Substituting the value of t

$$= \log\left|(x+1)+\sqrt{(x+1)^2+1}\right| + C$$

So we get

$$= \log\left|(x+1)+\sqrt{x^2+2x+2}\right| + C$$

11.
$\frac{1}{9x^2+6x+5}$
Solution:

It is given that

$$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{(3x+1)^2+(2)^2}dx$$

Take (3x + 1) = t

We get 3dx = dt

Integrating both sides

$$\int \frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int \frac{1}{t^2+2^2}dt$$

On further calculation

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right] + C$$

Substituting the value of t

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$$

12.

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Solution:

It is given that

$$\frac{1}{\sqrt{7-6x-x^2}}$$

We can write it as

$7 - 6x - x^2 = 7 - (x^2 + 6x + 9 - 9)$

By further calculation

$= 16 - (x^2 + 6x - 9)$

We get

$= 16 - (x + 3)^2$

$= 4^2 - (x + 3)^2$

Here

$$\int \frac{1}{\sqrt{7-6x-x^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx$$

Consider $x + 3 = t$

We get $dx = dt$

Integrating both sides

$$\int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (t)^2}} dt$$

We get

$$= \sin^{-1}\left(\frac{t}{4}\right) + C$$

Substituting the value of t

$$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

13.

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Solution:

It is given that

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

We can write it as

$(x - 1)(x - 2) = x^2 - 3x + 2$

By further calculation

$= x^2 - 3x + 9/4 - 9/4 + 2$

We get

$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

Here

$$\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

Consider $x - 3/2 = t$

We get $dx = dt$

Integrating both sides

$$\int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx = \int \frac{1}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} dt$$

We get

$$= \log\left|t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

Substituting the value of t

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

14.

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Solution:

It is given that

$$\frac{1}{\sqrt{8+3x-x^2}}$$

We can write it as

$8 + 3x - x^2 = 8 - (x^2 - 3x + 9/4 - 9/4)$

By further calculation

$$= \frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

Here

$$\int\frac{1}{\sqrt{8+3x-x^2}}dx = \int\frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Consider $x - 3/2 = t$

We get $dx = dt$

Integrating both sides

$$\int\frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx = \int\frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$$

We get

$$= \sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

Substituting the value of t

$$= \sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

On further calculation

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

15.

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Solution:

It is given that

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

We can write it as

$(x - a)(x - b) = x^2 - (a + b)x + ab$

By further calculation

$$= x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

Here

$$= \left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

Integrating both sides

$$\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$$

Consider

$$x-\left(\frac{a+b}{2}\right)=t$$

We get $dx = dt$

$$\int \frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx = \int \frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

It can be written as

$$= \log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

Substituting the value of t

$$= \log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

16.

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Solution:

Consider

$4x + 1 = A\ d/dx\ (2x^2 + x - 3) + B$

So we get

$4x + 1 = A\ (4x + 1) + B$

On further calculation

$4x + 1 = 4\ Ax + A + B$

By equating the coefficients of x and constant term on both sides

$4A = 4$

$A = 1$

$A + B = 1$

$B = 0$

Take $2x^2 + x - 3 = t$

By differentiation

$(4x + 1)\ dx = dt$

Integrating both sides

$$\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{1}{\sqrt{t}}dt$$

We get

$= 2\sqrt{t} + C$

Substituting the value of t

$$= 2\sqrt{2x^2+x-3}+C$$

17.

$$\frac{x+2}{\sqrt{x^2-1}}$$

Solution:

Consider

$$x+2 = A\frac{d}{dx}\left(x^2-1\right)+B \qquad ...(1)$$

It can be written as

x + 2 = A (2x) + B

Now equating the coefficients of x and constant term on both sides

2A = 1

A = ½

B = 2

Using equation (1) we get

$$(x+2) = \frac{1}{2}(2x)+2$$

Integrating both sides

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

Separating the terms

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$

Take

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx$$

If $x^2 - 1 = t$ we get $2x\,dx = dt$

So we get

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

By integration

$$= \frac{1}{2}\left[2\sqrt{t}\right]$$

$$= \sqrt{t}$$

Substituting the value of t

$$= \sqrt{x^2-1}$$

We can write it as

$$\int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$$

Using equation (2) we get

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$$

18.

$$\frac{5x-2}{1+2x+3x^2}$$

Solution:

Consider

$$5x-2 = A\frac{d}{dx}\left(1+2x+3x^2\right)+B$$

It can be written as

5x - 2 = A (2 + 6x) + B

Now equating the coefficients of x and constant term on both sides

5 = 6A

A = 5/6

2A + B = - 2

B = - 11/3

Using equation (1) we get

$$5x-2 = \frac{5}{6}(2+6x)+\left(-\frac{11}{3}\right)$$

Integrating both sides

$$\int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$$

Separating the terms

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$

We know that

$$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx \text{ and } I_2 = \int \frac{1}{1+2x+3x^2}dx$$

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \qquad ...(1)$$

Take

$$I_1 = \int \frac{2+6x}{1+2x+3x^2} dx$$

If $1 + 2x + 3x^2 = t$ we get $(2 + 6x)\, dx = dt$

So we get

$$I_1 = \int \frac{dt}{t}$$

By integration

$$I_1 = \log|t|$$

Substituting the value of t

$$I_1 = \log\left|1+2x+3x^2\right| \qquad ...(2)$$

Take

$$I_2 = \int \frac{1}{1+2x+3x^2} dx$$

$1 + 2x + 3x^2 = 1 + 3\,(x^2 + 2/3\, x)$

By addition and subtraction of 1/9

$$= 1 + 3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right)$$

We get

$$= 1 + 3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3}$$

On further calculation

$$= \frac{2}{3} + 3\left(x + \frac{1}{3}\right)^2$$

Here

$$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}\right]$$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]$$

By integration

$$I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]} dx$$

So we get

$$= \frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

By taking LCM

$$= \frac{1}{3}\left[\frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

On further calculation

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad \ldots(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{5x-2}{1+2x+3x^2} dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$$

We get

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

19.

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Solution:

It is given that

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

Consider

$$6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right) + B$$

It can be written as

6x + 7 = A (2x - 9) + B

Now equating the coefficients of x and constant term on both sides

2A = 6

A = 3

-9A + B = 7

B = 34

Using equation (1) we get

6x + 7 = 3 (2x – 9) + 34

Integrating both sides

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

Separating the terms

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

We know that

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2 \qquad ...(1)$$

Take

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

If $x^2 – 9x + 20 = t$ we get $(2x - 9)\, dx = dt$

So we get

$$I_1 = \frac{dt}{\sqrt{t}}$$

By integration

$$I_1 = 2\sqrt{t}$$

Substituting the value of t

$$I_1 = 2\sqrt{x^2-9x+20} \qquad ...(2)$$

Take

$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$$

By addition and subtraction of 81/4

$x^2 - 9x + 20 = x^2 - 9x + 20 + 81/4 - 81/4$

$$= \left(x - \frac{9}{2}\right)^2 - \frac{1}{4}$$

We get

$$= \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

By integration

$$I_2 = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

So we get

$$I_2 = \log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| \qquad ...(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx = 3\left[2\sqrt{x^2 - 9x + 20}\right] + 34 \log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$$

We get

$$= 6\sqrt{x^2 - 9x + 20} + 34 \log\left[\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right] + C$$

20.

$$\frac{x+2}{\sqrt{4x - x^2}}$$

Solution:

Consider

$$x + 2 = A\frac{d}{dx}\left(4x - x^2\right) + B$$

It can be written as

x + 2 = A (4 – 2x) + B

Now equating the coefficients of x and constant term on both sides

-2A = 1
A = -1/2
4A + B = 2
B = 4
Using equation (1) we get

$$(x+2) = -\frac{1}{2}(4-2x)+4$$

Integrating both sides

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

Separating the terms

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

We know that

$$I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx \text{ and } I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \qquad \ldots(1)$$

Take

$$I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$$

If $4x - x^2 = t$ we get $(4 – 2x)\, dx = dt$

So we get

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t}$$

Substituting the value of t

$$= 2\sqrt{4x-x^2} \text{ ...... (2)}$$

Take

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$$

$4x - x^2 = - (-4x + x^2)$

By addition and subtraction of 4

$4x - x^2 = (-4x + x^2 + 4 - 4)$

It can be written as

$= 4 - (x - 2)^2$

$= (2)^2 - (x - 2)^2$

By integration

$$I_2 = \int \frac{1}{\sqrt{(2)^2 - (x-2)^2}} dx$$

So we get

$$= \sin^{-1}\left(\frac{x-2}{2}\right) \quad ...(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

We get

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

21.
$$\frac{(x+2)}{\sqrt{x^2+2x+3}}$$
Solution:

It is given that

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}} dx$$

By multiplying and dividing by 2

$$= \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}} dx$$

Multiplying the terms

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}} dx$$

Separating the terms

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}} dx$$

We get

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

We know that

$$I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \qquad ...(1)$$

Take

$$I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx$$

Here $x^2 + 2x + 3 = t$

We get $(2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t}$$

Substituting the value of t

$$= 2\sqrt{x^2+2x+3} \qquad ...(2)$$

Take

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

We can write it as

$x^2 + 2x + 3 = x^2 + 2x + 1 + 2$

$$= (x+1)^2 + \left(\sqrt{2}\right)^2$$

So we get

$$I_2 = \int \frac{1}{\sqrt{(x+1)^2 + \left(\sqrt{2}\right)^2}}\,dx$$

By integration

$$= \log\left|(x+1) + \sqrt{x^2+2x+3}\right| \qquad ...(3)$$

By using equations (2) and (3) in (1) we get

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

So we get

$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

22.
$\frac{x+3}{x^2-2x-5}$

Solution:

Consider

$$(x+3) = A\frac{d}{dx}(x^2-2x-5)+B$$

It can be written as

x + 3 = A (2x – 2) + B

Now equating the coefficients of x and constant term on both sides

2A = 1

A = 1/2

-2A + B = 3

B = 4

Using equation (1) we get

$$(x+3) = \frac{1}{2}(2x-2)+4$$

Integrating both sides

$$\int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

Separating the terms

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

We know that
$I_1 = \int \frac{2x-2}{x^2-2x-5}dx$ and $I_2 = \int \frac{1}{x^2-2x-5}dx$

$$\int \frac{x+3}{(x^2-2x-5)}dx = \frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

Take

$$I_1 = \int \frac{2x-2}{x^2-2x-5}dx$$

If $x^2 - 2x - 5 = t$ we get $(2x - 2)\, dx = dt$

So we get

$$I_1 = \int \frac{dt}{t} = \log|t|$$

Substituting the value of t

$$= \log|x^2-2x-5| \quad \text{..... (2)}$$

Take

$$I_2 = \int \frac{1}{x^2-2x-5}dx$$

We can write it as

$$= \int \frac{1}{(x^2-2x+1)-6}dx$$

By separating the terms

$$= \int \frac{1}{(x-1)^2-(\sqrt{6})^2}dx$$

By integration

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \quad \text{...(3)}$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log|x^2-2x-5| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

We get

$$= \frac{1}{2}\log|x^2-2x-5| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

23.

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Solution:

Consider

$$5x+3=A\frac{d}{dx}\left(x^2+4x+10\right)+B$$

It can be written as

5x + 3 = A (2x + 4) + B

Now equating the coefficients of x and constant term on both sides

2A = 5

A = 5/2

4A + B = 3

B = -7

Using equation (1) we get

$$5x+3=\frac{5}{2}(2x+4)-7$$

Integrating both sides

$$\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int\frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

Separating the terms

$$=\frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx-7\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

We know that

$$I_1=\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx \text{ and } I_2=\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

$$\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\frac{5}{2}I_1-7I_2 \qquad \ldots(1)$$

Take

$$I_1=\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$$

If $x^2 + 4x + 10 = t$ we get $(2x + 4)\, dx = dt$

So we get

$$I_1=\int\frac{dt}{t}=2\sqrt{t}$$

Substituting the value of t

$$=2\sqrt{x^2+4x+10} \quad \ldots\ldots (2)$$

Take

$$I_2=\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

We can write it as

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+6}} dx$$

By separating the terms

$$= \int \frac{1}{(x+2)^2+(\sqrt{6})^2} dx$$

By integration

$$= \log\left|x+2+\sqrt{x^2+4x+10}\right| \quad ...(3)$$

Now substituting the equations (2) and (3) in equation (1)

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

We get

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

Choose the correct answer in Exercises 24 and 25.

24.

$\int \frac{dx}{x^2+2x+2}$ equals

(A) $x \tan^{-1}(x+1) + C$

(B) $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1} x + C$

(D) $\tan^{-1} x + C$

Solution:

It is given that

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

By separating the terms

$$= \int \frac{1}{(x+1)^2+(1)^2} dx$$

By integrating we get

$$= \left[\tan^{-1}(x+1)\right] + C$$

Therefore, B is the correct answer.

25.

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution:

It is given that

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

We can write it as

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

By further calculation we get

$$= \int \frac{1}{\sqrt{-4\left(x^2-\frac{9x}{4}+\frac{81}{64}-\frac{81}{64}\right)}}dx$$

Separating the terms we get

$$= \int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

On further simplification

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$

Using the formula

$$\int \frac{dy}{\sqrt{a^2-y^2}} = \sin^{-1}\frac{y}{a}+C$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C$$

Taking LCM

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$$

Therefore, B is the correct answer.

EXERCISE 7.5 PAGE: 322

Integrate the rational functions in Exercises 1 to 21.

1.

$$\frac{x}{(x+1)(x+2)}$$

Solution:

Consider

$$\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

We get

x = A (x + 2) + B (x + 1)

Now by equating the coefficients of x and constant term, we get

A + B = 1

2A + B = 0

By solving the equations we get

A = - 1 and B = 2

Substituting the values of A and B

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$$

By integrating both sides w.r.t x

$$\int\frac{x}{(x+1)(x+2)}dx = \int\frac{-1}{(x+1)} + \frac{2}{(x+2)}dx$$

So we get

= - log |x + 1| + 2 log |x + 2| + c

We can write it as

= log $(x + 2)^2$ – log |x + 1| + c

$$= \log\frac{(x+2)^2}{(x+1)} + C$$

2.

$$\frac{1}{(x+3)(x-3)}$$

Solution:

Consider

$$\frac{1}{(x+3)(x-3)}=\frac{A}{(x+3)}+\frac{B}{(x-3)}$$

We get

1 = A (x - 3) + B (x + 3)

Now by equating the coefficients of x and constant term, we get

A + B = 1

-3A + 3B = 0

By solving the equations we get

A = - 1/6 and B = 1/6

Substituting the values of A and B

$$\frac{1}{(x+3)(x-3)}=\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}$$

By integrating both sides w.r.t x

$$\int\frac{1}{(x^2-9)}dx=\int\left(\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}\right)dx$$

So we get

$$=-\frac{1}{6}\log|x+3|+\frac{1}{6}\log|x-3|+C$$

We can write it as

$$=\frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right|+C$$

3.

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Solution:

Consider

$$\frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

We get

3x - 1 = A (x - 2) (x – 3) + B (x - 1) (x – 3) + C (x – 1) (x – 2) ….. (1)

By substituting the value of x in equation (1), we get

A = 1, B = -5 and C = 4

Substituting the values of A, B and C

$$\frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}$$

By integrating both sides w.r.t x

$$\int\frac{3x-1}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}\right\}dx$$

So we get

= log |x – 1| - 5 log |x – 2| + 4 log |x – 3| + c

4.

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Solution:

Consider

$$\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

We get

x = A (x - 2) (x – 3) + B (x - 1) (x – 3) + C (x – 1) (x – 2) ….. (1)

By substituting the value of x in equation (1), we get

A = 1/2, B = -2 and C = 3/2

Substituting the values of A, B and C

$$\frac{x}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}$$

By integrating both sides w.r.t x

$$\int\frac{x}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}\right\}dx$$

So we get

= 1/2 log |x – 1| - 2 log |x – 2| + 3/2 log |x – 3| + c

5.

$$\frac{2x}{x^2+3x+2}$$

Solution:

Consider

$$\frac{2x}{x^2+3x+2}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$$

We get

2x = A (x + 2) + B (x + 1) ….. (1)

By substituting the value of x in equation (1), we get

A = - 2 and B = 4

Substituting the values of A and B

$$\frac{2x}{(x+1)(x+2)}=\frac{-2}{(x+1)}+\frac{4}{(x+2)}$$

By integrating both sides w.r.t x

$$\int\frac{2x}{(x+1)(x+2)}dx=\int\left\{\frac{4}{(x+2)}-\frac{2}{(x+1)}\right\}dx$$

So we get

= 4 log |x + 2| - 2 log |x + 1| + c

6.

$$\frac{1-x^2}{x(1-2x)}$$

Solution:

Consider

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$$

We know that

$$\frac{2-x}{x(1-2x)}=\frac{A}{x}+\frac{B}{(1-2x)}$$

We get

(2 – x) = A (1 – 2x) + Bx ….. (1)

By substituting the value of x in equation (1), we get

A = 2 and B = 3

Substituting the values of A and B

$$\frac{2-x}{x(1-2x)}=\frac{2}{x}+\frac{3}{1-2x}$$

We get

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left\{\frac{2}{x}+\frac{3}{(1-2x)}\right\}$$

By integrating both sides w.r.t x

$$\int\frac{1-x^2}{x(1-2x)}dx=\int\left\{\frac{1}{2}+\frac{1}{2}\left(\frac{2}{x}+\frac{3}{1-2x}\right)\right\}dx$$

By further calculation

$$=\frac{x}{2}+\log|x|+\frac{3}{2(-2)}\log|1-2x|+C$$

So we get

$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$$

7.

$$\frac{x}{(x^2+1)(x-1)}$$

Solution:

We know that

$$\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-1)}$$

It can be written as

$x = (Ax + B)(x – 1) + C(x^2 + 1)$

By multiplying the terms

$x = Ax^2 – Ax + Bx – B + Cx^2 + C$

Now by equating the coefficients of x^2, x and constant terms we get

$A + C = 0$

$-A + B = 1$

$-B + C = 0$

By solving the equations

$A = -½$, $B = ½$ and $C = ½$

Using equation (1)

$$\frac{x}{(x^2+1)(x-1)}=\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1}+\frac{\frac{1}{2}}{(x-1)}$$

By integrating both sides w.r.t. x

$$\int\frac{x}{(x^2+1)(x-1)}=-\frac{1}{2}\int\frac{x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$$

We get

$$=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

Here

$\int \frac{2x}{x^2+1} dx$, let $(x^2+1) = t$

We get

2x dx = dt

Substituting the values

$\int \frac{2x}{x^2+1} dx = \int \frac{dt}{t}$

By integrating w.r.t t

= log |t|

Substituting the value of t

= log $|x^2 + 1|$

So we get

$$\int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{4}\log\left|x^2+1\right| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log\left|x-1\right| + C$$

We can write it as

$$= \frac{1}{2}\log\left|x-1\right| - \frac{1}{4}\log\left|x^2+1\right| + \frac{1}{2}\tan^{-1}x + C$$

8.

$$\frac{x}{(x-1)^2(x+2)}$$

Solution:

We know that

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

It can be written as

$x = A (x - 1) (x + 2) + B (x + 2) + C (x - 1)^2$

Taking x = 1 we get

B = 1/3

Now by equating the coefficients of x^2 and constant terms we get

A + C = 0

-2A + 2B + C = 0

By solving the equations

A = 2/9 and C = -2/9

We get

$$\frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

By integrating both sides w.r.t. x

$$\int \frac{x}{(x-1)^2(x+2)} dx = \frac{2}{9}\int \frac{1}{(x-1)} dx + \frac{1}{3}\int \frac{1}{(x-1)^2} dx - \frac{2}{9}\int \frac{1}{(x+2)} dx$$

Here

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + C$$

By further calculation

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

9.

$$\frac{3x+5}{x^3-x^2-x+1}$$

Solution:

It is given that

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$$

We know that

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

It can be written as

$3x + 5 = A (x – 1) (x + 1) + B (x + 1) + C (x – 1)^2$

We get

$3x + 5 = A (x^2 – 1) + B (x + 1) + C (x^2 + 1 – 2x)$ ….. (1)

By substituting the value of x = 1 in equation (1)

B = 4

Now by equating the coefficients of x^2 and x we get

A + C = 0

B - 2C = 3

By solving the equations

A = -1/2 and C = 1/2

We get

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

By integrating both sides w.r.t. x

$$\int\frac{3x+5}{(x-1)^2(x+1)}dx = -\frac{1}{2}\int\frac{1}{x-1}dx + 4\int\frac{1}{(x-1)^2}dx + \frac{1}{2}\int\frac{1}{(x+1)}dx$$

Here

$$= -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

By further calculation

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$$

10.

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Solution:

It is given that

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$$

We know that

$$\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$$

It can be written as

$(2x – 3) = A (x – 1) (2x + 3) + B (x + 1) (2x + 3) + C (x + 1) (x - 1)$

$(2x – 3) = A (2x^2 + x – 3) + B (2x^2 + 5x + 3) + C (x^2 – 1)$

We get

$(2x – 3) = (2A + 2B + C) x^2 + (A + 5B) x + (-3A + 3B – C)$ ….. (1)

Now by equating the coefficients of x^2 and x we get

B = -1/10, A = 5/2 and C = -24/5

We get

$$\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

By integrating both sides w.r.t. x

$$\int \frac{2x-3}{(x^2-1)(2x+3)}dx = \frac{5}{2}\int \frac{1}{(x+1)}dx - \frac{1}{10}\int \frac{1}{x-1}dx - \frac{24}{5}\int \frac{1}{(2x+3)}dx$$

Here

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$

By further calculation

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

11.

$$\frac{5x}{(x+1)(x^2-4)}$$

Solution:

It is given that

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

We know that

$$\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$$

It can be written as
5x = A (x + 2) (x – 2) + B (x + 1) (x – 2) + C (x + 1) (x + 2) ….. (1)

By substituting x = - 1, - 2 and 2 in equation (1)
A = 5/3, B = - 5/2 and C = 5/6

We get
$$\frac{5x}{(x+1)(x+2)(x-2)}=\frac{5}{3(x+1)}-\frac{5}{2(x+2)}+\frac{5}{6(x-2)}$$

By integrating both sides w.r.t. x
$$\int\frac{5x}{(x+1)(x^2-4)}dx=\frac{5}{3}\int\frac{1}{(x+1)}dx-\frac{5}{2}\int\frac{1}{(x+2)}dx+\frac{5}{6}\int\frac{1}{(x-2)}dx$$

By further calculation
$$=\frac{5}{3}\log|x+1|-\frac{5}{2}\log|x+2|+\frac{5}{6}\log|x-2|+C$$

12.
$$\frac{x^3+x+1}{x^2-1}$$
Solution:

It is given that
$$\frac{x^3+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}$$
We know that
$$\frac{2x+1}{x^2-1}=\frac{A}{(x+1)}+\frac{B}{(x-1)}$$
It can be written as
2x + 1 = A (x - 1) + B (x + 1) ….. (1)

By substituting x = 1 and -1 in equation (1)
A = 1/2 and B = 3/2

We get
$$\frac{x^3+x+1}{x^2-1}=x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$$

By integrating both sides w.r.t. x
$$\int\frac{x^3+x+1}{x^2-1}dx=\int x\,dx+\frac{1}{2}\int\frac{1}{(x+1)}dx+\frac{3}{2}\int\frac{1}{(x-1)}dx$$

By further calculation
$$=\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1|+C$$

13.

$$\frac{2}{(1-x)(1+x^2)}$$

Solution:

We know that

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

It can be written as

$2 = A(1 + x^2) + (Bx + C)(1 - x)$

$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$ ….. (1)

Now by equating the coefficient of x^2, x and constant terms

$A - B = 0$

$B - C = 0$

$A + C = 2$

Solving the equations

$A = 1$, $B = 1$ and $C = 1$

We get

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

By integrating both sides w.r.t. x

$$\int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

Multiplying and dividing by 2 in the second term

$$= -\int \frac{1}{x-1} dx + \frac{1}{2}\int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

By further calculation

$$= -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1} x + C$$

14.

$$\frac{3x-1}{(x+2)^2}$$

Solution:

We know that

$$\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

It can be written as

$3x - 1 = A(x + 2) + B$ ….. (1)

Now by equating the coefficient of x and constant terms
A = 3
2A + B = - 1

Solving the equations
B = - 7

We get

$$\frac{3x-1}{(x+2)^2}=\frac{3}{(x+2)}-\frac{7}{(x+2)^2}$$

By integrating both sides w.r.t. x

$$\int\frac{3x-1}{(x+2)^2}dx=3\int\frac{1}{(x+2)}dx-7\int\frac{x}{(x+2)^2}dx$$

So we get

$$=3\log|x+2|-7\left(\frac{-1}{(x+2)}\right)+C$$

By further calculation

$$=3\log|x+2|+\frac{7}{(x+2)}+C$$

15.

$$\frac{1}{(x^4-1)}$$

Solution:

It is given that

$$\frac{1}{(x^4-1)}=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{(x+1)(x-1)(1+x^2)}$$

We know that

$$\frac{1}{(x+1)(x-1)(1+x^2)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{Cx+D}{(x^2+1)}$$

So we get
$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx + D)(x^2-1)$
By multiplying the terms
$1 = A(x^3 + x - x^2 - 1) + B(x^3 + x + x^2 + 1) + Cx^3 + Dx^2 - Cx - D$
It can be written as
$1 = (A + B + C)x^3 + (-A + B + D)x^2 + (A + B - C)x + (-A + B - D)$ ….. (1)

Now by equating the coefficient of x^3, x^2, x and constant terms
A + B + C = 0
-A + B + D = 0
A + B - C = 0
- A + B – D = 1

Solving the equations
A = -1/4, B = 1/4, C = 0 and D = - 1/2

We get

$$\frac{1}{x^4-1}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}-\frac{1}{2(x^2+1)}$$

By integrating both sides w.r.t. x

$$\int\frac{1}{x^4-1}dx=-\frac{1}{4}\log|x+1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$$

So we get

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

16.

$$\frac{1}{x(x^n+1)}$$

Solution:

By multiplying both numerator and denominator by x^{n-1}

$$\frac{1}{x(x^n+1)}=\frac{x^{n-1}}{x^{n-1}x(x^n+1)}=\frac{x^{n-1}}{x^n(x^n+1)}$$

Here $x^n = t$ we get
$nx^{n-1}\,dx = dt$
So we get

$$\int\frac{1}{x(x^n+1)}dx=\int\frac{x^{n-1}}{x^n(x^n+1)}dx=\frac{1}{n}\int\frac{1}{t(t+1)}dt$$

We know that

$$\frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{(t+1)}$$

It can be written as
1 = A (1 + t) + Bt ….. (1)

By substituting t = 0, -1 in equation (1)
A = 1 and B = -1

We get

$$\frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

By integrating both sides w.r.t. x

$$\int \frac{1}{x(x^n+1)}dx = \frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(t+1)}\right\}dx$$

So we get

$$= \frac{1}{n}\left[\log|t| - \log|t+1|\right] + C$$

Substituting the value of t

$$= -\frac{1}{n}\left[\log|x^n| - \log|x^n+1|\right] + C$$

It can be written as

$$= \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$$

17.

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Solution:

It is given that

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Consider

sin x = t

By differentiating w.r.t t

cos x dx = dt

Integrating w.r.t x

$$\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{dt}{(1-t)(2-t)}$$

Here we can write it as

$$\frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$$

We get

1 = A (2 – t) + B (1 – t) …….. (1)

By substituting t = 2 and t = 1 in equation (1)

A = 1 and B = - 1

$$\frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

Integrating w.r.t t

$$\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int\left\{\frac{1}{1-t}-\frac{1}{(2-t)}\right\}dt$$

So we get
= - log |1 – t| + log |2 – t| + C

It can be written as
$$= \log\left|\frac{2-t}{1-t}\right| + C$$
Substituting the value of t
$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

18.
$$\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)}$$
Solution:

We know that
$$\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)} = 1 - \frac{\left(4x^2+10\right)}{\left(x^2+3\right)\left(x^2+4\right)}$$
It can be written as
$$\frac{4x^2+10}{\left(x^2+3\right)\left(x^2+4\right)} = \frac{Ax+B}{\left(x^2+3\right)} + \frac{Cx+D}{\left(x^2+4\right)}$$
So we get
$4x^2 + 10 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 3)$
Multiplying the terms
$4x^2 + 10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$
Grouping the terms
$4x^2 + 10 = (A + C)x^3 + (B + D)x^2 + (4A + 3C)x + (4B + 3D)$

Now by equating the coefficients of x^3, x^2, x and constant terms
A + C = 0
B + D = 4
4A + 3C = 0
4B + 3D = 10
By solving these equations
A = 0, B = - 2, C = 0 and D = 6

Substituting the values
$$\frac{4x^2+10}{\left(x^2+3\right)\left(x^2+4\right)} = \frac{-2}{\left(x^2+3\right)} + \frac{6}{\left(x^2+4\right)}$$

We can write it as
$$\frac{\left(x^2+1\right)\left(x^2+2\right)}{\left(x^2+3\right)\left(x^2+4\right)} = 1 - \left(\frac{-2}{\left(x^2+3\right)} + \frac{6}{\left(x^2+4\right)}\right)$$

Integrating both sides w.r.t x

$$\int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int\left\{1+\frac{2}{(x^2+3)}-\frac{6}{(x^2+4)}\right\}dx$$

So we get

$$= \int\left\{1+\frac{2}{x^2+(\sqrt{3})^2}-\frac{6}{x^2+2^2}\right\}$$

Here

$$= x+2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)-6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C$$

By further calculation

$$= x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$$

19.

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Solution:

It is given that

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Consider $x^2 = t$

So we get

2x dx = dt

Integrating both sides

$$\int\frac{2x}{(x^2+1)(x^2+3)}dx = \int\frac{dt}{(t+1)(t+3)}$$

We can write it as

$$\frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)}+\frac{B}{(t+3)}$$

1 = A (t + 3) + B (t + 1) ….. (1)

Now by substituting t = - 3 and t = - 1 in equation (1)

A = 1/2 and B = - 1/2

Substituting the values

$$\frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)}-\frac{1}{2(t+3)}$$

Integrating w.r.t t

$$\int \frac{2x}{(x^2+1)(x^2+3)}dx = \int\left\{\frac{1}{2(t+1)} - \frac{1}{2(t+3)}\right\}dt$$

So we get

$$= \frac{1}{2}\log\left|(t+1)\right| - \frac{1}{2}\log\left|t+3\right| + C$$

It can be written as

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C$$

Substituting the value of t

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

20.

$$\frac{1}{x(x^4-1)}$$

Solution:

It is given that

$$\frac{1}{x(x^4-1)}$$

By multiplying both numerator and denominator by x^3

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

Integrating both sides

$$\int \frac{1}{x(x^4-1)}dx = \int \frac{x^3}{x^4(x^4-1)}dx$$

Consider $x^4 = t$

So we get $4x^3\,dx = dt$

We can write it as

$$\int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\int \frac{dt}{t(t-1)}$$

So we get

$$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$$

$1 = A\,(t - 1) + Bt$ ….. (1)

Now by substituting $t = 0$ in equation (1)

$A = -1$ and $B = 1$

Substituting the values

$$\frac{1}{t(t+1)} = \frac{-1}{t} + \frac{1}{t-1}$$

Integrating w.r.t t

$$\int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt$$

So we get

$$= \frac{1}{4}\left[-\log|t| + \log|t-1| \right] + C$$

It can be written as

$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + \text{C}$$

Substituting the value of t

$$= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + \text{C}$$

21.

$$\frac{1}{(e^x-1)}$$

Solution:

It is given that

$$\frac{1}{(e^x-1)}$$

Consider $e^x = t$

So we get $e^x \, dx = dt$

We can write it as

$$\int \frac{1}{e^x-1} dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$$

So we get

$$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$

1 = A (t - 1) + Bt ….. (1)

Now by substituting t = 1 and t = 0 in equation (1)

A = -1 and B = 1

Substituting the values

$$\frac{1}{t(t+1)} = \frac{-1}{t} + \frac{1}{t-1}$$

Integrating w.r.t t

$$\int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$$

Substituting the value of t

$$= \log\left|\frac{e^x-1}{e^x}\right| + C$$

Choose the correct answer in each of the Exercises 22 and 23.

22. $\int \frac{xdx}{(x-1)(x-2)}$ equals

(A) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

(B) $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

(C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

(D) $\log|(x-1)(x-2)| + C$

Solution:

We know that

$$\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

It can be written as

x = A (x – 2) + B (x – 1) …… (1)

Now by substituting x = 1 and 2 in equation (1)

A = - 1 and B = 2

Substituting the value of A and B

$$\frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$

Integrating both sides w.r.t x

$$\int \frac{x}{(x-1)(x-2)} dx = \int \left\{\frac{-1}{(x-1)} + \frac{2}{(x-2)}\right\} dx$$

We get

$= -\log|x-1| + 2\log|x-2| + C$

We can write it as

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Therefore, B is the correct answer.

23. $\int \frac{dx}{x(x^2+1)}$ *equals*

$(A) log|x| - \frac{1}{2}log(x^2+1) + C$

$(B) log|x| + \frac{1}{2}log(x^2+1) + C$

$(C) - log|x| + \frac{1}{2}log(x^2+1) + C$

$(D) \frac{1}{2}log|x| + log(x^2+1) + C$

Solution:

We know that

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

It can be written as

$1 = A(x^2 + 1) + (Bx + C)x$ …… (1)

Now by equating the coefficients of x^2, x and constant terms

A + B = 0

C = 0

A = 1

By solving the equations we get

A = 1, B = - 1 and C = 0

Substituting the value of A and B

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

Integrating both sides w.r.t x

$$\int \frac{1}{x(x^2+1)}dx = \int \left\{\frac{1}{x} - \frac{x}{x^2+1}\right\}dx$$

We get

$$= \log|x| - \frac{1}{2}\log|x^2+1| + C$$

Therefore, A is the correct answer.

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

1. x sin x

Solution:

It is given that

$I = \int x \sin x \, dx$

Here by taking x as first function and sin x as second function

Now integrating by parts we get

$I = x \int \sin x \; dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin x \; dx \right\} dx$

So we get

$= x(-\cos x) - \int 1 \cdot (-\cos x) dx$

It can be written as

= - x cos x + sin x + C

2. x sin 3x

Solution:

It is given that

$I = \int x \sin 3x \, dx$

Here by taking x as first function and 3x as second function

Now integrating by parts we get

$I = x \int \sin 3x \; dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin 3x \; dx \right\}$

So we get

$= x\left(\frac{-\cos 3x}{3} \right) - \int 1 \cdot \left(\frac{-\cos 3x}{3} \right) dx$

By multiplying the terms

$= \frac{-x \cos 3x}{3} + \frac{1}{3} \int \cos 3x \; dx$

It can be written as

$= \frac{-x \cos 3x}{3} + \frac{1}{9} \sin 3x + C$

3. x^2 e^x

Solution:

It is given that

$I = \int x^2 e^x dx$

Here by taking x^2 as first function and e^x as second function

Now integrating by parts we get

$$I = x^2 \int e^x\,dx - \int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx$$

So we get

$$= x^2e^x - \int 2x \cdot e^x dx$$

It can be written as

$$= x^2e^x - 2\int x \cdot e^x dx$$

Now integrating by parts we get

$$= x^2e^x - 2\left[x \cdot \int e^x dx - \int\left\{\left(\frac{d}{dx}x\right) \cdot \int e^x dx\right\}dx\right]$$

On further calculation

$$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$$

So we get

$$= x^2e^x - 2\left[xe^x - e^x\right]$$

By multiplying the terms

$$= x^2e^x - 2xe^x + 2e^x + C$$

Taking the common terms

$$= e^x\left(x^2 - 2x + 2\right) + C$$

4. x log x
Solution:

It is given that

$$I = \int x \log x dx$$

Here by taking x as first function and x as second function
Now integrating by parts we get

$$I = \log x \int x\ dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x\ dx\right\}dx$$

So we get

$$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

By multiplying the terms

$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$

It can be written as

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

5. x log 2x
Solution:

It is given that

x log 2x

Here by taking 2x as first function and x as second function

Now integrating by parts we get

$$I = \log 2x \int x\, dx - \int\left\{\left(\frac{d}{dx} 2\log 2x\right)\int x\, dx\right\}dx$$

So we get

$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx$$

By multiplying the terms

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$

It can be written as

$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

6. x^2 log x

Solution:

It is given that

$$I = \int x^2 \log x\ dx$$

Here by taking x as first function and x^2 as second function

Now integrating by parts we get

$$I = \log x \int x^2\, dx - \int\left\{\left(\frac{d}{dx} \log x\right)\int x^2 dx\right\}dx$$

So we get

$$= \log x\left(\frac{x^3}{3}\right) - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$$

By multiplying the terms

$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$

It can be written as

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

7. x $\sin^{-1}$ x

Solution:

It is given that

I = x $\sin^{-1}$ x

Here by taking $\sin^{-1}$ x as first function and x as second function

Now integrating by parts we get

$$I = \sin^{-1} x \int x\, dx - \int\left\{\left(\frac{d}{dx} \sin^{-1} x\right)\int x\, dx\right\}dx$$

So we get

$$= \sin^{-1} x\left(\frac{x^2}{2}\right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2}\, dx$$

By multiplying the terms

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}\, dx$$

Addition and subtraction of 1 in the numerator

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int \left\{\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right\} dx$$

On further simplification

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\int \left\{\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right\} dx$$

Integrating the terms

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\left\{\int \sqrt{1-x^2}\, dx - \int \frac{1}{\sqrt{1-x^2}}\, dx\right\}$$

So we get

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x - \sin^{-1} x\right\} + C$$

By further calculation

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + C$$

Taking the common terms

$$= \frac{1}{4}\left(2x^2 - 1\right)\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + C$$

8. x $\tan^{-1}$ x
Solution:

We know that

$$I = \int x \tan^{-1} x\ dx$$

Consider $\tan^{-1}$ x as the first function and x as the second function

Here integrating by parts we get

$$I = \tan^{-1} x \int x\, dx - \int \left\{\left(\frac{d}{dx}\tan^{-1} x\right)\int x\, dx\right\} dx$$

By further calculation

$$= \tan^{-1} x\left(\frac{x^2}{2}\right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\, dx$$

Multiplying the terms

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2} dx$$

Again integrating by parts

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right) dx$$

So we get

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

On further simplification

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

We get

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

9. x cos $^{-1}$ x
Solution:

We know that

$$I = \int x \cos^{-1} x dx$$

Consider cos $^{-1}$ x as the first function and x as the second function

Here integrating by parts we get

$$I = \cos^{-1} x \int x\, dx - \int \left\{\left(\frac{d}{dx} \cos^{-1} x\right) \int x\, dx\right\} dx$$

By further calculation

$$= \cos^{-1} x \frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$$

By adding and subtracting 1 to the numerator

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

It can be written as

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2}\int \left\{ \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) \right\} dx$$

Separating the terms

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2}\int \sqrt{1-x^2}\, dx - \frac{1}{2}\int \left(\frac{-1}{\sqrt{1-x^2}} \right) dx$$

We get

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2}\cos^{-1} x \qquad \ldots(1)$$

We know that

$$I_1 = \int \sqrt{1-x^2}\, dx$$

Integrating by parts we get

$$I_1 = x\sqrt{1-x^2} - \int \frac{d}{dx}\sqrt{1-x^2} \int x\, dx$$

On further calculation

$$I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}} . x\, dx$$

So we get

$$I_1 = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

Addition and subtraction of 1 to numerator

$$I_1 = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

By separating the terms

$$I_1 = x\sqrt{1-x^2} - \left\{ \int \sqrt{1-x^2}\, dx + \int \frac{-dx}{\sqrt{1-x^2}} \right\}$$

We get

$$I_1 = x\sqrt{1-x^2} - \left\{ I_1 + \cos^{-1} x \right\}$$

On further calculation

$$2I_1 = x\sqrt{1-x^2} - \cos^{-1} x$$

We can write it as

$$I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x$$

Now by substituting the value in equation (1)

$$I = x^2\frac{\cos^{-1} x}{2} - \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x\right) - \frac{1}{2}\cos^{-1} x$$

We get

$$= \frac{\left(2x^2-1\right)}{4}\cos^{-1} x - \frac{x}{4}\sqrt{1-x^2} + C$$

10. $(\sin^{-1} x)^2$
Solution:

We know that

$$I = \int\left(\sin^{-1} x\right)^2 \cdot 1\, dx$$

Consider $(\sin^{-1} x)^2$ as the first function and 1 as the second function

Here integrating by parts we get

$$I = \left(\sin^{-1} x\right)^2 \int 1\, dx - \int\left\{\frac{d}{dx}\left(\sin^{-1} x\right)^2 \cdot \int 1 \cdot dx\right\} dx$$

By further calculation

$$= \left(\sin^{-1} x\right)^2 \cdot x - \int \frac{2\sin^{-1} x}{\sqrt{1-x^2}} \cdot x\, dx$$

Multiplying the terms

$$= x\left(\sin^{-1} x\right)^2 + \int \sin^{-1} x \cdot \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx$$

Again integrating by parts

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \int \frac{-2x}{\sqrt{1-x^2}}\, dx - \int\left\{\left(\frac{d}{dx}\sin^{-1} x\right)\int \frac{-2x}{\sqrt{1-x^2}} dx\right\} dx\right]$$

So we get

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}\, dx\right]$$

On further simplification

$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\sin^{-1} x - \int 2\,dx$

We get

$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x + C$

11.

$\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}\,dx$

Solution:

We know that

$I = \int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}\,dx$

By multiplying and dividing by -2

$I = \frac{-1}{2}\int \frac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1} x\,dx$

Consider cos $^{-1}$ x as the first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as the second function

Here integrating by parts we get

$I = \frac{-1}{2}\left[\cos^{-1} x \int \frac{-2x}{\sqrt{1-x^2}}\,dx - \int\left\{\left(\frac{d}{dx}\cos^{-1} x\right)\int \frac{-2x}{\sqrt{1-x^2}}\,dx\right\}dx\right]$

By further calculation

$= \frac{-1}{2}\left[\cos^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}\,dx\right]$

Multiplying the terms

$= \frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1} x + \int 2\,dx\right]$

So we get

$= \frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1} x + 2x\right] + C$

On further simplification

$= -\left[\sqrt{1-x^2}\cos^{-1} x + x\right] + C$

12. x $\sec^2$ x
Solution:

It is given that

$$I = \int x\sec^2 x dx$$

Consider x as the first function and $\sec^2$ x as the second function

Integrating by parts we get

$$I = x\int \sec^2 x\, dx - \int\left\{\left\{\frac{d}{dx}x\right\}\int \sec^2 x\, dx\right\}dx$$

By further calculation

$$= x\tan x - \int 1\cdot \tan x dx$$

So we get

= x tan x + log |cos x| + C

13. $\tan^{-1}$ x
Solution:

It is given that

$$I = \int 1\cdot \tan^{-1} x dx$$

Consider $\tan^{-1}$ x as the first function and 1 as the second function

Integrating by parts we get

$$I = \tan^{-1} x\int 1 dx - \int\left\{\left(\frac{d}{dx}\tan^{-1} x\right)\int 1\cdot dx\right\}dx$$

By further calculation

$$= \tan^{-1} x\cdot x - \int \frac{1}{1+x^2}\cdot x\, dx$$

Multiplying and dividing by 2

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

We get

$$= x\tan^{-1} x - \frac{1}{2}\log\left|1+x^2\right| + C$$

$$= x\tan^{-1}x - \frac{1}{2}\log\left(1+x^2\right) + C$$

14. x (log x)^2
Solution:

It is given that

$$I = \int x\left(\log x\right)^2 dx$$

Consider (log x)^2 as the first function and x as the second function

Integrating by parts we get

$$I = (\log x)^2 \int x\,dx - \int\left[\left\{\left(\frac{d}{dx}(\log x)^2\right\} \int x\,dx\right]dx$$

By further calculation

$$= \frac{x^2}{2}(\log x)^2 - \left[\int 2\log x \cdot \frac{1}{x}\cdot\frac{x^2}{2}\,dx\right]$$

It can be written as

$$= \frac{x^2}{2}(\log x)^2 - \int x\log x\,dx$$

Now integrating by parts

$$I = \frac{x^2}{2}(\log x)^2 - \left[\log x\int x\,dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x\,dx\right\}dx\right]$$

So we get

$$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2}\log x - \int\frac{1}{x}\cdot\frac{x^2}{2}\,dx\right]$$

On further simplification

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\,dx$$

We get

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$$

15. (x^2 + 1) log x
Solution:

Consider

$$I = \int (x^2+1)\log x\,dx$$

It can be written as

$$= \int x^2 \log x\,dx + \int \log x\,dx$$

We know that

I = $I_1 + I_2$ …… (1)

Here

$I_1 = \int x^2 \log x\,dx$ and $I_2 = \int \log x\,dx$

Take

$$I_1 = \int x^2 \log x dx$$

Consider log x as the first function and x^2 as the second function

Now integrating by parts

$$I_1 = \log x \int x^2 dx - \int \left\{ \left(\frac{d}{dx}\log x \right) \int x^2 dx \right\} dx$$

On further calculation

$$= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}dx$$

It can be written as

$$= \frac{x^3}{3}\log x - \frac{1}{3}\left(\int x^2 dx \right)$$

So we get

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 \qquad \ldots (2)$$

Take

$$I_2 = \int \log x\,dx$$

Consider log x as the first function and 1 as the second function

Now integrating by parts

$$I_2 = \log x \int 1\cdot dx - \int \left\{ \left(\frac{d}{dx}\log x \right) \int 1\cdot dx \right\}$$

On further calculation

$$= \log x \cdot x - \int \frac{1}{x} \cdot x dx$$

It can be written as

$$= x \log x - \int 1 dx$$

So we get

$$= x \log x - x + C_2 \quad \ldots (3)$$

By using equations (2) and (3) in (1) we get

$$I = \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 + x \log x - x + C_2$$

We can write it as

$$= \frac{x^3}{3} \log x - \frac{x^3}{9} + x \log x - x + (C_1 + C_2)$$

We get

$$= \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C$$

16. e^x (sin x + cos x)
Solution:

Consider

$$I = \int e^x (\sin x + \cos x) dx$$

We know that

f (x) = sin x

So we get

f' (x) = cos x

Here

$$I = \int e^x \{ f(x) + f'(x) \} dx$$

It can be written as

$$\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$$

$$I = e^x \sin x + C$$

17.
$\frac{xe^x}{(1+x)^2}$

Solution:

It is given that

$$I = \int \frac{xe^x}{(1+x)^2} dx$$

We can write it as

$$= \int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx$$

By addition and subtraction of 1 to the numerator

$$= \int e^x \left\{ \frac{1+x-1}{(1+x)^2} \right\} dx$$

Separating the terms we get

$$= \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$$

Consider

$$f(x) = \frac{1}{1+x}$$

By differentiation

$$f'(x) = \frac{-1}{(1+x)^2}$$

So we get

$$\int \frac{xe^x}{(1+x)^2} dx = \int e^x \{ f(x) + f'(x) \} dx$$

We know that

$$\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + \mathrm{C}$$

We get

$$\int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + \mathrm{C}$$

18.
$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

Solution:

It is given that

$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

We can write it as

$$=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

Using the formula we can write it as

$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

By further simplification

$$=\frac{1}{2}e^x\cdot\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

So we get

$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$

$$=\frac{1}{2}e^2\left(1+\tan\frac{x}{2}\right)^2$$

By expanding using formula

$$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

We know that

$$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

So we get

$$\frac{e^x(1+\sin x)dx}{(1+\cos x)} = e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right] \quad ...(1)$$

Consider tan x/2 = f (x)

By differentiation

$$f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$$

Here

$$\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+C$$

Using equation (1) we get

$$\int \frac{e^x(1+\sin x)}{(1+\cos x)}dx = e^x\tan\frac{x}{2}+C$$

19.
$$e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]$$
Solution:

It is given that

$$I=\int e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]dx$$

Here if f (x) = 1/x we get

f' (x) = - 1/x²

We know that

$$\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+C$$

So we get

$$I=\frac{e^x}{x}+C$$

20.
$$\frac{(x-3)\,e^x}{(x-1)^3}$$
Solution:

It is given that

$$\int e^x\left\{\frac{x-3}{(x-1)^3}\right\}dx = \int e^x\left\{\frac{x-1-2}{(x-1)^3}\right\}dx$$

By separating the terms

$$= \int e^x\left\{\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right\}dx$$

We know that

$$f(x)=\frac{1}{(x-1)^2}$$

By differentiation

$$f'(x)=\frac{-2}{(x-1)^3}$$

Here

$$\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+\mathrm{C}$$

We get

$$\int e^x\left\{\frac{(x-3)}{(x-1)^2}\right\}dx = \frac{e^x}{(x-1)^2}+\mathrm{C}$$

21. e^{2x} sin x
Solution:

It is given that

$$I = \int e^{2x}\sin x\,dx \qquad \ldots(1)$$

Now integrating by parts we get

$$I = \sin x\int e^{2x}dx - \int\left\{\left(\frac{d}{dx}\sin x\right)\int e^{2x}dx\right\}dx$$

So we get

$$I = \sin x\cdot\frac{e^{2x}}{2} - \int\cos x\cdot\frac{e^{2x}}{2}dx$$

We can write it as

$$I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\int e^{2x}\cos x\,dx$$

Here again integrating by parts we get

$$I=\frac{e^{2x}\cdot\sin x}{2}-\frac{1}{2}\left[\cos x\int e^{2x}dx-\int\left\{\left(\frac{d}{dx}\cos x\right)\int e^{2x}dx\right\}dx\right]$$

So we get

$$I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x\cdot\frac{e^{2x}}{2}-\int(-\sin x)\frac{e^{2x}}{2}dx\right]$$

On further simplification

$$I=\frac{e^{2x}\cdot\sin x}{2}-\frac{1}{2}\left[\frac{e^{2x}\cos x}{2}+\frac{1}{2}\int e^{2x}\sin x dx\right]$$

By using equation (1) we get

$$I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}-\frac{1}{4}I$$

It can be written as

$$I+\frac{1}{4}I=\frac{e^{2x}\cdot\sin x}{2}-\frac{e^{2x}\cos x}{4}$$

We get

$$\frac{5}{4}I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}$$

By cross multiplication

$$I=\frac{4}{5}\left[\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}\right]+C$$

So we get

$$I=\frac{e^{2x}}{5}\left[2\sin x-\cos x\right]+C$$

22.

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Solution:

Take x = tan θ we get dx = $\sec^2\theta\, d\theta$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

So we get

$$= \sin^{-1}(\sin 2\theta) = 2\theta$$

By integrating both sides w.r.t x

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta \cdot \sec^2\theta\, d\theta$$

We get

$$= 2\int \theta \cdot \sec^2\theta\, d\theta$$

Now integrating by parts we get

$$2\left[\theta \cdot \int \sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta d\theta\right\}d\theta\right]$$

On further calculation

$$= 2\left[\theta \cdot \tan\theta - \int \tan\theta d\theta\right]$$

By integration of second term

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right] + C$$

Now by substituting the value of θ

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

We get

$$= 2x\tan^{-1}x + 2\log(1+x^2)^{-\frac{1}{2}} + C$$

It can be written as

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log(1+x^2)\right] + C$$

By further calculation

$$= 2x\tan^{-1}x - \log(1+x^2) + C$$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3}\, dx$ equals

(A) $\frac{1}{3}e^{x^3} + C$

(B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$

(D) $\frac{1}{2}e^{x^2} + C$

Solution:

It is given that

$$I = \int x^2 e^{x^3} dx$$

Take $x^3 = t$ we get

$3x^2\, dx = dt$

Here

$$I = \frac{1}{3}\int e^t dt$$

By integrating w.r.t t

$$= \frac{1}{3}\left(e^t\right) + C$$

Substituting the value of t

$$= \frac{1}{3}e^{x^3} + C$$

Therefore, A is the correct answer.

24. $\int e^x \sec x\,(1 + \tan x)\, dx$ **equals**

(A) $e^x \cos x + C$

(B) $e^x \sec x + C$

(C) $e^x \sin x + C$

(D) $e^x \tan x + C$

Solution:

It is given that

$$I = \int e^x \sec x\left(1 + \tan x\right) dx$$

Multiplying the terms we get

$$= \int e^x\left(\sec x + \sec x \tan x\right) dx$$

Take sec x = f (x)

So we get sec x tan x = f' (x)

We know that

$$\int e^x \{f(x) + f'(x)\}\, dx = e^x f(x) + C$$

Here

$I = e^x \sec x + C$

Therefore, B is the correct answer.

EXERCISE 7.7

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Integrate the functions in exercise 1 to 9

1. $\sqrt{4-x^2}$

Solution:

Given:

$\sqrt{4-x^2}$

Upon integration we get,

$$\int\sqrt{4-x^2}\,dx = \int\sqrt{(2)^2-(x)^2}\,dx$$

By using the formula,

$$\int\sqrt{a^2-x^2}\,dx = \frac{\pi}{2}\sqrt{a^2-x^2}\;\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$$

So,

$$\int\sqrt{4-x^2}\,dx = \frac{\pi}{2}\sqrt{4-x^2}\;\frac{4}{2}\sin^{-1}\frac{x}{a}+C$$

$$= \frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{a}+C$$

2. $\sqrt{1-4x^2}$

Solution:

Given:

$\sqrt{1-4x^2}$

Upon integration we get,

$$\sqrt{1-4x^2}\,dx = \int\sqrt{(1)^2-(2x)^2}\,dx$$

Let $2x = t$

So,

$2dx = dt$

$dx = dt/2$

Then,

$$I = \frac{1}{2}\int\sqrt{(1)^2-(t)^2}\,dt$$

By using the formula,

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}\,\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$$

So,

$$I=\frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+C$$

$$=\frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t+C$$

$$=\frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$$

$$=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$$

3. $\sqrt{x^2+4x+6}$

Solution:

Given:

$\sqrt{x^2+4x+6}$

Upon integration we get,

$$I=\int\sqrt{x^2+4x+6}\,dx$$

$$=\int\sqrt{x^2+4x+4+2}\,dx$$

$$=\int\sqrt{(x+2)^2+\left(\sqrt{2}\right)^2}\,dx$$

By using the formula,

$$\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C}$$

So,

$$I=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$

$$=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$

4. $\sqrt{x^2+4x+1}$

Solution:

Given:

$\sqrt{x^2+4x+1}$

Upon integration we get,

$$I = \int \sqrt{x^2+4x+1}\,dx$$

$$= \int \sqrt{\left(x^2+4x+4\right)-3}\,dx$$

$$= \int \sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\,dx$$

By using the formula,

$$\int \sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

So,

$$I = \frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$$

5. $\sqrt{1-4x-x^2}$

Solution:

Given:

$\sqrt{1-4x-x^2}$

Upon integration we get,

$$I = \int \sqrt{1-4x-x^2}\,dx$$

$$= \int \sqrt{1-\left(x^2+4x+4-4\right)}\,dx$$

$$= \int \sqrt{1+4-(x+2)^2}\,dx$$

$$= \int \sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}\,dx$$

By using the formula,

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

So,

$$I = \frac{(x+2)}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$

6. $\sqrt{x^2 + 4x - 5}$

Solution:

Given:

$\sqrt{x^2 + 4x - 5}$

Upon integration we get,

$$I = \sqrt{x^2 + 4x - 5}\, dx$$

$$= \int \sqrt{(x^2 + 4x + 4) - 9}\, dx$$

$$= \int \sqrt{(x+2)^2 - (3)^2}\, dx$$

By using the formula,

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

So,

$$I = \frac{(x+2)}{2}\sqrt{x^2 + 4x - 5} - \frac{9}{2}\log\left|(x+2) + \sqrt{x^2 + 4x - 5}\right| + C$$

7. $\sqrt{1 + 3x - x^2}$

Solution:

Given:

$\sqrt{1 + 3x - x^2}$

Upon integration we get,

$$I = \int \sqrt{1 + 3x - x^2}\, dx$$

$$= \int \sqrt{1 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)}\, dx$$

$$= \int \sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\, dx$$

$$= \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\, dx$$

By using the formula,

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$$

So,

$$I = \frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2}+\frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right)+C$$

$$= \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$$

8. $\sqrt{x^2+3x}$

Solution:

Given:

$\sqrt{x^2+3x}$

Upon integration we get,

$$I = \int \sqrt{x^2+3x}\, dx$$

$$= \int \sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}\, dx$$

$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\, dx$$

By using the formula,

$$\int \sqrt{x^2-a^2x}\, dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

So,

$$I = \frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2-3x} - \frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2-3x}\right|+C$$

$$= \frac{(2x+3)}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

9. $\sqrt{1+\frac{x^2}{9}}$

Solution:

Given:

$$\sqrt{1+\frac{x^2}{9}}$$

Upon integration we get,

$$I = \int\sqrt{1+\frac{x^2}{9}}\,dx$$

$$= \frac{1}{3}\int\sqrt{9+x^2}\,dx$$

$$= \frac{1}{3}\int\sqrt{(3)^2+x^2}\,dx$$

By using the formula,

$$\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x^2+a^2\right|+C$$

So,

$$I = \frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right]+C$$

$$= \frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$$

Choose the correct answer in Exercises 10 to 11

10. $\int \sqrt{1+x^2}\ dx$ is equal to

A. $\frac{x}{2}\sqrt{1+x^2+\frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right|+C}$

B. $\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}+C$

C. $\frac{2}{3}x\left(1+x^2\right)^{\frac{3}{2}}+C$

D. $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|\left(x+\sqrt{1+x^2}\right)\right|+C$

Solution:

Given:

$\int \sqrt{1+x^2}\ dx$

By using the formula,

$\int \sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

So,

$\int \sqrt{1+x^2}dx = \frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

Hence the correct option is A.

11. $\int \sqrt{x^2-8x+7}\ dx$ is equal to

A. $\frac{1}{2}(x-4)\sqrt{x^2-2x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Solution:

Given:

$$\int\sqrt{x^2-8x+7}\,dx$$

Upon integration we get,

$$I=\int\sqrt{x^2-8x+7}\,dx$$

$$=\int\sqrt{(x^2-8x+16)-9}\,dx$$

$$=\int\sqrt{(x-4)^2-(3)^2}\,dx$$

By using the formula,

$$\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

So,

$$I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\sqrt{x^2-8x+7}\right|+C$$

Hence the correct option is D.

EXERCISE 7.8

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Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x\,dx$

Solution:

Given:

$$\int_a^b x\,dx$$

We know that f(x) is continuous in [a, b]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \text{ where } h = \frac{b-a}{n}$$

By substituting the value of h in the above expression we get

$$\int_a^b (x)dx = \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{r=0}^{n-1} f\left(a+\frac{(b-a)r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{r=0}^{n-1}\left(\frac{(b-a)r}{n}\right)+a$$

By expanding the summation we get,

$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left(\frac{(b-a)(n-1)(n)}{2n}+a(n-1)\right)$$

Upon simplification we get,

$$= \lim_{n\to\infty}\frac{(b-a)}{n}\cdot\frac{(b-a)(n^2-n)+2an^2-2an}{2n}$$

$$= \lim_{n\to\infty}\frac{(b-a)}{n}\cdot\frac{(b+a)n^2-(b+a)n}{2n}$$

$$= \lim_{n\to\infty} \frac{(b+a)(b-a)n^2 - (b+a)(b-a)n}{2n^2}$$

On computing we get,

$$= \lim_{n\to\infty} \left(\frac{(b+a)(b-a)}{2} - \frac{(b+a)(b-a)}{n} \right)$$

$$= \frac{(b+a)(b-a)}{2}$$

$$= \frac{b^2 - a^2}{2}$$

2. $\int_0^5 (x+1)\, dx$

Solution:

Given:

$$\int_0^5 (x+1)\, dx$$

We know that f(x) is continuous in [a, b] i.e., [0, 5]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh), \text{ where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_0^5 (x+1)dx = \lim_{n\to\infty} \left(\frac{5}{n}\right) \sum_{r=0}^{n-1} f\left(\frac{5r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty} \left(\frac{5}{n}\right) \sum_{r=0}^{n-1} \left(\frac{5r}{n}\right) + 1$$

By expanding the summation we get,

$$= \lim_{n\to\infty} \left(\frac{5}{n}\right) \left(\frac{5(n-1)(n)}{2n} + (n-1) \right)$$

Upon simplification we get,

$$= \lim_{n\to\infty} \frac{5}{n} \cdot \frac{5n^2 - 5n + 2n^2 - 2n}{2n}$$

$$= \lim_{n\to\infty} \frac{5}{n} \cdot \frac{7n^2 - 7n}{2n}$$

$$= \lim_{n\to\infty} \frac{35n^2 - 35n}{2n^2}$$

$$= \lim_{n\to\infty} \frac{35}{2} - \left(\frac{35}{2n}\right)$$

$$= \frac{35}{2}$$

3. $\int_{2}^{3} x^2\, dx$

Solution:

Given:

$\int_{2}^{3} x^2\, dx$

We know that f(x) is continuous in [a, b] i.e., [2, 3]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a + rh), \text{where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_2^3 (x^2)dx = \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{r=0}^{n-1} f\left(2 + \left(\frac{r}{n}\right)\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{r=0}^{n-1} \left(2 + \left(\frac{r}{n}\right)\right)^2$$

By expanding the summation we get,

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{r=0}^{n-1} \left(\frac{r^2}{n^2} + 4 + \frac{4r}{n}\right)$$

Upon simplification we get,

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(n-1)(n)(2n-1)}{6n^2}+4n+\frac{4(n-1)(n)}{2n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(n^2-n)(2n-1)}{6n^2}+4n+\frac{2(n^2-n)}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(2n^3-2n^2-n^2+n)}{6n^2}+4n+\frac{2(n^2-n)}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(2n^3-3n^2+n)+(24n^3)+(12n^3-12n^2)}{6n^2}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{38n^3-15n^2+n}{6n^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{38n^3-15n^2+n}{6n^3}\right)$$

On computing we get,

$$=\lim_{n\to\infty}\left(\frac{38}{6}\right)-\left(\frac{15}{6n}\right)+\left(\frac{1}{6n^2}\right)$$

$$=\frac{38}{6}$$

$$=\frac{19}{3}$$

4. $\int_1^4 (x^2-x)\,dx$

Solution:

Given:

$$\int_1^4 (x^2-x)\,dx$$

We know that f(x) is continuous in [a, b] i.e., [1, 4]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh), \text{where } h = (b-a)/n$$

Substituting the value of h in the above expression we get,

$$\int_1^4 (x^2 - x)dx = \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{r=0}^{n-1} f\left(\left(1 + \frac{3r}{n}\right)\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{r=0}^{n-1} (\left(1 + \frac{3r}{n}\right)^2 - \left(1 + \frac{3r}{n}\right)$$

By expanding the summation we get,

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{r=0}^{n-1} (1 + \frac{9r^2}{n^2} + \frac{6r}{n} - 1 - \frac{3r}{n})$$

$$= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{r=0}^{n-1} \left(\frac{9r^2}{n^2} + \frac{3r}{n}\right)$$

Upon simplification we get,

$$= \lim_{n\to\infty} \frac{3}{n} \left(\frac{9(n-1)(n)(2n-1)}{6n^2} + \frac{3n(n-1)}{2n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(\frac{9(n^2-n)(2n-1)}{6n^2} + \frac{3n(n-1)}{2n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(\frac{9(2n^3 - 2n^2 - n^2 + n)}{6n^2} + \frac{3n(n-1)}{2n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(\frac{(18n^3 - 27n^2 + 9n) + (9n^3 - 9n^2)}{6n^2}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(\frac{27n^3 - 36n^2 + 9n}{6n^2}\right)$$

On computing we get,

$$= \lim_{n\to\infty} \left(\frac{81n^3 - 108n^2 + 27n}{6n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{81}{6}\right) - \left(\frac{108}{6n}\right) + \left(\frac{27}{6n^2}\right)$$

= 27/2

5. $\int_{-1}^{1} e^x \, dx$

Solution:

Given:

$$\int_{-1}^{1} e^x \, dx$$

We know that f(x) is continuous in [a, b] i.e., [-1, 1]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a + rh), \text{ where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_0^2 (e^x)dx = \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{r=0}^{n-1} f\left(-1 + \frac{2r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{r=0}^{n-1} e^{\frac{2r}{n}-1}$$

By expanding the summation we get,

$$= \lim_{n\to\infty} \left(\frac{2}{ne}\right) (e^0 + e^h + e^{2h} + \cdots \ldots \ldots \ldots \ldots . +e^{nh}$$

sum of $= e^0 + e^h + e^{2h} + \cdots \ldots \ldots \ldots \ldots . +e^{nh}$

Whose g.p has common ratio with $e^{1/n}$.

Whose sum is:

$$= \frac{e^h(1-e^{nh})}{1-e^h}$$

Upon simplification we get,

$$= \lim_{n\to\infty}\left(\frac{2}{ne}\right)\left(\frac{e^h\left(1-e^{nh}\right)}{1-e^h}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{ne}\right).\frac{e^h\left(1-e^{nh}\right)}{\frac{1-e^h.h}{h}}$$

$$= \lim_{h\to 0}\frac{1-e^h}{h}$$

$= -1$

$$= \lim_{n\to\infty}\left(\frac{2}{ne}\right)\left(\frac{e^h\left(1-e^{nh}\right)}{-h}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{ne}\right)\left(\frac{e^{\left(\frac{2}{n}\right)}\left(1-e^{n\times\left(\frac{2}{n}\right)}\right)}{-\frac{2}{n}}\right) \text{[Since, h = 2/n]}$$

$$= \frac{e^2-1}{e}$$

$= e\text{-}e^{-1}$

6. $\int_0^4 (x+e^{2x})\,dx$

Solution:

Given:

$$\int_0^4 (x+e^{2x})\,dx$$

$h(x) = \int_0^4 x.\,dx$

$$g(x) = \int_0^4 e^{2x}.dx$$

So, f (x) = h (x) + g (x)

Now let us solve for h (x)

We know that h (x) is continuous in [0, 4]

Then we have,

$$\int_a^b h(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \text{where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_0^4 (x)dx = \lim_{n\to\infty}\left(\frac{4}{n}\right)\sum_{r=0}^{n-1} f\left(\frac{4r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\sum_{r=0}^{n-1}\left(\frac{4r}{n}\right)$$

By expanding the summation we get,

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\left(\frac{2(n-1)(n)}{n}\right)$$

Upon simplification we get,

$$= \lim_{n\to\infty}\frac{4}{n}.\frac{2n^2-2n}{n}$$

$$= \lim_{n\to\infty}\frac{4}{n}\frac{2n^2-2n}{n}$$

$$= \lim_{n\to\infty}\frac{8n^2-8n}{n^2}$$

$$= \lim_{n\to\infty} 8-\left(\frac{8}{n}\right)$$

=8

Now let us solve for g(x)

We know that g(x) is continuous in [0, 4]

Then we have,

$$\int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh), \text{where } h = \frac{b-a}{n}$$

Substituting the value of h in the above expression we get,

$$\int_0^4 (e^{2x})dx = \lim_{n\to\infty}\left(\frac{4}{n}\right)\sum_{r=0}^{n-1} f\left(\frac{4r}{n}\right)$$

Since, f (a) = a

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\sum_{r=0}^{n-1} e^{\frac{4r}{n}}$$

By expanding the summation we get,

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)(e^0 + e^h + e^{2h} + \cdots \ldots\ldots\ldots\ldots. + e^{nh}$$

sum of $= e^0 + e^h + e^{2h} + \cdots \ldots\ldots\ldots\ldots. + e^{nh}$

Whose g.p is common with ratio $e^{1/n}$

Whose sum is:

$$= \frac{e^h(1-e^{nh})}{1-e^h}$$

Upon simplification we get,

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\left(\frac{e^h(1-e^{nh})}{1-e^h}\right)$$

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\left(\frac{e^h(1-e^{nh})}{\frac{1-e^h.h}{h}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\left(\frac{e^h(1-e^{nh})}{-h}\right) \text{[Since, } \lim_{h\to 0}\frac{1-e^h}{h} = -1\text{]}$$

$$= \lim_{n\to\infty}\left(\frac{4}{n}\right)\left(\frac{e^{\left(\frac{4}{n}\right)}\left(1-e^{n\times\left(\frac{4}{n}\right)}\right)}{-\frac{4}{n}}\right) \text{[Since, h = 4/n]}$$

$$= (e^8 - 1)$$

On computing we get,

f (x) = h (x) + g (x)

$= 8 + e^8 - 1$

EXERCISE 7.9

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Evaluate the definite integrals in Exercises 1 to 20.

1. $\int_{-1}^{1}(x+1)\,dx$

Solution:

Let I = $\int_{-1}^{1}(x+1)dx$

So,

$$I=\int_{-1}^{1}(x+1)dx$$

On splitting the integrals, we have

$$I=\int_{-1}^{1}x\,dx+\int_{-1}^{1}1\times dx \qquad \left[\int x^{n}dx=\frac{x^{n+1}}{n+1}\right]$$

Applying the limits after integration,

$$I=\left[\frac{x^{2}}{2}\right]_{-1}^{1}+\left[x\right]_{-1}^{1}$$

$$I=\left[\frac{1^{2}}{2}-\frac{(-1)^{2}}{2}\right]+\left[1-(-1)\right]$$

$$I=\left[\frac{1}{2}-\frac{1}{2}\right]+\left[1+1\right]=0+2$$

I = 2

Therefore, $\int_{-1}^{1}(x+1)dx=2$

2. $\int_{2}^{3}\frac{1}{x}\,dx$

Solution:

Let I = $\int_2^3 \frac{1}{x} dx$

$$I = \int_2^3 \frac{1}{x} dx \qquad [\int \frac{1}{x} dx = \log x]$$

Applying the limits after integration,

$$I = \left[\log|x|\right]_2^3$$

I = log |3| - log |2|

I = log 3/2

Therefore,

$$\int_2^3 \frac{1}{x} dx = \log \frac{3}{2}$$

3. $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx$

Solution:

Let I = $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

$$I = \int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$$

Splitting the integrals, we have

$$I = \int_1^2 4x^3 dx - \int_1^2 5x^2 dx + \int_1^2 6x dx + \int_1^2 9 dx$$

$$I = 4\int_1^2 x^3 dx - 5\int_1^2 x^2 dx + 6\int_1^2 x dx + 9\int_1^2 dx$$

Performing integration separately, we get

$$I = 4\times\left[\frac{x^{3+1}}{3+1}\right]_1^2 - 5\times\left[\frac{x^{2+1}}{2+1}\right]_1^2 + 6\times\left[\frac{x^{1+1}}{1+1}\right]_1^2 + 9\times\left[\frac{x^{0+1}}{0+1}\right]_1^2$$

$$\left[\int x^n dx = \frac{x^{n+1}}{n+1}\right]$$

Applying the limits after integration,

$$I = 4\times\left[\frac{x^4}{4}\right]_1^2 - 5\times\left[\frac{x^3}{3}\right]_1^2 + 6\times\left[\frac{x^2}{2}\right]_1^2 + 9\times[x]_1^2$$

$$= 2^4 - 1^4 - 5\left[\frac{2^3}{3} - \frac{1^3}{3}\right] + 6\left[\frac{2^2}{2} - \frac{1^2}{2}\right] + 9[2-1]$$

$$= 16 - 1 - 5\left[\frac{7}{3}\right] + 3(3) + 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99-35}{3} = \frac{64}{3}$$

Therefore, $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right)dx = 64/3$

4. $\int_0^{\frac{\pi}{4}} \sin 2x\, dx$

Solution:

Let I = $\int_0^{\frac{\pi}{4}} \sin 2x\, dx$

I = $\int_0^{\frac{\pi}{4}} \sin 2x\, dx$

Applying limits after integration, we have

$$I = \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}} \qquad [\int \sin x\, dx = -\cos x]$$

I = - (cos 2 × π/4 – cos 0)/2

I = - (cos π/2 – cos 0)/2 = - (0 – 1)/2

I = ½

Therefore, $\int_0^{\frac{\pi}{4}} \sin 2x\, dx$ = ½

5. $\int_0^{\frac{\pi}{2}} \cos 2x\, dx$

Solution:

Let I = $\int_0^{\frac{\pi}{2}} \cos 2x\, dx$

I = $\int_0^{\frac{\pi}{2}} \cos 2x\, dx$

Integrating cos 2x and applying limits, we have

$$I = \left[\frac{\sin 2x}{2}\right]_0^{\pi/2} \qquad [\int \cos x dx = \sin x + c]$$

$$I = \frac{1}{2}\left(\sin 2 \times \frac{\pi}{2} - \sin 2 \times 0\right)$$

$$I = \frac{1}{2}(\sin \pi - \sin 0)$$

I = 1/2 × (0 – 0) = 0

Therefore, $\int_0^{\frac{\pi}{2}} \cos 2x\, dx$ = 0

6. $\int_4^5 e^x dx$

Solution:

Let I = $\int_4^5 e^x\, dx$

I = $\int_4^5 e^x\, dx$

Applying the limits after integration, we get

I = $\left[e^x\right]_4^5$ = $e^5 - e^4$ $\qquad [\int e^x dx = e^x + c]$

I = e^4 (e – 1)

Therefore, $\int_4^5 e^x\,dx = e^4(e-1)$

7. $\int_0^{\frac{\pi}{4}} \tan x\, dx$

Solution:

Let I = $\int_0^{\frac{\pi}{4}} \tan x\,dx$

I = $\int_0^{\frac{\pi}{4}} \tan x\,dx$ [Using $\int \tan x\,dx = -\log|\cos x| + c$]

I = $\left[-\log|\cos x|\right]_0^{\pi/4}$

Applying limits after integrating, we have

I = $-\left(\log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right)$

I = $-\left(\log\left|\frac{1}{\sqrt{2}}\right| - \log|1|\right) = -\log(2)^{-\frac{1}{2}} + 0$

I = $\frac{1}{2}\log 2$

Therefore, $\int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{1}{2}\log 2$

8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$

Solution:

Let I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

I = $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

Performing integration, we have

$$I = \left[\log\left|\operatorname{cosec} x - \cot x\right|\right]_{\pi/6}^{\pi/4}$$

[Using $\int \operatorname{cosec} x dx = \log\left|\operatorname{cosec} x - \cot x\right| + c$]

Applying limits after integration, we get

I = log |cosec π/4 – cot π/4| - log |cosec π/6 – cot π/6|

I = log |√2 – 1| - log |2 - √3|

$$I = \log\left|\frac{\sqrt{2}-1}{2-\sqrt{3}}\right|$$

Therefore, $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx = \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$

9. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Solution:

Let I = $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Performing integration,

I = $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ [Using $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$]

Applying limits after integration, we have

I = $\left[\sin^{-1} x\right]_0^1$

$I = \sin^{-1}(1) - \sin^{-1}(0) = \pi/2 - 0$
$I = \pi/2$

Therefore, $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$

10. $\int_0^1 \frac{dx}{1+x^2}$
Solution:

Let I = $\int_0^1 \frac{dx}{1+x^2}$

I = $\int_0^1 \frac{dx}{1+x^2}$

We know that,

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

Hence, on integrating we get

I = $\left[\tan^{-1}x\right]_0^1$

Applying limits, we have

$I = \tan^{-1}(1) - \tan^{-1}(0) = \pi/4 - 0$
$I = \pi/4$

Therefore, $\int_0^1 \frac{dx}{1+x^2} = \pi/4$

11. $\int_2^3 \frac{dx}{x^2-1}$
Solution:

Let I = $\int_2^3 \frac{dx}{x^2-1}$

On integrating, we have

I = $\int_2^3 \frac{dx}{x^2-1}$ [w.k.t $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a} + c$]

Applying limits after integration, we get

I = $\left[\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_2^3 = \frac{1}{2}\left(\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right)$

I = $\frac{1}{2}\left(\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right) = \frac{1}{2}\log\frac{1/2}{1/3}$

I = ½ log 3/2

Therefore, $\int_2^3 \frac{dx}{x^2-1}$ = ½ log 3/2

12. $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

Solution:

Let I = $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

We know that,

cos 2x = 2cos^{2}x – 1

So, cos^{2}x = $\frac{1+\cos 2x}{2}$

Putting the value cos^{2}x in I and splitting the integrals, we have

I = $\int_0^{\pi/2} \frac{1+\cos 2x}{2} dx = \frac{1}{2}\int_0^{\pi/2} dx + \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx$ [$\int \cos x dx = \sin x + c$]

Applying limits after integration, we get

I = $\frac{1}{2}[x]_0^{\pi/2} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \frac{1}{4}\left(\sin 2\times\frac{\pi}{2} - \sin 2\times 0\right)$

I = $\frac{\pi}{4}+\frac{1}{4}(0-0)$ = π/4

Therefore, $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ = π/4

13. $\int_2^3 \frac{x\,dx}{x^2+1}$

Solution:

Let I = $\int_2^3 \frac{x\,dx}{x^2+1}$

Let's assume $x^2 + 1 = t$

So,

$d(x^2 + 1) = dt$

$2x\,dx = dt$

$x\,dx = dt/2$

When $x = 2$; $t = 2^2 + 1 = 5$

When $x = 3$; $t = 3^2 + 1 = 10$

Substituting $(x^2 + 1)$ and $x\,dx$ in I, we have

I = $\int_5^{10} \frac{dt}{2t} = \frac{1}{2}\int_5^{10} \frac{dt}{t}$ $\left[\text{w.k.t } \int \frac{1}{x}dx = \log x\right]$

Applying limits after integration, we get

I = $\frac{1}{2}[\log t]_5^{10} = \frac{1}{2}(\log 10 - \log 5) = \frac{1}{2}\log\frac{10}{5}$

I = ½ log 2

Therefore, $\int_2^3 \frac{x\,dx}{x^2+1}$ = ½ log 2

14. $\int_0^1 \frac{2x+3}{5x^2+1}\,dx$

Solution:

Let I = $\int_0^1 \frac{2x+3}{5x^2+1}$

Multiplying by 5 in numerator and denominator:

I = $\frac{1}{5}\int_0^1 \frac{5(2x+3)}{5x^2+1}dx = \frac{1}{5}\int_0^1 \frac{10x+15}{5x^2+1}dx$

Splitting the fraction into two fractions, we have

I = $\frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}dx + 3\int_0^1 \frac{1}{5x^2+1}$

Now, I = $I_1 + I_2$

Where, $I_1 = \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}dx$

Let us take $5x^2 + 1 = t$ .... (1)

$d(5x^2 + 1) = dt$

$10x\ dx = dt$ ..... (2)

When $x = 0$; $t = 5 \times 0^2 + 1 = 1$

When $x = 1$; $t = 5 \times 1^2 + 1 = 6$

Substituting (1) and (2) in I_1, we have

$I_1 = \frac{1}{5}\int_1^6 \frac{dt}{t} = \frac{1}{5}\left[\log|t|\right]_1^6$ $\left[\text{w.k.t } \int \frac{1}{x}dx = \log x\right]$

Applying limits to integrals, we get

$I_1 = \frac{1}{5}\left(\log|6| - \log|1|\right) = \frac{1}{5}(\log 6 - 0)$

$I_1 = \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}dx = \frac{\log 6}{5}$

Next,

$I_2 = 3\int_0^1 \frac{1}{5x^2+1}dx = \frac{3}{5}\int_0^1 \frac{1}{x^2+\frac{1}{5}}dx$ $\left[\text{w.k.t } \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

Applying limits to integrals, we get

$I_1 = \frac{1}{5}(\log|6| - \log|1|) = \frac{1}{5}(\log 6 - 0)$

$I_1 = \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}dx = \frac{\log 6}{5}$

Next,

$I_2 = 3\int_0^1 \frac{1}{5x^2+1}dx = \frac{3}{5}\int_0^1 \frac{1}{x^2+\frac{1}{5}}dx$

[w.k.t $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$]

$I_2 = \frac{3}{5} \times \frac{1}{\frac{1}{\sqrt{5}}}\left[\tan^{-1}\sqrt{5}x\right]_0^1 = \frac{3}{5} \times \sqrt{5}\left(\tan^{-1}\sqrt{5} - \tan^{-1}0\right)$

$I_2 = 3/\sqrt{5} \tan^{-1}5$

$I_2 = \frac{3}{5} \times \frac{1}{\frac{1}{\sqrt{5}}}\left[\tan^{-1}\sqrt{5}x\right]_0^1 = \frac{3}{5} \times \sqrt{5}\left(\tan^{-1}\sqrt{5} - \tan^{-1}0\right)$

$I_2 = 3/\sqrt{5} \tan^{-1}5$

Hence, $I = I_1 + I_2$

$I = 1/5 \log 6 + 3/\sqrt{5} \tan^{-1}5$

Therefore, $\int_0^1 \frac{2x+3}{5x^2+1}dx = 1/5 \log 6 + 3/\sqrt{5} \tan^{-1}5$

15. $\int_0^1 x e^{x^2} dx$

Solution:

Let $I = \int_0^1 xe^{x^2} dx$

On taking $x^2 = t \Rightarrow 2x\, dx = dt$

When $x = 0$; $t = 0$

When $x = 1$; $t = 1$

Substituting t and dt in I,

I = $\int_0^1 \frac{e^t dt}{2}$ = ½ $\int_0^1 e^t dt$ $[\int e^x dx = e^x + c]$

I = $\frac{1}{2}\left[e^t\right]_0^1 = \frac{1}{2}\left(e - e^0\right) = \frac{1}{2}(e-1)$

Therefore, $\int_0^1 xe^{x^2} dx$ = ½ (e – 1)

16. $\int_1^2 \frac{5x^2}{x^2+4x+3}$

Solution:

Let I = $\int_1^2 \frac{5x^2}{x^2+4x+3}$

On dividing $5x^2$ by $x^2 + 4x + 3$ we get 5 as quotient and – (20x + 15) as remainder

So, I = $\int_1^2 \left(5 - \frac{20x+15}{x^2+4x+3}\right) dx$

Splitting the integrals, we have

I = $\int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3}$ = $5[x]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3}$

I = 5 (2 – 1) - $\int_1^2 \frac{20x+15}{x^2+4x+3}$

I = 5 – I_1

Now,

I_1 = $\int_1^2 \frac{20x+15}{x^2+4x+3}$

Adding and subtracting 25 in the numerator, we get

$I_1 = \int_1^2 \frac{20x+15+25-25}{x^2+4x+3}dx = \int_1^2 \frac{20x+40}{x^2+4x+3}dx - \int_1^2 \frac{25}{x^2+4x+3}dx$

$I_1 = 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx - 25\int_1^2 \frac{1}{x^2+4x+3}dx$

Let us assume $x^2 + 4x + 3 = t$

Then, $(2x + 4)\, dx = dt$

So,

$I_1 = 10\int \frac{dt}{t} - 25\int \frac{1}{x^2+4x+3+1-1}dx = 10\log t + 25\int \frac{1}{x^2+4x+4-1}dx$

$I_1 = 10 \log t - 25\int \frac{1}{(x+2)^2-1^2}dx$ [w.k.t $\int \frac{1}{x}dx = \log x$]

$I_1 = 10 \log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right]$ [w.k.t $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a} + c$]

Applying limits after integration, we get

$I_1 = 10\left[\log\left(x^2+4x+3\right)\right]_1^2 - \frac{25}{2}\left[\log\left(\frac{x+1}{x+3}\right)\right]_1^2$

$I_1 = 10 \times$

$\left[\log\left(2^2+4\times2+3\right) - \log\left(1^2+4\times1+3\right)\right] - \frac{25}{2}\left[\log\left(\frac{2+1}{2+3}\right) - \log\left(\frac{1+1}{1+3}\right)\right]$

$I_1 = 10\left[\log 15 - \log 8\right] - \frac{25}{2}\left[\log\frac{3}{5} - \log\frac{2}{4}\right]$

$I_1 = 10\left[\log(5\times3) - \log(4\times2)\right] - \frac{25}{2}\left[\log 3 - \log 5 - \log 2 + \log 4\right]$

$I_1 =$

$10\log 5 + 10\log 3 - 10\log 4 - 10\log 2 - \frac{25}{2}\log 3 + \frac{25}{2}\log 5 + \frac{25}{2}\log 2 - \frac{25}{2}\log 4$

$I_1 =$

$$\left(10+\frac{25}{2}\right)\log 5-\left(10+\frac{25}{2}\right)\log 4+\left(10-\frac{25}{2}\right)\log 3+\left(-10+\frac{25}{2}\right)\log 2$$

$I_1 = \frac{45}{2}\log 5-\frac{45}{2}\log 4-\frac{5}{2}\log 3+\frac{5}{2}\log 2=\frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}$

As, $I = 5 - I_1$

On substituting I_1 in I we get,

$I = 5 - \frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}$

Therefore, $\int_1^2 \frac{5x^2}{x^2+4x+3} = 5 - \frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}$

17. $\int_0^{\frac{\pi}{4}}(2\sec^2 x + x^3 + 2)\, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{4}}(2\sec^2 x + x^3 + 2)dx$

Splitting the given integral, we have

$I = \int_0^{\pi/4}\left(2\sec^2 x + x^3 + 2\right)dx = 2\int_0^{\pi/4}\sec^2 x dx + \int_0^{\pi/4}x^3 dx + 2\int_0^{\pi/4}dx$

Now, integration separately and applying limits, we get

$I = 2\left[\tan x\right]_0^{\pi/4} + \left[\frac{x^4}{4}\right]_0^{\pi/4} + 2\left[x\right]_0^{\pi/4}$ $\left[\text{w.k.t} \int \sec^2 x dx = \tan x + c\right]$

$I = 2 (\tan \pi/4 - \tan 0) + \frac{1}{4} ((\pi/4)^4 - 0) + 2 (\pi/4 - 0)$

$I = 2\times 1+\frac{1}{4}\times\left(\frac{\pi}{4}\right)^4+2\times\frac{\pi}{4}$

Expanding the exponents, we have

$I = 2+\frac{\pi}{2}+\frac{\pi^4}{1024}$

Therefore, $\int_0^{\pi/4} \left(2\sec^2 x + x^3 + 2\right)dx = 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$

18. $\int_0^{\pi} (\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2})\, dx$

Solution:

Let I = $\int_0^{\pi} (\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2})dx$

We know that,

$\cos x = \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}$

So, substituting $\cos x = \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}$ in I, we have

Applying the limits after integration, we get

I = $\int_0^{\pi} \cos x dx = [\sin x]_0^{\pi}$ $\quad$ $[\text{w.k.t} \int \cos x dx = \sin x + c]$

I = $\sin \pi - \sin 0 = 0 - 0 = 0$

Therefore, $\int_0^{\pi} (\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2})dx = 0$

19. $\int_0^{2} \frac{6x+3}{x^2+4}\, dx$

Solution:

Let I = $\int_0^{2} \frac{6x+3}{x^2+4} dx$

I = $3\int_0^{2} \frac{2x+1}{x^2+4} = 3\int_0^{2} \frac{2x}{x^2+4} dx + 3\int_0^{2} \frac{1}{x^2+4} dx$

Now, we have $I = I_1 + I_2$

Where $I_1 = 3\int_0^2 \frac{2x}{x^2+4}dx$

Let $x^2 + 4 = t$

$2x\, dx = dt$

When $x = 0$; $t = 4$

When $x = 2$; $t = 2^2 + 4 = 8$

Substituting t and dt in I_1

$I_1 = 3\int_4^8 \frac{dt}{t} = 3\left[\log|t|\right]_4^8$ $\left[\text{w.k.t } \int \frac{1}{x}dx = \log x\right]$

$I_1 = 3\,[\log|8| - \log|4|] = 3 \log 8/4$

$I_1 = 3 \log ½ = -3 \log 2$

And, $I_2 = 3\int_0^2 \frac{1}{x^2+4}dx = 3\int_0^2 \frac{1}{x^2+2^2}dx$ $\left[\text{w.k.t } \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

$I_2 = 3\times\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{3}{2}\left[\tan^{-1}\frac{2}{2} - \tan^{-1}\frac{0}{2}\right] = \frac{3}{2}\left[\tan^{-1}1 - \tan^{-1}0\right]$

$I_2 = \frac{3}{2}\times\frac{\pi}{4} = 3\pi/8$

Now, $I = I_1 + I_2$

$I = 3 \log ½ + 3\pi/8$

Therefore, $\int_0^2 \frac{6x+3}{x^2+4}dx = 3 \log ½ + 3\pi/8$

20. $\int_0^1 (x\, e^x + \sin\frac{\pi x}{4})\, dx$

Solution:

Let $I = \int_0^1 (x\, e^x + \sin\frac{\pi x}{4})dx$

Splitting the integrals, we have

I = $\int_0^1 xe^x\,dx + \int_0^1 \sin\frac{\pi x}{4}\,dx$

Now, I = $I_1 + I_2$

$I_1 = \int_0^1 xe^x\,dx$ [Using u-v integral form: u = x and v = e^x]

$I_1 = x\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx$

$I_1 = xe^x - \int e^x dx$ $\left[\text{w.k.t} \int e^x dx = e^x + c\right]$

Now, integrating the reduced form and applying the limits, we get

$I_1 = \left[xe^x - e^x\right]_0^1 = \left[\left(1\times e^1 - e^1\right) - \left(0\times e^0 - e^0\right)\right]$

$I_1 = e - e - 0 + 1$

$I_1 = 1$

Next, taking I_2

$I_2 = \int_0^1 \sin\frac{\pi x}{4}\,dx$ $\left[\text{w.k.t} \int \sin x\,dx = -\cos x\right]$

Applying the limits after integration, we get

$I_2 = \left[-\frac{\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right]_0^1 = -\frac{4}{\pi}\left[\cos\frac{\pi}{4}\times 1 - \cos\frac{\pi}{4}\times 0\right] = -\frac{4}{\pi}\left[\cos\frac{\pi}{4} - \cos 0\right]$

$I_2 = \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$

As, I = $I_1 + I_2$

Hence, I = $1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$

Therefore, $\int_0^1 (xe^x + \sin\frac{\pi x}{4})dx = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$

21. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

Solution:

Let I = $\int_1^{\sqrt{3}} \frac{dx}{x^2+1}$

I = $\int_1^{\sqrt{3}} \frac{dx}{x^2+1}$

On integrating using standard form and applying limits, we get

I = $\left[\tan^{-1} x\right]_1^{\sqrt{3}} = \left[\tan^{-1}\sqrt{3} - \tan^{-1} 1\right] = \frac{\pi}{3} - \frac{\pi}{4}$

$\left[\text{w.k.t } \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$

I = $\frac{4\pi - 3\pi}{12} = \pi/12$

Therefore, $\int_1^{\sqrt{3}} \frac{dx}{x^2+1} = \pi/12$

Hence, option (D) is correct.

22.

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

Solution:

Let I = $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$

I = $\int_0^{3} \frac{dx}{4+9x^2}$

Now, taking 9 common from Denominator in I, we have

I = $\frac{1}{9}\int_0^{\frac{2}{3}} \frac{dx}{\frac{4}{9}+x^2} = \frac{1}{9}\int_0^{\frac{2}{3}} \frac{dx}{\left(\frac{2}{3}\right)^2+x^2}$ $\left[\text{w.k.t } \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$

Using the standard form for integrating and applying the limits, we get

I = $\frac{1}{9}\times\frac{3}{2}\left[\tan^{-1}\frac{x}{\frac{2}{3}}\right]_0^{\frac{2}{3}} = \frac{1}{9}\times\frac{3}{2}\left[\tan^{-1}\frac{3x}{2}\right]_0^{\frac{2}{3}}$

I = $\frac{1}{6}\left[\tan^{-1}\frac{3}{2}\times\frac{2}{3} - \tan^{-1}0\right] = \frac{1}{6}\left[\tan^{-1}1 - \tan^{-1}0\right]$

I = $\frac{1}{6}\times\left(\frac{\pi}{4}-0\right)$ = π/24

Therefore, $\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$ = π/24

Hence, option (C) is correct.

EXERCISE 7.10

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Evaluate the integrals in Exercise 1 to 8 by substitution.

1. $\int_0^1 \frac{x}{x^2+1} dx$

Solution:

Given integral: $\int_0^1 \frac{x}{x^2+1} dx$

Let's take $x^2 + 1 = t$

Then, $2x\, dx = dt$

$x\, dx = \frac{1}{2}\, dt$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

Now,

$$\int_0^1 \frac{x}{x^2+1} dx = \int_1^2 \frac{dt}{2t}$$

$$= \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}[\log|t|]_1^2$$

$$= \frac{1}{2}[\log 2 - \log 1]$$

$$= \frac{1}{2}\log 2$$

2. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\, \cos^5\phi\, d\phi$

Solution:

Given integral: $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Let's consider $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\, \cos\phi\, d\phi$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,(\cos^2\phi)^2 \ \cos\phi \ d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\left(1-\sin^2\phi\right)^2 \ \cos\phi \ d\phi$$

Also, let $\sin\phi = t \Rightarrow \cos\phi d\phi = dt$

So when, $\phi = 0,\ t = 0$ and when $\phi = \frac{\pi}{2},\ t = 1$

Hence,

$$I = \int_0^1 \sqrt{t}\left(1 - t^2\right)^2 \ dt$$

Expanding and splitting the integrals, we have

$$= \int_0^1 t^{\frac{1}{2}}\left(1 + t^4 - 2t^2\right) \ dt$$

$$= \int_0^1 (t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}})dt$$

Integrating the terms individually by standard form, we get

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} + \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154 + 42 - 132}{231} = \frac{64}{231}$$

Therefore, $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi \ d\phi$ = 64/231

3. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Solution:

Given integral: $\int_0^1 \sin^{-1}\left(\frac{2x}{x^2+1}\right)dx$

Let us take $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

So when, $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \pi/4$

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{x^2+1}\right)dx$

Now, by substitution I becomes

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{\tan^2\theta+1}\right)\sec^2\theta d\theta$$

Transforming the trigonometric ratio into its simple form, we have

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

Applying the inverse trigonometric ratio, we get

$$I = \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta d\theta$$

$$I = 2\int_0^{\frac{\pi}{4}} \theta\sec^2\theta d\theta$$

Now, by applying product rule as:

$$\int u.v dx = u.\int v dx - \int \frac{du}{dx}.\left\{\int v dx\right\}dx$$

$$I = 2\left[\theta\int\sec^2\theta d\theta - \int\frac{d}{d\theta}\theta.\left\{\int\sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int 1.\tan\theta d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \log|\sec\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} - \log\left|\sec\frac{\pi}{4}\right| - 0 + \log\left|\sec 0\right|\right]$$

$$= 2\left[\frac{\pi}{4} - \log\left(\sqrt{2}\right) + \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log(2)\right]$$

$$= \frac{\pi}{2} + \log(2)$$

Therefore, $\int_0^1 \sin^{-1}\left(\frac{2x}{x^2+1}\right)dx = \frac{\pi}{2} + \log(2)$

4. $\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$)

Solution:

Given integral: $\int_0^2 x\sqrt{x+2}\,dx$

Let's take $x + 2 = t^2 \Rightarrow dx = 2t\,dt$

And, $x = t^2 - 2$

So when, $x = 0$, $t = \sqrt{2}$ and when $x = 2$, $t = 2$

Hence, after substitution the given integral can be written as:

$$\int_0^2 x\sqrt{x+2}\,dx = \int_{\sqrt{2}}^2 \left(t^2 - 2\right)\sqrt{t^2}\,2t\,dt$$

Taking the square root we have,

$$= 2\int_{\sqrt{2}}^2 \left(t^2 - 2\right)t.t\,dt$$

$$= 2\int_{\sqrt{2}}^2 \left(t^2 - 2\right)t^2\,dt$$

$$= 2\int_{\sqrt{2}}^{2}\left(t^4 - 2t^2\right)dt$$

On integrating the terms separately, we get

$$= 2\left[\frac{t^5}{5} - \frac{2t^3}{3}\right]_{\sqrt{2}}^{2}$$

Applying the limits after integration, we have

$$= 2\left[\frac{(2)^5}{5} - \frac{2(2)^3}{3} - \frac{\left(\sqrt{2}\right)^5}{5} + \frac{2\left(\sqrt{2}\right)^3}{3}\right]_{\sqrt{2}}^{2}$$

$$= 2\left[\frac{32}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3}\right]$$

$$= 2\left[\frac{96 - 80 - 12\sqrt{2} + 20\sqrt{2}}{15}\right]$$ [Taking L.C.M for addition]

$$= 2\left[\frac{16 + 8\sqrt{2}}{15}\right]$$

$$= \left[\frac{16(2 + \sqrt{2}}{15}\right]$$ [After taking common terms]

$$= \frac{16\sqrt{2}\left(\sqrt{2} + 1\right)}{15}$$

Therefore, $\int_0^2 x\sqrt{x + 2}dx = \frac{16\sqrt{2}\left(\sqrt{2} + 1\right)}{15}$

5. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\, dx$

Solution:

Given integral: $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$

Let cos x = t

On differentiating,

-sin xdx = dt

sin xdx = -dt

So, when x = 0, t = 1 and when x = π /2, t = 0

Hence, the given integration upon substitution will change as

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$

On integrating, we have

$$-\int_1^0 \frac{dt}{1+t^2} = -\left[\frac{1}{1}.\tan^{-1} t\right]_1^0 \qquad \left[\text{As w.k.t } \int \frac{dt}{x^2+a^2} = \frac{1}{a}.\tan^{-1}\frac{x}{a} + C\right]$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[0 - \frac{\pi}{4}\right]$$

$$= -\left[-\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

Therefore, $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = \frac{\pi}{4}$

6. $\int_0^2 \frac{dx}{x+4-x^2}$

Solution:

Given integral: $\int_0^2 \frac{dx}{x+4-x^2}$

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-(x^2-x-4)}$$

The given integral can be written as,

$$\int_0^2 \frac{dx}{-(x^2-x+\frac{1}{4}-\frac{1}{4}-4)}$$

[By completing its square method]

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left[\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2\right]}$$

Now, taking suitable substitution

Let $x-\frac{1}{2} = t \Rightarrow dx = dt$

So when $x = 0,\ t = -\frac{1}{2}$ and when $x = 2,\ t = \frac{3}{2}$

After substitution, the integral changes as:

$$\int_0^2 \frac{dx}{\left[\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2\right]} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left[\left(\frac{\sqrt{17}}{2}\right)^2-(t)^2\right]}$$

[As w.k.t, $\int \frac{dx}{\left[(a)^2-(x)^2\right]} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$]

On integrating, we have

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left[\left(\frac{\sqrt{17}}{2}\right)^2 - (t)^2\right]} = \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\left(\frac{\sqrt{17}}{2} + t\right)}{\frac{\sqrt{17}}{2} - t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

Applying limits,

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\left(\frac{\sqrt{17}}{2} + \frac{3}{2}\right)}{\frac{\sqrt{17}}{2} - \frac{3}{2}} - \log \frac{\left(\frac{\sqrt{17}}{2} - \frac{1}{2}\right)}{\frac{\sqrt{17}}{2} + \frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\left(\sqrt{17} + 3\right)}{\sqrt{17} - 3} - \log \frac{\left(\sqrt{17} - 1\right)}{\sqrt{17} + 1}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \left\{\frac{\left(\sqrt{17} + 3\right)}{\sqrt{17} - 3} \times \frac{\left(\sqrt{17} + 1\right)}{\sqrt{17} - 1}\right\}\right]$$

[Using logarithmic properties]

$$= \frac{1}{\sqrt{17}}\left[\log \left\{\frac{\left(\sqrt{17} + 3\right)\left(\sqrt{17} + 1\right)}{\left(\sqrt{17} - 3\right)\left(\sqrt{17} - 1\right)}\right\}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17 + 3 + 4\sqrt{17}}{17 + 3 - 4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{5 + \sqrt{17}}{5 - \sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{\left(5-\sqrt{17}\right)\left(5+\sqrt{17}\right)}\right]$$

[Rationalising the surd]

$$= \frac{1}{\sqrt{17}} \log\left[\frac{\left(25+17+10\sqrt{17}\right)}{25-17}\right]$$

$$= \frac{1}{\sqrt{17}} \log\left[\frac{\left(42+10\sqrt{17}\right)}{8}\right] = \frac{1}{\sqrt{17}} \log\left[\frac{\left(21+5\sqrt{17}\right)}{4}\right]$$

7. $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

Solution:

Given integral: $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$$= \int_{-1}^{1} \frac{dx}{(x^2+2x+1)+4}$$

$$= \int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2}$$

[By completing the square]

Taking substitution, x + 1 = t

So, dx = dt

When x = -1, t = 0 and when x = 1, t = 2

Hence, the given integral is now changed as

$$\int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2} = \int_{0}^{2} \frac{dt}{(t)^2+(2)^2}$$

[As w.k.t $\int \frac{dt}{x^2+a^2} = \frac{1}{a}.\tan^{-1}\frac{x}{a} + C$]

$$\int_0^2 \frac{dt}{(t)^2 + (2)^2} = \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

Therefore, $\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5} = \frac{\pi}{8}$

8. $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

Solution:

Given integral: $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

Taking substitution, $2x = t \Rightarrow 2\,dx = dt$

So when $x = 1$, $t = 2$ and when $x = 2$, $t = 4$

Hence, the given integral will change as:

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \int_2^4 \left(\frac{1}{\left(\frac{t}{2}\right)} - \frac{1}{2\left(\frac{t}{2}\right)^2}\right) e^t \left(\frac{dt}{2}\right)$$

$$= \frac{1}{2}\int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t dt$$

$$= \int_2^4 \frac{1}{2}.(2)\left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$$

[Taking common and simplifying]

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt$$

Further, let $1/t = f(t)$

Then we have, $f'(t) = -1/t^2$

Converting the integral into the required form,

$$\int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt = \int_2^4 (f(t) + f'(t)) e^t dt$$

$$\left[\text{As, w.k.t } \int (f(x) + f'(x)) e^x dx = e^x f(x) + C\right]$$

Up to integration, we get

$$\int_2^4 (f(t) + f'(t)) e^t dt = \left[e^t f(t)\right]_2^4$$

$$= \left[e^t . \frac{1}{t}\right]_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^4 - 2e^2}{4} = \frac{e^2(e^2 - 2)}{4}$$

Therefore, $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \frac{e^2(e^2 - 2)}{4}$

Choose the correct answer in Exercise 9 and 10.

9. The value of the integral $\int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{\frac{1}{3}}}{x^4} dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

Solution:

Given integral: $\int_{\frac{1}{3}}^{1} \left(\frac{(x - x^3)^{\frac{1}{3}}}{x^4}\right) dx$

Let $I = \int_{\frac{1}{3}}^{1} \left(\frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} \right) dx$

Now, taking x = sin θ ⇒ dx = cos θ d θ

So when, $x = \frac{1}{3},\ \theta = \sin^{-1}\left(\frac{1}{3}\right)$ and when x = 1, θ = π/2

Hence, after substitution the given integral will become:

$$I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{\left(\sin\theta - \sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \right) \cos\theta d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(1 - \sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \right) \cos\theta d\theta$$

[Taking common]

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta} \right) \cos\theta d\theta$$

[Simplifying by using trigonometric identity]

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^2\theta . \sin^2\theta} \right) \cos\theta d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{\left(\cos\theta\right)^{\frac{2}{3}+1}}{\left(\sin\theta\right)^{2-\frac{1}{3}}} \right) . \frac{1}{\sin^2\theta} d\theta$$

[Simplifying by using exponents properties]

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left(\frac{\left(\cos\theta\right)^{\frac{5}{3}}}{\left(\sin\theta\right)^{\frac{5}{3}}} \right) .\mathrm{cosec}^2\theta d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \left((\cot\theta)^{\frac{5}{3}}\right).\operatorname{cosec}^2\theta d\theta$$

.... (i)

Now, let $\cot\theta = t \Rightarrow -\operatorname{cosec}^2\theta\, d\theta$

So when, $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}$, $t = 0$

After substitution, (i) becomes:

$$= \int_{2\sqrt{2}}^{0} -(t)^{\frac{5}{3}}.dt$$

On integrating and applying limits, we have

$$= -\left[\frac{(t)^{\frac{5}{3}+1}}{\frac{5}{3}+1}\right]_{2\sqrt{2}}^{0}$$

$$= -\left[\frac{(t)^{\frac{8}{3}}}{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[(0)^{\frac{8}{3}} - \left(2\sqrt{2}\right)^{\frac{8}{3}}\right]$$

$$= -\frac{3}{8}\left[-\left(\sqrt{8}\right)^{\frac{8}{3}}\right] = \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$= \frac{3}{8}[16]$$

$= 6$

Therefore, the correct option is (A).

10. If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is

(A) $\cos x + x \sin x$ (B) $x \sin x$

(C) $x \cos x$ (D) $\sin x + x \cos x$

Solution:

Given integral function: $f(x) = \int_0^x t \sin t dt$

Applying product rule, we have

$$\int u.v dx = u.\int v dx - \int \frac{du}{dx}.\left\{\int v dx\right\} dx$$

So,

$$f(x) = t\int_0^x \sin t dt - \int_0^x \left\{\left(\frac{d}{dt}t\right).\int \sin t dt\right\} dt = \left[t(-\cos t)\right]_0^x - \int_0^x (-\cos t) dt$$

Applying the limits, we get

$$= \left[-t(\cos t) + \sin t\right]_0^x$$

= -x cos x + sin x – 0

Thus, f(x) = -x cos x + sin x

On differentiating, we have

$$f'(x) = -\left[x.\frac{d}{dx}\cos x + \cos x.\frac{d}{dx}x + \frac{d}{dx}\sin x\right]$$

f'(x) = -[{x (-sinx)} + cosx] + cosx

= x sin x – cos x + cos x

= x sin x

Therefore, the correct option is (B).

EXERCISE 7.11 PAGE NO: 347

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1. $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

Solution:

Given, $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

Let, $I = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx \ldots..(1)$

We know that, $\left\{ \int_0^a f(x)dx = \int_0^a f(a-x)dx \right\}$

By using above formula, the given question can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx$$

From the standard integration formulae we have

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2(x)\ dx \ldots.(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left[\sin^2(x) + \cos^2(x)\right] dx$$

By using standard identities the above equation can be written as

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} [1] dx$$

Now by applying the limits we get

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} - 0$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Let, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \ldots..(1)$

As we know that, $\left\{ \int_0^a f(x) dx = \int_0^a f(a-x) dx \right\}$

By using the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\, dx$$

By substituting the standard identities we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \,(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} [1]\, dx$$

Integrating the above equation and applying the limits we get

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} - 0$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

3. $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$

Solution:

Given $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} \, dx$

$$\text{let, } I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} \, dx \ldots\ldots(1)$$

As we know that

$$\left\{ \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right\}$$

By substituting the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + \cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)} \, dx$$

Again by substituting the standard identities we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} \, dx \,(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}\, dx$$

The above equation can be written as

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} [1] dx$$

Integrating and applying the limit we get

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} - 0$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x\, dx}{\sin^5 x + \cos^5 x}$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x}\, dx$

$$\text{let, } I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x}\, dx \ldots\ldots(1)$$

As we know that

$$\left\{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right\}$$

By substituting the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}\,dx$$

The above equation can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x}\,dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x}\,dx$$

The above equation becomes

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} [1]\,dx$$

Now by integrating and applying the limits we get

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} - 0$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

5. $\int_{-5}^{5} |x+2| \, dx$

Solution:

Given: $\int_{-5}^{5} |x+2| \, dx$

As we can see that (x+2) ≤ 0 on [-5, -2] and (x + 2) ≥ 0 on [-2, 5]

As we know that

$$\left\{ \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \right\}$$

Now by substituting the formula we get

$$\Rightarrow I = \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx$$

Integrating and applying the limits we get

$$\Rightarrow I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

On simplifying

$$\Rightarrow I = -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$\Rightarrow I = -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

On computing we get

$$\Rightarrow I = -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$$

$$\Rightarrow I = 29$$

6. $\int_2^8 |x-5|\, dx$

Solution:

Given $\int_2^8 |x-5|\ dx$

As we can see that (x - 5) ≤ 0 on [2, 5] and (x + 2) ≥ 0 on [5, 8]

As we know that

$$\left\{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right\}$$

By applying the above formula we get

$$\Rightarrow I = \int_2^5 -(x-5)dx + \int_5^8 (x-5)dx$$

Now by integrating the above equation

$$\Rightarrow I = -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

Now by applying the limits we get

$$\Rightarrow I = -\left[\frac{(5)^2}{2} - 5(5) - \frac{(2)^2}{2} + 5(2)\right] + \left[\frac{(8)^2}{2} - 5(8) - \frac{(5)^2}{2} + 5(5)\right]$$

On computing

$$\Rightarrow I = -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[\frac{64}{2} - 40 - \frac{25}{2} + 25\right]$$

$$\Rightarrow I = -\frac{25}{2} + 17 + 32 - 15 - \frac{25}{2}$$

On simplifying we get

$$\Rightarrow I = 34 - 25$$

$$\Rightarrow I = 9$$

7. $\int_0^1 x\,(1-x)^n\,dx$

Solution:

Given: $\int_0^1 x(1-x)^n\ dx$

$$\text{let, } I = \int_0^1 x(1-x)^n\ dx$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^1 (1-x)(1-(1-x))^n \ dx$$

The above equation can be written as

$$\Rightarrow I = \int_0^1 (1-x)(x)^n \ dx$$

By multiplying we get

$$\Rightarrow I = \int_0^1 (x)^n - (x)^{n+1} dx$$

On integrating

$$\Rightarrow I = \left[\frac{(x)^{n+1}}{n+1} - \frac{(x)^{n+2}}{n+2}\right]_0^1$$

Now by applying the limits we get

$$\Rightarrow I = \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$\Rightarrow I = \left[\frac{(n+2)-(n+1)}{(n+1)(n+2)}\right]$$

On simplification

$$\Rightarrow I = \left[\frac{1}{(n+1)(n+2)}\right]$$

8. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

Solution:

Given: $\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$

$$\text{let, } I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx \ldots..(1)$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

Again we know the standard formula

$$\left\{\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}\right\}$$

By substituting the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{\tan\left(\frac{\pi}{4}\right)-\tan(x)}{1+\tan\left(\frac{\pi}{4}\right)\tan(x)}\right]dx$$

Applying the values we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan(x)}{1+\tan(x)}\right]dx$$

On simplification the above equation can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan(x)}\right]dx$$

Now by applying log formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log[2]dx - \int_0^{\frac{\pi}{4}} \log[1+\tan(x)]dx$$

From equation (1) we can write as

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log[2]dx - I$$

On integration

$$\Rightarrow 2I = [x\log 2]_0^{\frac{\pi}{4}}$$

Now by applying the limits we get

$$\Rightarrow 2I = \frac{\pi}{4}\log 2 - 0$$

$$\Rightarrow I = \frac{\pi}{8}\log 2$$

9. $\int_0^2 x\sqrt{2-x}\,dx$

Solution:

Given: $\int_0^2 x\sqrt{2-x}\,dx$

$$\text{let, } I=\int_0^2 x\sqrt{2-x}\,dx \ldots..(1)$$

As we know that

$$\left\{\int_0^a f(x)dx=\int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I=\int_0^2 (2-x)\sqrt{2-(2-x)}dx$$

On simplification the above equation can be written as

$$\Rightarrow I=\int_0^2 (2-x)\sqrt{(x)}dx$$

On multiplication we get

$$\Rightarrow I=\int_0^2 \left(2x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)dx$$

On integration

$$\Rightarrow I=\left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$\Rightarrow I=\left[\frac{4}{3}\left(x^{\frac{3}{2}}\right)-\frac{2}{5}\left(x^{\frac{5}{2}}\right)\right]_0^2$$

Now by applying the limits the above equation can be written as

$$\Rightarrow I = \left[\frac{4}{3}\left((2)^{\frac{3}{2}}\right) - \frac{2}{5}\left((2)^{\frac{5}{2}}\right)\right]$$

By computing

$$\Rightarrow I = \frac{4}{3} \times 2\sqrt{2} - \frac{2}{5} \times 4\sqrt{2}$$

$$\Rightarrow I = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

On simplification

$$\Rightarrow I = \frac{40\sqrt{2} - 24\sqrt{2}}{15}$$

$$\Rightarrow I = \frac{16\sqrt{2}}{15}$$

10. $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$

Solution:

Given: $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$

$$\text{let, } I = \int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$$

Now by applying Sin 2x formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log(2\sin x \cos x)\}\, dx$$

Applying log formula we can write above equation as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{2\log \sin x - \log(2) - \log(\sin x) - \log(\cos x)\}dx$$

On simplification

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \sin x - \log 2 - \log \cos x\}dx \ldots(1)$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left\{\log \sin\left(\frac{\pi}{2} - x\right) - \log 2 - \log \cos\left(\frac{\pi}{2} - x\right)\right\}dx$$

Using allied angles formulae, the above equation becomes

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \{\log \cos x - \log 2 - \log \sin x\}dx \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2)\ dx$$

By taking common

$$2I = -2\log 2\int_0^{\frac{\pi}{2}}(1)\ dx$$

On integrating we get

$$\Rightarrow 2I = -2\log 2\left[x\right]_0^{\frac{\pi}{2}}$$

Now by applying the limits

$$\Rightarrow 2I = -2\log 2\left[\frac{\pi}{2} - 0\right]$$

$$\Rightarrow 2I = -2\log 2\left(\frac{\pi}{2}\right)$$

On simplification we get

$$\Rightarrow I = \frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I = \frac{\pi}{2}\left(\log\frac{1}{2}\right)$$

11. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

Solution:

As we can see f (x) = $\sin^2$x and f (-x) = $\sin^2$ (-x) = (sin (-x))^2 = (-sin x)^2 = $\sin^2$x.

That is f (x) = f (-x)

So, $\sin^2$x is an even function.

It is also known that if f(x) is an even function then, we have

$$\left\{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\right\}$$

Now by using this formula the given question can be written as

$$\Rightarrow I = 2.\int_{0}^{\frac{\pi}{2}} \left(\sin^2 x\right) dx$$

Now by substituting $\sin^2 x$ formula we get

$$\Rightarrow I = 2.\int_{0}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$$

$$\Rightarrow I = \int_{0}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

On integrating we get

$$\Rightarrow I = \left[x - \frac{\sin 2x}{2}\right]_{0}^{\frac{\pi}{2}}$$

Now by applying the limits

$$\Rightarrow I = \frac{\pi}{2} - \sin\pi - 0 + \sin 0$$

$$\Rightarrow I = \frac{\pi}{2}$$

12. $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

Solution:

Given: $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$

$$\text{let, } I = \int_0^{\pi} \frac{x}{1+\sin x}\, dx \ldots..(1)$$

As we know that

$$\left\{ \int_0^a f(x)dx = \int_0^a f(a-x)dx \right\}$$

By using above formula we get

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)}\, dx$$

Now by multiplying and simplifying the equation we get

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x}\, dx \ldots\ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{(\pi - x)+x}{1+\sin x}\, dx$$

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin x}\, dx$$

Now by multiplying and dividing the above equation by (1 – sin x) we get

$$2I = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}\, dx$$

On simplification we get

$$2I = \pi\int_0^{\pi}\frac{(1-\sin x)}{\cos^2 x}\,dx$$

By splitting the numerator we get

$$2I = \pi\int_0^{\pi}\left\{\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right\}dx$$

The above equation can be written as

$$2I = \pi\int_0^{\pi}\left\{\sec^2 x - \tan x\sec x\right\}dx$$

$$\Rightarrow 2I = \pi\left[\tan x - \sec x\right]_0^{\pi}$$

$$\Rightarrow 2I = \pi[2]$$

$\Rightarrow I = \pi$

13. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx$

Solution:

Given: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^7 x)\,dx$

$$\text{let, } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^7 x)\,dx$$

As we can see $f(x) = \sin^7 x$ and $f(-x) = \sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$.

That is f (x) = -f (-x)

So, $\sin^2 x$ is an odd function.

It is also known that if f(x) is an odd function then,

$$\left\{\int_{-a}^{a} f(x)\,dx = 0\right\}$$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin^7 x\right)dx = 0$$

14. $\int_0^{2\pi} \cos^5 x\, dx$

Solution:

$$\text{let, } I = \int_0^{2\pi} \left(\cos^5 x\right)dx$$

As we see, $f(x) = \cos^5 x$ and $f(2\pi - x) = \cos^5(2\pi - x) = \cos^5 x = f(x)$

$$\text{because, } \int_0^{2a} f(x)\,dx = 2.\int_0^{a} f(x)\,dx, \text{if } f(2a - x) = f(x)$$

$$\text{and } \int_0^{2a} f(x)\,dx = 0, \text{if } f(2a - x) = -f(x)$$

$$\Rightarrow I = 2.\int_0^{\pi} \left(\cos^5 x\right)dx$$

$$\text{Now}\left\{\cos^5(\pi - x) = -\cos^5 x\right\}$$

$$\Rightarrow I = 0$$

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

Solution:

Given: $\int\limits_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x}\ dx$

$$\text{let, } I = \int\limits_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x}\ dx \ldots..(1)$$

As we know that

$$\left\{ \int\limits_0^a f(x)dx = \int\limits_0^a f(a-x)dx \right\}$$

By using the above formula in given equation it can be written as

$$\Rightarrow I = \int\limits_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\ dx$$

Now by applying allied angle formula we get

$$\Rightarrow I = \int\limits_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\cos x \sin x}\ dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int\limits_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1+\sin x \cos x}\ dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{0}{1+\sin x \cos x} \, dx$$

$$\Rightarrow I = 0$$

16. $\int_0^{\pi} \log(1+\cos x)\, dx$

Solution:

Given: $\int_0^{\pi} \log(1+\cos x) \, dx$

$$\text{let, } I = \int_0^{\pi} \log(1+\cos x) \, dx \ldots..(1)$$

As we know that

$$\left\{ \int_0^a f(x)dx = \int_0^a f(a-x)dx \right\}$$

Now by using the above formula we get

$$\Rightarrow I = \int_0^{\pi} \log(1+\cos(\pi - x)) \, dx$$

Here by allied angle formula we get

$$\Rightarrow I = \int_0^{\pi} \log(1-\cos x) \, dx \ldots\ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \{\log(1+\cos x) + \log(1-\cos x)\} \, dx$$

The above equation can be written as

$$2I = \int_0^{\pi} \log\left(1-\cos^2 x\right)\ dx$$

By using trigonometric identities we get

$$2I = \int_0^{\pi} \log\left(\sin^2 x\right)\ dx$$

$$2I = \int_0^{\pi} 2.\log\left(\sin x\right)\ dx$$

$$2I = 2.\int_0^{\pi} \log\left(\sin x\right)\ dx$$

$$I = \int_0^{\pi} \log\left(\sin x\right)\ dx \ldots..(3)$$

$$\text{because, } \int_0^{2a} f(x)dx = 2.\int_0^{a} f(x)dx, \text{if } f(2a-x) = f(x)$$

Here, if f (x) = log (sin x) and f (π – x) = log (sin (π – x)) = log (sin x) = f (x)

$$\Rightarrow I = 2.\int_0^{\frac{\pi}{2}} \log \sin x dx \ldots(4)$$

$$\Rightarrow I = 2.\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2}-x\right)dx$$

By using trigonometric equation we get

$$\Rightarrow I = 2.\int_0^{\frac{\pi}{2}} \log \cos x dx \ldots..(5)$$

Adding (1) and (2), we get

$$\Rightarrow 2I = 2.\int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x)dx$$

Now by adding and subtracting log 2 we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x + \log 2 - \log 2)dx$$

The above equation can be written as

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log(2 \sin x \cos x) - \log 2)dx$$

Now by splitting the integral we get

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log(\sin 2x)dx - \int_0^{\frac{\pi}{2}} \log 2dx$$

Let $2x = t \Rightarrow 2dx = dt$

When $x = 0$, $t = 0$ and when $x = \pi/2$, $t = \pi$

$$\Rightarrow I = \left[\frac{1}{2}\int_0^{\pi} (\log(\sin t)dt\right] - \left(\frac{\pi}{2}\log 2\right)$$

$$\Rightarrow I = \left[\frac{I}{2}\right] - \left(\frac{\pi}{2}\log 2\right)$$

$$\Rightarrow I = -\left(\frac{\pi}{2}\log 2\right)$$

$$\Rightarrow I = -(\pi \log 2)$$

17. $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

Solution:

Given: $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

$$\text{let, } I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx \ldots..(1)$$

As we know that

$$\left\{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx$$

The above equation becomes,

$$\Rightarrow 2I = \int_0^a [1]\,dx$$

On integrating we get

$$\Rightarrow 2I = [x]_0^a$$

Now by applying the limits

$$\Rightarrow 2I = a - 0$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

18. $\int_0^4 |x-1|\, dx$

Solution:

Given: $\int_0^4 |x-1|\, dx$

As we can see that (x-1) ≤ 0 when 0 ≤ x ≤ 1 and (x - 1) ≥ 0 when 1 ≤ x ≤ 4

As we know that

$$\left\{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right\}$$

By substituting the above formula we get

$$\Rightarrow I = \int_0^1 -(x-1)dx + \int_1^4 (x-1)dx$$

On integration

$$\Rightarrow I = -\left[\frac{x^2}{2} - x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$$

Now by applying the limit we get

$$\Rightarrow I = -\left[\frac{(1)^2}{2} - 1 - \frac{(0)^2}{2} + 0\right] + \left[\frac{(4)^2}{2} - 4 - \frac{(1)^2}{2} + 1\right]$$

$$\Rightarrow I = -\left[\frac{1}{2} - 1\right] + \left[8 - 4 - \frac{1}{2} + 1\right]$$

$$\Rightarrow I = \frac{1}{2} + 5 - \frac{1}{2}$$

$$\Rightarrow I = 5$$

19. Show that $\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Solution:

Given: $\int_0^a f(x)g(x)\,dx$

let, $I = \int_0^a f(x)g(x)\,dx \ldots..(1)$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I = \int_0^a f(a-x)g(a-x)\,dx$$

$$\Rightarrow I = \int_0^a f(x)g(a-x)\,dx \ldots..(2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \left\{ f(x)g(x) + f(x)g(a-x) \right\} dx$$

$$\Rightarrow 2I = \int_0^a f(x)\left\{ g(x) + g(a-x) \right\} dx$$

$$\Rightarrow 2I = \int_0^a f(x)\{4\} \ dx \text{ as, } \left\{ g(x) + g(a-x) = 4 \right\}$$

$$\Rightarrow I = \frac{1}{2}\int_0^a f(x) \times 4 \ dx$$

$$\Rightarrow I = 2.\int_0^a f(x) dx$$

Choose the correct answer in Exercises 20 and 21.

20. The value of $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$ is

(A) 0 (B) 2 (C) π (D) 1

Solution:
(C) π

Explanation:

Given: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 + x\cos x + \tan^5 x + 1 \right) dx$

$$\text{let, } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 + x\cos x + \tan^5 x + 1 \right) dx$$

Now by splitting the integrals we get

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x\cos x)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\tan^5 x)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1)dx$$

It is also known that if f(x) is an even function then,

$$\left\{ \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \right\}$$

It is also known that if f(x) is an odd function then,

$$\Rightarrow I = 0+0+0+2.\int_{0}^{\frac{\pi}{2}} (1)dx \quad \left\{ \int_{-a}^{a} f(x)dx = 0 \right\}$$

$$\Rightarrow I = 2.[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I = 2.\frac{\pi}{2}$$

$$\Rightarrow I = \pi$$

Correct answer is C

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

(A) 2 (B) 3/4 (C) 0 (D) –2

Solution:

(C) 0

Explanation:

Given: $\int_{0}^{2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$

$$\text{let, } I=\int_{0}^{\frac{\pi}{2}}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx\ldots..(1)$$

As we know that

$$\left\{\int_{0}^{a}f(x)dx=\int_{0}^{a}f(a-x)dx\right\}$$

By using the above formula we get

$$\Rightarrow I=\int_{0}^{\frac{\pi}{2}}\log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx$$

By applying allied angles formulae we get

$$\Rightarrow\ I=\int_{0}^{\frac{\pi}{2}}\log\left(\frac{4+3\cos x}{4+3\sin}\right)dx\ldots..(2)$$

Adding (1) and (2), we get

$$2I=\int_{0}^{\frac{\pi}{2}}\left\{\log\left(\frac{4+3\sin x}{4+3\cos x}\right)+\left(\frac{4+3\cos x}{4+3\sin}\right)\right\}dx$$

$$\Rightarrow 2I=\int_{0}^{\frac{\pi}{2}}\log 1dx$$

Substituting log 1 = 0 we get

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0.dx$$

$$\Rightarrow I = 0$$

Correct answer is (c)

MISCELLANEOUS EXERCISE

Integrate the functions in Exercises 1 to 24.

1. $\frac{1}{x-x^3}$

Solution:

Given: $\frac{1}{x-x^3}$

Let $I = \frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1+x)(1-x)}$

Using partial differentiation

Let $\frac{1}{x(1+x)(1-x)} = \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x}$ ... (1)

By taking LCM we get

$\Rightarrow \frac{1}{x(1+x)(1-x)} = \frac{A(1+x)(1-x)+B(x)(1-x)+C(x)(1+x)}{x(1+x)(1-x)}$

$\Rightarrow \frac{1}{x(1+x)(1-x)} = \frac{A(1-x^2)+Bx(1-x)+Cx(1+x)}{x(1+x)(1-x)}$

$\Rightarrow 1 = A - Ax^2 + Bx - Bx^2 + C x + Cx^2$

$\Rightarrow 1 = A + (B + C) x + (-A - B + C) x^2$

Equating the coefficients of x, x^2 and constant value. We get:

(a) A = 1

(b) B + C = 0 ⇒ B = -C

(c) -A – B + C =0

⇒ -1 – (-C) +C = 0

⇒ 2C = 1 ⇒ C = 1/2

So, B = -1/2

Put these values in equation (1)

$$\Rightarrow \frac{1}{x(1+x)(1-x)} = \frac{1}{x} + \frac{-\left(\frac{1}{2}\right)}{1+x} + \frac{\left(\frac{1}{2}\right)}{1-x}$$

$$\Rightarrow \int \frac{1}{x(1+x)(1-x)}dx = \int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{1+x}dx + \frac{1}{2}\int \frac{1}{1-x}dx$$

On integrating we get

$$= \log|x| - \frac{1}{2}\log|1+x| + \frac{1}{2}\log|1-x|$$

By using logarithmic formula the above equation can be written as

$$= \log|x| - \log\left|(1+x)^{\frac{1}{2}}\right| + \log\left|(1-x)^{\frac{1}{2}}\right|$$

$$= \log\left|\frac{x}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\right| + C$$

On simplification we get

$$= \log\left|\frac{(x^2)^{\frac{1}{2}}}{(1+x)(1-x)^{\frac{1}{2}}}\right| + C$$

$$= \log\left|\frac{(x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}\right| + C$$

$$= \log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right| + C$$

$$\Rightarrow I = \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$$

2. $\dfrac{1}{\sqrt{x+a}+\sqrt{x+b}}$

Solution:

Given: $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$

Let $I = \frac{1}{\sqrt{x+a}+\sqrt{x+b}}$

Multiply and divide by, $\sqrt{x+a} - \sqrt{x+b}$

$$\Rightarrow I = \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a})^2-(\sqrt{x+b})^2}$$

On simplification we get

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}$$

Applying integration

$$\Rightarrow \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b} dx$$

$$= \frac{1}{a-b} \int (\sqrt{x+a} - \sqrt{x+b}) dx$$

$$= \frac{1}{a-b} \int ((x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}) dx$$

On integrating we get

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$\Rightarrow I = \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$$

3. $\frac{1}{x\sqrt{ax-x^2}}$ [Hint: Put $x = \frac{a}{t}$]

Solution:

Given: $\frac{1}{x\sqrt{ax-x^2}}$

Let $I = \frac{1}{x\sqrt{ax-x^2}}$

Put $x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2}dt$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx = \int \frac{1}{\frac{a}{t}\sqrt{\frac{a.a}{t}-\left(\frac{a}{t}\right)^2}}.-\frac{a}{t^2}dt$$

By taking a common we get

$$= \int \frac{-1}{at}.\frac{1}{\sqrt{\frac{1}{t}-\left(\frac{1}{t}\right)^2}}dt$$

Now by multiplying t we get

$$= -\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t}-\left(\frac{t}{t}\right)^2}}dt$$

The above equation becomes

$$= -\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$$

$$= -\frac{1}{a}\int (t-1)^{-\frac{1}{2}}dt$$

On integrating we get

$$= -\frac{1}{a}\left[\frac{\sqrt{(t-1)}}{\frac{1}{2}}\right] + C$$

$$= -\frac{2}{a}\left[\sqrt{\left(\frac{a}{x}-1\right)}\right] + C \text{ because, } t = \frac{a}{x}$$

$$\Rightarrow I = -\frac{2}{a}\left[\sqrt{\left(\frac{a-x}{x}\right)}\right] + C$$

4. $\dfrac{1}{x^2(x^4+1)^{\frac{3}{4}}}$

Solution:

Given: $\dfrac{1}{x^2.(x^4+1)^{\frac{3}{4}}}$

Let $I = \dfrac{1}{x^2.(x^4+1)^{\frac{3}{4}}}$

Multiply and divide by x^{-3}, we get

$$\frac{x^{-3}}{x^2.x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}.(x^4+1)^{-\frac{3}{4}}}{x^2.x^{-3}}$$

$$= \frac{(x^4+1)^{-\frac{3}{4}}}{x^5.x^{-3\times\frac{4}{4}}}$$

On simplification the above equation can be written as

$$= \frac{(x^4+1)^{-\frac{3}{4}}}{x^5.(x^4)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5}.\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

On computing we get

$$= \frac{1}{x^5}.\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

let, $\frac{1}{x^4} = t = (x)^{-4} \Rightarrow \frac{-4}{x^5} dx = dt \Rightarrow \frac{1}{x^5} dx = -\frac{dt}{4}$

$$\Rightarrow \int \frac{1}{x^2.(x^4+1)^{\frac{3}{4}}}.dx = \int \frac{1}{x^5}.\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}.dx$$

Substituting the above values we get

$$= \int (1+t)^{-\frac{3}{4}}.\left(-\frac{dt}{4}\right)$$

$$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}.dt$$

On integrating

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C$$

Now by substituting the value of t we get

$$= -\frac{1}{4}\left[\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$$

5. $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ [Hint: $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$, put $x = t^6$]

Solution:

Given $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$

Given question can be written as,

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Let $x = t^6 \Rightarrow dx = 6t^5dt$

$$\Rightarrow \int \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}.dx = \int \frac{6t^5}{t^2(1+t)}.dt$$

On computing we get

$$= 6.\int \frac{t^3}{(1+t)}.dt$$

After division we get,

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = 6.\int \left[(t^2 - t + 1) - \frac{1}{(1+t)}\right].dt$$

Now by splitting the integrals and computing

$$= 6.\left\{\int t^2.dt - \int t.dt + \int 1.dt - \int \left[\frac{1}{(1+t)}\right].dt\right\}$$

On integrating

$$= 6\left[\left(\frac{t^3}{3}\right) - \left(\frac{t^2}{2}\right) + t - \log(1+t)\right]$$

Now by substituting the value of t we get

$$= 6\left[\left(\frac{\left(x^{\frac{1}{6}}\right)^3}{3}\right) - \left(\frac{\left(x^{\frac{1}{6}}\right)^2}{2}\right) + \left(x^{\frac{1}{6}}\right) - \log\left(1+\left(x^{\frac{1}{6}}\right)\right)\right] + C$$

$$= \left[\left(2x^{\frac{1}{2}}\right) - \left(3x^{\frac{1}{3}}\right) + 6.x^{\frac{1}{6}} - 6.\log\left(1+x^{\frac{1}{6}}\right)\right] + C$$

$= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1 + x^{\frac{1}{6}}\right) + C$

6. $\dfrac{5x}{(x+1)(x^2+9)}$

Solution:

Given: $\frac{5x}{(x+1)(x^2+9)}$

Let $I = \dfrac{5x}{(x+1)(x^2+9)}$

Using partial fraction

Let $\frac{5x}{(x+1)(x^2+9)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+9)}$ ... (1)

$\Rightarrow \dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A(x^2+9) + (Bx+C)(x+1)}{(x+1)(x^2+9)}$

$\Rightarrow 5x = A(x^2 + 9) + (Bx + C)(x + 1)$

$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

$\Rightarrow 5x = 9A + C + (B + C)x + (A + B)x^2$

Equating the coefficients of x, x^2 and constant value, we get

(a) $9A + C = 0 \Rightarrow C = -9A$

(b) $B+C = 5 \Rightarrow B = 5-C \Rightarrow B = 5 - (-9A) \Rightarrow B = 5 + 9A$

(c) $A + B = 0 \Rightarrow A = -B \Rightarrow A = -(5 + 9A) \Rightarrow 10A = -5 \Rightarrow A = -1/2$

And $C = 9/2$ and $B = 1/2$

Put these values in equation (1) we get

$\Rightarrow \dfrac{5x}{(x+1)(x^2+9)} = \dfrac{A}{(x+1)} + \dfrac{Bx+C}{(x^2+9)}$

$$\Rightarrow \frac{5x}{(x+1)(x^2+9)} = \frac{-\frac{1}{2}}{(x+1)} + \frac{\left(\frac{1}{2}\right)x + \frac{9}{2}}{(x^2+9)}$$

The above equation can be written as

$$\Rightarrow \frac{5x}{(x+1)(x^2+9)} = -\frac{1}{2}.\frac{1}{(x+1)} + \frac{1}{2}.\left(\frac{x+9}{(x^2+9)}\right)$$

Now by applying integrals on both sides we get

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}.\int \frac{1}{(x+1)}dx + \frac{1}{2}.\int \frac{x}{(x^2+9)}dx + \frac{9}{2}\int \frac{1}{(x^2+9)}dx$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}.\int \frac{1}{(x+1)}dx + I_1 + \frac{9}{2}\int \frac{1}{(x^2+(3^2)}dx$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}.\log|x+1| + I_1 + \frac{9}{2}.\left(\frac{1}{3}\tan^{-1}\frac{x}{3}\right) \quad \ldots.(2)$$

Now solving for I_1we get

$$I_1 = \frac{1}{2}.\int \frac{x}{(x^2+9)}dx$$

Put $x^2 = t \Rightarrow 2xdx = dt$

$$\Rightarrow I_1 = \frac{1}{2}.\int \frac{1}{(t+9)}.\frac{dt}{2}$$

$$\Rightarrow I_1 = \frac{1}{4}\log|t+9|$$

$$\Rightarrow I_1 = \frac{1}{4}\log\ |x^2+9|$$

Put the value in equation (2)

$$\Rightarrow \int \frac{5x}{(x+1)(x^2+9)}dx = -\frac{1}{2}.\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}.\left(\tan^{-1}\frac{x}{3}\right) + C$$

7. $\dfrac{\sin x}{\sin(x-a)}$

Solution:

Given: $\frac{\sin x}{\sin(x-a)}$

Let $I = \dfrac{\sin x}{\sin(x-a)}$

Let $x - a = t \Rightarrow x = t + a \Rightarrow dx = dt$

$$\Rightarrow \int \frac{\sin x}{\sin(x-a)} dx = \int \frac{\sin(t+a)}{\sin(t)} dt$$

As we know that, {sin (A+B) = sin A cos B + cos A sin B}

$$\Rightarrow \int \frac{\sin x}{\sin(x-a)} dx = \int \frac{\sin t \cos a + \cos t \sin a}{\sin(t)} dt$$

The above equation becomes

$$= \int \frac{\sin t \cos a}{\sin t} + \frac{\cos t \sin a}{\sin t} dt$$

On simplification

$$= \int (\cos a + \cot t \sin a)\, dt$$

Now by splitting the integrals we get

$$= \int (\cos a) dt + \int (\cot t \sin a) dt$$

$$= (\cos a) \int 1.\, dt + \sin a . \int (\cot t) dt$$

On integrating we get

$= (\cos a).t + \sin a . \log|\sin t| + C$

Now by substituting the value of t we get

$= (\cos a).(x - a) + \sin a . \log|\sin(x - a)| + C$

$= \sin a . \log|\sin(x - a)| + x.\cos a - a.\cos a + C$

$= \sin a . \log|\sin(x - a)| + x.\cos a + C_2$

8. $\dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

Solution:

Given $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

let, $I = \dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

Now by taking common and above equation can be written as

$\Rightarrow \dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \dfrac{e^{4\log x}(e^{\log x} - 1)}{e^{2\log x}(e^{\log x} - 1)}$

On simplification

$= e^{2\log x}$

$= e^{\log x^2}$

$= x^2$

Applying integrals

$\Rightarrow \int \dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx = \int x^2 dx$

$= \dfrac{x^3}{3} + C$

9. $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

Solution:

Given: $\frac{\cos x}{\sqrt{4-\sin^2 x}}$

$$\text{let } I = \frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Put sin x = t ⇒ cos x dx = dt

The given equation can be written as

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx = \int \frac{1}{\sqrt{4-t^2}}\,dt$$

$$= \int \frac{1}{\sqrt{(2^2-t^2)}}\,dt$$

On integrating we get

$$= \sin^{-1}\left(\frac{t}{2}\right) + C$$

$$\Rightarrow I = \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

10. $\dfrac{\sin^8 - \cos^8 x}{1-2\sin^2 x\cos^2 x}$

Solution:

Given: $\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x.\cos^2 x}$

$$\text{let, } I = \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x.\cos^2 x}$$

As we know that $a^2 - b^2 = (a + b)(a - b)$

Now by using this formula we get

$$\Rightarrow \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x.\cos^2 x} = \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{\sin^2 x + \cos^2 x - \sin^2 x.\cos^2 x - \sin^2 x.\cos^2 x}$$

$$= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)}{(\sin^2 x - \sin^2 x.\cos^2 x) + (\cos^2 x - \sin^2 x.\cos^2 x)}$$

We know that $\cos^2 + \sin^2 x = 1$, using this in above equation

$$= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x).(1)}{\sin^2 x(1 - \cos^2 x) + \cos^2 x(1 - \sin^2 x)}$$

$$= \frac{-(\sin^4 x + \cos^4 x)(\cos^2 x - \sin^2 x)}{\sin^2 x(\sin^2 x) + \cos^2 x(\cos^2 x)}$$

On simplification we get

$$= \frac{-(\sin^4 x + \cos^4 x)(\cos^2 x - \sin^2 x)}{(\sin^4 x + \cos^4 x)}$$

$$= (\sin^2 x - \cos^2 x)$$

$$= -\cos 2x$$

$$\Rightarrow \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x.\cos^2 x} dx = \int -\cos 2x \, dx$$

On integrating

$$\Rightarrow I = -\frac{\sin 2x}{2} + C$$

11. $\frac{1}{\cos(x+a)\cos(x+b)}$

Solution:

Given: $\frac{1}{\cos(x+a)\cos(x+b)}$

$$\text{let, } I = \frac{1}{\cos(x+a)\cos(x+b)}$$

Multiply and divide by sin (a - b), we get

$$I = \frac{1}{\sin(a-b)}.\left(\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right)$$

Now by adding and subtracting x from the numerator

$$= \frac{1}{\sin(a-b)}.\left(\frac{\sin(a-b+x-x)}{\cos(x+a)\cos(x+b)}\right)$$

By grouping we get

$$= \frac{1}{\sin(a-b)}.\left(\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\right)$$

As we know that {sin (A-B) = sin A cos B - cos A sin B}

By using this formula we get

$$\Rightarrow I = \frac{1}{\sin(a-b)}.\left(\frac{\sin(x+a).\cos(x+b)-\cos(x+a).\sin(x+b)}{\cos(x+a)\cos(x+b)}\right)$$

$$= \frac{1}{\sin(a-b)}.\left(\frac{\sin(x+a).\cos(x+b)}{\cos(x+a)\cos(x+b)} - \frac{\cos(x+a).\sin(x+b)}{\cos(x+a)\cos(x+b)}\right)$$

On simplification we get

$$= \frac{1}{\sin(a-b)}.\left(\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)}\right)$$

$$= \frac{1}{\sin(a-b)}.[\tan(x+a)-\tan(x+b)]$$

Taking integrals on both sides we get

$$\Rightarrow \int \frac{1}{\cos(x+a)\cos(x+b)}dx = \int \frac{1}{\sin(a-b)}.[\tan(x+a)-\tan(x+b)]dx$$

$$= \frac{1}{\sin(a-b)}\left\{\int \tan(x+a)\,dx - \int \tan(x+b)\,dx\right\}$$

On integrating we get

$$= \frac{1}{\sin(a-b)}[-\log|\cos(x+a)| - (-\log|\cos(x+a)|)]$$

$$= \frac{1}{\sin(a-b)}[-\log|\cos(x+a)| + \log|\cos(x+a)|]$$

$$\Rightarrow I = \frac{1}{\sin(a-b)}.\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right| + C$$

12. $\frac{x^3}{\sqrt{1-x^8}}$

Solution:

Given: $\frac{x^3}{\sqrt{1-x^8}}$

let $I = \frac{x^3}{\sqrt{1-x^8}}$

Now, let $x^4 = t \Rightarrow 4x^3\,dx = dt$

And $x^3\,dx = dt/4$

Substituting these values in given question we get

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \int \frac{1}{\sqrt{1-t^2}}\left(\frac{dt}{4}\right)$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{1^2-t^2}}.\,dt$$

On integrating we get

$$= \frac{1}{4}\sin^{-1} t + C$$

Now by substituting t value we get

$$\Rightarrow I = \frac{1}{4}\sin^{-1}(x^4) + C$$

13. $\dfrac{e^x}{(1+e^x)(2+e^x)}$

Solution:

Given: $\frac{e^x}{(1+e^x)(2+e^x)}$

$$\text{let}, I = \frac{e^x}{(1+e^x)(2+e^x)}$$

Let $e^x = t \Rightarrow e^x\,dx = dt$

Now substituting these values in given question we get

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)}dx = \int \frac{1}{(1+t)(2+t)}dt$$

$$= \int \left[\frac{1}{(1+t)} - \frac{1}{(2+t)}\right]dt$$

Now by splitting the integrals we get

$$= \int \left[\frac{1}{(1+t)}\right]dt - \int \left[\frac{1}{(2+t)}\right]dt$$

On integrating we get

$$= \log|(1+t)| - \log|(2+t)| + C$$

$$= \log\left|\frac{1+t}{2+t}\right| + C$$

$$\Rightarrow I = \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

14. $\dfrac{1}{(x^2+1)(x^2+4)}$

Solution:

Given: $\frac{1}{(x^2+1)(x^2+4)}$

$$\text{Let I} = \frac{1}{(x^2+1)(x^2+4)}$$

Using partial fraction method, we get

$$\text{let } \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)} \ldots (1)$$

$$\Rightarrow \frac{1}{(x+1)(x^2+9)} = \frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x+1)(x^2+9)}$$

$\Rightarrow 1 = (Ax + B)(x^2 + 4) + (C x + D)(x^2 + 1)$

$\Rightarrow 1 = Ax^3 + 4 Ax + Bx^2 + 4B + Cx^3 + C x + Dx^2 + D$

$\Rightarrow 1 = (A + C) x^3 + (B + D) x^2 + (4 A + C) x + (4B + D)$

Equating the coefficients of x, x^2, x^3 and constant value. We get:

(a) $A + C = 0 \Rightarrow C = -A$

(b) $B + D = 0 \Rightarrow B = -D$

(c) $4A + C = 0 \Rightarrow 4A = -C \Rightarrow 4A = A \Rightarrow 3A = 0 \Rightarrow A = 0 \Rightarrow C = 0$

(d) $4B + D = 1 \Rightarrow 4B - B = 1 \Rightarrow B = 1/3 \Rightarrow D = -1/3$

Put these values in equation (1)

$$\Rightarrow \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow \frac{1}{(x^2+1)(x^2+4)} = \frac{(0)x+\frac{1}{3}}{(x^2+1)} + \frac{(0)x+\left(-\frac{1}{3}\right)}{(x^2+4)}$$

$$\Rightarrow \frac{1}{(x^2+1)(x^2+4)} = \frac{\frac{1}{3}}{(x^2+1)} + \frac{\left(-\frac{1}{3}\right)}{(x^2+4)}$$

Now by taking integrals on both sides we get

$$\Rightarrow \int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3}.\int \frac{1}{(x^2+1)} dx - \frac{1}{3}.\int \frac{1}{(x^2+4)} dx$$

$$\Rightarrow \int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3}.\int \frac{1}{(x^2+1^2)} dx - \frac{1}{3}.\int \frac{1}{(x^2+2^2)} dx$$

On integrating we get

$$= \frac{1}{3}.\tan^{-1} x - \frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$\Rightarrow I = \frac{1}{3}.\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

15. $\cos^3 x \; e^{\log \sin x}$

Solution:

Given: $\cos^3 x e^{\log \sin x}$

Let $I = \cos^3 x e^{\log \sin x}$

Logarithmic and exponential functions cancels each other in above equation then we get

$= \cos^3 x.\sin x$

Let cos x = t ⇒ -sin x dx = dt ⇒ sin x dx = dt

Substituting these values in given question we get

$$\Rightarrow \int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x.\sin x\, dx$$

$$= \int t^3.(-dt)$$

$$= -\int t^3.dt$$

On integrating

$$= -\frac{t^4}{4} + C$$

Now by substituting the value of t we get

$$= -\frac{\cos^4 x}{4} + C$$

16. $e^{3\log x}(x^4 + 1)^{-1}$

Solution:

Given: $e^{3\log x}(x^4 + 1)^{-1}$

Let $I = e^{3\log x}(x^4 + 1)^{-1}$

$$= e^{\log x^3}(x^4 + 1)^{-1}$$

Logarithmic and exponential functions cancels each other in above equation then we get

$$= \frac{x^3}{x^4 + 1}$$

Let $x^4 = t \Rightarrow 4x^3\, dx = dt \Rightarrow x^3\, dx = dt/4$

Now by substituting these values in given question we get

$$\Rightarrow \int e^{3\log x}(x^4 + 1)^{-1} = \int \frac{x^3}{x^4 + 1} dx$$

$$= \int \frac{1}{t + 1}.\frac{dt}{4}$$

$$= \frac{1}{4}.\int \frac{1}{t+1}.dt$$

On integration we get

$$= \frac{1}{4}\log(t+1) + C$$

Now by substituting the values of t we get

$$\Rightarrow I = \frac{1}{4}\log(x^4+1) + C$$

17. $f'(ax+b)\,[f(ax+b)]^n$

Solution:

Given: $f'(ax + b)\ [f(ax + b)]^n$

Let $f(ax + b) = t \Rightarrow a.f'(ax + b)\,dx = dt$

Now by substituting these values in given question we get

$$\Rightarrow \int f'(ax+b)[f(ax+b)^n] = \int t^n\left(\frac{dt}{a}\right)$$

$$= \frac{1}{a}\int t^n dt$$

On integrating

$$= \frac{1}{a}.\frac{t^{n+1}}{n+1} + C$$

Here by substituting the value of t we get

$$= \frac{1}{a}.\frac{(f(ax+b))^{n+1}}{n+1} + C$$

$$= \frac{1}{a(n+1)}.(f(ax+b))^{n+1} + C$$

18. $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Solution:

Given: $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

$$\text{let I} = \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

As we know that, {sin (A+B) = sin A cos B + cos A sin B}

Using this formula we get

$$\Rightarrow \mathrm{I} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x \sin\alpha)}}$$

Multiplying and dividing by sin x to denominator we get

$$\Rightarrow \mathrm{I} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x . \frac{\sin x}{\sin x} \sin\alpha)}}$$

On rearranging we get

$$= \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \sin x . \frac{\cos x}{\sin x} \sin\alpha)}}$$

Simplifying we get

$$= \frac{1}{\sqrt{\sin^4 x(\cos\alpha + \cot x \sin\alpha)}}$$

$$= \frac{1}{\sin^2 x\sqrt{(\cos\alpha + \cot x \sin\alpha)}}$$

$$= \frac{\operatorname{cosec}^2 x}{\sqrt{(\cos\alpha + \cot x \sin\alpha)}}$$

now, let $(\cos\alpha + \cot x \sin\alpha) = t \Rightarrow -\operatorname{cosec}^2 x. \sin\alpha\, dx = dt$

Now by substituting these values in given question we get

$$\Rightarrow \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{(\cos\alpha + \cot x \sin\alpha)}} dx$$

$$= \int \frac{1}{\sqrt{t}} . -\frac{dt}{\sin\alpha}$$

$$= -\frac{1}{\sin\alpha}\int \frac{1}{\sqrt{t}} . dt$$

$$= -\frac{1}{\sin\alpha}\int t^{-\frac{1}{2}} . dt$$

On integrating we get

$$= -\frac{1}{\sin\alpha}\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$$

$$= -\frac{2}{\sin\alpha}\left[\sqrt{t}\right] + C$$

Now by substituting the value of t

$$= -\frac{2}{\sin\alpha}\left[\sqrt{(\cos\alpha + \cot x \sin\alpha)}\right] + C$$

Computing and simplifying

$$= -\frac{2}{\sin\alpha}\left[\sqrt{\left(\cos\alpha + \frac{\cos x}{\sin x}\sin\alpha\right)}\right] + C$$

$$= -\frac{2}{\sin\alpha}\left[\sqrt{\frac{(\cos\alpha \sin x + \cos x \sin\alpha)}{\sin x}}\right] + C$$

$$\Rightarrow I = -\frac{2}{\sin\alpha}\left[\sqrt{\frac{\sin(x+\alpha)}{\sin x}}\right] + C$$

19. $\dfrac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x \in [0, 1]$

Solution:

Given: $\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}$

Let $I = \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}} \dots (1)$

As we know, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x} = \frac{\pi}{2}$

Now using this identity we get

$$\Rightarrow I = \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}} = \frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\left(\frac{\pi}{2}\right)}$$

$$\Rightarrow \int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx = \int \frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\left(\frac{\pi}{2}\right)}dx$$

$$= \left(\frac{2}{\pi}\right)\int \left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

Now by splitting the integral we get

$$= \left(\frac{2}{\pi}\right)\int \left(\frac{\pi}{2}.dx\right) - \left(\frac{2}{\pi}\right)\int 2.\left(\cos^{-1}\sqrt{x}.dx\right)$$

$$= \int (1.dx) - \left(\frac{4}{\pi}\right)\int \left(\cos^{-1}\sqrt{x}.dx\right)$$

On integration we get

$$\Rightarrow I = x - \left(\frac{4}{\pi}\right)I_1 \dots .(2)$$

Now, first solve for I_1:

$$\text{as}, I_1 = \int \left(\cos^{-1}\sqrt{x}.dx\right)$$

let $\sqrt{x} = t \Rightarrow \frac{1}{2}x^{-\frac{1}{2}}dx = dt \ \Rightarrow \frac{dx}{\sqrt{x}} = 2.dt \Rightarrow dx = 2.tdt$

$$\Rightarrow I_1 = \int (\cos^{-1} t.2t.dt)$$

$$= 2\int t.\cos^{-1} t \ dt$$

Because, $\int u.v\,dx = u.\int v\,dx - \int \frac{du}{dx}.\{\int vdx\}\ dx$

$$\Rightarrow 2\int t.\cos^{-1} t \ dt = 2.\left[\cos^{-1} t.\int t\,dt - \int \frac{d(\cos^{-1} t)}{dt}.\left\{\int tdt\right\} dt\right]$$

$$= 2.\cos^{-1} t.\frac{t^2}{2} - 2.\int \left(-\frac{1}{\sqrt{1-t^2}}\right).\left\{\frac{t^2}{2}\right\} dt$$

$$= t^2.\cos^{-1} t - \int \left(\frac{-t^2}{\sqrt{1-t^2}}\right).\ dt$$

Now by adding and subtracting 1 to numerator we get

$$= t^2.\cos^{-1} t - \int \left(\frac{-1+1-t^2}{\sqrt{1-t^2}}\right).\ dt$$

Splitting the denominator

$$= t^2.\cos^{-1} t - \int \left(\frac{-1}{\sqrt{1-t^2}} + \frac{1-t^2}{\sqrt{1-t^2}}\right).\ dt$$

Splitting the integral we get

$$= t^2.\cos^{-1} t + \int \left(\frac{1}{\sqrt{1-t^2}}dt\right) - \int \left(\sqrt{1-t^2}\right).dt$$

$$= t^2.\cos^{-1} t + \int \left(\frac{1}{\sqrt{1-t^2}}dt\right) - \frac{t}{2}.\sqrt{1-t^2}$$

as, $\int \left(\sqrt{a^2-x^2}\right).dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow I_1 = t^2.\cos^{-1}t + \sin^{-1}t - \frac{t}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1}(t)$$

$$\Rightarrow I_1 = t^2.\cos^{-1}t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}t$$

Put it in equation. (2)

$$\Rightarrow I = x - \left(\frac{4}{\pi}\right)\left[t^2.\cos^{-1}t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}t\right] \text{ ....} (2)$$

Now substitute the value of t we get

$$\Rightarrow I = x - \left(\frac{4}{\pi}\right)\left[(\sqrt{x})^2.\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-(\sqrt{x})^2} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

Computing and simplifying we get

$$= x - \left(\frac{4}{\pi}\right)\left[x.\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - \left(\frac{4}{\pi}\right)\left[x.\left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\left(\sqrt{x-x^2}\right)}{2} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$\Rightarrow I = \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

20. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$

Solution:

Given: $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Let $I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Let x= $\cos^2\theta \Rightarrow dx = -2\sin\theta \cos\theta\, d\theta$

$\Rightarrow \sqrt{x} = \cos\theta \text{ or } \theta = \cos^{-1}\sqrt{x}$

Substituting these values in given question we get

$$\Rightarrow I = \int \sqrt{\frac{1-\sqrt{\cos^2\theta}}{1+\sqrt{\cos^2\theta}}}(-2\sin\theta\cos\theta)d\theta$$

$$= \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta$$

Substituting the standard formulae we get

$$= \int -\sqrt{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}}(2\sin\theta\cos\theta)d\theta$$

Multiplying and dividing by 2 we get

$$= \int -\sqrt{\frac{\sin^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)}}\left(2\sin 2\frac{\theta}{2}\cos 2\frac{\theta}{2}\right)d\theta$$

Using standard identities the above equation can be written as

$$= \int -\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}.(2).\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right).\left(2\cos^2\left(\frac{\theta}{2}\right)-1\right)d\theta$$

$$\Rightarrow \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx = \int -4.\left[\sin^2\left(\frac{\theta}{2}\right)\right]\left(2\cos^2\left(\frac{\theta}{2}\right)-1\right)d\theta$$

$= \int -4.\left\{\left[2.\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\right] - \sin^2\left(\frac{\theta}{2}\right)\right\}d\theta$

Splitting the integrals we get

$= \int -2.\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 d\theta + 4\int \sin^2\left(\frac{\theta}{2}\right)d\theta$

Again by using standard identities above equation can be written as

$= -2.\int \sin^2\theta d\theta + 4\int \sin^2\left(\frac{\theta}{2}\right)d\theta$

$= -2.\int \frac{1-\cos 2\theta}{2}d\theta + 4\int \frac{1-\cos\theta}{2}d\theta$

On integrating we get

$= -2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] + 4\left[\frac{\theta}{2} - \frac{\sin\theta}{2}\right] + C$

$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin\theta + C$

Computing and simplifying

$= \theta + \frac{2.\sin\theta.\cos\theta}{2} - 2\sin\theta + C$

$= \theta + \frac{2.\sqrt{1-\cos^2\theta}.\cos\theta}{2} - 2\sqrt{1-\cos^2\theta} + C$

Substituting the values we get

$= \cos^{-1}\sqrt{x} + \sqrt{1-x}.\sqrt{x} - 2\sqrt{1-x} + C$

$= \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} - 2\sqrt{1-x} + C$

$\Rightarrow I = \cos^{-1}\sqrt{x} + \sqrt{x-x^2} - 2\sqrt{1-x} + C$

21. $\frac{2+\sin 2x}{1+\cos 2x}e^x$

Solution:

$$\text{let I} = \frac{2 + \sin 2x}{1 + \cos 2x} e^x$$

Subsisting the sin 2x = 2 sin x cos x formula we get

$$= \left(\frac{2 + 2\sin x \cos x}{2\cos^2 x}\right) e^x$$

Now by taking 2 common

$$= 2.\left(\frac{1 + \sin x \cos x}{2\cos^2 x}\right) e^x$$

On simplification

$$= \left(\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x}\right) e^x$$

$$= (\sec^2 x + \tan x) e^x$$

Substituting integrals both the sides we get

$$\Rightarrow \int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx = \int (\sec^2 x + \tan x) e^x dx$$

Now let tan x = f(x)

$\Rightarrow f'(x) = \sec^2 x \, dx$

$$\Rightarrow \int \frac{2 + \sin 2x}{1 + \cos 2x} e^x dx = \int (f(x) + f'(x)) e^x dx$$

On integrating we get

$$= e^x f(x) + C$$

$$\Rightarrow I = e^x \tan x + C$$

22. $\dfrac{x^2 + x + 1}{(x+1)^2 (x+2)}$

Solution:

Given: $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Let $I = \frac{x^2+x+1}{(x+1)^2(x+2)}$

Using partial fraction we get

Let $\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$ ... (1)

$$\Rightarrow \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A(x+1)(x+2)+B(x+2)+C(x+1)^2}{(x+1)^2(x+2)}$$

$$\Rightarrow \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A(x^2+3x+2)+B(x+2)+C(x^2+2x+1)}{(x+1)^2(x+2)}$$

$\Rightarrow x^2 + x + 1 = Ax^2 + 3Ax + 2A + Bx + 2B + Cx^2 + 2Cx + C$

$\Rightarrow x^2 + x + 1 = (2A+2B+C) + (3A+B+2C)x + (A+C)x^2$

Equating the coefficients of x, x^2 and constant value. We get:

(a) $A + C = 1$

(b) $3A + B + 2C = 1$

(c) $2A+2B+C = 1$

After solving the above equations we get

$A = -2$, $B = 1$ and $C = 3$

Substituting the values of A, B and C we get

$$\Rightarrow \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{(x+2)}$$

Taking integrals on both sides

$$\Rightarrow \int \frac{x^2+x+1}{(x+1)^2(x+2)} dx = \int \left(\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{(x+2)}\right) dx$$

Splitting the integrals we get

$$= -2.\int\left(\frac{1}{x+1}\right)dx + \int\left(\frac{1}{(x+1)^2}\right)dx + 3.\int\left(\frac{1}{(x+2)}\right)dx$$

$$= -2.\int\left(\frac{1}{x+1}\right)dx + \int((x+1)^{-2})dx + 3.\int\left(\frac{1}{(x+2)}\right)dx$$

On integrating we get

$$= -2\log|x+1| + \left(\frac{(x+1)^{-1}}{(-1)}\right) + 3\log|x+1| + C$$

$$= -2\log|x+1| - \frac{1}{(x+1)} + 3\log|x+1| + C$$

23. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Solution:

Given: $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

let $I = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Let x= cos θ ⇒ dx = -sin θ d θ

$\Rightarrow \theta = \cos^{-1}x$

Now by substituting these values in given question we get

$$\Rightarrow I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx = \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta)d\theta$$

Using standard identities the above equation can be written as

$$= -\int \tan^{-1} \sqrt{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}} (\sin\theta)\, d\theta$$

$$= -\int \tan^{-1} \sqrt{\tan^2\left(\frac{\theta}{2}\right)} (\sin\theta)\, d\theta$$

On simplification we get

$$= -\int \tan^{-1} \tan\frac{\theta}{2}. (\sin\theta)\, d\theta$$

$$= -\frac{1}{2}\int \theta. (\sin\theta)\, d\theta$$

Now by using product rule

$$\int u. v\, dx = u. \int v\, dx - \int \frac{du}{dx}. \left\{\int v dx\right\} dx$$

$$= -\frac{1}{2}\int \theta. (\sin\theta)\, d\theta \ = -\frac{1}{2}\left[\theta. \int \sin\theta\, d\theta - \int \frac{d\theta}{d\theta}. \left\{\int \sin v\, d\theta\right\} d\theta\right]$$

Computing and integrating we get

$$= -\frac{1}{2}\left[\theta. (-\cos\theta) - \int 1. (-\cos\theta)\ d\theta\right]$$

$$= -\frac{1}{2}[-\theta. \cos\theta + \sin\theta\,]$$

Substituting the values we get

$$= \frac{1}{2}\theta. \cos\theta - \frac{1}{2}\sqrt{(1-\cos^2\theta}$$

$$= \frac{1}{2}\cos^{-1} x. x - \frac{1}{2}\sqrt{(1-x^2} + C$$

$$= \frac{1}{2}\left(x. \cos^{-1} x - \sqrt{(1-x^2)}\right) + C$$

24. $\dfrac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}$

Solution:

Given: $\frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}$

$$\text{let } I = \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}$$

$$= \frac{\sqrt{x^2+1}}{x^4}[\log(x^2+1)-\log x^2]$$

Using logarithmic identities we get

$$= \frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

On computing

$$= \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\left[\log\left(1+\frac{1}{x^2}\right)\right]$$

$$\text{now let } 1+\frac{1}{x^2} = t \Rightarrow \ -\frac{2}{x^3}dx = dt$$

Substituting these values in given question we get

$$\Rightarrow \int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx = \int \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\left[\log\left(1+\frac{1}{x^2}\right)\right]dx$$

$$= \int -\frac{1}{2}.\sqrt{t}[\log(t)]dt$$

By using product rule

$$\int u.v\,dx = u.\int v\,dx - \int \frac{du}{dx}.\left\{\int vdx\right\}dx$$

$$= \int -\frac{1}{2}.\sqrt{t}[\log(t)]dt = -\frac{1}{2}\left[\log t.\int \sqrt{t}\,dt - \int \frac{d}{dt}\log t.\left\{\int \sqrt{t}dt\right\} dt\right]$$

Computing and simplifying we get

$$= -\frac{1}{2}\left[\log t.\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t}.\left\{\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right\}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \int \left\{\frac{t^{\frac{3}{2}-1}}{\frac{3}{2}}\right\}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}\,dt\right]$$

On integration we get

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$= \left[-\frac{1}{2}.\frac{2}{3}t^{\frac{3}{2}}\log t + \frac{1}{2}.\frac{2}{3}.\frac{2}{3}.t^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

Substituting the value of t we get

$$\Rightarrow I = -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C$$

Evaluate the definite integrals in Exercises 25 to 33.

25. $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Solution:

Given: $\int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$\text{let, } I = \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Substituting the standard identities for 1 – sin x and 1 – cos x we get

$$= \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\left(\frac{x}{2}\right)}\right)dx$$

Now splitting the denominator

$$= \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1}{2\sin^2\left(\frac{x}{2}\right)} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\left(\frac{x}{2}\right)}\right)dx$$

$$= \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right) - \cot\frac{x}{2}\right)dx$$

$$\text{now let } f(x) = -\cot\frac{x}{2}$$

Substituting these values we get

$$\Rightarrow f'(x) = -\left(-\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\pi}(e^x\left(\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right) - \cot\frac{x}{2}\right)dx = \int_{-\frac{\pi}{2}}^{\pi}(f(x) + f'(x))e^x dx$$

On integration we get

$$= [e^x f(x)]_{-\frac{\pi}{2}}^{\pi}$$

$$= \left[e^x\left(-\cot\frac{x}{2}\right)\right]_{-\frac{\pi}{2}}^{\pi}$$

Now by applying the limits we get

$$= -\left[e^{\pi}\left(\cot\frac{\pi}{2}\right) - e^{\frac{\pi}{2}}\left(\cot\frac{\pi}{4}\right)\right]$$

$$= -\left[e^{\pi}(0) - e^{\frac{\pi}{2}}(1)\right]$$

$$= -\left[0 - e^{\frac{\pi}{2}}\right]$$

On simplification we get

$$\Rightarrow I = e^{\frac{\pi}{2}}$$

26. $\int_0^{\frac{\pi}{4}} \frac{\sin x \ \cos x}{\cos^4 x + \sin^4 x} dx$

Solution:

Given: $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$

$$\text{let, } I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Taking $\cos^4 x$ as common we get

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x\left(1 + \frac{\sin^4 x}{\cos^4 x}\right)} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{(1 + \tan^4 x)} dx$$

Now let $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x \, dx = dt$

And when x=0 then t=0 and when x=π /4 then t=1

Now by substituting these values in above equation we get

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{(1 + \tan^4 x)} dx = \int_0^1 \frac{1}{(1 + t^2)}\left(\frac{dt}{2}\right)$$

On integration

$$\Rightarrow \mathrm{I} = \frac{1}{2}[\tan^{-1} t]_0^1$$

Now by applying the limits we get

$$= \frac{1}{2}[\tan^{-1} 1 - \tan^{-1} 0]$$

$$\Rightarrow \mathrm{I} = \frac{1}{2} . \frac{\pi}{4}$$

$$\Rightarrow \mathrm{I} = \frac{\pi}{8}$$

27. $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos^2 x + 4\sin^2 x}$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

$$\text{let, } \mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx \;\dots..(1)$$

Substituting $\sin^2 x$ formula we get

$$\Rightarrow \mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1) - (4\cos^2 x)}\, dx$$

On computing we get

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{4 - 3\cos^2 x}\, dx$$

Now multiplying and dividing by 3 to the numerator we get

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{3}.3\cos^2 x}{4-3\cos^2 x}\,dx$$

Again by adding and subtracting 4 to the numerator we get

$$= -\frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{-3\cos^2 x+4-4}{4-3\cos^2 x}\,dx$$

The above equation can be written as

$$= -\frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x-4}{4-3\cos^2 x}\,dx$$

Now splitting the integrals we get

$$= -\frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x}{4-3\cos^2 x}\,dx + \frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4}{4-3\cos^2 x}\,dx$$

$$= -\frac{1}{3}.\int_0^{\frac{\pi}{2}} (1)\,dx + \frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4}{4-3\left(\frac{1}{\sec^2 x}\right)}\,dx$$

Applying the limits we get

$$= -\frac{1}{3}.[x]_0^{\frac{\pi}{2}} + \frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x-3}\,dx$$

$$= -\frac{1}{3}.\left[\frac{\pi}{2}\right] + \frac{1}{3}.\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1+\tan^2 x)-3}\,dx$$

$$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3}.\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}\,dx$$

$$\Rightarrow I = -\frac{\pi}{6} + I_1 \;\ldots .(2)$$

First solve for I_1:

$$I_1 = \frac{2}{3} \cdot \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x}\, dx$$

Let 2 tan x = t ⇒ 2 sec²x dx dt

When x = 0 then t = 0 and when x = π /2 then t = ∞

Substituting these values for above equation we get

$$\Rightarrow \frac{2}{3} \cdot \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x}\, dx = \frac{2}{3} \cdot \int_0^{\infty} \frac{1}{1 + t^2}\, dt$$

Integrating and applying the limits we get

$$\Rightarrow I_1 = \frac{2}{3}[\tan^{-1} t]_0^{\infty}$$

$$= \frac{2}{3}[\tan^{-1}\infty - \tan^{-1}0]$$

$$\Rightarrow I_1 = \frac{2}{3} \cdot \frac{\pi}{2}$$

$$\Rightarrow I_1 = \frac{\pi}{3}$$

Put this value in equation (2)

$$\Rightarrow I = -\frac{\pi}{6} + \frac{\pi}{3}$$

$$\Rightarrow I = \frac{\pi}{6}$$

28. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$

Solution:

Given: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\, dx$

$$\text{let, } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$$

On rearranging we get

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-\sin 2x)}}\, dx$$

Now by substituting the sin 2x formula we get

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1 + 1 - 2\sin x \cos x)}}\, dx$$

1 can be written as $\sin^2 x + \cos^2 x$

Substituting this in above equation we get

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{1 - (\sin^2 x + \cos^2 x - 2\sin x \cos x)}}\, dx$$

As we know $(a + b)^2 = a^2 + b^2$ using this in above equation we get

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{(1 - (\sin x - \cos x)^2)}}\, dx$$

Now let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

$$\text{when } x = \frac{\pi}{6} \Rightarrow t = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} \text{ and when } x = \frac{\pi}{3} \Rightarrow t = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

Substituting these values in above equation we get

$$\Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{(1 - (\sin x - \cos x)^2)}}\, dx = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{(1 - (t)^2)}}\, dt$$

$$= \int_{-\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{(1-(t)^2)}}\, dt$$

$$\text{let } f(x) = \frac{1}{\sqrt{(1-(t)^2)}} \text{ and } f(-x) = \frac{1}{\sqrt{(1-(-t)^2)}} = \frac{1}{\sqrt{(1-(t)^2)}} = f(x)$$

That is f (x) = f (-x)

So, f(x) is an even function.

It is also known that if f(x) is an even function then, we have

$$\left\{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\right\}$$

By using the above formula we get

$$\Rightarrow I = 2.\int_{0}^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{(1-(t)^2)}}\, dt$$

On integrating

$$\Rightarrow I = [2.\sin^{-1} t]_0^{\frac{\sqrt{3}-1}{2}}$$

Now by applying the limits

$$\Rightarrow I = 2.\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

29. $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

Solution:

Given: $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$\text{let, } I = \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Now multiply and divide $\sqrt{1+x}+\sqrt{x}$ to the above equation we get

$$= \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} \times \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} dx$$

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$$

On simplification

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1} dx$$

Now by splitting the integrals we get

$$= \int_0^1 \sqrt{1+x}dx + \int_0^1 \sqrt{x}dx$$

$$= \int_0^1 ((1+x)^{\frac{1}{2}})dx + \int_0^1 (x)^{\frac{1}{2}}dx$$

On integrating we get

$$\Rightarrow I = \left[\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 + \left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$

Now by applying the limits we get

$$= \frac{2}{3}.[(1+1)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}}] + \frac{2}{3}.[(1)^{\frac{3}{2}}]$$

Computing and simplifying we get

$$= \frac{2}{3}.[(2)^{\frac{3}{2}} - (1)^{\frac{3}{2}}] + \frac{2}{3}.[(1)^{\frac{3}{2}}]$$

$$= \frac{2}{3}.[(2)^{\frac{3}{2}} - 1] + \frac{2}{3}.[1]$$

$$= \frac{2}{3}.\left[(2)^{\frac{3}{2}}\right] - \frac{2}{3}.[1] + \frac{2}{3}.[1]$$

$$= \frac{2}{3}.[2\sqrt{2}]$$

$$\Rightarrow I = \frac{4\sqrt{2}}{3}$$

30. $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

Also, let sin x – cos x = t

Differentiating both sides, we get,

(Cos x + sin x) dx = dt

When x = 0, t = -1

And when $x = \frac{\pi}{4}$, t = 0

Now, $(\text{Sin } x - \cos x)^2 = t^2$

$1 - 2 \sin x \cos x = t^2$

$\text{Sin } 2x = 1 - t^2$

Putting all the values, we get the integral,

$$I = \int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)}$$

$$I = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$$

The above equation can be written as

$$I = \int_{-1}^{0} \frac{dt}{(5)^2 - (4t)^2}$$

On integrating we get

$$I = \frac{1}{4}\left[\frac{1}{2(5)}\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$$

Now by applying the limits we get

$$I = \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right]$$

$$I = \frac{1}{40}\log 9$$

31. $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\ dx$

$$\text{let, } I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\ dx$$

$$= \int_0^{\frac{\pi}{2}} 2\sin x \cos x \,.\, \tan^{-1}(\sin x)\ dx$$

Let sin x = t ⇒ cos x dx = dt

When x =0 then t = 0 and when x = π /2 then t = 1

Now by substituting these values in above equation we get

$$\Rightarrow \int_0^{\frac{\pi}{2}} 2\sin x \cos x \, . \tan^{-1}(\sin x)\ dx = \int_0^1 2t. \tan^{-1}(t)\ dt$$

Using product rule

$$\int u. v\,dx = u. \int v\,dx - \int \frac{du}{dx}. \left\{\int vdx\right\} dx$$

$$\Rightarrow 2\int_0^1 t. \tan^{-1}(t)\ dt = 2\left[\tan^{-1}(t). \int t\,dt - \int \frac{d}{dt}(\tan^{-1}(t)). \left\{\int t. dt\right\} dt\right]$$

Computing using product rule we get

$$= 2\left[\tan^{-1}(t). \frac{t^2}{2} - \int \frac{1}{1+t^2}. \frac{t^2}{2}\ dt\right]$$

$$= 2\left[\tan^{-1}(t). \frac{t^2}{2} - \frac{1}{2}. \int \frac{-1+1+t^2}{1+t^2}\ dt\right]$$

Splitting the integrals we get

$$= 2\left[\tan^{-1}(t). \frac{t^2}{2} - \frac{1}{2}. \left\{\int -\frac{1}{1+t^2}\ dt + \int \frac{1+t^2}{1+t^2}\ dt\right\}\right]$$

On simplification we get

$$= 2\left[\tan^{-1}(t). \frac{t^2}{2} - \frac{1}{2}. \left\{\int -\frac{1}{1+t^2}\ dt + \int 1\ dt\right\}\right]$$

$$= 2\left[\tan^{-1}(t). \frac{t^2}{2} - \frac{1}{2}. \{-\tan^{-1}(t) + t\}\right]$$

$$= [t^2.\tan^{-1}(t). -\{-\tan^{-1}(t) + t\}]$$

Computing we get

$$\Rightarrow 2\int_0^1 t. \tan^{-1}(t)\ dt = [t^2.\tan^{-1}(t). -\{-\tan^{-1}(t) + t\}]_0^1$$

Now by applying the limits

$$= [1^2.\tan^{-1}(1). -\{-\tan^{-1}(1) + 1\}] - [0^2.\tan^{-1}(0). -\{-\tan^{-1}(0) + 0\}]$$

$$= \left[1.\frac{\pi}{4}. - \left\{-\frac{\pi}{4} + 1\right\}\right]$$

$$= \left[\frac{\pi}{4} + \frac{\pi}{4} - 1\right]$$

$$\Rightarrow I = \left[\frac{\pi}{2} - 1\right]$$

32. $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$

Solution:

Given: $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx$

$$\text{let}, I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx \quad .... (1)$$

As we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a - x)dx\right\}$$

Using this in above equation we get

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx$$

Using standard allied angles the above equation can be written as

$$= \int_0^{\pi} \frac{(\pi - x)(-\tan(x))}{(-\sec x) + (-\tan x)} dx$$

$$= \int_0^{\pi} \frac{-(\pi - x)(\tan(x))}{-[(\sec x) + (\tan x)]} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)(\tan(x))}{\sec x + \tan x} dx \quad .... (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} + \frac{(\pi - x)(\tan (x))}{\sec x + \tan x}\, dx$$

Now by adding we get

$$2I = \int_0^{\pi} \frac{\pi\tan x}{\sec x + \tan x}\, dx$$

Tan x can be written as

$$= \int_0^{\pi} \frac{\pi.\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}\, dx$$

$$2I = \pi. \int_0^{\pi} \frac{(\sin x)}{(1 + \sin x)}\, dx$$

$$= \pi. \int_0^{\pi} \frac{(-1 + 1 + \sin x)}{(1 + \sin x)}\, dx$$

Now by splitting the integrals we get

$$= \pi. \int_0^{\pi} \frac{(-1)}{(1 + \sin x)}\, dx + \pi. \int_0^{\pi} \frac{(1 + \sin x)}{(1 + \sin x)}\, dx$$

Again by multiplying and dividing above equation by 1 – sin x we get

$$= \pi. \int_0^{\pi} \frac{(-1)}{(1 + \sin x)} \times \frac{(1 - \sin x)}{(1 - \sin x)} dx + \pi. \int_0^{\pi} 1.\, dx$$

Splitting the integrals

$$= -\pi. \int_0^{\pi} \frac{(1 - \sin x)}{(1 - \sin^2 x} dx + \pi. \int_0^{\pi} 1.\, dx$$

$$2I = -\pi. \int_0^{\pi} \frac{(1 - \sin x)}{\cos^2 x}\, dx + \pi. \int_0^{\pi} 1.\, dx$$

$$2I = -\pi. \int_0^{\pi} \left\{\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right\} dx + \pi. \int_0^{\pi} 1.\, dx$$

$$2I = -\pi.\int_0^{\pi}\{\sec^2 x - \tan x \sec x\}\,dx + \pi.\int_0^{\pi} 1.\,dx$$

On integrating we get

$$\Rightarrow 2I = -\pi.[\tan x - \sec x]_0^{\pi} + [x]_0^{\pi}$$

Now by applying the limits we get

$$\Rightarrow 2I = -\pi.[\tan\pi - \sec\pi - \tan 0 + \sec 0] + \pi.[\pi - 0]$$

$$\Rightarrow 2I = -\pi.[0 - (-1) - 0 + 1] + \pi.[\pi]$$

$$\Rightarrow 2I = \pi.[-2 + \pi]$$

$$\Rightarrow I = \frac{\pi}{2}.[\pi - 2]$$

33. $\int_1^4 [|x-1|+|x-2|+|x-3|]\,dx$

Solution:

Given: $\int_1^4 [|x-1| + |x-2| + |x-3|]\,dx$

Let,

$$\Rightarrow I = \int_1^4 [|x-1| + |x-2| + |x-3|]\,dx$$

Now by splitting the integrals we get

$$\Rightarrow I = \int_1^4 [|x-1|]\,dx + \int_1^4 [|x-2|]\,dx + \int_1^4 [|x-3|]\,dx$$

let $I = I_1 + I_2 + I_3$

First solve for I_1:

$$I_1 = \int_1^4 [|x-1|]\,dx$$

As we can see that (x - 1) ≥ 0 when 1 ≤ x ≤ 4

$$\Rightarrow I_1 = \int_1^4 (x-1)\,dx$$

On integrating we get

$$\Rightarrow I_1 = \left[\frac{x^2}{2} - x\right]_0^1$$

Now by applying the limits we get

$$\Rightarrow I_1 = \left[\frac{(4)^2}{2} - 4 - \frac{(1)^2}{2} + 1\right]$$

$$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1\right]$$

$$\Rightarrow I_1 = \left[5 - \frac{1}{2}\right]$$

$$\Rightarrow I_1 = \frac{9}{2}$$

Now solve for I_2:

$$I_2 = \int_1^4 [|x-2|]\,dx$$

As we can see that (x - 2) ≤ 0 when 1 ≤ x ≤ 2 and (x – 2) ≥ 0 when 2 ≤ x ≤ 4

As we know that

$$\left\{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right\}$$

By using this we get

$$\Rightarrow I_2 = \int_1^2 -(x-2)dx + \int_2^4 (x-2)dx$$

On integrating

$$\Rightarrow I_2 = -\left[\frac{x^2}{2} - 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

Now by applying the limits we get

$$\Rightarrow I_2 = -\left[\frac{(2)^2}{2} - 2(2) - \frac{(1)^2}{2} + 2(1)\right] + \left[\frac{(4)^2}{2} - 2(4) - \frac{(2)^2}{2} + 2(2)\right]$$

$$\Rightarrow I_2 = -\left[2 - 4 - \frac{1}{2} + 2\right] + [8 - 8 - 2 + 4]$$

$$\Rightarrow I_2 = \left[\frac{1}{2} + 2\right]$$

$$\Rightarrow I_2 = \frac{5}{2}$$

Now solve for I_3:

$$I_3 = \int_1^4 [|x - 3|]\, dx$$

As we can see that $(x - 3) \le 0$ when $1 \le x \le 3$ and $(x - 3) \ge 0$ when $3 \le x \le 4$

As we know that

$$\left\{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right\}$$

By using above formula we get

$$\Rightarrow I_3 = \int_1^3 -(x - 3)dx + \int_3^4 (x - 3)dx$$

On integrating we get

$$\Rightarrow I_3 = -\left[\frac{x^2}{2} - 3x\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

Now by applying the limits

$$\Rightarrow I_3 = -\left[\frac{(3)^2}{2} - 3(3) - \frac{(1)^2}{2} + 3(1)\right] + \left[\frac{(4)^2}{2} - 3(4) - \frac{(3)^2}{2} + 3(3)\right]$$

$$\Rightarrow I_3 = -\left[\frac{9}{2} - 9 - \frac{1}{2} + 3\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$$

$$\Rightarrow I_3 = \left[2 + \frac{1}{2}\right]$$

$$\Rightarrow I_3 = \frac{5}{2}$$

as $I = I_1 + I_2 + I_3$

Substituting the above all values we get

$$\Rightarrow I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2}$$

$$\Rightarrow I = \frac{19}{2}$$

Prove the following (Exercises 34 to 39)

34. $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Solution:

Given: $\int_1^3 \frac{dx}{(x^2)(x+1)}$

To Prove : $\int_1^3 \frac{dx}{(x^2)(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Let I $= \frac{dx}{(x^2)(x+1)}$

Using partial fraction

let $\frac{1}{(x^2)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ ... (1)

$$\Rightarrow \frac{1}{(x^2)(x+1)} = \frac{A(x)(x+1) + B(x+1) + C(x^2)}{(x+1)(x^2)}$$

$$\Rightarrow 1 = A(x^2 + x) + (Bx + B) + Cx^2$$

$\Rightarrow 1 = Ax^2 + Ax + B + Bx + Cx^2$

$\Rightarrow 1 = B + (A + B)\, x + (A + C)\, x^2$

Equating the coefficients of x, x^2 and constant value. We get

(a) B = 1

(b) $A + B = 0 \Rightarrow A = -B \Rightarrow A = -1$

(c) $A + C = 0 \Rightarrow C = -A \Rightarrow C = 1$

Put these values in equation (1)

$$\Rightarrow \frac{1}{(x^2)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow \frac{1}{(x^2)(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

Taking integrals on both side we get

$$\Rightarrow \int \frac{1}{(x^2)(x+1)}\,dx = \int -\frac{1}{x}\,dx + \int \frac{1}{(x^2)}\,dx + \int \frac{1}{(x+1)}\,dx$$

$$\Rightarrow \int_1^3 \frac{1}{(x^2)(x+1)}\,dx = [-\log|x| - x^{-1} + \log|x+1|]_1^3$$

$$\Rightarrow \int_1^3 \frac{1}{(x^2)(x+1)}\,dx = \left[-\frac{1}{x} + \log\left|\frac{x+1}{x}\right|\right]_1^3$$

Now by applying the limits we get

$$= \left[-\frac{1}{3} + \log\left|\frac{3+1}{3}\right| - \left(-\frac{1}{1} + \log\left|\frac{1+1}{1}\right|\right)\right]$$

$$= \left[-\frac{1}{3} + \log\left|\frac{4}{3}\right| + \left(1 - \log\left|\frac{2}{1}\right|\right)\right]$$

Computing and simplifying we get

$$= \left[-\frac{1}{3} + 1 + \log\left|\frac{4}{3} \times \frac{1}{2}\right|\right]$$

$$\Rightarrow I = \left[\frac{2}{3} + \log\left|\frac{2}{3}\right|\right]$$

⇒ L.H.S = R.H.S

Hence proved.

35. $\int_0^1 x\, e^x dx = 1$

Solution:

Given: $\int_0^1 xe^x dx$

To Prove : $\int_0^1 xe^x dx = 1$

Let $I = \int_0^1 xe^x dx$

Using product rule we get

$$\int u.v\, dx = u.\int v\, dx - \int \frac{du}{dx}.\left\{\int vdx\right\} dx$$

$$\Rightarrow \int_0^1 xe^x dx = x.\int_0^1 e^x dx - \int_0^1 \frac{dx}{dx}.\left\{\int e^x dx\right\}.dx$$

On integrating

$$\Rightarrow \int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 1.e^x dx$$

Now by applying the limits we get

$$\Rightarrow \int_0^1 xe^x dx = [xe^x]_0^1 - [e^x]_0^1$$

$$\Rightarrow \int_0^1 xe^x dx = [1.e^1 - 0.e^0] - [e^1 - e^0]$$

$$\Rightarrow \int_0^1 xe^x dx = e - 0 - e + 1$$

$$\Rightarrow \int_0^1 xe^x dx = 1$$

Therefore L.H.S = R.H.S

Hence Proved.

36. $\int_{-1}^{1} x^{17} \cos^4 x \, dx = 0$

Solution:

Given: $\int_{-1}^{1} x^{17}.\cos^4 x dx$

To Prove : $\int_{-1}^{1} x^{17}.\cos^4 x dx = 0$

Let I $= \int_{-1}^{1} x^{17}.\cos^4 x dx$

As we can see f(x) $=x^{17}$ $.\cos^4 x$ and f (-x) = $(-x)^{17}$ $.\cos^4$ (-x) = $-x^{17}$ $.\cos^4 x$

That is f (x) = -f (-x)
so, it is an odd function.

It is also known that if f(x) is an odd function then we have

$$\left\{ \int_{-a}^{a} f(x) dx = 0 \right\}$$

$$\Rightarrow I = \int_{-1}^{1} x^{17}.\cos^4 x dx = 0$$

Hence proved.

37. $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$

Solution:

Given: $\int_0^{\frac{\pi}{2}} \sin^3 x dx$

To Prove : $\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$

Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x dx \ \ldots (1)$

Above equation can be written as

$$= \int_0^{\frac{\pi}{2}} \sin x. \sin^2 x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x. (1 - \cos^2 x) dx$$

Now by splitting the integrals

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin x\, dx - \int_0^{\frac{\pi}{2}} \sin x. \cos^2 x dx$$

$$\Rightarrow I = [-\cos x]_0^{\pi/2} - I_1 \ \ldots (2)$$

First solve for I_1:

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{2}} \sin x. \cos^2 x dx$$

Let cos x = t ⇒ -sin x dx = dt ⇒ sin x dx = - dt

When x = 0 then t = 1 and when x = π /2 then t = 0

$$\Rightarrow I_1 = \int_1^0 t^2(-dt)$$

$$= -\int_1^0 t^2(dt)$$

On integrating we get

$$= -\left[\frac{t^3}{3}\right]_1^0$$

Now by applying the limits we get

$$= -\left\{-\frac{1}{3}\right\}$$

$$\Rightarrow I_1 = \frac{1}{3}$$

Substitute in equation (2)

$$\Rightarrow I = [-\cos x]_0^{\pi/2} - \frac{1}{3}$$

$$\Rightarrow I = -\left\{\cos\frac{\pi}{2} - \cos 0\right\} - \frac{1}{3}$$

$$\Rightarrow I = 1 - \frac{1}{3}$$

$$\Rightarrow I = \frac{2}{3}$$

L.H.S = R.H.S

Hence Proved.

38. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx = 1 - \log 2$

Solution:

Given: $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx$

To Prove : $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$

Let I $= \int_0^{\frac{\pi}{4}} 2\tan^3 x dx$ ... (1)

The above equation can be written as

$$= \int_0^{\frac{\pi}{4}} 2.\tan x.\tan^2 x dx$$

Substituting $\tan^2 x$ formula we get

$$= 2.\int_0^{\frac{\pi}{4}} \tan x.(\sec^2 x - 1) dx$$

Now by splitting the integral we get

$$\Rightarrow I = 2\left\{-\int_0^{\frac{\pi}{4}} \tan x\, dx + \int_0^{\frac{\pi}{4}} \tan x.\sec^2 x\, dx\right\}$$

$$\Rightarrow I = -[2\log\sec x]_0^{\pi/4} + 2.I_1 \quad ...(2)$$

First solve for I_1:

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{4}} \tan x.\sec^2 x\, dx$$

Let tan x = t ⇒ $\sec^2$ x dx = dt

When x=0 then t= 0 and when x = π /2 then t = 1

$$\Rightarrow I_1 = \int_0^1 t.\, dt$$

On integrating we get

$$= \left[\frac{t^2}{2}\right]_0^1$$

Applying the limits we get

$$\Rightarrow I_1 = \frac{1}{2}$$

Substitute in equation (2)

$$\Rightarrow I = [2\log\cos x]_0^{\pi/4} + 2.\frac{1}{2}$$

On simplification we get

$$\Rightarrow I = 2\left\{\log\cos\frac{\pi}{4} - \log\cos 0\right\} + 1$$

Substituting the values of cos 0 = 1 we get

$$\Rightarrow I = 2\left\{\log\frac{1}{\sqrt{2}} - \log 1\right\} + 1$$

$$\Rightarrow I = \left\{\log\left(\frac{1}{\sqrt{2}}\right)^2 - \log(1)^2\right\} + 1$$

$$\Rightarrow I = 1 - \log 2 + \log 1$$

$$\Rightarrow I = 1 - \log 2$$

L.H.S = R.H.S

Hence the proof.

39. $\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1$

Solution:

Given: $\int_0^1 \sin^{-1} x \, dx$

To Prove : $\int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - 1$

Let I $= \int_0^1 \sin^{-1} x . 1dx$

Using product rule we get

$$\int u.v\,dx = u.\int v\,dx - \int \frac{du}{dx}.\left\{\int vdx\right\} dx$$

$$\Rightarrow \int_0^1 xe^x dx = \sin^{-1} x . \int_0^1 1.dx - \int_0^1 \frac{d}{dx}\sin^{-1} x . \left\{\int 1.dx\right\}.dx$$

On integrating we get

$$\Rightarrow \int_0^1 xe^x dx = [\sin^{-1} x . x]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}}.x\,dx$$

$$\Rightarrow I = [\sin^{-1} x . x]_0^1 - I_1 \;\; \ldots (2)$$

First solve for I_1:

$$\Rightarrow I_1 = \int_0^1 \frac{1}{\sqrt{1-x^2}}.x\,dx$$

Let $1 - x^2 = t \Rightarrow -2\,x\,dx = dt$

When x = 0 then t = 1 and when x = 1 then t = 0

$$\Rightarrow I_1 = \int_1^0 \frac{1}{\sqrt{t}}.\frac{-dt}{2}$$

$$= -\frac{1}{2}\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^0$$

$$\Rightarrow I_1 = \sqrt{1}$$

$\Rightarrow I_1 = 1$

Substitute in equation (2)

$\Rightarrow I = [\sin^{-1} x \,.\, x]_0^1 - 1$

$\Rightarrow I = \sin^{-1}(1) - 0 - 1$

$\Rightarrow I = \frac{\pi}{2} - 1$

L.H.S = R.H.S

Hence Proved.

40. Evaluate $\int_0^1 e^{2-3x} dx$ as a limit of a sum.

Solution:

Given: $\int_0^1 e^{2-3x} dx$

$$\text{Let } I = \int_0^1 e^{2-3x} dx$$

$$\text{because, } \int_a^b f(x) dx = (b-a) \lim_{n\to\infty} \frac{1}{n} [f(a) + f(a+h) + \cdots + f(a+(n-1)h)]$$

$$\text{where, } h = \frac{b-a}{n}$$

Here, a = 0, b = 1, and f (x) = e^{2-3x} and h

$$= \lim_{n\to\infty} \frac{1}{n} [e^2 + e^2 . e^{3h} + + e^2 . e^{-6h} \ldots + e^2 . e^{-3(n-1)h}]$$

$$= \lim_{n\to\infty} \frac{1}{n} [e^2 \{1 + e^{3h} + e^{-6h} + \cdots + . e^{-3(n-1)h}\}]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[e^2 \left\{ \frac{1 - \left(e^{-3h}\right)^n}{1 - \left(e^{-3h}\right)} \right\} \right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-\frac{3}{n}}\right)^n}{1-\left(e^{-\frac{3}{n}}\right)}\right\}\right] \text{ as, } h = \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^2\left\{\frac{(e^{-3})-1}{\left(e^{-\frac{3}{n}}\right)-1}\right\}\right]$$

$$= e^2.(e^{-3}-1)\lim_{n\to\infty} \frac{1}{n}.\left(-\frac{n}{3}\right)\left[\left\{\frac{-\frac{3}{n}}{\left(e^{-\frac{3}{n}}\right)-1}\right\}\right]$$

On simplification we get

$$= -\frac{(e^2.(e^{-3}-1))}{3}\lim_{n\to\infty}\left[\left\{\frac{-\frac{3}{n}}{\left(e^{-\frac{3}{n}}\right)-1}\right\}\right]$$

We know that

$$\lim_{n\to\infty}\left[\frac{x}{(e^x)-1}\right] = 1$$

Substituting this in above equation we get

$$= \frac{-e^{-1}+e^2}{3}(1)$$

$$\Rightarrow I = \frac{1}{3}\left(e^2 - \frac{1}{e}\right)$$

Choose the correct answers in Exercises 41 to 44.

41. $\int \frac{dx}{e^x + e^{-x}}$ is equal to

(A) $\tan^{-1}(e^x) + C$ (B) $\tan^{-1}(e^{-x}) + C$

(C) $\log(e^x - e^{-x}) + C$ (D) $\log(e^x + e^{-x}) + C$

Solution:

(A) $\tan^{-1}(e^x) + C$

Explanation:

Given: $\int \frac{dx}{e^x+e^{-x}}$

$$\text{let } I = \int \frac{dx}{e^x + e^{-x}}$$

The above equation can be written as

$$= \int \frac{dx}{e^{-x}(e^{2x}+1)}$$

$$= \int \frac{e^x dx}{(e^{2x}+1)}$$

Put $e^x = t \Rightarrow e^x\, dx = dt$

$$\Rightarrow \int \frac{e^x dx}{(e^{2x}+1)} = \int \frac{dt}{(t^2+1)}$$

$$= \tan^{-1} t + C$$

$$= \tan^{-1}(e^x) + C$$

Hence, correct option is (A).

42. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to

(A) $\frac{-1}{\sin x + \cos x} + C$

(B) $\log|\sin x + \cos x| + C$

(C) $\log|\sin x - \cos x| + C$

(D) $\frac{1}{(\sin x + \cos x)^2}$

Solution:

(B) $\log|\sin x + \cos x| + C$

Explanation:

Given: $\int \frac{\cos 2x}{(\sin x+\cos x)^2} dx$

let $I = \int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$

Substituting cos 2x formula we get

$= \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2} dx$

By using $a^2 - b^2 = (a + b)(a - b)$ we get

$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\sin x + \cos x)^2} dx$

On simplification

$= \int \frac{(\cos x - \sin x)}{(\sin x + \cos x)} dx$

Put $\sin x + \cos x = t \Rightarrow \cos x - \sin x = dt$

$\Rightarrow \int \frac{(\cos x - \sin x)}{(\sin x + \cos x)} dx = \int \frac{dt}{t}$

$= \log|t| + C$

$= \log|\sin x + \cos x| + C$

Hence, correct option is (B).

43. If $f(a+b-x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to

(A) $\frac{a+b}{2}\int_a^b f(b-x)\, dx$

(B) $\frac{a+b}{2}\int_a^b f(b+x)\, dx$

(C) $\frac{b-a}{2}\int_a^b f(x)\, dx$

(D) $\frac{a+b}{2}\int_a^b f(x)\, dx$

Solution:

(D) $\frac{a+b}{2}\int_a^b f(x)\,dx$

Explanation:

Given: $\int_a^b x\,f(x)\,dx$

$$\text{let, } I = \int_a^b x\,f(x)\,dx$$

As we know that

$\{f(x) = f(a+b-x)\}$

Using this we get

$$\Rightarrow I = \int_a^b (a+b-x)\,f(a+b-x)\,dx$$

$$\Rightarrow I = \int_a^b (a+b-x)\,f(x)\,dx$$

Now by splitting the integral we get

$$\Rightarrow I = \int_a^b (a+b)\,f(x)\,dx - \int_a^b (x)\,f(x)\,dx$$

$$\Rightarrow I = \int_a^b (a+b)\,f(x)\,dx - I$$

$$\Rightarrow 2I = \int_a^b (a+b)\,f(x)\,dx$$

$$\Rightarrow I = \frac{(a+b)}{2}\int_a^b f(x)\,dx$$

Hence, correct option is (D).

44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

(A) 1 (B) 0 (C) –1 (D) π

Solution:

(B) 0

Explanation:

Given: $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$$\text{Let I} = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$$

The above equation can be written as

$$= \int_0^1 \tan^{-1}\left(\frac{x+x-1}{1+x(1-x)}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$$

As we know that

$$\tan^{-1}\left(\frac{A-B}{1+AB)}\right) = \tan^{-1}(A)\tan^{-1}(B)$$

By using this formula we get

$$= \int_0^1 [\tan^{-1}(x) - \tan^{-1}(1-x)]\,dx \;\ldots(1)$$

Again as we know that

$$\left\{\int_0^a f(x)dx = \int_0^a f(a-x)dx\right\}$$

By using this we can write as

$= \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(1-(1-x))]\, dx$

$= \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]\, dx \ \dots (2)$

Adding (1) and (2), we get

$2I = \int_0^1 [\tan^{-1}(x) - \tan^{-1}(1-x)]\, dx + \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]\, dx$

$2I = \int_0^1 [\tan^{-1}(x) - \tan^{-1}(1-x) + \tan^{-1}(1-x) - \tan^{-1}(x)]\, dx$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

Hence, correct option is (B).