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CS 302.1003

CS302
Assignment 3

1. (15 points) Determine, for the typical algorithms that you use to perform calculations by hand, the running time to do the following (also explain the reasoning behind your answers)
   (a) Subtract two n-digit integers
   (b) Compute a^n where a is a 3-digit number
   (c) Divide an n-digit number with a 1-digit number that is greater than 1. (i.e. n-digit number / 1-digit number)

a) $f(n) = b - c$ // where b and c are n-digit #s

runtime is $\boxed{O(n)}$ because if b and c have 3 digits, there will be doing 3 different subtractions, with a possible borrow. There is a linear pattern, thus O(n).

$$\begin{array}{r} \#_3\ \#_2\ \#_1 \\ -\ \#_3\ \#_2\ \#_1 \\ \hline \end{array}$$
sub_1 sub_2 sub_3

b. $f(n) = a^n$ // a is a 3 digit

Each multiplication is 1 runtime.

n=2 a * a is $O(3^2)$ run time

The product is +3 more digits max

n = 3 → 6 digits × 3 digits = 18 multiplications

n = 4 → 9 digits × 3 = 27 multiplications

$$\begin{array}{r} \#_3\ \#_2\ \#_1 \\ \times\ \#_3\ \#_2\ \#_1 \\ \hline \#_{13}\ \#_{12}\ \#_{11} \\ \#_{33}\ \#_{22}\ \#_{21}\ 0 \\ \dots\ 0\ 0 \end{array}$$

$3(\underset{k}{3} \cdot \underset{i}{1} + 3 \cdot 2 + 3 \cdot 3 \dots)$

$\rightarrow 3 \sum_{i=1}^{n} \cancel{3}i = 3\left(\frac{n(n-1)}{2}\right) = \frac{\cancel{3}n^2 - \cancel{n}}{\cancel{2}} = \boxed{O(n^2)}$

drop k drop lower powers

c) $k \overline{) a\,b\,c\,d}$ div_1 n digits sub_1 div_2 sub_2 div_3

Search 0 - 9 to see if that fits into first digit. → does not depend on n → O(9)

divide and subtract → O(2)

Repeat n times → O(n)

run time is $\boxed{O(n)}$ because constants are dropped

2. (10 points) Show for any constant $k > 0$ the following always must be true $O(\log^k n) = O(n)$. You can use the natural log, its derivative is

$$\left(\frac{\partial}{\partial n}\right) \ln^k(n) = \frac{k \ln^{k-1}(n)}{n}$$

$$\lim_{n \to \infty} \left(\frac{\frac{\partial}{\partial n} \ln^k n}{n} \right) = \lim_{n \to \infty} \frac{\frac{k \ln^{k-1}(n)}{n}}{1}$$

$$\lim_{n \to \infty} \frac{k(k-1) \ln(n)}{n} =$$

$$= k(k-1) \lim_{n \to \infty} \frac{\ln(n)}{n} = \boxed{0}$$

Big O means $f(n) \leq g(n)$, where $k > 0$

$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$ means $f(n) \leq g(n)$

3. (5 points) Prove or disprove $2^n = \Theta(3^n)$

Prove $2^n = O(3^n)$ and $2^n = \Omega(3^n)$

$2^n = O(3^n) \Rightarrow \lim_{n \to \infty} \frac{2^n}{k \cdot 3^n} = 0$ so $O(3^n) \geq 2^n$

$2^n = \Omega\, 3^n \Rightarrow \lim_{n \to \infty} \frac{2^n}{0 \cdot 3n} = \lim_{n \to \infty} \frac{2^n}{0} = \infty \Rightarrow 2^n > \Omega(3^n)$

False. Θ is lower and upper bound. For $2^n = \Theta(3^n)$ to be true $\Omega(3^n) \leq 2^n \leq O(3^n)$; however $\Omega(3^n)$ is less than 2^n, but not less than or equal to.

Thus, $\boxed{2^n \neq \Theta(3^n)}$

4. (15 points) Rearrange the following functions from smallest asymptotic growth rate to the largest asymptotic growth, if any functions have the same asymptotic growth rate, please indicate that (use < to denote the function on the right has a larger growth rate and use = to denote same growth rate, for example $n < n^2$ and $3n = 4n$)

$$1.0001^n,\ 2^{64},\ 4\log_2 n,\ n\sqrt{n},\ \frac{n}{2^{64}},\ 2n^{1.5} - n\log_2 n,\ 4000n + 22,\ n\log_2 n,\ \frac{n^2}{2} + 20n - 4,\ \log_{10} n$$

constant

$n(n^{1/2}) \to n^{3/2}$

log n

n

$n^{1.5} > n\log_2 n \to n^{1.5}$

linear = n

n log n

n^2

log n

$n\sqrt{n}$ and $n\log_2 n \to \lim_{n\to\infty} \frac{n\log_2 n}{n\sqrt{n}} = 0 \to n\log_2 n < n\sqrt{n}$

$1.0001^n \to \lim_{n\to\infty} \frac{1.0001^n}{n\sqrt{n}} = \infty \to 1.0001^n > n\sqrt{n}$

$4\log_2 n \to \lim_{n\to\infty} \frac{4\log_2(n)}{\log(n)} = \frac{4}{\log(2)} \to 4\log_2 n = \log n$

$n\sqrt{n}$ and $2n^{1.5} - n\log_2 n \to \lim_{n\to\infty} \frac{2n^{1.5} - n\log_2 n}{n\sqrt{2}} = 2 \to n\sqrt{n} = 2n^{1.5} - n\log_2 n$

$\hookrightarrow n^{3/2}$

$\frac{n}{2^{64}} \le c\cdot n \to \frac{n}{2^{64}} = n$

$4000n + 22 \to \lim_{n\to\infty} \frac{4000n+22}{n} = 4000 \to 4000n + 22 = n$

1.0001^n and $\frac{n^2}{2} + 20n - 4 \to \lim_{n\to\infty} \frac{1.0001^n}{\frac{n^2}{2} + 20n + 4} = \infty \to 1.0001^n > \frac{n^2}{2} + 20n - 4$

$\log_{10} n \to \lim_{n\to\infty} \frac{\log_{10} n}{\log n} = \frac{1}{\log(10)} \to \log_{10} n = \log n$

Note: $k < \log n < n < n\log(n) < n^2\log(n) < n^{3\ldots} < 2^n$

$$\boxed{2^{64} < 4\log_2 n = \log_{10} n < \frac{n}{2^{64}} = 4000n + 22 < n\log_2 n < n\sqrt{n} = 2n^{1.5} - n\log_2 < \frac{n^2}{2} + 20n - 4 < 1.0001^n}$$

5. (15 points) Perform insertion sort on the following list, show the steps of the inner loop and show when an iteration of the outer loop finishes.

5, 3, 10, 2, 9, 1

5 3 10 2 9 1

$i = 1$
key @ 3
inner: sorted
outer: finishes → get next

5 3

$j = \overset{i-1}{0} \to 1$
inner: swap ×1

3 5

$i = 2$
key @ 10
outer: finishes → get next

3 5 10

inner: sorted

$i = 3$
key @ 2
outter: finishes get next

3 5 10 2

$j = 3-1 = 2$
$2 \to 1 \to 0 \to -1$
inner: swap ×3

2 3 5 10

outer: finishes → get next

$i = 4$
key @ 9
$j = 4-1 = 3$

2 3 5 10 9

inner: swap ×1

2 3 5 9 10

$i = 5$
key @ 1
outer: finishes → get next

$j = 4 \to 3 \to 2 \to 1 \to 0 \to -1$

2 3 5 9 10 1

inner: swap × 5

1 2 3 5 9 10 ✓

inner: sorted

outer: finished

1 2 3 5 9 10

6. (15 points) Perform mergesort on the following list, draw the recursive tree to shows how the problem is broken up and how the merge is done.

1, 30, 29, 31, 11, 25, 32, 21, 35, 23, 13, 2, 4, 6, 5, 10.

1 30 29 31 11 25 32 21   35 23 13 2 4 6 5 10

1 30 29 31   11 25 32 21   35 23 13 2   4 6 5 10

1 30   29 31   11 25   32 21   35 23   13 2   4 6   5 10

1   30   29   31   11   25   32   21   35   23   13   2   4   6   5   10

1 30   29 31   11 25   21 32   23 35   2 13   4 6   5 10

1 29 30 31   11 21 25 32   2 13 23 35   4 5 6 10

1 11 21 25 29 30 31 32   2 4 5 6 10 13 23 35

1 2 4 5 6 10 11 13 21 23 25 29 30 31 32 35

* ___ -> unsort.

⌊⌋ -> sorted *

7. (15 points) Perform quicksort on the following list, show the pivot and show how the left and right pointers are moving and how the elements are swapped after in each step, choose the leftmost element of each sublist as the pivot.

10, 23, 7, 6, 2, 5, 13, 8, 19, 11

10 23 7 6 2 5 13 8 19 11

P L R ← ~~R~~ ← ~~R~~

10 8 7 6 2 5 13 23 19 11

P ~~L~~ → ~~L~~ → ~~L~~ → ~~L~~ R ← ~~R~~ ← ~~R~~ R

→ ~~L~~ → L

5 8 7 6 2 $\boxed{10}$ 13 23 19 11

P L R P L R

5 2 7 6 8 $\boxed{10}$ 13 11 19 23

P ~~L~~ → L ~~R~~ ← ~~R~~ P ~~L~~ → L R

R ← ~~R~~ ← R ← ~~R~~ ←

2 $\boxed{5}$ 7 6 8 $\boxed{10}$ 11 $\boxed{13}$ 19 23

P L ~~R~~ ← ~~R~~ P L R

R ← ~~R~~ ← R ← R ←

$\boxed{2}$ $\boxed{5}$ 7 6 8 $\boxed{10}$ $\boxed{11}$ $\boxed{13}$ 19 23

P ~~L~~ ← R P L

R → L R ← ~~R~~

$\boxed{2}$ $\boxed{5}$ 6 $\boxed{7}$ 8 $\boxed{10}$ $\boxed{11}$ $\boxed{13}$ $\boxed{19}$ 23

P L P

← R L

R

$\boxed{2}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{10}$ $\boxed{11}$ $\boxed{13}$ $\boxed{19}$ $\boxed{23}$

P

$\boxed{2}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{10}$ $\boxed{11}$ $\boxed{13}$ $\boxed{19}$ $\boxed{23}$ ✓