EXERCISE 13.1

1. Given that E and F are events, such that P (E) = 0.6, P (F) = 0.3 and P (E ∩ F) = 0.2, find P (E|F) and P (F|E).

Solution:

Given P (E) = 0.6, P (F) = 0.3 and P (E ∩ F) = 0.2

We know that by the definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

By substituting the values we get

$$\Rightarrow P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.2}{0.3} = \frac{2}{3}$$

And $\Rightarrow P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.6} = \frac{2}{6} = \frac{1}{3}$

2. Compute P (A|B), if P (B) = 0.5 and P (A ∩ B) = 0.32

Solution:

Given: P (B) = 0.5 and P (A ∩ B) = 0.32

We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now by substituting the values we get

$$\Rightarrow P(A|B) = \frac{0.32}{0.5} = \frac{32}{50} = \frac{16}{25}$$

3. If P (A) = 0.8, P (B) = 0.5 and P (B|A) = 0.4, find
(i) P (A ∩ B)

(ii) P (A|B)

(iii) P (A ∪ B)

Solution:

Given P (A) = 0.8, P (B) = 0.5 and P (B|A) = 0.4

(i) We know that by definition of conditional probability,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$\Rightarrow$ P (A ∩ B) = P (B|A) P (A)

$\Rightarrow$ P (A ∩ B) = 0.4 × 0.8

$\Rightarrow$ P (A ∩ B) = 0.32

(ii) We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now substituting the values we get

$$\Rightarrow P(A|B) = \frac{0.32}{0.5} = 0.64$$

$\Rightarrow$ P (A|B) = 0.64

(iii) Now, ∵ P (A ∪ B) = P (A) + P (B) – P (A ∩ B)

Substituting the values we get

$\Rightarrow$ P (A ∪ B) = 0.8 + 0.5 – 0.32 = 1.3 – 0.32

$\Rightarrow$ P (A ∪ B) = 0.98

4. Evaluate P (A ∪ B), if 2P (A) = P (B) = 5/13 and P (A|B) = 2/5.

Solution:

Given $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$

$\Rightarrow P(B) = \frac{5}{13}, P(A) = \frac{5}{13\times2} = \frac{5}{26}, P(A|B) = \frac{2}{5}$ .... (i)

We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$\Rightarrow$ P (A ∩ B) = P (A|B) P (B)

$\Rightarrow P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}$ .... (ii)

Now, ∵ P (A * B) = P (A) + P (B) – P (A ∩ B)

$$\Rightarrow P(A \cup B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{5 + 10 - 4}{26} = \frac{15 - 4}{26}$$

$$\Rightarrow P(A \cup B) = \frac{11}{26}$$

5. If P (A) = 6/11, P (B) = 5/11 and P (A ∪ B) = 7/11, find
(i) P (A∩B)

(ii) P (A|B)

(iii) P (B|A)

Solution:

Given: $P(A) = \frac{6}{11}, P(B) = \frac{5}{11}, P(A \cup B) = \frac{7}{11}$

(i) We know that P (A * B) = P (A) + P (B) – P (A ∩ B)

$\Rightarrow$ P (A ∩ B) = P (A) + P (B) – P (A ∪ B)

$$\Rightarrow P(A \cap B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{11 - 7}{11}$$

$$\Rightarrow P(A \cap B) = \frac{4}{11}$$

(ii) Now, by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A|B) = \frac{4/11}{5/11}$$

$$\Rightarrow P(A|B) = \frac{4}{5}$$

(iii) Again, by definition of conditional probability,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(B|A) = \frac{4/11}{6/11} = \frac{4}{6} = \frac{2}{3}$$

$$\Rightarrow P(B|A) = \frac{2}{3}$$

Determine P (E|F) in Exercises 6 to 9.

6. A coin is tossed three times, where
(i) E : head on the third toss, F : heads on first two tosses
(ii) E : at least two heads, F : at most two heads
(iii) E : at most two tails, F : at least one tail

Solution:

The sample space of the given experiment will be:

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(i) Here, E: head on third toss

And F: heads on first two tosses

⇒ E = {HHH, HTH, THH, TTH} and F = {HHH, HHT}

⇒ E ∩ F = {HHH}

So, $P(E) = \frac{4}{8} = \frac{1}{2}, P(F) = \frac{2}{8} = \frac{1}{4}, P(E \cap F) = \frac{1}{8}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/8}{1/4} = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow P(E|F) = \frac{1}{2}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/8}{1/4} = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow P(E|F) = \frac{1}{2}$$

(ii) Here, E: at least two heads

And F: at most two heads

⇒ E = {HHH, HHT, HTH, THH} and F = {HHT, HTH, THH, HTT, THT, TTH, TTT}

⇒ E ∩ F = {HHT, HTH, THH}

So, $P(E) = \frac{4}{8} = \frac{1}{2}, P(F) = \frac{7}{8}, P(E \cap F) = \frac{3}{8}$

Now, we know that

By definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/8}{7/8} = \frac{3}{7}$$

(iii) Here, E: at most two tails

And F: at least one tail

⇒ E = {HHH, HHT, HTH, THH, HTT, THT, TTH}

And F = {HHT, HTH, THH, HTT, THT, TTH, TTT}

So, $P(E) = \frac{7}{8}, P(F) = \frac{7}{8}, P(E \cap F) = \frac{6}{8} = \frac{3}{4}$

Now, we know that

By definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/4}{7/8} = \frac{6}{7}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/4}{7/8} = \frac{6}{7}$$

7. Two coins are tossed once, where
(i) E: tail appears on one coin, F: one coin shows the head
(ii) E: no tail appears, F: no head appears

Solution:

The sample space of the given experiment is S = {HH, HT, TH, TT}

(i) Here, E: tail appears on one coin

And F: one coin shows head

⇒ E = {HT, TH} and F = {HT, TH}

⇒ E ∩ F = {HT, TH}

So, $P(E) = \frac{2}{4} = \frac{1}{2}, P(F) = \frac{2}{4} = \frac{1}{2}, P(E \cap F) = \frac{2}{4} = \frac{1}{2}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/2}{1/2}$$

⇒ P (E|F) = 1

(ii) Here, E: no tail appears

And F: no head appears

⇒ E = {HH} and F = {TT}

⇒ E ∩ F = ϕ

So, $P(E) = \frac{1}{4}, P(F) = \frac{1}{4}, P(E \cap F) = \frac{0}{4} = 0$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Substituting the values we get

$$\Rightarrow P(E|F) = \frac{0}{1/4}$$

$\Rightarrow$ P (E|F) = 0

8. A die is thrown three times, E: 4 appears on the third toss, F: 6 and 5 appear, respectively, on the first two tosses.

Solution:

The sample space has 216 outcomes, where each element of the sample space has 3 entries and is of the form (x, y, z) where $1 \le x, y, z \le 6$.

Here, E: 4 appears on the third toss

$$\Rightarrow E = \left\{\begin{matrix} (1,1,4),(1,2,4),(1,3,4),(1,4,4),(1,5,4),(1,6,4), \\ (2,1,4),(2,2,4),(2,3,4),(2,4,4),(2,5,4),(2,6,4), \\ (3,1,4),(3,2,4),(3,3,4),(3,4,4),(3,5,4),(3,6,4), \\ (4,1,4),(4,2,4),(4,3,4),(4,4,4),(4,5,4),(4,6,4), \\ (5,1,4),(5,2,4),(5,3,4),(5,4,4),(5,5,4),(5,6,4), \\ (6,1,4),(6,2,4),(6,3,4),(6,4,4),(6,5,4),(6,6,4) \end{matrix}\right\}$$

Now, F: 6 and 5 appears respectively on first two tosses

$\Rightarrow$ F = {(6, 5, 1), (6, 5, 2), (6, 5, 3), (6, 5, 4), (6, 5, 5), (6, 5, 6)}

$\Rightarrow$ E ∩ F = {(6, 5, 4)}

So, $P(E) = \frac{36}{216}, P(F) = \frac{6}{216}, P(E \cap F) = \frac{1}{216}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/216}{6/216} = \frac{1}{6}$$

$$\Rightarrow P(E|F) = \frac{1}{6}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Now by substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/216}{6/216} = \frac{1}{6}$$

$$\Rightarrow P(E|F) = \frac{1}{6}$$

9. Mother, father and son line up at random for a family picture
E: son on one end, F: father in the middle.

Solution:

Let M denotes mother, F denotes father, and S denotes son.

Then, the sample space for the given experiment will be

S = {MFS, SFM, FSM, MSF, SMF, FMS}

Here, E: Son on one end

And F: Father in the middle

⇒ E = {MFS, SFM, SMF, FMS} and F = {MFS, SFM}

⇒ E ∩ F = {MFS, SFM}

So, $P(E) = \frac{4}{6} = \frac{2}{3}, P(F) = \frac{2}{6} = \frac{1}{3}, P(E \cap F) = \frac{2}{6} = \frac{1}{3}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Now by substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/3}{1/3} = 1$$

⇒ P (E | F) = 1

10. A black and a red dice are rolled.
(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.
(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution:

Let B denote black coloured die and R denote red coloured die.

Then, the sample space for the given experiment will be:

$$S = \begin{Bmatrix} (B1, R1), (B1, R2), (B1, R3), (B1, R4), (B1, R5), (B1, R6), \\ (B2, R1), (B2, R2), (B2, R3), (B2, R4), (B2, R5), (B2, R6), \\ (B3, R1), (B3, R2), (B3, R3), (B3, R4), (B3, R5), (B3, R6), \\ (B4, R1), (B4, R2), (B4, R3), (B4, R4), (B4, R5), (B4, R6), \\ (B5, R1), (B5, R2), (B5, R3), (B5, R4), (B5, R5), (B5, R6), \\ (B6, R1), (B6, R2), (B6, R3), (B6, R4), (B6, R5), (B6, R6) \end{Bmatrix}$$

(a) Let A be the event of 'obtaining a sum greater than 9' and B be the event of 'getting a 5 on black die'.

Then, A = {(B4, R6), (B5, R5), (B5, R6), (B6, R4), (B6, R5), (B6, R6)}

And B = {(B5, R1), (B5, R2), (B5, R3), (B5, R4), (B5, R5), (B5, R6)}

⇒ A ∩ B = {(B5, R5), (B5, R6)}

So, $P(A) = \frac{6}{36} = \frac{1}{6}, P(B) = \frac{6}{36} = \frac{1}{6}, P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

Now, we know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now by substituting the values we get

$$\Rightarrow P(A|B) = \frac{1/18}{1/6} = \frac{6}{18} = \frac{1}{3}$$

$$\Rightarrow P(A|B) = \frac{1}{3}$$

(b) Let A be the event of 'obtaining a sum 8' and B be the event of 'getting a number less than 4 on red die'.

Then, A = {(B2, R6), (B3, R5), (B4, R4), (B5, R3), (B6, R2)}

And $B = \begin{Bmatrix} (B1, R1)(B2, R1), (B3, R1), (B4, R1), (B5, R1), (B6, R1), \\ (B1, R2), (B2, R2), (B3, R2), (B4, R2), (B5, R2), (B6, R2), \\ (B1, R3), (B2, R3), (B3, R3), (B4, R3), (B5, R3), (B6, R3) \end{Bmatrix}$

⇒ A ∩ B = {(B5, R3), (B6, R2)}

So, $P(A) = \frac{5}{36}, P(B) = \frac{18}{36} = \frac{1}{2}, P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

Now, we know that

By definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now by substituting the values we get

$$\Rightarrow P(A|B) = \frac{1/18}{1/2} = \frac{2}{18} = \frac{1}{9}$$

$$\Rightarrow P(A|B) = \frac{1}{9}$$

11. A fair die is rolled. Consider events E = {1,3,5}, F = {2,3} and G = {2,3,4,5}
Find
(i) P (E|F) and P (F|E)
(ii) P (E|G) and P (G|E)
(iii) P ((E ∪ F)|G) and P ((E ∩ F)|G)

Solution:

The sample space for the given experiment is S = {1, 2, 3, 4, 5, 6}

Here, E = {1, 3, 5}, F = {2, 3} and G = {2, 3, 4, 5} .... (i)

$\Rightarrow P(E) = \frac{3}{6} = \frac{1}{2}, P(F) = \frac{2}{6} = \frac{1}{3}, P(G) = \frac{4}{6} = \frac{2}{3}$ .... (ii)

Now, E ∩ F = {3}, F ∩ G = {2, 3}, E ∩ G = {3, 5} .... (iii)

$\Rightarrow P(E \cap F) = \frac{1}{6}, P(F \cap G) = \frac{2}{6} = \frac{1}{3}, P(E \cap G) = \frac{2}{6} = \frac{1}{3}$ .... (iv)

(i) We know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/6}{1/3} = \frac{3}{6} = \frac{1}{2} \text{ [Using (II) and (IV)]}$$

$$\Rightarrow P(E|F) = \frac{1}{2}$$

Similarly, we have

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/6}{1/2} = \frac{2}{6} = \frac{1}{3} \text{ [Using (ii) and (iv)]}$$

$$\Rightarrow P(F|E) = \frac{1}{3}$$

(ii) We know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|G) = \frac{P(E \cap G)}{P(G)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$\Rightarrow P(E|G) = \frac{1}{2}$$

Similarly, we have

$$P(G|E) = \frac{P(G \cap E)}{P(E)} = \frac{1/3}{1/2} = \frac{2}{3}$$

$$\Rightarrow P(G|E) = \frac{2}{3}$$

(iii) Clearly, from (i), we have

E = {1, 3, 5}, F = {2, 3} and G = {2, 3, 4, 5}

⇒ E ∪ F = {1, 2, 3, 5}

⇒ (E ∪ F) ∩ G = {2, 3, 5}

(iii) Clearly, from (i), we have

E = {1, 3, 5}, F = {2, 3} and G = {2, 3, 4, 5}

⇒ E ∪ F = {1, 2, 3, 5}

⇒ (E ∪ F) ∩ G = {2, 3, 5}

$\Rightarrow P((E \cup F) \cap G) = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow P((E \cup F) \cap G) = \frac{1}{2}$ .... (v)

Now, we know that

By definition of conditional probability,

$P(E|F) = \frac{P(E \cap F)}{P(F)}$

$\Rightarrow P((E \cup F)|G) = \frac{P((E \cup F) \cap G)}{P(G)} = \frac{1/2}{2/3} = \frac{3}{4}$ [Using (ii) and (v)]

$\Rightarrow P((E \cup F)|G) = \frac{3}{4}$

Similarly, we have $E \cap F = \{3\}$ [Using (iii)]

And $G = \{2, 3, 4, 5\}$ [Using (i)]

$\Rightarrow (E \cap F) \cap G = \{3\}$

$\Rightarrow P((E \cap F) \cap G) = \frac{1}{6}$ .... (vi)

So,

$P((E \cap F)|G) = \frac{P((E \cap F) \cap G)}{P(G)} = \frac{1/6}{2/3} = \frac{1}{4}$ [Using (ii) and (vi)]

$\Rightarrow P((E \cap F)|G) = \frac{1}{4}$

12. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?

Solution:

Let B denote boy and G denote girl.

Then, the sample space of the given experiment is S = {GG, GB, BG, BB}

Let E be the event that ‘both are girls’.

$\Rightarrow$ E = {GG}

$$\Rightarrow P(E) = \frac{1}{4}$$

(i) Let F be the event that 'the youngest is a girl'.

$\Rightarrow$ F = {GG, BG}

$\Rightarrow P(F) = \frac{2}{4} = \frac{1}{2}$ .... (i)

Now, E ∩ F = {GG}

$\Rightarrow P(E \cap F) = \frac{1}{4}$ .... (ii)

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$\Rightarrow P(E|F) = \frac{1/4}{1/2} = \frac{2}{4} = \frac{1}{2}$ [Using (i) and (ii)]

$$\Rightarrow P(E|F) = \frac{1}{2}$$

(ii) Let H be the event that 'at least one is a girl'.

$\Rightarrow$ H = {GG, GB, BG}

$\Rightarrow P(H) = \frac{3}{4}$ .... (iii)

Now, E ∩ H = {GG}

$\Rightarrow P(E \cap H) = \frac{1}{4}$ .... (iv)

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

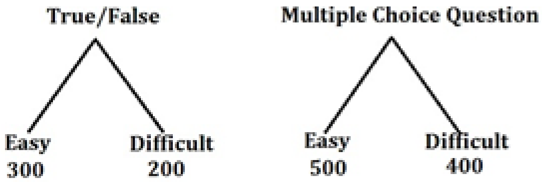
$$\Rightarrow P(E|H) = \frac{P(E \cap H)}{P(H)} = \frac{1/4}{3/4} = \frac{1}{3} \text{ [Using (iii) and (iv)]}$$

$$\Rightarrow P(E|H) = \frac{1}{3}$$

13. An instructor has a question bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question, given that it is a multiple-choice question?

Solution:

Here, there are two types of questions, True/False or Multiple Choice Questions (T/F or MCQ), and each of them is divided into Easy and Difficult type, as shown below in the tree diagram.



So, in all, there are, 500 T/F questions and 900 MCQs.

Also, there are 800 Easy questions and 600 difficult questions.

⇒ the sample space of this experiment has 500 + 900 = 1400 outcomes.

Now, let E be the event of 'getting an Easy question' and F be the event of 'getting an MCQ'.

$\Rightarrow \mathrm{P(E)} = \frac{800}{1400} = \frac{8}{14}$ And $\mathrm{P(F)} = \frac{900}{1400} = \frac{9}{14}$ .... (i)

Now, E ∩ F is the event of getting an MCQ which is Easy.

Clearly, from the diagram, we know that there are 500 MCQs that are easy.

So $\mathrm{P(E \cap F)} = \frac{500}{1400} = \frac{5}{14}$.......... (ii)

Now, we know that by definition of conditional probability,

$$\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$$

$\Rightarrow \mathrm{P(E|F)} = \frac{5/14}{9/14} = \frac{5}{9}$ [Using (i) and (ii)]

$$\Rightarrow \mathrm{P(E|F)} = \frac{5}{9}$$

$$\mathrm{P(E|F)} = \frac{\mathrm{P(E \cap F)}}{\mathrm{P(F)}}$$

$\Rightarrow \mathrm{P(E|F)} = \frac{5/14}{9/14} = \frac{5}{9}$ [Using (i) and (ii)]

$$\Rightarrow \mathrm{P(E|F)} = \frac{5}{9}$$

14. Given that the two numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

Solution:

The sample space of the given experiment is given below

$$S = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

Let E be the event that 'the sum of numbers on the dice is 4' and F be the event that 'the two numbers appearing on throwing the two dice are different'.

$\Rightarrow$ E = {(1, 3), (2, 2), (3, 1)}

$$F = \begin{Bmatrix} (1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,4),(3,5),(3,6), \\ (4,1),(4,2),(4,3),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5) \end{Bmatrix}$$

And

$\Rightarrow$ E ∩ F = {(1, 3), (3, 1)}

$$\Rightarrow P(E) = \frac{3}{36} = \frac{1}{12}, P(F) = \frac{30}{36} = \frac{5}{6}, P(E \cap F) = \frac{2}{36} = \frac{1}{18} \text{ .... (i)}$$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

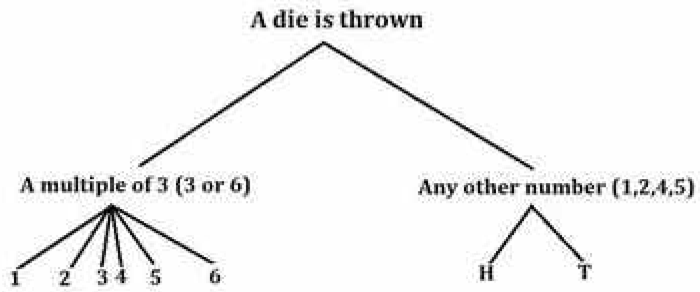
$$\Rightarrow P(E|F) = \frac{1/18}{5/6} = \frac{1}{15}$$

$$\Rightarrow P(E|F) = \frac{1}{15}$$

15. Consider the experiment of throwing a die; if a multiple of 3 comes up, throw the die again and if any other number comes up, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Solution:

The experiment is explained below in the tree diagram:



The sample space of the given experiment is given below

$$S = \begin{Bmatrix} (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6), \\ 1H, 2H, 4H, 5H, 1T, 2T, 4T, 5T \end{Bmatrix}$$

Let E be the event that 'the coin shows a tail' and F be the event that 'at least one die shows a 3'.

$\Rightarrow E = \{1T, 2T, 4T, 5T\}$ and $F = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (6, 3)\}$

$\Rightarrow E \cap F = \varphi \Rightarrow P(E \cap F) = 0$ .... (i)

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$\Rightarrow P(E|F) = \frac{0}{P(F)} = 0$ [Using (i)]

$\Rightarrow P(E|F) = 0$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{0}{P(F)} = 0 \text{ [Using (i)]}$$

$\Rightarrow$ P (E | F) = 0

16. If P (A) = 1/2, P (B) = 0, then P (A|B) is
A. 0
B. ½
C. not defined
D. 1

Solution:

C. Not defined

Explanation:

We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ .... (i)}$$

Given: P (A) = ½

And P (B) = 0

$\Rightarrow$ Using (i), we have

$$P(A|B) = \frac{P(A \cap B)}{0} = (A \cap B) \times \frac{1}{0}, \text{ which is not defined.}$$

17. If A and B are events such that P (A|B) = P (B|A), then
A. A ⊂ B but A ≠ B
B. A = B
C. A ∩ B = φ
D. P (A) = P (B)

Solution:

D. P (A) = P (B)

Explanation:

Given: P (A|B) = P (B|A) .... (i)

Now, we know that by definition of conditional probability,

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ .... (ii)

$P(B|A) = \frac{P(A \cap B)}{P(A)}$ .... (iii)

Using (i), we have

P (A|B) = P (B|A)

$\Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)}$

⇒ P (A) = P (B)

EXERCISE 13.2 PAGE NO: 546

1. If P (A) = 3/5 and P (B) = 1/5, find P (A ∩ B) if A and B are independent events.

Solution:

Given P (A) = 3/5 and P (B) = 1/5

As A and B are independent events.

⇒ P (A ∩ B) = P (A).P (B)

= 3/5 × 1/5 = 3/25

2. Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution:

Given, a pack of 52 cards.

As we know, there are 26 cards in total, which are black. Let A and B denote, respectively, the events that the first and second drawn cards are black.

Now, P (A) = P (black card in first draw) = 26/52 = ½

Because the second card is drawn without replacement, now the total number of black cards will be 25, and the total number of cards will be 51, which is the conditional probability of B, given that A has already occurred.

Now, P (B/A) = P (black card in second draw) = 25/51

Thus, the probability that both cards are black

⇒ P (A ∩ B) = ½ × 25/51 = 25/102

Hence, the probability that both the cards are black = 25/102

3. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges, out of which 12 are good, and 3 are bad ones will be approved for sale.

Solution:

Given, a box of oranges.

Let A, B and C denotes the events, respectively, that the first, second and third drawn orange is good.

Now, P (A) = P (good orange in first draw) = 12/15

Because the second orange is drawn without replacement, now the total number of good oranges will be 11, and the total number of oranges will be 14, which is the conditional probability of B, given that A has already occurred.

Now, P (B/A) = P (good orange in second draw) = 11/14

Because the third orange is drawn without replacement, now the total number of good oranges will be 10, and the total oranges will be 13, which is the conditional probability of C, given that A and B have already occurred.

Now, P (C/AB) = P (good orange in third draw) = 10/13

Thus, the probability that all the oranges are good

⇒ P (A ∩ B ∩ C) = 12/15 × 11/14 × 10/13 = 44/91

Hence, the probability that a box will be approved for sale = 44/91

4. A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

Solution:

Given, a fair coin and an unbiased die are tossed.

We know that the sample space S.

S = {(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)}

Let A be the event head that appears on the coin.

⇒ A = {(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)}

⇒ P (A) = 6/12 = ½

Now, Let B be the event 3 on the die.

⇒ B = {(H, 3), (T, 3)}

⇒ P (B) = 2/12 = 1/6

As, A ∩ B = {(H, 3)}

⇒ P (A ∩ B) = 1/12 …… (1)

And P (A). P (B) = ½ × 1/6 = 1/12 …… (2)

From (1) and (2) P (A ∩ B) = P (A). P (B)

Therefore, A and B are independent events.

5. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even', and B be the event, 'the number is red'. Are A and B independent?

Solution:

The sample space for the dice will be

S = {1, 2, 3, 4, 5, 6}

Let A be the event, and the number is even.

⇒ A = {2, 4, 6}

⇒ P (A) = 3/6 = ½

Now, let B be the event, and the number is red.

⇒ B = {1, 2, 3}

⇒ P (B) = 3/6 = 1/2

As, A ∩ B = {2}

⇒ P (A ∩ B) = 1/6 …….. (1)

And P (A). P (B) = ½ × ½ = ¼ ….. (2)

From (1) and (2) P (A ∩ B) ≠ P (A). P (B)

Therefore, A and B are not independent events.

6. Let E and F be events with P (E) = 3/5, P (F) = 3/10 and P (E ∩ F) = 1/5. Are E and F independent?

Solution:

Given P (E) = 3/5, P (F) = 3/10 and P (E ∩ F) = 1/5

P (E). P (F) = 3/5 × 3/10 = 9/50 ≠ 1/5

⇒ P (E ∩ F) ≠ P (E). P (F)

Therefore, E and F are not independent events.

7. Given that the events A and B are such that P (A) = 1/2, P (A ∪ B) = 3/5 and P (B) = p. Find p if they are (i) mutually exclusive (ii) independent.

Solution:

Given, P (A) = ½, P (A ∪ B) = 1/5 and P (B) = p

(i) Mutually exclusive

When A and B are mutually exclusive.

Then, (A ∩ B) = ϕ

⇒ P (A ∩ B) = 0

As we know, P (A ∪ B) = P (A) + P (B) – P (A ∩ B)

⇒ 3/5 = ½ + p -0

⇒ P = 3/5 – ½ = 1/10

(ii) Independent

When A and B are independent.

⇒ P (A ∩ B) = P (A). P (B)

⇒ P (A ∩ B) = ½ p

As we know, P (A ∪ B) = P (A) + P (B) – P (A ∩ B)

⇒ 3/5 = ½ + 2 – p/2

⇒ p/2 = 3/5 – ½

⇒ p = 2 × 1/10 = 1/5

8. Let A and B be independent events with P (A) = 0.3 and P (B) = 0.4. Find
(i) P (A ∩ B)

(ii) P (A ∪ B)
(iii) P (A|B)

(iv) P (B|A)

Solution:

Given, P (A) = 0.3 and P(B) = 0.4

(i) P (A ∩ B)

When A and B are independent.

⇒ P (A ∩ B) = P (A). P (B)

⇒ P (A ∩ B) = 0.3 × 0.4

⇒ P (A ∩ B) = 0.12

(ii) P (A ∪ B)

As we know, P (A ∪ B) = P (A) + P (B) – P (A ∩ B)

⇒ P (A ∪ B) = 0.3 + 0.4 – 0.12

⇒ P (A ∪ B) = 0.58

(iii) P (A|B)

As we know $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A|B) = \frac{0.12}{0.4}$

$\Rightarrow$ P (A|B) = 0.3

(iv) P (B|A)

As we know $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$\Rightarrow P(B|A) = \frac{0.12}{0.3}$

$\Rightarrow$ P (B|A) = 0.4

9. If A and B are two events such that P (A) = 1/4 , P (B) = 1/2 and P (A ∩ B) = 1/8, find P (not A and not B).

Solution:

Given P (A) = ¼, P (B) = ½ and P (A ∩ B) = 1/8

P (not A and not B) = P (A' ∩ B')

As, { A' ∩ B' = (A ∪ B)' }

⇒ P (not A and not B) = P ((A ∪ B)')

= 1 – P (A ∪ B)

= 1- [P (A) + P (B) – P (A ∩ B)]

$$= 1 - \left[\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right]$$

$$= 1 - \left[\frac{5}{8}\right] = \frac{3}{8}$$

10. Events A and B are such that P (A) = 1/2, P (B) = 7/12 and P (not A or not B) = 1/4. State whether A and B are independent.

Solution:

Given P (A) = ½, P (B) =7/12 and P (not A or not B) = 1/4

⇒ P $(A' \cup B')$ = 1/4

⇒ P $(A \cap B)'$ = 1/4

⇒ 1 – P $(A \cap B)$ = 1/4

⇒ P $(A \cap B)$ = 1 – 1/4

⇒ P $(A \cap B)$ = 3/4…….. (1)

And P (A). P (B) = ½ × 7/12 = 7/24 …. (2)

From (1) and (2) P $(A \cap B) \neq$ P (A). P (B)

Therefore, A and B are not independent events.

11. Given two independent events A and B, such that P (A) = 0.3, P(B) = 0.6.
Find
(i) P (A and B)

(ii) P (A and not B)
(iii) P (A or B)

(iv) P (neither A nor B)

Solution:

Given, P (A) = 0.3, P (B) = 0.6.

(i) P (A and B)

As A and B are independent events.

⇒ P (A and B) = P $(A \cap B)$ = P (A). P (B)

= 0.3 × 0.6

= 0.18

(ii) P (A and not B)

⇒ P (A and not B) = P $(A \cap B')$ = P (A) – P $(A \cap B)$

= 0.3 – 0.18

= 0.12

(iii) P (A or B)

⇒ P (A or B) = P $(A \cup B)$

As we know, P $(A \cup B)$ = P (A) + P (B) – P $(A \cap B)$

⇒ P (A ∪ B) = 0.3 + 0.6 – 0.18

⇒ P (A ∪ B) = 0.72

(iv) P (neither A nor B)

P (neither A nor B) = P (A' ∩ B')

As, { A' ∩ B' = (A ∪ B)' }

⇒ P (neither A nor B) = P ((A ∪ B)')

= 1 – P (A ∪ B)

= 1 – 0.72

= 0.28

12. A die is tossed thrice. Find the probability of getting an odd number at least once.

Solution:

Given a die is tossed thrice.

Then the sample space S = {1, 2, 3, 4, 5, 6}

Let P (A) = probability of getting an odd number in the first throw.

⇒ P (A) = 3/6 = ½.

Let P (B) = probability of getting an even number

⇒ P (B) =3/6 = ½.

Now, the probability of getting an even number three times = ½ × ½ × ½ = 1/8

So, the probability of getting an odd number at least once

= 1 – the probability of getting an odd number in no throw

= 1 – the probability of getting an even number in three times

= 1 – 1/8

∴ Probability of getting an odd number at least once = 7/8

13. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that
(i) both balls are red.
(ii) the first ball is black, and the second is red.
(iii) one of them is black, and the other is red.

Solution:

Given, A box containing 10 black and 8 red balls.

Total number of balls in box = 18

(i) Both balls are red.

Probability of getting a red ball in the first draw = 8/18 = 4/9

As the ball is replaced after the first throw,

Hence, the probability of getting a red ball in the second draw = 8/18 = 4/9

Now, the probability of getting both balls red = 4/9 × 4/9 = 16/81

(ii) the First ball is black, and the second is red.

Probability of getting a black ball in the first draw = 10/18 = 5/9

As the ball is replaced after the first throw,

Hence, the probability of getting a red ball in the second draw = 8/18 = 4/9

Now, the probability of getting the first ball is black, and the second is red = 5/9 × 5/9 = 20/81

(iii) One of them is black, and the other is red.

The probability of getting a black ball in the first draw = 10/18 = 5/9

As the ball is replaced after the first throw,

Hence, the probability of getting a red ball in the second draw = 8/18 = 4/9

Now, the probability of getting the first ball is black, and the second is red = 5/9 × 4/9 = 20/81

Probability of getting a red ball in the first draw = 8/18 = 4/9

As the ball is replaced after the first throw,

Hence, the probability of getting a black ball in the second draw = 10/18 = 5/9

Now, the probability of getting the first ball is red, and the second is black = 5/9 × 4/9 = 20/81

Therefore, the probability of getting one of them is black, and the other is red:

= Probability of getting the first ball is black and second is red + Probability of getting the first ball is red and second is black

= 20/81 + 20 /81 = 40/81

14. Probability of solving a specific problem independently by A and B are 1/2 and 1/3, respectively. If both try to solve the problem independently, find the probability that
(i) The problem is solved.

(ii) Exactly one of them solves the problem.

Solution:

Given,

P (A) = Probability of solving the problem by A = 1/2

P (B) = Probability of solving the problem by B = 1/3

Because both A and B are independent.

⇒ P (A ∩ B) = P (A). P (B)

⇒ P (A ∩ B) = ½ × 1/3 = 1/6

P (A') = 1 – P (A) = 1 – 1/2 = 1/2

P (B') = 1 – P (B) = 1 – 1/3 = 2/3

(i) The problem is solved.

The problem is solved, i.e., it is either solved by A, or it is solved by B.

= P (A ∪ B)

As we know, P (A ∪ B) = P (A) + P (B) – P (A ∩ B)

⇒ P (A ∪ B) = ½ + 1/3 – 1/6 = 4/6

⇒ P (A ∪ B) = 2/3

(ii) Exactly one of them solves the problem.

That is, either problem is solved by A but not by B or vice versa.

That is P (A).P (B') + P (A').P (B)

= ½ (2/3) + ½ (1/3)

= 1/3 + 1/6 = 3/6

⇒ P (A).P (B') + P (A').P (B) = ½

15. One card is drawn at random from a well-shuffled deck of 52 cards. In which of the following cases are the events E and F independent?
(i) E: 'the card drawn is a spade' F: 'the card drawn is an ace'
(ii) E: 'the card drawn is black' F: 'the card drawn is a king'
(iii) E: 'the card drawn is a king or queen' F: 'the card drawn is a queen or jack'.

Solution:

Given, a deck of 52 cards.

(i) In a deck of 52 cards, 13 cards are spade, 4 cards are ace, and only one card is there, which is spade and ace both.

Hence, P (E) = the card drawn is a spade = 13/52 = 1/4

P (F) = the card drawn is an ace = 4/52 = 1/13

P (E ∩ F) = the card drawn is a spade and ace both = 1/52….. (1)

And P (E). P (F)

= ¼ × 1/13 = 1/52…. (2)

From (1) and (2)

⇒ P (E ∩ F) = P (E). P (F)

Hence, E and F are independent events.

(ii) In a deck of 52 cards, 26 cards are black, 4 cards are king, and only two cards are there, which are black and king both.

Hence, P (E) = the card drawn is of black = 26/52 = ½

P (F) = the card drawn is a king = 4/52 = 1/13

P (E ∩ F) = the card drawn is a black and king both = 2/52 = 1/26…. (1)

And P (E). P (F)

= ½ × 1/13 = 1/26…. (2)

From (1) and (2)

⇒ P (E ∩ F) = P (E). P (F)

Hence, E and F are independent events.

(iii) In a deck of 52 cards, 4 cards are queen, 4 cards are king, and 4 cards are jack.

Hence, P (E) = the card drawn is either king or queen = 8/52 = 2/13

P (F) = the card drawn is either queen or jack = 8/52 = 2/13

There are 4 cards which are either king or queen and either queen or jack.

P (E ∩ F) = the card drawn is either king or queen, and either queen or jack = 4/52 = 1/13 … (1)

And P (E). P (F)

= 2/13 × 2/13 = 4/169…. (2)

From (1) and (2)

⇒ P (E ∩ F) ≠ P (E). P (F)

Hence, E and F are not independent events.

16. In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.
(a) Find the probability that she reads neither Hindi nor English newspapers.
(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.
(c) If she reads English newspaper, find the probability that she reads Hindi newspaper

Solution:

Given:

Let H and E denote the number of students who read Hindi and English newspapers, respectively.

Hence, P (H) = Probability of students who read Hindi newspaper= 60/100 = 3/5

P (E) = Probability of students who read English newspaper = 40/100 = 2/5

P (H ∩ E) = Probability of students who read Hindi and English both newspaper = 20/100 = 1/5

(a) Find the probability that she reads neither Hindi nor English newspapers.

P (neither H nor E)

P (neither H nor E) = P (H' ∩ E')

As, { H' ∩ E' = (H ∪ E)'}

⇒ P (neither A nor B) = P ((H ∪ E)')

= 1 – P (H ∪ E)

= 1- [P (H) + P (E) – P (H ∩ E)]

$$= 1 - \left[\frac{3}{5} + \frac{2}{5} - \frac{1}{5}\right]$$

$$= 1 - \left[\frac{4}{5}\right] = \frac{1}{5}$$

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

P (E|H) = Hindi newspaper reading has already occurred and the probability that she reads English newspaper is to find.

As we know $P(E|H) = \frac{P(H \cap E)}{P(H)}$

$\Rightarrow P(E|H) = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{5} \times \frac{5}{3}$

⇒ P (E|H) = 1/3

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

P (H|E) = English newspaper reading has already occurred and the probability that she reads Hindi newspaper is to find.

As we know $P(H|E) = \frac{P(H \cap E)}{P(E)}$

$\Rightarrow P(H|E) = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{5} \times \frac{5}{2}$

⇒ P (H|E) = ½

17. The probability of obtaining an even prime number on each die when a pair of dice is rolled is
A. 0
B. 1/3
C. 1/12
D. 1/36

Solution:

D. 1/36

Explanation:

Given, A pair of dice is rolled.

Hence, the number of outcomes = 36

Let P (E) be the probability of getting an even prime number on each die.

As we know, the only even prime number is 2.

So, E = {2, 2}

⇒ P € = 1/36

18. Two events, A and B, will be independent, if
(A) A and B are mutually exclusive
(B) P (A′B′) = [1 – P (A)] [1 – P (B)] (C) P (A) = P (B)
(D) P (A) + P (B) = 1

Solution:

(B) P (A′B′) = [1 – P (A)] [1 – P (B)]

Explanation:

P (A′B′) = [1 – P (A)] [1 – P (B)]

⇒ P (A′∩ B′) = 1 – P (A) – P (B) + P (A) P (B)

⇒ 1 – P (A ∪ B) =1 – P (A) – P (B) + P (A) P (B)

= – [P (A) + P (B) – P (A ∩ B)] = – P (A) – P (B) + P (A) P (B)

= – P (A) – P (B) + P (A ∩ B) = – P (A) – P (B) + P (A) P (B)

⇒ P (A ∩ B) = P (A). P (B)

Hence, it shows A and B are Independent events.

EXERCISE 13.3 PAGE NO: 555

1. An urn contains 5 red and 5 black balls. A ball is drawn at random; its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn, and then a ball is drawn at random. What is the probability that the second ball is red?

Solution:

Given, the urn contains 5 red and 5 black balls.

Let in the first attempt, the ball drawn is of red colour.

⇒ P (probability of drawing a red ball) = 5/10 = ½

Now, the two balls of the same colour (red) are added to the urn then the urn contains 7 red and 5 black balls.

⇒ P (probability of drawing a red ball) = 7/12

Now, let in the first attempt, the ball drawn is of black colour.

⇒ P (probability of drawing a black ball) = 5/10 = ½

Now, the two balls of the same colour (black) are added to the urn then the urn contains 5 red and 7 black balls.

⇒ P (probability of drawing a red ball) = 5/12

Therefore, the probability of drawing the second ball as of red colour is

$$= \left(\frac{1}{2} \times \frac{7}{12}\right) + \left(\frac{1}{2} \times \frac{5}{12}\right) = \frac{1}{2}\left(\frac{7}{12} + \frac{5}{12}\right) = \frac{1}{2} \times 1 = \frac{1}{2}$$

2. A bag contains 4 red and 4 black balls, and another bag contains 2 red and 6 black balls. One of the two bags is selected at random, and a ball is drawn from the bag, which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution:

Let E_1 be the event of choosing bag I, E_2 be the event of choosing the bag, say bag II and A be the event of drawing a red ball.

Then, P (E_1) = P (E_2) = 1/2

Also, P (A|E_1) = P (drawing a red ball from bag I) = 4/8 = ½

And P (A|E_2) = P (drawing a red ball from bag II) = 2/8 = ¼

Now, the probability of drawing a ball from bag I, being given that it is red, is P (E_1|A).

By using Bayes' theorem, we have

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4}}$$

$$= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{8}}$$

$$= \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}$$

$$\Rightarrow P(E_1|A) = \frac{2}{3}$$

3. Of the students in a college, it is known that 60% reside in the hostel, and 40% are day scholars (not residing in the hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college, and he has an A grade; what is the probability that the student is a hostlier?

Solution:

Let E_1 be the event that student is a hostler, E_2 be the event that student is a day scholar and A be the event of getting A grade.

Then $P(E_1) = 60\% = \frac{60}{100} = 0.6$

And $P(E_2) = 40\% = \frac{40}{100} = 0.4$

Also P $(A|E_1)$ = P (students who attain A grade reside in hostel) = 30% = 0.3

And P $(A|E_2)$ = P (students who attain A grade is day scholar) = 20% = 0.2

Now the probability of students who reside in hostel, being given he attain A grade, is P $(E_1|A)$.

By using Bayes' theorem, we have:

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

By using Bayes' theorem, we have:

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Substituting the values we get

$$= \frac{0.6 \times 0.3}{0.6 \times 0.3 + 0.4 \times 0.2}$$

$$= \frac{0.18}{0.18 + 0.08}$$

$$= \frac{0.18}{0.26} = \frac{18}{26} = \frac{9}{13}$$

$$\Rightarrow P(E_1|A) = \frac{9}{13}$$

4. In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let 3/4 be the probability that he knows the answer and 1/4 be the probability that he guesses. Assuming that a student who guesses the answer will be correct with probability 1/4. What is the probability that the student knows the answer, given that he answered it correctly?

Solution:

Let E_1 be the event that the student knows the answer, E_2 be the event that the student guesses the answer, and A be the event that the answer is correct.

Then, P (E_1) = ¾

And P (E_2) = ¼

Also, P $(A|E_1)$ = P (correct answer given that he knows) = 1

And P $(A|E_2)$ = P (correct answer given that he guesses) = ¼

Now, the probability that he knows the answer, being given that answer is correct, is P $(E_1|A)$.

By using Bayes' theorem, we have

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Substituting the values we get

$$= \frac{\frac{3}{4}.1}{\frac{3}{4}.1 + \frac{1}{4}.\frac{1}{4}}$$

$$= \frac{\frac{3}{4}}{\frac{3}{4} + \frac{1}{16}}$$

$$= \frac{\frac{3}{4}}{\frac{13}{16}} = \frac{12}{13}$$

$$\Rightarrow P(E_1|A) = \frac{12}{13}$$

5. A laboratory blood test is 99% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e., if a healthy person is tested, then, with a probability 0.005, the test will imply he has the disease). If 0.1 per cent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?

Solution:

Let E_1 be the event that person has a disease, E_2 be the event that the person does not have a disease, and A be the event that the blood test is positive.

As E_1 and E_2 are the events which are complimentary to each other.

Then, P (E_1) + P (E_2) = 1

⇒ P (E_2) = 1 – P (E_1)

Then, P (E_1) = 0.1% = 0.1/100 = 0.001 and P (E_2) = 1 – 0.001 = 0.999

Also, P (A|E_1) = P (result is positive given that person has disease) = 99% = 0.99

And P (A|E_2) = P (result is positive given that person has no disease) = 0.5% = 0.005

Now, the probability that a person has a disease, given that his test result is positive, is P (E_1|A).

By using Bayes' theorem, we have

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Substituting the values we get

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005}$$

$$= \frac{0.00099}{0.00099 + 0.004995}$$

$$= \frac{0.00099}{0.005985} = \frac{990}{5985} = \frac{110}{665}$$

$$\Rightarrow P(E_1|A) = \frac{22}{133}$$

Substituting the values we get

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005}$$

$$= \frac{0.00099}{0.00099 + 0.004995}$$

$$= \frac{0.00099}{0.005985} = \frac{990}{5985} = \frac{110}{665}$$

$$\Rightarrow P(E_1|A) = \frac{22}{133}$$

6. There are three coins. One is a two-headed coin (having heads on both faces), another is a biased coin that comes up heads 75% of the time, and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads; what is the probability that it was the two-headed coin?

Solution:

Let E_1 be the event of choosing a two-headed coin, E_2 be the event of choosing a biased coin, and E_3 be the event of choosing an unbiased coin. Let A be the event that the coin shows the head.

Then, P (E_1) = P (E_2) = P (E_3) = 1/3

As a coin has both sides, so it will show the head.

Also P (A|E_1) = P (correct answer given that he knows) = 1

And P (A|E_2) = P (coin shows head given that the coin is biased) = 75% = 75/100 = ¾

And P (A|E_3) = P (coin shows head given that the coin is unbiased) = ½

Now, the probability that the coin is two-headed, given that it shows a head, is P (E_1|A).

By using Bayes' theorem, we have

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + P(E_3).P(A|E_3)}$$

By substituting the values we get

$$= \frac{\frac{1}{3}.1}{\frac{1}{3}.1 + \frac{1}{3}.\frac{3}{4} + \frac{1}{3}.\frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}\left(1 + \frac{3}{4} + \frac{1}{2}\right)}$$

$$= \frac{1}{\frac{9}{4}} = \frac{4}{9}$$

$$\Rightarrow P(E_1|A) = \frac{4}{9}$$

7. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of accidents are 0.01, 0.03 and 0.15, respectively. One of the insured persons met with an accident. What is the probability that he is a scooter driver?

Solution:

Let E_1 be the event that the driver is a scooter driver, E_2 be the event that the driver is a car driver and E_3 be the event that the driver is a truck driver. Let A be the event that the person meets with an accident.

Total number of drivers = 2000 + 4000 + 6000 = 12000

Then P (E_1) = 2000/12000 = 1/6

P (E_2) = 4000/12000 = 1/3

P (E_3) = 6000/12000 = ½

As a coin has both sides, so it will show the head.

Also, P $(A|E_1)$ = P (accident of a scooter driver) = 0.01 = 1/100

And P $(A|E_2)$ = P (accident of a car driver) = 0.03 = 3/100

And P $(A|E_3)$ = P (accident of a truck driver) = 0.15 = 15/100 = 3/20

Now, the probability that the driver is a scooter driver, being given that he met with an accident, is P $(E_1|A)$.

By using Bayes' theorem, we have

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + P(E_3).P(A|E_3)}$$

Now by substituting the values we get

$$= \frac{\frac{1}{6}\cdot\frac{1}{100}}{\frac{1}{6}\cdot\frac{1}{100} + \frac{1}{3}\cdot\frac{3}{100} + \frac{1}{2}\cdot\frac{3}{20}}$$

$$= \frac{\frac{1}{600}}{\frac{1}{20}\left(\frac{1}{30} + \frac{1}{5} + \frac{3}{2}\right)}$$

$$= \frac{\frac{1}{30}}{\frac{52}{30}} = \frac{1}{52}$$

$$\Rightarrow P(E_1|A) = \frac{1}{52}$$

8. A factory has two machines, A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile, and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

Solution:

Let E_1 be the event that the item is produced by A, E_2 be the event that the item is produced by B, and X be the event that produced product is found to be defective.

Then, P (E_1) = 60% = 60/100 = 3/5

P (E_1) = 40% = 40/100 = 2/5

Also, P $(X|E_1)$ = P (item is defective given that it is produced by machine A) = 2% = 2/100 = 1/50

And P $(X|E_2)$ = P (item is defective given that it is produced by machine B) = 1% = 1/100

Now, the probability that an item is produced by B, being given that the item is defective, is P $(E_2|A)$.

By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2).P(X|E_2)}{P(E_1).P(X|E_1) + P(E_2).P(X|E_2)}$$

By substituting the values we get

$$= \frac{\frac{2}{5} \times \frac{1}{100}}{\frac{3}{5} \times \frac{2}{100} + \frac{2}{5} \times \frac{1}{100}}$$

$$= \frac{\frac{2}{5} \times \frac{1}{100}}{\frac{1}{500}(6+2)} = \frac{2}{8}$$

$$\Rightarrow P(E_2|A) = \frac{1}{4}$$

9. Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4, respectively. Further, if the first group wins, the probability of introducing a new product is 0.7, and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

Solution:

Let E_1 be the event that the first group wins the competition, E_2 be the event that the second group wins the competition and A be the event of introducing a new product.

Then, P (E_1) = 0.6 and P (E_2) = 0.4

Also, P ($A|E_1$) = P (introducing a new product given that the first group wins) = 0.7

And P ($A|E_2$) = P (introducing a new product given that the second group wins) = 0.3

Now, the probability of that new product introduced by the second group, being given that a new product was introduced, is P ($E_2|A$).

By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2).P(A|E_2)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Now by substituting the values we get

$$= \frac{0.4 \times 0.3}{0.6 \times 0.7 + 0.4 \times 0.3}$$

$$= \frac{0.12}{0.42 + 0.12}$$

$$= \frac{0.12}{0.54} = \frac{12}{54} = \frac{2}{9}$$

$$\Rightarrow P(E_2|A) = \frac{2}{9}$$

P ($E_2|A$) = 2/9

10. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

Solution:

let E_1 be the event that the outcome on the die is 5 or 6, E_2 be the event that the outcome on the die is 1, 2, 3 or 4 and A be the event getting exactly head.

Then, P (E_1) = 2/6 = 1/3

P (E_2) = 4/6 = 2/3

As in throwing a coin three times, we get 8 possibilities.

{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

⇒ P ($A|E_1$) = P (Obtaining exactly one head by tossing the coin three times if she gets 5 or 6) = 3/8

And P $(A|E_2)$ = P (Obtaining exactly one head by tossing the coin three times if she gets 1, 2, 3 or 4) = ½

Now, the probability that the girl threw 1, 2, 3 or 4 with a die, being given that she obtained exactly one head, is P $(E_2|A)$.

By using Bayes' theorem, we have

$$P(E_2|A) = \frac{P(E_2).P(A|E_2)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Now by substituting the values we get

$$= \frac{\frac{2}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{3}{8} + \frac{2}{3}\cdot\frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{8} + \frac{1}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{3+8}{24}} = \frac{8}{11}$$

$$\Rightarrow P(E_2|A) = \frac{8}{11}$$

P $(E_2|A)$ = 8/11

11. A manufacturer has three machine operators, A, B and C. The first operator, A produces 1% defective items, whereas the other two operators, B and C, produce 5% and 7% defective items, respectively. A is on the job for 50% of the time, B is on the job for 30% of the time, and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

Solution:

Let E_1 be the event of time consumed by machine A, E_2 be the event of time consumed by machine B and E_3 be the event of time consumed by machine C. Let X be the event of producing defective items.

Then, P (E_1) = 50% = 50/100 = ½

P (E_2) = 30% = 30/100 = 3/10

P (E_3) = 20% = 20/100 = 1/5

As a coin has both sides, it will show head.

Also, P $(X|E_1)$ = P (defective item produced by A) = 1% = 1/100

And P $(X|E_2)$ = P (defective item produced by B) = 5% = 5/100

And P $(X|E_3)$ = P (defective item produced by C) = 7% = 7/100

Now, the probability that an item produced by machine A, being given that a defective item is produced, is P $(E_1|A)$.

By using Bayes' theorem, we have

$$P(E_1|X) = \frac{P(E_1).P(X|E_1)}{P(E_1).P(X|E_1) + P(E_2).P(X|E_2) + P(E_3).P(X|E_3)}$$

Now by substituting the values we get

$$= \frac{\frac{1}{2}\cdot\frac{1}{100}}{\frac{1}{2}\cdot\frac{1}{100} + \frac{3}{10}\cdot\frac{5}{100} + \frac{1}{5}\cdot\frac{7}{100}}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{100}}{\frac{1}{100}\left(\frac{1}{2} + \frac{3}{2} + \frac{7}{5}\right)}$$

$$= \frac{\frac{1}{2}}{\frac{17}{5}} = \frac{5}{34}$$

$$\Rightarrow P(E_1|X) = \frac{5}{34}$$

P $(E_1|A)$ = 5/34

12. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

Solution:

Let E1 be the event that the drawn card is a diamond, E2 be the event that the drawn card is not a diamond and A be the event that the card is lost.

As we know, out of 52 cards, 13 cards are diamond, and 39 cards are not diamond.

Then P (E_1) = 13/52 and P (E_2) = 39/52

Now, when a diamond card is lost, then there are 12 diamond cards out of a total of 51 cards.

Two diamond cards can be drawn out of 12 diamond cards in ${}^{12}C_2$ ways.

Similarly, two diamond cards can be drawn out of a total of 51 cards in ${}^{51}C_2$ ways.

Then the probability of getting two cards when one diamond card is lost is P $(A|E_1)$.

Also, P $(A|E_1) = {}^{12}C_2 / {}^{51}C_2$

Also P $(A|E_1) = {}^{12}C_2 / {}^{51}C_2$

$$= \frac{12!}{2! \times 10!} \times \frac{2! \times 49!}{51!}$$

$$= \frac{12 \times 11 \times 10!}{2 \times 1 \times 10!} \times \frac{2 \times 1 \times 49!}{51 \times 50 \times 49!}$$

$$= \frac{12 \times 11}{51 \times 50} = \frac{22}{425}$$

Now, when not a diamond card is lost, then there are 13 diamond cards out of a total of 51 cards.

Two diamond cards can be drawn out of 13 diamond cards in ${}^{13}C_2$ ways.

Similarly, two diamond cards can be drawn out of a total of 51 cards in ${}^{51}C_2$ ways.

Then, the probability of getting two cards when the card is lost, which is not a diamond is P $(A|E_2)$.

Also, P $(A|E_2) = {}^{13}C_2 / {}^{51}C_2$

$$= \frac{13!}{2! \times 11!} \times \frac{2! \times 49!}{51!}$$

$$= \frac{13 \times 12 \times 11!}{2 \times 1 \times 10!} \times \frac{2 \times 1 \times 49!}{51 \times 50 \times 49!}$$

$$= \frac{13 \times 12}{51 \times 50} = \frac{26}{425}$$

Now the probability that the lost card is diamond, being given that the card is lost, is P $(E_1|A)$.

By using Bayes' theorem, we have:

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

Now by substituting the values we get

$$= \frac{\frac{1}{4}\cdot\frac{22}{425}}{\frac{1}{4}\cdot\frac{22}{425} + \frac{3}{4}\cdot\frac{26}{425}}$$

$$= \frac{\frac{1}{425}\cdot\frac{22}{4}}{\frac{1}{425}\left(\frac{22}{4} + \frac{26\times 3}{4}\right)}$$

$$= \frac{\frac{11}{2}}{\frac{100}{4}} = \frac{11}{50}$$

$$\Rightarrow P(E_1|A) = \frac{11}{50}$$

13. Probability that A speaks the truth is 4/5. A coin is tossed. A reports that a head appears. The probability that actually there was a head is
A. 4/5
B. 1/2
C. 1/5
D. 2/5

Solution:

A. 4/5

Explanation:

Let E_1 be the event that A speaks the truth, E_2 be the event that A lies and X be the event that it appears head.

Therefore, P (E_1) = 4/5

As E_1 and E_2 are the events which are complimentary to each other.

Then, P (E_1) + P (E_2) = 1

$\Rightarrow$ P (E_2) = 1 – P (E_1)

$\Rightarrow$ P (E_2) = 1 – 4/5 = 1/5

If a coin is tossed, it may show a head or tail.

Hence, the probability of getting head is 1/2 whether A speaks the truth or A lies.

P ($X|E_1$) = P ($X|E_2$) = ½

Now, the probability that actually there was a head given that A speaks truth, is P ($E_1|X$).

By using Bayes' theorem, we have

$$P(E_1|X) = \frac{P(E_1).P(X|E_1)}{P(E_1).P(X|E_1) + P(E_2).P(X|E_2)}$$

Now substituting the values we get

$$= \frac{\frac{4}{5}\cdot\frac{1}{2}}{\frac{4}{5}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{2}}$$

$$= \frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{10}} = \frac{\frac{2}{5}}{\frac{5}{10}} = \frac{4}{5}$$

$$\Rightarrow P(E_1|X) = \frac{4}{5}$$

Therefore correct answer is (A).

14. If A and B are two events such that A ⊂ B and P(B) ≠ 0, then which of the following is correct?
A. P (A| B) = P (B)/ P (A)
B. P(A|B) < P(A)
C. P(A|B) ≥ P(A)
D. None of these

Solution:

C. P (A|B) ≥ P (A)

Explanation:

A and B are two events such that A ⊂ B and P (B) ≠ 0

As A ⊂ B ⇒ A ∩ B = A

⇒ P (A ∩ B) = P (A)

As A ⊂ B ⇒ P (A) < P (B)

As we know $P\ (A|B) = \frac{P\ (A \cap B)}{P(B)} = \frac{P(A)}{P(B)} \neq \frac{P(B)}{P(A)}\ ..(1)$

Consider

$$P\ (A|B) = \frac{P\ (A\ \cap\ B)}{P(B)} = \frac{P(A)}{P(B)}\ ..(2)$$

It is also known that P (B) ≤ 1

$$\Rightarrow \frac{1}{P(B)} \geq 1$$

$$\Rightarrow \frac{P(A)}{P(B)} \geq P(A)$$

⇒ P (A|B) ≥ P (A) … (3)

Hence, the correct answer is (C).

EXERCISE 13.4 PAGE NO: 569

1. State which of the following are not the probability distributions of a random variable. Give reasons for your answer.

(i)

X	0	1	2
P(X)	0.4	0.4	0.2

Solution:

Here, we have a table with values for X and P(X).

As we know, the sum of all the probabilities in a probability distribution of a random variable must be one.

$$\text{i.e.} \sum_{i=1}^{n} p_i = 1, \text{where } p_i > 1 \text{ and } i = 0,1,2,\dots,n$$

Hence, the sum of probabilities of the given table = 0.4 + 0.4 + 0.2

= 1

Hence, the given table is the probability distribution of a random variable.

(ii)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	– 0.1	0.3

Solution:

Here, we have a table with values for X and P(X).

As we see from the table that P(X) = -0.1 for X = 3.

It is known that the probability of any observation must always be positive so that it can’t be negative.

Hence, the given table is not the probability distribution of a random variable.

(iii)

Y	– 1	0	1
P(Y)	0.6	0.1	0.2

Solution:

Here, we have a table with values for X and P(X).

As we know, the sum of all the probabilities in a probability distribution of a random variable must be one.

$$\text{i.e.} \sum_{i=1}^{n} p_i = 1, \text{where } p_i > 1 \text{ and } i = 0,1,2,\ldots,n$$

Hence, the sum of probabilities of the given table = 0.6 + 0.1 + 0.2

= 0.9 ≠ 1

Hence, the given table is not the probability distribution of a random variable.

(iv)

Z	3	2	1	0	–1
P(Z)	0.3	0.2	0.4	0.1	0.05

Solution:

Here, we have a table with values for X and P(X).

As we know, the sum of all the probabilities in a probability distribution of a random variable must be one.

$$\text{i.e.} \sum_{i=1}^{n} p_i = 1, \text{where } p_i > 1 \text{ and } i = 0,1,2,\ldots,n$$

Hence, the sum of probabilities of given table = 0.3 + 0.2 + 0.4 + 0.1 + 0.05

= 1.05 ≠ 1

Hence, the given table is not the probability distribution of a random variable.

2. An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represent the number of black balls. What are the possible values of X? Is X a random variable?

Solution:

Given, the urn contains 5 red and 2 black balls.

Let R represent the red ball and B represent the black ball.

Two balls are drawn randomly.

Hence, the sample space of the experiment is S = {BB, BR, RB, RR}

X represents the number of black balls.

⇒ X (BB) = 2

X (BR) = 1

X (RB) = 1

X (RR) = 0

Therefore, X is a function on sample space whose range is {0, 1, 2}.

Thus, X is a random variable which can take the values 0, 1 or 2.

3. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are the possible values of X?

Solution:

Given, a coin is tossed 6 times.

X represents the difference between the number of heads and the number of tails.

⇒ X (6H, 0T) = |6-0| = 6

X (5H, 1T) = |5-1| = 4

X (4H, 2T) = |4-2| = 2

X (3H, 3T) = |3-3| = 0

X (2H, 4T) = |2-4| = 2

X (1H, 5T) = |1-5| = 4

X (0H, 6T) = |0-6| = 6

Therefore, X is a function on sample space whose range is {0, 2, 4, 6}.

Thus, X is a random variable which can take the values 0, 2, 4 or 6.

4. Find the probability distribution of
(i) number of heads in two tosses of a coin.

Solution:

Given, a coin is tossed twice.

Hence, the sample space of the experiment is S = {HH, HT, TH, TT}

X represents the number of heads.

⇒ X (HH) = 2

X (HT) = 1

X (TH) = 1

X (TT) = 0

Therefore, X is a function on sample space whose range is {0, 1, 2}.

Thus, X is a random variable which can take the values 0, 1 or 2.

As we know,

P (HH) = P (HT) = P (TH) = P (TT) = 1/4

P (X = 0) = P (TT) = 1/4

P (X = 1) = P (HT) + P (TH) = 1/4 + 1/4 = 1/2

P (X = 2) = P (HH) = 1/4

Hence, the required probability distribution is

X	0	1	2
P (X)	1/4	1/2	1/4

(ii) Number of tails in the simultaneous tosses of three coins.

Solution:

Given three coins are tossed simultaneously. Hence, the sample space of the experiment is S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

X represents the number of tails.

As we see, X is a function on sample space whose range is {0, 1, 2, 3}.

Thus, X is a random variable which can take the values 0, 1, 2 or 3.

P (X = 0) = P (HHH) = 1/8

P (X = 1) = P (HHT) + P (HTH) + P (THH) = 1/8 + 1/8 + 1/8 = 3/8

P (X = 2) = P (HTT) + P (THT) + P (TTH) = 1/8 + 1/8 + 1/8 = 3/8

P (X = 3) = P (TTT) = 1/8

Hence, the required probability distribution is

X	0	1	2	3

P (X)	1/8	3/8	3/8	1/8

(iii) Number of heads in four tosses of a coin.

Solution:

Given, four tosses of a coin.

Hence, the sample space of the experiment is

S = {HHHH, HHHT, HHTH, HTHH, HTTH, HTHT, HHTT, HTTT, THHH, TTHH, THTH, THHT, THTT, TTHT, TTTH, TTTT}

X represents the number of heads.

As we see, X is a function on sample space whose range is {0, 1, 2, 3, 4}.

Thus, X is a random variable which can take the values 0, 1, 2, 3 or 4.

P (X = 0) = P (TTTT) = 1/16

P (X = 1) = P (HTTT) + P (TTTH) + P (THTT) + P (TTHT) = 1/16 + 1/16 + 1/16 + 1/16 = ¼

P(X = 2) = P (HHTT) + P (THHT) + P (TTHH) + P (THTH) + P (HTHT) + P(HTTH)= 1 /16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 = 6/16 = 3/8

P(X = 3) = P (THHH) + P (HHHT) + P (HTHH) + P (HHTH) = 1/16 + 1/16 + 1/16 + 1/16 = ¼

P(X = 4) = P (HHHH) = 1/16

Hence, the required probability distribution is

X	0	1	2	3	4
P (X)	1/16	1/4	3/8	1/4	1/16

5. Find the probability distribution of the number of successes in two tosses of a die, where success is defined as
(i) number greater than 4
(ii) six appears on at least one die

Solution:

Given, a die is tossed two times.

When a die is tossed two times, then the number of observations will be (6 × 6) = 36.

Now, let X is a random variable which represents success.

(i) Here, success is given as a number greater than 4.

Now

P (X = 0) = P (number ≤ 4 in both tosses) = 4/6 × 4/6 = 4/9

P (X = 1) = P (number ≤ 4 in first toss and number ≥ 4 in second case) + P (number ≥ 4 in first toss and number ≤ 4 in second case) is

= (4/6 × 2/6) + (2/6 × 4/6) = 4/9

P (X = 2) = P (number ≥ 4 in both tosses) = 2/6 × 2/6 = 1/9

Hence, the required probability distribution is

X	0	1	2
P (X)	4/9	4/9	1/9

(ii) Here, success is given as six appears on at least one die.

Now, P (X = 0) = P (six does not appear on any of die) = 5/6 × 5/6 = 25/36

P (X = 1) = P (six appears at least once of the die) = (1/6 × 5/6) + (5/6 × 1/6) = 10/36 = 5/18

Hence, the required probability distribution is

X	0	1
P (X)	25/36	5/18

6. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Solution:

Given, a lot of 30 bulbs which include 6 defectives.

Then, the number of non-defective bulbs = 30 – 6 = 24

As 4 bulbs are drawn at random with replacement.

Let X denotes the number of defective bulbs from the selected bulbs.

Clearly, X can take the value of 0, 1, 2, 3 or 4.

P (X = 0) = P (4 are non-defective and 0 defective)

$$= {}^4C_0 \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} = \frac{256}{625}$$

P (X = 1) = P (3 are non-defective and 1 defective)

$$= 4_{C_1}.\frac{1}{5}.\left(\frac{4}{5}\right)^3 = \frac{256}{625}$$

P (X = 2) = P (2 are non-defective and 2 defective)

$$= 4_{C_2}.\left(\frac{1}{5}\right)^2.\left(\frac{1}{5}\right)^2 = \frac{96}{625}$$

P (X = 3) = P (1 are non-defective and 3 defective)

$$= 4_{C_3}.\left(\frac{1}{5}\right)^3.\frac{4}{5} = \frac{16}{625}$$

P (X = 4) = P (0 are non-defective and 4 defective)

$$= 4_{C_4}.\left(\frac{1}{5}\right)^4 = \frac{1}{625}$$

Hence, the required probability distribution is

X	0	1	2	3	4
P (X)	256/625	256/625	96/625	16/625	1/625

7. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of the number of tails.

Solution:

Given, head is 3 times as likely to occur as tail.

Now, let the probability of getting a tail in the biased coin be x.

⇒ P (T) = x

And P (H) = 3x

For a biased coin, P (T) + P (H) = 1

⇒ x + 3x = 1

⇒ 4x = 1

⇒ x = 1/4

Hence, P (T) = 1/4 and P (H) = 3/4

As the coin is tossed twice, the sample space is {HH, HT, TH, TT}

Let X be a random variable representing the number of tails.

Clearly, X can take the value of 0, 1 or 2.

P(X = 0) = P (no tail) = P (H) × P (H) = ¾ × ¾ = 9/16

P(X = 1) = P (one tail) = P (HT) × P (TH) = ¾. ¼ × ¼. ¾ = 3/8

P(X = 2) = P (two tail) = P (T) × P (T) = ¼ × ¼ = 1/16

Hence, the required probability distribution is

X	0	1	2
P (x)	9/16	3/8	1/16

8. A random variable X has the following probability distribution.

X	0	1	2	3	4	5	6	7
P(X)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Determine

(i) k

(ii) P (X < 3)

(iii) P (X > 6)

(iv) P (0 < X < 3)

Solution:

Given, a random variable X with its probability distribution.

(i) As we know, the sum of all the probabilities in a probability distribution of a random variable must be one.

$$\text{i.e.} \sum_{i=1}^{n} p_i = 1, \text{where } p_i > 1 \text{ and } i = 0,1,2,\ldots,n$$

Hence, the sum of probabilities of the given table

⇒ $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7K^2 + k = 1$

⇒ $10K^2 + 9k = 1$

⇒ $10K^2 + 9k - 1 = 0$

⇒ (10K-1) (k + 1) = 0

k = -1, 1/10

It is known that the probability of any observation must always be positive so that it can't be negative.

So k = 1/10

(ii) Now, we have to find P(X < 3)

P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)

= 0 + k + 2k

= 3k

P (X < 3) = 3 × 1/10 = 3/10

(iii) Now, we have to find P(X > 6)

P(X > 6) = P(X = 7)

= $7K^2$ + k

= 7 × $(1/10)^2$ + 1/10

= 7/100 + 1/10

P (X > 6) = 17/100

(iv) Consider P (0 < X < 3)

P (0 < X < 3) = P(X = 1) + P(X = 2)

= k + 2k

= 3k

P (0 < X < 3) = 3 × 1/10 = 3/10

9. The random variable X has a probability distribution P(X) of the following form, where k is some number:

$$P(X) = \begin{cases} k, & \text{If } x = 0 \\ 2k, & \text{If } x = 1 \\ 3k, & \text{If } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the value of k.
(b) Find P (X < 2), P (X ≤ 2), P(X ≥ 2).

Solution:

Given: A random variable X with its probability distribution.

(a) As we know, the sum of all the probabilities in a probability distribution of a random variable must be one.

$$\text{i.e.} \sum_{i=1}^{n} p_i = 1, \text{where } p_i > 1 \text{ and } i = 0,1,2,\ldots,n$$

Hence, the sum of probabilities of the given table

⇒ k + 2k + 3k + 0 = 1

⇒ 6k = 1

k = 1/6

(b) Now, we have to find P(X < 2)

P (X < 2) = P (X = 0) + P (X = 1)

= k + 2k

= 3k

P (X < 2) = 3 × 1/6 = ½

Consider P (X ≤ 2)

P (X ≤ 2) = P (X = 0) + P (X = 1) + P (X = 2)

= k + 2k + 3k

= 6k

P (X ≤ 2) = 6 × 1/6 = 1

Now, we have to find P(X ≥ 2)

P(X ≥ 2) = P(X = 2) + P(X > 2)

= 3k + 0

= 3k

P (X ≥ 2) = 3 × 1/6 = ½

10. Find the mean number of heads in three tosses of a fair coin.

Solution:

Given, a coin is tossed three times.

Three coins are tossed simultaneously. Hence, the sample space of the experiment is S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

X represents the number of heads.

As we see, X is a function on sample space whose range is {0, 1, 2, 3}.

Thus, X is a random variable which can take the values 0, 1, 2 or 3.

P (X = 0) = P (TTT) = 1/8

P (X = 1) = P (TTH) + P (THT) + P (HTT) = 1/8 +1/8 + 1/8 = 3/8

P (X = 2) = P (THH) + P (HTH) + P (HHT) = 1/8 + 1/8 + 1/8 = 3/8

P (X = 3) = P (HHH) = 1/8

Hence, the required probability distribution is

X	0	1	2	3
P (X)	1/8	3/8	3/8	1/8

Therefore, mean μ is

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i$$

Now by substituting the values we get

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= \frac{3}{8} + \frac{3}{4} + \frac{3}{8} = \frac{3 + 6 + 3}{8} = \frac{12}{8}$$

$$\Rightarrow \mu = E(X) = \frac{3}{2} = 1.5$$

11. Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

Solution:

Given, a die is thrown two times.

When a die is tossed two times, then the number of observations will be (6 × 6) = 36

Now, let X is a random variable which represents the success and is given as six appears on at least one die.

Now,

P (X = 0) = P (six does not appear on any of die) = 5/6 × 5/6 = 25/36

P (X = 1) = P (six appears at least once of the die) = (1/6 × 5/6) + (5/6 × 1/6) = 10/36 = 5/18

P (X = 2) = P (six does appear on both of die) = 1/6 × 1/6 = 1/36

Hence, the required probability distribution is

X	0	1	2
P (X)	25/36	5/18	1/36

Therefore, Expectation of X E (X)

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

Now by substituting the values we get

$$= 0 \times \frac{25}{36} + 1 \times \frac{5}{18} + 2 \times \frac{1}{36}$$

$$= \frac{5}{18} + \frac{1}{18} = \frac{6}{18} = \frac{1}{3}$$

$$\Rightarrow E(X) = \frac{1}{3}$$

12. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E(X).

Solution:

Given, the first six positive integers.

Two numbers can be selected at random (without replacement) from the first six positive integers in 6 × 5 = 30 ways.

X denote the larger of the two numbers obtained. Hence, X can take any value of 2, 3, 4, 5 or 6.

For X = 2, the possible observations are (1, 2) and (2, 1)

P (X) = 2/30 = 1/15

For X = 3, the possible observations are (1, 3), (3, 1), (2, 3) and (3, 2).

P (X) = 4/30 = 2/15

For X = 4, the possible observations are (1, 4), (4, 1), (2,4), (4,2), (3,4) and (4,3).

P (X) = 6/30 = 1/5

For X = 5, the possible observations are (1, 5), (5, 1), (2,5), (5,2), (3,5), (5,3) (5, 4) and (4,5).

P (X) = 8/30 = 4/15

For X = 6, the possible observations are (1, 6), (6, 1), (2,6), (6,2), (3,6), (6,3) (6, 4), (4,6), (5,6) and (6,5).

P (X) = 10/30 = 1/3

Hence, the required probability distribution is

X	2	3	4	5	6
P (X)	1/15	2/15	1/5	4/15	1/3

Therefore E(X) is

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

Now by substituting the values we get

$$= 2 \times \frac{1}{15} + 3 \times \frac{2}{15} + 4 \times \frac{1}{5} + 5 \times \frac{4}{15} + 6 \times \frac{1}{3}$$

$$= \frac{2}{15} + \frac{6}{15} + \frac{4}{5} + \frac{20}{15} + 2 = \frac{2 + 6 + 12 + 20 + 30}{15} = \frac{70}{15}$$

$$\Rightarrow E(X) = \frac{14}{3}$$

13. Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of X.

Solution:

Given, two fair dice are rolled

When two fair dice are rolled, then the number of observations will be $6 \times 6 = 36$.

X denote the sum of the numbers obtained when two fair dice are rolled. Hence, X can take any value of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

For X = 2, the possible observations are (1, 1).

P (X) = 1/36

For X = 3, the possible observations are (1, 2) and (2, 1).

P (X) = 2/36 = 1/18

For X = 4, the possible observations are (1, 3), (2, 2) and (3, 1).

P (X) = 3/36 = 1/12

For X = 5, the possible observations are (1, 4), (4, 1), (2, 3) and (3, 2).

P (X) = 4/39 = 1/9

For X = 6, the possible observations are (1, 5), (5, 1), (2, 4), (4, 2) and (3, 3).

P (X) = 5/36

For X = 7, the possible observations are (1, 6), (6, 1), (2,5), (5,2),(3,4) and (4,3).

P (X) = 6/36 = 1/6

For X = 8, the possible observations are (2, 6), (6, 2), (3, 5), (5, 3) and (4, 4).

P (X) = 5/36

For X = 9, the possible observations are (5, 4), (4, 5), (3, 6) and (6, 3).

P (X) = 4/36 = 1/9

For X = 10, the possible observations are (5, 5), (4, 6) and (6, 4).

P (X) = 3/36 = 1/12

For X = 11, the possible observations are (6, 5) and (5, 6).

P (X) = 2/36 = 1/18

For X = 12, the possible observations are (6, 6).

P (X) = 1/36

Hence, the required probability distribution is

X	2	3	4	5	6	7	8	9	10	11	12
P (X)	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

Therefore E(X) is

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

Now by substituting the values we get

$$= 2\times\frac{1}{36} + 3\times\frac{1}{18} + 4\times\frac{1}{12} + 5\times\frac{1}{9} + 6\times\frac{5}{36} + 7\times\frac{1}{6} + 8\times\frac{5}{36} + 9\times\frac{1}{9} + 10\times\frac{1}{12} + 11\times\frac{1}{18} + 12\times\frac{1}{36}$$

$$= \frac{1}{18} + \frac{1}{6} + \frac{1}{3} + \frac{5}{9} + \frac{5}{6} + \frac{7}{6} + \frac{10}{9} + 1 + \frac{5}{6} + \frac{11}{18} + \frac{1}{3}$$

$$= \frac{1+3+6+10+15+21+20+18+15+11+6}{18} = \frac{126}{18}$$

$$\Rightarrow E(X) = 7$$

E (X) = 7

And $E(X^2)$ is:

$$E(X^2) = \sum_{i=1}^{n} x_i^2 . p(x_i)$$

$$= 4 \times \frac{1}{36} + 9 \times \frac{1}{18} + 16 \times \frac{1}{12} + 25 \times \frac{1}{9} + 36 \times \frac{5}{36} + 49 \times \frac{1}{6} + 64 \times \frac{5}{36} + 81 \times \frac{1}{9} + 100 \times \frac{1}{12} + 121 \times \frac{1}{18} + 144 \times \frac{1}{36}$$

$$= \frac{1}{9} + \frac{1}{2} + \frac{4}{3} + \frac{25}{9} + 5 + \frac{49}{6} + \frac{80}{9} + 9 + \frac{25}{3} + \frac{121}{18} + 4$$

$$= \frac{2 + 9 + 24 + 50 + 90 + 147 + 160 + 162 + 150 + 121 + 72}{18}$$

$$= \frac{987}{18}$$

$\Rightarrow E(X^2) = 54.833$

Then Variance, Var (X) = $E(X^2) - (E(X))^2$

$= 54.833 - (7)^2$

$= 54.833 - 49 = 5.833$

And Standard deviation $= \sqrt{\text{Var}(X)}$

$= \sqrt{5.833}$

$= 2.415$

14. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen, and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find the mean, variance and standard deviation of X.

Solution:

Given, the class of 15 students, with their ages.

From the given data, we can draw a table.

X	14	15	16	17	18	19	20	21
f	2	1	2	3	1	2	3	1

P(X = 14) = 2/15

P(X = 15) = 1/15

P(X = 16) = 2/15

P(X = 17) = 3/15

P(X = 18) = 1/15

P(X = 19) = 2/15

P(X = 20) = 3/15

P(X = 21) = 1/15

Hence, the required probability distribution is

X	14	15	16	17	18	19	20	21
P (X)	2/15	1/15	2/15	3/15	1/15	2/15	3/15	1/15

Therefore E(X) is

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

$$= 14 \times \frac{2}{15} + 15 \times \frac{1}{15} + 16 \times \frac{2}{15} + 17 \times \frac{3}{15} + 18 \times \frac{1}{15} + 19 \times \frac{2}{15} + 20 \times \frac{3}{15} + 21 \times \frac{1}{15}$$

$$= \frac{28 + 15 + 32 + 21 + 18 + 38 + 60 + 21}{15} = \frac{263}{15}$$

$\Rightarrow$ E(X) = 17.53

And $E(X^2)$ is:

$$E(X^2) = \sum_{i=1}^{n} x_i^2 . p(x_i)$$

$$= (14)^2 \times \frac{2}{15} + (15)^2 \times \frac{1}{15} + (16)^2 \times \frac{2}{15} + (17)^2 \times \frac{3}{15} + (18)^2 \times \frac{1}{15} + (19)^2 \times \frac{2}{15} + (20)^2 \times \frac{3}{15} + (21)^2 \times \frac{1}{15}$$

$$= \frac{392 + 225 + 512 + 867 + 324 + 722 + 1200 + 441}{15} = \frac{4683}{15}$$

$\Rightarrow E(X^2) = 312.2$

Then Variance, $Var(X) = E(X^2) – (E(X))^2$

$= 312.2 – (17.53)^2$

$= 312.2 – 307.417 \approx 4.78$

And Standard deviation $= \sqrt{Var(X)}$

$= \sqrt{4.78}$

$\Rightarrow$ Standard deviation = 2.19

15. In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random, and we take X = 0 if he is opposed and X = 1 if he is in favour. Find E(X) and Var (X).

Solution:

Given, X = 0 if members oppose, and X = 1 if members are in favour.

P(X = 0) = 30% = 30/100 = 0.3

P(X = 1) = 70% = 70/100 = 0.7

Hence, the required probability distribution is

X	0	1
P (X)	0.3	0.7

Therefore E(X) is

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

= 0 × 0.3 + 1 × 0.7

⇒ E(X) = 0.7

And $E(X^2)$ is:

$$E(X^2) = \sum_{i=1}^{n} x_i^2 . p(x_i)$$

= $(0)^2 \times 0.3 + (1)^2 \times 0.7$

⇒ $E(X^2) = 0.7$

Then Variance, $Var(X) = E(X^2) – (E(X))^2$

= $0.7 – (0.7)^2$

= 0.7 – 0.49 = 0.21

16. The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is
A. 1
B. 2
C. 5
D. 8/3

Solution:

B. 2

Explanation:

Given, a die having written 1 on three faces, 2 on two faces and 5 on one face.

Let X be the random variable representing a number on the given die.

Then, X can take any value of 1, 2 or 5.

The total number is six.

Now,

P(X = 1) = 3/6 = ½

P(X = 2) = 1/3

P(X = 5) = 1/6

Hence, the required probability distribution is

X	1	2	5
P (X)	1/2	1/3	1/6

Therefore, Expectation of X E(X):

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

Now by substituting the values we get

$$= 1 \times \frac{1}{2} + 2 \times \frac{1}{3} + 5 \times \frac{1}{6}$$

$$= \frac{1}{2} + \frac{2}{3} + \frac{5}{6} = \frac{3 + 4 + 5}{6} = \frac{12}{6}$$

$$= 1 \times \frac{1}{2} + 2 \times \frac{1}{3} + 5 \times \frac{1}{6}$$

$$= \frac{1}{2} + \frac{2}{3} + \frac{5}{6} = \frac{3 + 4 + 5}{6} = \frac{12}{6}$$

$\Rightarrow E(X) = 2$

Hence, the correct answer is (B).

17. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is
A. 37/221
B. 5/13
C. 1/13
D. 2/13

Solution:

D. 2/13

Explanation:

Given, a deck of cards.

X be the number of aces obtained.

Hence, X can take the value of 0, 1 or 2.

As we know, in a deck of 52 cards, 4 cards are aces. Therefore, 48 cards are non-ace cards.

P (X = 0) = P (0 ace and 2 non ace cards) $= \frac{{}^4C_0 \times {}^{48}C_2}{{}^{52}C_2}$

$$= \frac{1128}{1326}$$

P (X = 1) = P (1 ace and 1 non ace cards) $= \frac{4_1 \times {}^{48}C_1}{{}^{52}C_2}$

$$= \frac{192}{1326}$$

P (X = 2) = P (2 ace and 0 non ace cards) $= \frac{{}^4C_2 \times {}^{48}C_0}{{}^{52}C_2}$

$$= \frac{6}{1326}$$

= 6/1326

Hence, the required probability distribution is

X	0	1	2
P (X)	1128/1326	192/1326	6/1326

Therefore, Expectation of X E(X)

$$E(X) = \sum_{i=1}^{n} x_i p_i$$

$$= 0 \times \frac{1128}{1326} + 1 \times \frac{192}{1326} + 2 \times \frac{6}{1326}$$

$$= \frac{204}{1326}$$

$$\Rightarrow E(X) = \frac{2}{13}$$

Hence, the correct answer is (D).

EXERCISE 13.5 PAGE NO: 576

1. A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of
(i) 5 successes?

(ii) At least 5 successes?
(iii) At most 5 successes?

Solution:

We know that the repeated tosses of dice are known as the Bernoulli trials.

Let the number of successes of getting an odd number in an experiment of 6 trials be x.

Probability of getting an odd number in a single throw of a dice (p)

$$= \frac{\text{number of odd numbers on a dice}}{\text{total number of numbers on a dice}} = \frac{3}{6} = \frac{1}{2}$$

Thus, q = 1 – p = ½

Now, here x has a binomial distribution.

Thus, $P(X = x) = {}^nC_x q^{n-x} p^x$, where x = 0, 1, 2 …n

$= {}^6C_x (1/2)^{6-x} (1/2)^x$

$= {}^6C_x (1/2)^6$

(i) Probability of getting 5 successes = P(X = 5)

$= {}^6C_5 (1/2)^6$

= 6 ×1/64

= 3/32

(ii) Probability of getting at least 5 successes = $P(X \geq 5)$

= P(X = 5) + P(X = 6)

$= {}^6C_5 (1/2)^6 + {}^6C_5 (1/2)^6$

= 6 ×1/64 + 6 ×1/64

= 6/64 + 1/64

= 7/64

(iii) Probability of getting at most 5 successes = $P(X \leq 5)$

We can also write it as 1 – P(X>5)

= 1 – P(X = 6)

= 1 – ${}^6C_6$ (1/2)^6

= 1 – 1/64

= 63/64

2. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

Solution:

We know that the repeated tosses of a pair of dice are known as Bernoulli trials.

Let the number of times of getting doublets in an experiment of throwing two dice simultaneously four times be x.

$$\text{Probability of getting doublets in a single throw of a a pair of dice(p)}$$
$$= \frac{\text{number of doublets possible in a pair of dice}}{\text{total number of posible pairs when two dice thrown}} = \frac{6}{36}$$
$$= \frac{1}{6}$$

Thus, q = 1 – p = 1 – 1/6 = 5/6

Now, here x has a binomial distribution, where n = 4, p = 1/6, q = 5/6

Thus, P(X = x) = ${}^nC_x q^{n-x} p^x$, where x = 0, 1, 2, … n

= ${}^4C_x$ (5/6)$^{4-x}$ (1/6)^x

= ${}^4C_x$ ($5^{4-x}/6^6$)

Hence, probability of getting 2 successes = P(X = 2)

= ${}^4C_2$ ($5^{4-2}/6^4$)

= 6 ($5^2/6^4$)

= 6 × (25/1296)

= 25/216

3. There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution:

Let there be x number of defective items in a sample of ten items drawn successively.

Now, as we can see that the drawing of the items is done with replacement. Thus, the trials are Bernoulli trials.

Now, the probability of getting a defective item, p = 5/100 = 1/20

Thus, q = 1 – 1/20 = 19/20

∴ We can say that x has a binomial distribution, where n = 10 and p = 1/20

Thus, $P(X = x) = {}^{n}C_{x}\, q^{n-x} p^{x}$, where x = 0, 1, 2 …n

$$= {}^{10}C_x \left(\frac{19}{20}\right)^{10-x} \left(\frac{1}{20}\right)^{x}$$

Probability of getting not more than one defective item = P(X ≤1)

= P(X = 0) + P(X = 1)

$= {}^{10}C_0\, (19/20)^{10}(1/20)^0 + {}^{10}C_1\, (19/20)^9(1/20)^1$

$$= \left(\frac{19}{20}\right)^{10} + 10 \times \left(\frac{19}{20}\right)^{9} \left(\frac{1}{20}\right)^{1}$$

$$= \left(\frac{19}{20}\right)^{9} \left[\frac{19}{20} + \frac{10}{20}\right]$$

$$= \left(\frac{19}{20}\right)^{9} \times \left(\frac{29}{20}\right)$$

4. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that
(i) All five cards are spades?
(ii) Only 3 cards are spades?
(iii) None is a spade?

Solution:

Let the number of spade cards among the five drawn cards be x.

As we can observe that the drawing of cards is with replacement; thus, the trials will be Bernoulli trials.

Now, we know that in a deck of 52 cards, there are total of 13 spade cards.

Thus, the probability of drawing a spade from a deck of 52 cards

= 13/52 = ¼

q = 1 – ¼ = 3/4

Thus, x has a binomial distribution with n = 5 and p = ¼

Thus, $P(X = x) = {}^{n}C_{x}\, q^{n-x} p^{x}$, where x = 0, 1, 2, …n

$$= {}^{5}C_{x}\left(\frac{3}{4}\right)^{5-x}\left(\frac{1}{4}\right)^{x}$$

(i) Probability of drawing all five cards as spades = P(X = 5)

$$= {}^{5}C_{5}\left(\frac{3}{4}\right)^{0}\left(\frac{1}{4}\right)^{5}$$

$$= 1 \times \frac{1}{1024}$$

$$= \frac{1}{1024}$$

= 1/1024

(ii) Probability of drawing three out five cards as spades = P(X = 3)

$$= {}^{5}C_{3}\left(\frac{3}{4}\right)^{2}\left(\frac{1}{4}\right)^{3}$$

$$= 10 \times \frac{9}{16} \times \frac{1}{64}$$

$$= \frac{45}{512}$$

(iii) Probability of drawing all five cards as non-spades = P(X = 0)

$$= {}^{5}C_{0}\left(\frac{3}{4}\right)^{5}\left(\frac{1}{4}\right)^{0}$$

$$= 1 \times \frac{243}{1024}$$

$$= \frac{243}{1024}$$

5. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs

(i) none
(ii) not more than one
(iii) more than one
(iv) at least one will fuse after 150 days of use.

Solution:

Let us assume that the number of bulbs that will fuse after 150 days of use in an experiment of 5 trials be x.

As we can see that the trial is made with replacement; thus, the trials will be Bernoulli trials.

It is already mentioned in the question that p = 0.05

Thus, q = 1 – p = 1 – 0.05 = 0.95

Here, we can clearly observe that x has a binomial representation with n = 5 and p = 0.05

Thus, $P(X = x) = {}^nC_x q^{n-x} p^x$, where x = 0, 1, 2… n

$= {}^5C_x (0.95)^{5-x}(0.05)^x$

(i) Probability of no such bulb in a random drawing of 5 bulbs = P(X = 0)

$= {}^5C_0 (0.95)^{5-0}(0.05)^0$

$= 1\times 0.95^5$

$= (0.95)^5$

(ii) Probability of not more than one such bulb in a random drawing of 5 bulbs = P (X≤ 1)

= P(X = 0) + P(X = 1)

$= {}^5C_0 (0.95)^{5-0}(0.05)^0 + {}^5C_1(0.95)^{5-1}(0.05)^1$

$= 1\times 0.95^5 + 5 \times (0.95)^4 \times 0.05$

$= (0.95)^4 (0.95 + 0.25)$

$= (0.95)^4 \times 1.2$

(iii) Probability of more than one such bulb in a random drawing of 5 bulbs = P (X>1)

= 1 – P(X ≤ 1)

$= 1 – [(0.95)^4 \times 1.2]$

(iv) Probability of at least one such bulb in a random drawing of 5 bulbs = P (X ≥ 1)

= 1 – P(X < 1)

= 1 – P(X = 0)

$= 1 – (0.95)^5$

6. A bag consists of 10 balls, each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Solution:

Let us assume that number of balls with a digit marked as zero among the experiment of 4 balls drawn simultaneously is x.

As we can see that the balls are drawn with replacement; thus, the trial is a Bernoulli trial.

Probability of a ball drawn from the bag to be marked as digit 0 = 1/10

It can be clearly observed that X has a binomial distribution with n = 4 and p = 1/10

Thus, q = 1 – p = 1 – 1/10 = 9/10

Thus, $P(X = x) = {}^nC_x q^{n-x} p^x$, where x = 0, 1, 2, …n

$$= {}^4C_x\left(\frac{9}{10}\right)^{4-x}\left(\frac{1}{10}\right)^x$$

Probability of no ball marked with zero among the 4 balls = P(X = 0)

$$= {}^4C_0\left(\frac{9}{10}\right)^{4-0}\left(\frac{1}{10}\right)^0$$

$$= {}^4C_x\left(\frac{9}{10}\right)^{4}\left(\frac{1}{10}\right)^0$$

$$= 1 \times \left(\frac{9}{10}\right)^4$$

$$= \left(\frac{9}{10}\right)^4$$

7. In an examination, 20 questions of the true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.

Solution:

Let us assume that the number of correctly answered questions out of twenty questions be x.

'Head' on the coin shows the true answer, and the 'tail' on the coin shows the false answer. Thus, the repeated tosses or the correctly answered questions are Bernoulli trails.

Thus, $p = ½$ and $q = 1 – p = 1 – ½ = ½$

Here, it can be clearly observed that x has a binomial distribution, where n = 20 and p = ½

Thus, $P(X = x) = {}^nC_x q^{n-x} p^x$, where x = 0, 1, 2 … n

$$= {}^{20}C_x \left(\frac{1}{2}\right)^{20-x} \left(\frac{1}{2}\right)^x$$

$$= {}^{20}C_x \left(\frac{1}{2}\right)^{20}$$

Probability of at least 12 questions answered correctly = $p(X \geq 12)$

$= P(X = 12) + P(X = 13) +…+ P(X = 20)$

$$= {}^{20}C_{12} \left(\frac{1}{2}\right)^{20} + {}^{20}C_{13} \left(\frac{1}{2}\right)^{20} + \cdots + {}^{20}C_{20} \left(\frac{1}{2}\right)^{20}$$

$$= \left(\frac{1}{20}\right)^{20} \left({}^{20}C_{12} + {}^{20}C_{13} + \cdots + {}^{20}C_{20}\right)$$

8. Suppose X has a binomial distribution B (6, ½). Show that X = 3 is the most likely outcome.
(Hint: P(X = 3) is the maximum among all P(xi), xi = 0,1,2,3,4,5,6)

Solution:

Given, X is any random variable whose binomial distribution is B (6, ½)

Thus, n = 6 and p = ½

$q = 1 – p = 1 – ½ = ½$

Thus, $P(X = x) = {}^nC_x q^{n-x} p^x$, where x = 0, 1, 2 …n

$$= {}^{6}C_x \left(\frac{1}{2}\right)^{6-x} \left(\frac{1}{2}\right)^x$$

$$= {}^{6}C_x \left(\frac{1}{2}\right)^{6}$$

It can be clearly observed that $P(X = x)$ will be the maximum if ${}^6c_x$ will be the maximum.

$\therefore {}^6c_x = {}^6c_6 = 1$

${}^6c_1 = {}^6c_5 = 6$

${}^6C_2 = {}^6C_4 = 15$

${}^6C_3 = 20$

Hence, we can clearly see that ${}^6C_3$ is the maximum.

∴ for x = 3, P(X = x) is maximum.

Hence, proved that the most likely outcome is x = 3.

9. On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution:

In this question, we have the repeated correct answer guessing from the given multiple choice questions, using the Bernoulli trials.

Let us now assume that X represents the number of correct answers by guessing in the multiple-choice set.

Now, the probability of getting a correct answer, p = 1/3

Thus, q = 1 – p = 1 – 1/3 = 2/3

Clearly, we X is a binomial distribution, where n = 5 and P = 1/3

$$\therefore P(X = x) = {}^nC_x\, q^{n-x} p^x$$

$$= {}^5C_x \left(\frac{2}{3}\right)^{5-x} . \left(\frac{1}{3}\right)^x$$

Hence, probability of guessing more than 4 correct answer = P (X ≥ 4)

= P (X = 4) + P (X = 5)

$$= {}^5C_4 \left(\frac{2}{3}\right) . \left(\frac{1}{3}\right)^4 + {}^5C_5 \left(\frac{1}{3}\right)^5$$

$$= 5 . \frac{2}{3} . \frac{1}{81} + 1 . \frac{1}{243}$$

$$= \frac{11}{243}$$

10. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is 1/100. What is the probability that he will win a prize
(a) At least once
(b) Exactly once
(c) At least twice?

Solution:

(a) Let X represents the number of prizes won in 50 lotteries, and the trials are Bernoulli trials.

Here, X is a binomial distribution where n = 50 and p = 1/100

Thus, q = 1 – p = 1 – 1/100 = 99/100

$$\therefore P(X = x) = {}^{n}C_{x}\, q^{n-x} p^{x}$$

$$= {}^{50}C_{x}\left(\frac{99}{100}\right)^{50-x}.\left(\frac{1}{100}\right)^{x}$$

Hence, probability of winning in lottery at least once = P (X ≥ 1)

= 1 – P (X <1)

= 1 – P (X = 0)

$$= 1 - {}^{50}C_{0}\left(\frac{99}{100}\right)^{50}$$

$$= 1 - 1.\left(\frac{99}{100}\right)^{50}$$

$$= 1 - \left(\frac{99}{100}\right)^{50}$$

(b) Probability of winning in lottery exactly once = P (X = 1)

$$= {}^{50}C_{1}\left(\frac{99}{100}\right)^{49}.\left(\frac{1}{100}\right)^{1}$$

$$= 50\left(\frac{1}{100}\right)\left(\frac{99}{100}\right)^{49}$$

$$= \frac{1}{2}\left(\frac{99}{100}\right)^{49}$$

(c) Probability of winning in lottery at least twice = P (X ≥ 2)

= 1 – P (X < 2)

$$= 1 - P(X \leq 1)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= [1 - P(X = 0)] - P(X = 1)$$

$$= 1 - \left(\frac{99}{100}\right)^{50} - \frac{1}{2}.\left(\frac{99}{100}\right)^{49}$$

$$= 1 - \left(\frac{99}{100}\right)^{49}.\left[\frac{99}{100} + \frac{1}{2}\right]$$

$$= 1 - \left(\frac{99}{100}\right)^{49}.\left(\frac{149}{100}\right)$$

$$= 1 - \left(\frac{149}{100}\right)\left(\frac{99}{100}\right)^{49}$$

$$= 1 - \left(\frac{99}{100}\right)^{49}.\left(\frac{149}{100}\right)$$

$$= 1 - \left(\frac{149}{100}\right)\left(\frac{99}{100}\right)^{49}$$

11. Find the probability of getting 5 exactly twice in 7 throws of a die.

Solution:

Let us assume X represent the number of times of getting 5 in 7 throws of the die.

Also, the repeated tossing of a die is the Bernoulli trials.

Thus, the probability of getting 5 in a single throw, p = 1/6

And, q = 1 – p

= 1 – 1/6

= 5/6

Clearly, X has the binomial distribution where n = 7 and p = 1/6

$$\therefore P(X = x) = {}^nC_x\, q^{n-x} p^x$$

$$= {}^7C_x \left(\frac{5}{6}\right)^{7-x} . \left(\frac{1}{6}\right)^x$$

Hence, probability of getting 5 exactly twice in a die = P (X = 2)

$$= {}^7C_2 \left(\frac{5}{6}\right)^5 . \left(\frac{1}{6}\right)^2$$

$$= 21 . \left(\frac{5}{6}\right)^5 . \frac{1}{36}$$

$$= \left(\frac{7}{12}\right)\left(\frac{5}{6}\right)^5$$

12. Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution:

Let us assume X represent the number of times of getting sixes in 6 throws of a die.

Also, the repeated tossing of die selection is the Bernoulli trials

Thus, the probability of getting six in a single throw of the die, p = 1/6

Clearly, X has the binomial distribution, where n = 6 and p = 1/6

And, q = 1 – p = 1 – 1/6 = 5/6

$$\therefore P\ (X\ =\ x) = {}^{n}C_{x}\, q^{n-x} p^{x}$$

$$= {}^{6}C_{x} \left(\frac{5}{6}\right)^{6-x} \left(\frac{1}{6}\right)^{x}$$

Hence, probability of throwing at most 2 sixes = P (X ≤2)

= P (X = 0) + p (X = 1) + P (X = 2)

$$= {}^{6}C_{0}.\left(\frac{5}{6}\right)^{6} + {}^{6}C_{1}.\left(\frac{5}{6}\right)65\,.\left(\frac{1}{6}\right) + {}^{6}C_{2}\left(\frac{5^{4}}{6}\right).\left(\frac{1}{6}\right)^{2}$$

$$= 1.\left(\frac{5}{6}\right)^{6} + 6\,.\frac{1}{6}.\left(\frac{5}{6}\right)^{5} + 15\,.\frac{1}{36}.\left(\frac{5}{6}\right)^{4}$$

$$= \left(\frac{5}{6}\right)^{6} + \left(\frac{5}{6}\right)^{5} + \frac{5}{12}.\left(\frac{5}{6}\right)^{4}$$

$$= \left(\frac{5}{6}\right)^{4}\left[\left(\frac{5}{6}\right)^{2} + \left(\frac{5}{6}\right) + \left(\frac{5}{12}\right)\right]$$

$$= \left(\frac{5}{6}\right)^{4}.\left[\frac{25+30+15}{36}\right]$$

$$= \frac{70}{36}.\left(\frac{5}{6}\right)^{4}$$

$$= \frac{35}{18}.\left(\frac{5}{6}\right)^{4}$$

13. It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

Solution:

Let us assume X represent the number of times selecting defective articles in a random sample space of 12 given articles.

Also, the repeated articles in a random sample space are the Bernoulli trials.

Clearly, X has the binomial distribution where n = 12 and p = 10% = 1/10

And, q = 1 – p = 1 – 1/10 = 9/10

$$\therefore P\,(X = x) = {}^{n}C_x\, q^{n-x} p^x$$

$$= {}^{12}C_x \left(\frac{9}{10}\right)^{12-x} . \left(\frac{1}{10}\right)^x$$

Hence, probability of selecting 9 defective articles $= {}^{12}C_9 \left(\frac{9}{10}\right)^3 \left(\frac{1}{10}\right)^9$

$$= 220 . \frac{9^3}{10^3} . \frac{1}{10^9}$$

$$= \frac{22 \times 9^3}{10^{11}}$$

14. In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is
A. 10^{-1}
B. $(1/2)^5$
C. $(9/10)^5$
D. 9/10

Solution:

C. $(9/10)^5$

Explanation:

Let us assume X represent the number of times selecting defective bulbs in a random sample of given 5 bulbs.

Also, the repeated selection of defective bulbs from a box is the Bernoulli trials.

Clearly, X has the binomial distribution where n = 5 and p = 1/10

And, q = 1 – p = 1 – 1/10

$$\therefore P\,(X = x) = {}^{n}C_x\, q^{n-x} p^x$$

$$= {}^{5}C_x \left(\frac{9}{10}\right)^{5-x} \left(\frac{1}{10}\right)^x$$

Hence, probability that none bulb is defective = P (X = 0)

$$= {}^{5}C_0 . \left(\frac{9}{10}\right)^5$$

$$= 1.\left(\frac{9}{10}\right)^5$$

$$= \left(\frac{9}{10}\right)^5$$

∴ Option C is correct

15. The probability that a student is not a swimmer is 1/5. Then the probability that out of five students, four are swimmers is
A. $^5C_4$ 1/5 $(4/5)^4$
B. $(4/5)^4$ (1/5)
C. $^5C_1$ 1/5 $(4/5)^4$
D. None of these

Solution:

A. $^5C_4$ 1/5 $(4/5)^4$

Explanation:

Let us assume X represent the number of students out of 5 who are swimmers.

Also, the repeated selection of students who are swimmers is the Bernoulli trials.

Thus, the probability of students who are not swimmers = q = 1/5

Clearly, we have X has the binomial distribution where n = 5

And, p = 1 – q

= 1 – 1/5

= 4/5

$$\therefore P\ (X\ =\ x) = {}^nC_x\ q^{n-x}p^x$$

$$= {}^5C_x\left(\frac{1}{5}\right)^{5-x}.\left(\frac{4}{5}\right)^x$$

Hence, probability that four students are swimmers = P (X = 4)

$$= {}^5C_4\ \left(\frac{1}{5}\right).\left(\frac{4}{5}\right)^4$$

∴ Option A is correct

MISCELLANEOUS EXERCISE

1. A and B are two events such that P (A) ≠ 0. Find P (B|A), if:
(i) A is a subset of B
(ii) A ∩ B = φ

Solution:

It is given that,

A and B are two events such that P (A) ≠ 0

We have, $A \cap B = A$

$\therefore P\ (A \cap B) = P\ (B \cap A) = P\ (A)$

Hence, $P(B|A) = \frac{P\ (B \cap A)}{P\ (A)}$

$= \frac{P\ (A)}{P\ (A)}$

= 1

(ii) We have,

$P\ (A \cap B) = 0$

$\therefore P(B|A) = \frac{P\ (A \cap B)}{P\ (A)}$

= 0

2. A couple has two children,
(i) Find the probability that both children are males, if it is known that at least one of the children is male.
(ii) Find the probability that both children are females, if it is known that the elder child is a female.

Solution:

(i) According to the question, if the couple has two children, then the sample space is

S = {(b, b), (b, g), (g, b), (g, g)}

Assume that A denote the event of both children having male and B denote the event of having at least one of the male children.

Thus, we have

$A \cap B = \{(b, b)\}$

$P(A \cap B) = \frac{1}{4}$

P (A) = ¼

P (B) = 3/4

Hence, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

By substituting the values we get

$= \frac{\frac{1}{4}}{\frac{3}{4}}$

= 1/3

(ii) Assume that C denotes the event of having both children as females and D denotes the event of having an elder child as female.

∴ C = {(g, g)}

P (C) = ¼

And, D = {(g, b), (g, g)}

P (D) = (2/4)

Hence, $P(C|D) = \frac{P(C \cap D)}{P(D)}$

$= \frac{\frac{1}{4}}{\frac{2}{4}}$

= ½

3. Suppose that 5% of men and 0.25% of women have grey hair. A grey-haired person is selected at random. What is the probability of this person being male? Assume that there is an equal number of males and females.

Solution:

Given that 5% of men and 0.25% of women have grey hair.

∴ Total % of people having grey hair = 5 + 0.25

= 5.25 %

Hence, the probability of having a selected male person having grey hair, P = 5/25 = 20/21

4. Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right-handed?

Solution:

Given that 90% of the people are right-handed.

Let p denotes the probability of people that are right-handed, and q denotes the probability of people that are left-handed.

p = 9/10 and q = 1 – 9/10 = 1/10

Now, by using the binomial distribution probability of having more than 6 right-handed people can be given as:

$$\sum_{r=7}^{10} 10\ C_r p^r q^{n-r} = \sum_{r=7}^{10} 10\, C_r \left(\frac{9}{10}\right)^r \left(\frac{1}{10}\right)^{10-r}$$

Hence, the probability of having more than 6 right handed people:

= 1 – P (More than 6 people are right handed)

$= 1 - \sum_{r=7}^{10} 10\, C_r (0.9)^r (0.1)^{10-r}$

5. An urn contains 25 balls of which 10 balls bear the mark 'X' and the remaining 15 bear the mark 'Y'. A ball is drawn at random from the urn, its mark is noted down, and it is replaced. If 6 balls are drawn in this way, find the probability that:
(i) All will bear 'X' mark.
(ii) Not more than 2 will bear 'Y' mark.
(iii) At least one ball will bear 'Y' mark.
(iv) The number of balls with 'X' mark and 'Y' mark will be equal.

Solution:

(i) It is given in the question that

Total number of balls in the urn = 25

Number of balls bearing mark 'X' = 10

Number of balls bearing mark 'Y' = 15

Let p denotes the probability of balls bearing the mark 'X', and q denotes the probability of balls bearing the mark 'Y'.

p = 10/25 = 2/5 and q = 15/25 = 3/5

Now, 6 balls are drawn with replacement. Hence, the number of trials are Bernoulli triangle.

Assume Z be the random variable that represents the number of balls bearing the 'Y' mark in the trials.

∴ Z has a binomial distribution where n = 6 and p = 2/5

$$P\ (Z = z) = {}^{n} C_z\, p^{n-z} q^z$$

Hence, P (All balls will bear mark 'X') = P (Z = 0)

$$= {}^{6} C_0 \left(\frac{2}{5}\right)^6$$

$$= \left(\frac{2}{5}\right)^6$$

(ii) Probability (Not more than 2 will bear 'Y' mark) = P (Z ≤ 2)

= P (Z = 0) + P (Z = 1) + P (Z = 2)

$$= {}^{6}C_0\ (p)^6 (q)^0 + {}^{6}C_1 (p)^5 (q)^1 + {}^{6}C_2 (p)^4 (q)^2$$

$$= \left(\frac{2}{5}\right)^6 + 6\ \left(\frac{2}{5}\right)^5 \left(\frac{3}{5}\right) + 15\ \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^2$$

$$= \left(\frac{2}{5}\right)^4 \left[\left(\frac{2}{5}\right)^2 + 6 \left(\frac{2}{5}\right)\left(\frac{3}{5}\right) + 15\ \left(\frac{3}{5}\right)^2\right]$$

$$= \left(\frac{2}{5}\right)^4 \left[\frac{175}{25}\right]$$

$$= 7\ \left(\frac{2}{5}\right)^4$$

(iii) Now, Probability (At least one ball will bear 'Y' mark) = P (Z ≥ 1)

= 1 – P (Z = 0)

$= 1 - \left(\frac{2}{5}\right)^6$

(iv) Probability (Having equal number of balls with 'X' mark and 'Y' mark) = P (Z = 3)

$= {}^6C_3\left(\frac{2}{54}\right)^3\left(\frac{3}{5}\right)^3$

$= \frac{20\times8\times27}{15625}$

$= \frac{864}{3125}$

6. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is 5/6. What is the probability that he will knock down fewer than 2 hurdles?

Solution:

Assume that p be the probability that the player will clear the hurdle, while q be the probability that the player will knock down the hurdle.

∴ p = 5/6 and q = 1 – 5/6 = 1/6

Let us also assume X be the random variable that represents the number of times the player will knock down the hurdle.

∴ By binomial distribution, $P\ (X = x) = {}^n C_x\, p^{n-x} q^x$

Hence, probability (players knocking down less than 2 hurdles) = P (X < 2)

= P (X = 0) + P (X = 1)

$= {}^{10}C_0\ (q)^0 (p)^{10} + {}^{10}C_1 (q)(p)^9$

$= \left(\frac{5}{6}\right)^{10} \times \left[\frac{5}{6} + \frac{10}{6}\right]$

$= \frac{5}{2} \times \left(\frac{5}{6}\right)^9$

$= \frac{(5)^{10}}{2\times(6)^6}$

$= \frac{5}{2} \times \left(\frac{5}{6}\right)^9$

$= \frac{(5)^{10}}{2\times(6)^9}$

7. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.

Solution:

From the given question, it is clear that

Probability of getting a six in a throw of die = 1/6

And, the probability of not getting a six = 5/6

Let us assume, p = 1/6 and q = 5/6

Now, we have

The probability that the 2 sixes come in the first five throws of the die

$$= {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

$$= \frac{10\times(5)^3}{(6)^5}$$

Also, Probability that the six come in the sixth throw $= \frac{10\times(5)^3}{(6)^5}\times\frac{1}{6}$

$$= \frac{10\times 125}{(6)^6}$$

$$= \frac{625}{23328}$$

8. If a leap year is selected at random, what is the chance that it will contain 53 Tuesdays?

Solution:

We know that in a leap year, there are a total of 366 days, 52 weeks and 2 days.

Now, in 52 weeks, there is a total of 52 Tuesdays.

∴ The probability that the leap year will contain 53 Tuesdays is equal to the probability of remaining 2 days will be Tuesdays.

Thus, the remaining two days can be

(Monday and Tuesday), (Tuesday and Wednesday), (Wednesday and Thursday), (Thursday and Friday), (Friday and Saturday), (Saturday and Sunday) and (Sunday and Monday)

∴ Total Number of cases = 7

Cases in which Tuesday can come = 2

Hence, probability (leap year having 53 Tuesdays) = 2/7

9. An experiment succeeds twice as often as it fails. Find the probability that in the next six trials, there will be at least 4 successes.

Solution:

Given that probability of failure = x

And, probability of success = 2x

∴ x + 2x = 1

3x = 1

X = 1/3

2x = 2/3

Assume p = 1/3 and q = 2/3

Also, X be the random variable that represents the number of trials.

Hence, by binomial distribution, we have

$$P\ (X = x) = {}^n C_x\, p^{n-x} q^x$$

∴ Probability of having at least 4 successes $= P\ (X \geq 4)$

= P (X = 4) + P (X = 5) + P (X = 6)

$$= {}^6 C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 + {}^6 C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right) + {}^6 C_6 \left(\frac{2}{3}\right)^6$$

$$= \frac{15\ (2)^4}{3^6} + \frac{6\ (2)^5}{3^6} + \frac{(2)^6}{3^6}$$

$$= \frac{31 \times (2)^4}{(3)^6}$$

$$= \frac{31}{9} \left(\frac{2}{3}\right)^4$$

10. How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

Solution:

Let us assume that the man tosses the coin n times. Thus, n tosses are the Bernoulli trials.

∴ Probability of getting head at the toss of the coin = ½

Let us assume, p = ½ and q = ½

$\therefore P\ (X = x) = {}^{n}C_{x}\, p^{n-x} q^{x}$

$= {}^{n}C_{x} \left(\frac{1}{2}\right)^{n-x} \left(\frac{1}{2}\right)^{x}$

$= {}^{n}C_{x} \left(\frac{1}{2}\right)^{n}$

It is given in the question that,

Probability of getting at least one head > 90/100

$\therefore P\ (x\ \geq 1) > 0.9$

1 – P (x = 0) > 0.9

$1 - {}^{n}C_{0} . \frac{1}{2^{n}} > 0.9$

$\frac{1}{2^{n}} < 0.1$

$2^{n} > \frac{1}{0.1}$

$2^{n} > 10$

Hence, the minimum value of n satisfying the given inequality = 4

∴ The man have to toss the coin 4 or more times

11. In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decides to throw a die thrice but quits as and when he gets a six. Find the expected value of the amount he wins/loses.

Solution:

For the situation given in the equation, we have

Probability of getting a six in a throw of a die = 1/6

Also, the probability of not getting a 6 = 5/6

Now, there are three cases from which the expected value of the amount he wins can be calculated.

(i) The first case is that if he gets a six on his first through, then the required probability will be 1/6

∴ Amount received by him = Rs. 1

(ii) Secondly, if he gets six on his second throw, then the probability = (5/6 × 1/6)

= 5/36

∴ Amount received by him = – Rs. 1 + Rs. 1

= 0

(iii) Lastly, if he does not get six in his first two throws and gets six in his third throw, then the probability = 5/6 × 5/6 × 1/6

= 25/216

∴ Amount received by him = – Rs. 1 – Rs. 1 + Rs. 1

= – 1

Hence, the expected value that he can win = 1/6 – 25/216

= (36 – 25)/216

= 11/216

12. Suppose we have four boxes A, B, C and D containing coloured marbles as given below.

Box	Marble colour		
	Red	White	Black
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random, and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from box A, box B and box C?

Solution:

Let us assume R be the event of drawing the red marbles.

Let us also assume E_A, E_B and E_C denote the boxes A, B and C, respectively.

Given that,

Total number of marbles = 40

Also, total number of red marbles = 15

P (R) = 15/40

= 3/8

Probability of taking out the red marble from box A,

$$P\ (E_A|R) = \frac{P\ (E_A \cap R)}{P\ (R)}$$

$$= \frac{\frac{1}{40}}{\frac{3}{8}}$$

= 1/15

Also, probability of taking out the red marble from box B,

$$P\ (E_B|R) = \frac{P\ (E_B \cap R)}{P\ (R)}$$

$$= \frac{\frac{6}{40}}{\frac{3}{8}}$$

= 2/5

And, Probability of taking out the red marble from box C,

$$P\ (E_C|R) = \frac{P\ (E_C \cap R)}{P\ (R)}$$

$$= \frac{\frac{8}{40}}{\frac{3}{8}}$$

= 8/15

13. Assume that the chances of a patient having a heart attack are 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30%, and a prescription of a certain drug reduces its chances by 25%. At a time, a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga.

Solution:

Let us assume X denotes the events having a person's heart attack.

A_1 denote events having the selected person follow the course of yoga and meditation.

And A_2 denote the events having the person adopted the drug prescription.

It is given in the question that,

P (X) = 0.40

And, $P\ (A_1) = \ P\ (A_2) = \frac{1}{2}$

$$P\ (X|A_1) = 0.40 \times 0.70 = 0.28$$

$$P\ (X|A_2) = 0.40 \times 0.75 = 0.30$$

∴ Probability (The patient suffering from a heart attack and followed a course of meditation and yoga):

$$P\ (A_1|X) = \frac{P(A_1)\ P\ (X|A_1)}{P(A_1)P\ (X|A_1) + \ P\ (A_2)P\ (X|A_2)}$$

$$= \frac{\frac{1}{2} \times 0.28}{\frac{1}{2} \times 0.28 + \frac{1}{2} \times 0.30}$$

$$= \frac{14}{29}$$

14. If each element of a second-order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assume that the individual entries of the determinant are chosen independently, each value being assumed with probability 1/2).

Solution:

From the question, we have

Total number of determinants of second order where the element being or 1 = $(2)^4$

= 16

Now, we have the value of determinants is positive in the following cases:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 11 \\ 0 & 11 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

∴ Required probability = 3/16

15. An electronic assembly consists of two subsystems, say, A and B. From previous testing procedures, the following probabilities are assumed to be known:
P (A fails) = 0.2
P (B fails alone) = 0.15
P (A and B fail) = 0.15
Evaluate the following probabilities:
(i) P (A fails | B has failed)
(ii) P (A fails alone)

Solution:

(i) Let us assume the event which is failed by A is denoted by $E_{A.}$

And the event which is failed by B is denoted by $E_{B.}$

It is given in the question that,

The event failed by A, P (E_A) = 0.2

Event failed by both, $P\ (E_A \cap E_B) = \ 0.15$

And, event failed by B alone $= \ P\ (E_B) - \ P\ (E_A \cap E_B)$

0.15 = P (E_B) – 0.15

∴ P (E_B) = 0.30

Hence, $P\ (E_A|E_B) = \frac{P\ (E_A \cap E_B)}{P\ (E_4)}$

$= \frac{0.15}{0.3}$

= 0.5

(ii) We have, probability where A fails alone $= P\ (E_A) - \ P\ (E_A \cap E_B)$

$= 0.2 - 0.15$

$= 0.05$

(ii) We have, the probability where A fails alone
$= P\ (E_A) - \ P\ (E_A \cap E_B)$

= 0.2 – 0.15

= 0.05

16. Bag I contains 3 red and 4 black balls, and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II, and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.
Choose the correct answer in each of the following.

Solution:

Let us first assume A_1 denote the events that a red ball is transferred from bags I to II.

And A_2 denote the event that a black ball is transferred from bags I to II.

∴ P (A_1) = 3/7

And, P (A_2) = 4/7

Let X be the event that the drawn ball is red

∴ when the red ball is transferred from bags I to II,

$$P(X|A_1) = \frac{5}{10}$$

$$= \frac{1}{2}$$

And, when black ball is transferred from bag I to II,

$$P(X|A_2) = \frac{4}{10}$$

$$= \frac{2}{5}$$

Hence, $P\ (A_2|X) = \frac{P(A_2)\ P(X|A_2)}{P(A_1)\ P(X|A_1)+P(A_2)\ P(X|A_2)}$

$$= \frac{\frac{4}{7}\times\frac{2}{5}}{\frac{3}{7}\times\frac{1}{2}+\frac{4}{7}\times\frac{2}{5}}$$

$$= \frac{16}{31}$$

17. If A and B are two events such that P(A) ≠ 0 and P(B | A) = 1, then
A. A ⊂ B
B. B ⊂ A

C. B = φ
D. A = φ

Solution:

A. A ⊂ B

Explanation:

It is given in the question that,

A and B are two events where,

$P\ (A) \neq 0$

And, P (B|A) = 1

$\therefore P\ (B|A) = \frac{P\ (B \cap A)}{P\ (A)}$

$1 = \frac{P\ (B\ \cap A)}{P\ (A)}$

$P\ (A) = \ P\ (B \cap A)$

$\therefore A\ \subset B$

Hence, option A is correct

18. If P (A|B) > P (A), then which of the following is correct.
A. P (B|A) < P (B)
B. P (A ∩ B) < P (A) . P (B)
C. P (B|A) > P (B)
D. P (B|A) = P (B)

Solution:

C. P (B|A) > P (B)

Explanation:

Given that,

P (A|B) > P (A)

$\therefore \frac{P\ (A \cap B)}{P\ (B)} > P\ (A)$

$P\ (A\ \cap B) >\ P\ (A).P\ (B)$

$\frac{P\ (A\ \cap B)}{P\ (A)} > P\ (B)$

P (B|A) > P (B)

Hence, option C is correct

19. **If A and B are any two events such that P(A) + P(B) – P(A and B) = P(A), then**
A. P(B|A) = 1
B. P(A|B) = 1
C. P(B|A) = 0
D. P(A|B) = 0

Solution:

B. P (A|B) = 1
Explanation:

Given that,

A and B are any two events where,

P (A) + P (B) – P (A and B) = P (A)

$P\ (A) +\ P\ (B) -\ P\ (A\ \cap B) = P\ (A)$

$P\ (B) -\ P\ (A \cap B) =\ 0$

$P\ (A \cap B) =\ P\ (B)$

$\therefore P(A|B) = \frac{P\ (A \cap B)}{P\ (B)}$

= 1

Hence, option B is correct