The Stone-Čech Compactification

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G. Eric Moorhouse, University of Wyoming

We construct the Stone-Čech compactification βX of a topological space X—the most universal or general compactification possible for X. The main theorem describing its existence and uniqueness is stated in Section 2. We first provide examples in Section 1 illustrating the universal property enjoyed by βX, thus clarifying our description of βX as the most general compactification of X available. The main theorem is proved in two parts, beginning in Section 3 with the special case that X is discrete. Here we give the construction via ultrafilters; see [C, CN] for more general results in this direction. This special case is not only the hardest, but also the key to proving the general case. Although the existence of βX follows from Tychonoff's embedding theorem (also using the Axiom of Choice) which we have already seen, that construction is less explicit in some cases—in particular it is not easy to actually identify the closed subspace $[0,1]^F$ into which X is embedded, unless X is particularly nice, in which case Tychonoff's embedding is not needed anyway. The hardest case is the case when X is discrete; and in this case the points of βX are "inescapably" just ultrafilters on X; see Section 3. The construction of βX in the general case (see Section 4) is then obtained by taking an appropriate quotient. In Section 5 we present Hindman's Theorem, a celebrated application of the Stone-Čech compactification to additive combinatorics. This is more than "yet another" application of ultrafilters in combinatorial set theory—it is the topological structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, that is used.

Marshall Stone (1903–1989) | Eduard Čech (1893–1960)

1. Embeddings

Recall that a subset $S \subseteq X$ is **dense** if its closure is $\overline{S} = X$; in other words, every nonempty open set in X contains a point of S. We need the following quickie.

1.1 Lemma. Let X and Y be topological spaces, and suppose Y is Hausdorff. Then any continuous function $f : X \to Y$ is uniquely determined by its values on a dense subset of X.

Proof. Suppose $f, g : X \to Y$ are continuous, and $f(s) = g(s)$ for all $s \in S$, where $S \subseteq X$ is dense. Suppose also that $f(x) \neq g(x)$ for some $x \in X$. Let $U, V \subset Y$ be disjoint open neighbourhoods of $f(x)$ and $g(x)$, respectively. Then $f^{-1}(U)$ and $g^{-1}(V)$ are open neighbourhoods of x, so we can find $s \in S \cap f^{-1}(U) \cap g^{-1}(V)$. Then $f(s) = g(s)$ is a point in $U \cap V = \varnothing$, a contradiction. $\square$

Now let X be a topological space Y. An **embedding** of X into another space Y is an injective continuous map $\iota : X \to Y$ such that $X \simeq \iota(X)$. In this case we usually identify X with its image $\iota(X)$. We would like to embed X in a space that has some desirable property (such as completeness or compactness) which X itself might lack. In order that Y be the *smallest* such space (with the desired property) in which X embeds, we can usually assume X (or rather, $\iota(X)$) is dense in Y; otherwise X would embed in the smaller space $\overline{\iota(X)}$ which often also has the desired property (and in the case of completeness or compactness, this is true).

An important example of an embedding, which we have already considered, is the **completion** $\widehat{X}$ of an arbitrary metric space X. By definition, $\widehat{X}$ is a complete metric space in which X embeds as a dense subspace, i.e. we have an isometric embedding $\iota : X \to \widehat{X}$. Here, ι is an **isometric embedding** if it satisfies the stronger condition that it preserves distance. The pair $(\widehat{X}, \iota)$ enjoys the following universal property: If $f : X \to Y$ is an isometric embedding of X in a complete metric space Y, then there is a unique isometric embedding $\widehat{f} : \widehat{X} \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & \widehat{X} \arrow[d, dashed, "\widehat{f}"] \\ & Y \end{tikzcd}$$

and this universal property suffices to define the completion. The uniqueness of the completion follows from this universal property, while its existence is supplied by our construction

$$\widehat{X} = \mathrm{Cauchy}(X)/\!\sim, \quad \iota(x) = (x, x, x, \ldots)$$

where $\mathrm{Cauchy}(X) \subseteq X^\omega$ is the space of all Cauchy sequences in X.

In a similar way, given a topological space X, we would like to embed a topological space X in a compact Hausdorff space Y. We will see that as long as X satisfies some minimal necessary conditions, then such compactifications of X do indeed exist; and there is a 'most general' such compactification (the Stone-Čech compactification βX), from which all other compactifications of X are obtainable.

We introduce βX, beginning with the example of the non-compact space $X = (0, 1)$. There are two rather natural compactifications of $(0, 1)$, including the 'one-point compactification'

(1) $\qquad (0,1) \to S^1, \quad t \mapsto e^{2\pi t i}$, and

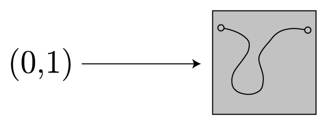
(2) $\qquad (0,1) \to [0,1], \quad t \mapsto t.$

Here, (1) is known as the 'one-point compactification' of $(0,1)$. In general, the **one-point compactification** of X has point set $X \cup \{\infty\}$ where we have added just one new point, denoted '∞'. Open sets in $X \cup \{\infty\}$ are of two types:

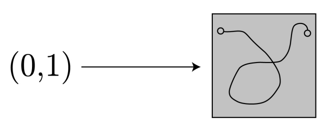
- open subsets of X remain open in $X \cup \{\infty\}$; also
- subsets of the form $(X \smallsetminus K) \cup \{\infty\}$ where $K \subseteq X$ is closed and compact.

The resulting topological space $X \cup \{\infty\}$ is compact whenever X itself is Hausdorff and locally compact but not compact. (We say that X is **locally compact** if every point has an open neighbourhood which is contained in some compact subset.) As further examples, the one-point compactification of $\mathbb{R}^2$ is S^2 (alternatively, we view the Riemann sphere as the one-point compactification of $\mathbb{C}$) and $[0,\omega] = \omega \cup \{\omega\}$ is the one-point compactification of $\omega = \{0,1,2,\ldots\}$.

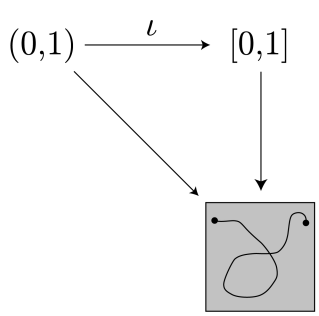
The compactification (2) above is the Stone-Čech compactification of $(0,1) \simeq \mathbb{R}$; that is, $\beta\mathbb{R} \simeq [0,1]$. This is the most general compactification of $\mathbb{R}$ in a sense that we proceed to describe. Any embedding of $(0,1)$ into a compact space may be viewed as a parameterized path without endpoints:



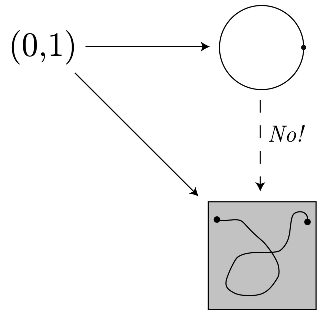
This embedding is not universal, in the sense that we have added more points than strictly necessary in order to compactify X; the problem is that the embedded copy of $(0,1)$ is not dense in the new space. Henceforth we ask that X be embedded as a *dense* subspace of some compact Hausdorff space. Note that this requirement is satisfied by the embeddings (1) and (2) above. Next, we observe that $[0,1]$ is the most general compactification of $(0,1)$ in the sense that for every continuous map $f : (0,1) \to Y$ where Y is compact Hausdorff, there is a unique extension to $\widehat{f} : [0,1] \to Y$. For example, a continuous map of the form



extends to $[0,1]$:



Such an extension is not available for the one-point compactification (1):



We are ready to show that every completely regular Hausdorff space X has a Stone-Čech compactification: an embedding of X as a dense subspace of a compact Hausdorff space βX. Moreover, this compactification is essentially unique.

2. The Stone-Čech Compactification

A **Stone-Čech compactification** of a topological space X is an embedding $\iota : X \to \beta X$ where βX is compact Hausdorff, such that for every continuous map from X to a compact Hausdorff space Y, say $f : X \to Y$, there is a *unique* continuous map $\widehat{f} : \beta X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed, "\widehat{f}"] \\ & Y \end{tikzcd}$$

Here 'embedding' means that ι is a homeomorphism between X and its image $\iota(X) \subseteq \beta X$. We may then identify X with its image under this embedding, namely $\iota(X) \subseteq \beta X$. With this interpretation, $\widehat{f}$ is merely an extension of f; and the defining property of the Stone-Čech compactification is that the correspondence $f \leftrightarrow \widehat{f}$ constitutes a bijection

$$C(X,Y) \leftrightarrow C(\beta X, Y)$$

where $C(X,Y)$ is the set of all continuous functions $X \to Y$.

2.1 Theorem. A topological space X can have a Stone-Čech compactification only if X is completely regular and Hausdorff. If the Stone-Čech compactification exists, then it is unique up to homeomorphism; and in this case the embedded subspace $\iota(X) \simeq X$ is dense in βX.

Proof. If βX is a Stone-Čech compactification of X, then βX is compact Hausdorff and hence normal; and so by Urysohn's Lemma, βX is completely regular. Since $X \simeq \iota(X)$ is a subspace of βX, it is also completely regular and Hausdorff. As indicated above, we identify X with the embedded subspace $\iota(X)$. If this subspace is not dense, there exists a point $y \in \beta X$ such that $y \notin \overline{\iota(X)}$; so there exists a continuous function $\widehat{f} : \beta X \to [0,1]$ such that $\widehat{f}|_{\iota(X)} = 0$ but $\widehat{f}(y) = 1$. Now the constant zero function $X \to [0,1]$, $x \mapsto 0$ has more than one continuous extension to βX, namely $\widehat{f}$ and the constant zero function on βX; but this violates the uniqueness property of $\widehat{f}$. We conclude that (the embedded copy of) X is dense in βX.

To prove the uniqueness of the Stone-Čech compactification, suppose both $\iota : X \to Z$ and $\iota' : X \to Z'$ satisfy the defining property of the Stone-Čech compactification. Then there exist unique continuous maps $\phi : Z \to Z'$ and $\phi' : Z' \to Z$ such that $\phi \circ \iota = \iota'$ and $\phi' \circ \iota' = \iota$.

$$\begin{tikzcd}
 & Z \arrow[d, "\phi"] \arrow[dd, bend left=60, dashed, "\mathrm{id}_Z"] \\
X \arrow[ur, "\iota"] \arrow[r, "\iota'"] \arrow[dr, "\iota"'] & Z' \arrow[d, "\phi'"] \\
 & Z
\end{tikzcd}$$

But now there are two continuous maps $Z \to Z$ completing a commutative diagram above, i.e. $\mathrm{id}_Z \circ \iota = \iota$ and $(\phi' \circ \phi) \circ \iota = \iota$. By the uniqueness condition in the definition of Stone-Čech compactification, we have $\phi' \circ \phi = \iota$. A simlar argument shows that $\phi \circ \phi' = \mathrm{id}_{Z'}$. Thus $Z \simeq Z'$ with ϕ and ϕ' forming an inverse pair of homeomorphisms, and the Stone-Čech compactification $Z \simeq Z'$ is unique. $\square$

It remains for us to prove the *existence* of the Stone-Čech compactification, assuming X is completely regular and Hausdorff. The construction is given in Sections 3 (when X is discrete) and 4 (for general X).

3. Construction of βX when X is discrete

Here we assume X is a discrete topological space, this being the hardest case. Here we simply define

$$\beta X = \{\mathfrak{U} : \mathfrak{U} \text{ is an ultrafilter on } X\}.$$

Thus points of βX are just ultrafilters on X. We must specify the topology of βX and show that it is compact Hausdorff. A basis for this topology is obtained as follows: for every subset $A \subseteq X$, define

$$\langle A \rangle = \{\mathfrak{U} \in \beta X \;:\; A \in \mathfrak{U}\}.$$

Before proceeding further, recall that in the case of a singleton set $A = \{x\}$, there is a *unique* filter containing $\{x\}$, namely the *principal ultrafilter* which we have denoted

$$\mathfrak{F}_x = \{B \subseteq X \,:\, x \in B\}$$

and so in this case, $\langle\{x\}\rangle = \{\mathfrak{F}_x\}$ is a singleton point in βX.

3.1 Lemma. The map $A \mapsto \langle A\rangle$, $\mathcal{P}(X) \to \mathcal{P}(\beta X)$ gives an embedding of partially ordered sets under '$\subseteq$'. In particular,

(a) $\langle\varnothing\rangle = \varnothing$ and $\langle X\rangle = \beta X$.
(b) If $A \subseteq B \subseteq X$ then $\langle A\rangle \subseteq \langle B\rangle$.
(c) $\langle A \cup B\rangle = \langle A\rangle \cup \langle B\rangle$ and $\langle A \cap B\rangle = \langle A\rangle \cap \langle B\rangle$.
(d) $\langle X \smallsetminus A\rangle = \beta X \smallsetminus \langle A\rangle$.
(e) The map $\mathcal{P}(X) \to \mathcal{P}(\beta X)$, $A \mapsto \langle A\rangle$ is one-to-one.

Proof. I will just prove (e), and leave the remaining parts as an exercise. Suppose $A \neq B$ are distinct subsets of X. If there exists $a \in A$ with $a \notin B$, then $B \subseteq X \smallsetminus \{a\}$ so $\langle B\rangle \subseteq \langle X \smallsetminus \{a\}\rangle$. However $\langle A\rangle \supseteq \langle\{a\}\rangle \neq \varnothing$ is disjoint from $\langle X \smallsetminus \{a\}\rangle$ by (d). Thus $\langle A\rangle \neq \langle B\rangle$. If there exists $b \in B$ with $b \notin A$, the argument is similar. $\square$

3.2 Corollary. The subsets $\langle A\rangle \subseteq \beta X$ form a basis for a topology on βX. The map $\iota : X \to \beta X$, $x \mapsto \mathfrak{F}_x$ embeds X as a discrete dense subspace of βX. Thus the restriction of ι to $X \to \iota(X)$ is a homeomorphism.

Proof. The fact that the subsets $\langle A\rangle \subseteq \beta X$ form a basis for a topology on βX follows directly from Lemma 3.1. The map $\iota : X \to \beta X$, $x \mapsto \mathfrak{F}_x$ is clearly one-to-one; and since the singletons $\langle\{x\}\rangle = \{\mathfrak{F}_x\}$ are open, the image of X under this embedding is discrete. For every nonempty subset $A \subseteq X$, the basic open set $\langle A\rangle \subseteq \beta X$ satisfies $\iota(x) = \mathfrak{F}_x \in \langle A\rangle$ whenever $x \in A$; thus the image $\iota(A) \subseteq \beta X$ is dense. The result follows. $\square$

Since $\langle X \smallsetminus A\rangle = \beta X \smallsetminus \langle A\rangle$, the sets $\langle A\rangle$ are clopen; they constitute a family of basic closed sets, as well as a family of basic open sets, in the terminology of Section 1.

3.3 Theorem. For a discrete space X, the space βX as defined above is compact and Hausdorff. Moreover it is the Stone-Čech compactification of X.

Proof. Let $\mathfrak{U}, \mathfrak{U}' \in \beta X$ be distinct points, i.e. distinct ultrafilters on X. For some subset $A \subseteq X$, we have $A \in \mathfrak{U}$ but $A \notin \mathfrak{U}'$, so that $A' \in \mathfrak{U}'$ where $A' = X \smallsetminus A$; then $\langle A\rangle$ and $\langle A'\rangle$ are disjoint open neighbourhoods of the points $\mathfrak{U}, \mathfrak{U}'$ respectively. Thus βX is Hausdorff.

We prove compactness using the fact that $\{\langle A\rangle : A \subseteq X\}$ is a family of basic closed sets for βX. Consider an indexed family $\{\langle A_\alpha\rangle\}_\alpha$ of basic closed sets with the finite intersection property; we must show that $\bigcap_\alpha \langle A_\alpha\rangle \neq \varnothing$. Since $\langle A_{\alpha_1} \cap \cdots \cap A_{\alpha_n}\rangle = \langle A_{\alpha_1}\rangle \cap \cdots \cap \langle A_{\alpha_n}\rangle \neq \varnothing = \langle\varnothing\rangle$, by Lemma 3.1(e) we have $A_{\alpha_1} \cap \cdots \cap A_{\alpha_n} \neq \varnothing$. Thus the sets A_α themselves satisfy the finite intersection property; and so they generate a filter on X. Extending this filter to an ultrafilter, we obtain $\mathfrak{U} \in \beta X$ such that $A_\alpha \in \mathfrak{U}$ for all α, i.e. $\mathfrak{U} \in \bigcap_\alpha \langle A_\alpha\rangle$. This shows that βX is compact.

It remains to be checked that ι satisfies the required universal property. Let $f : X \to Y$ be any map, where Y is an arbitrary compact Hausdorff space. (Since X is discrete, every map defined on X is continuous.) Given $\mathfrak{U} \in \beta X$, the push-forward ultrafilter $f_*(\mathfrak{U})$ on Y must converge to a unique point by the compactness of Y, and we denote this point $\widehat{f}(\mathfrak{U}) \in Y$. This gives a well-defined function $\widehat{f} : \beta X \to Y$. For all $x \in X$, $\widehat{f}(\iota(x)) = \widehat{f}(\mathfrak{F}_x) = f(x)$ by definition of $\widehat{f}$, since $\mathfrak{F}_x \searrow x$; thus $\widehat{f} \circ \iota = f$.

Next we show that $\widehat{f} : \beta X \to Y$ is continuous: Let $\mathfrak{U} \in \beta X$ and let $V \subseteq Y$ be an open neighbourhood of $\widehat{f}(\mathfrak{U}) \in Y$; it suffices to find a basic open set $\langle U\rangle$ in βX such that

$$\mathfrak{U} \in \langle U\rangle \subseteq \widehat{f}^{-1}(V).$$

Since Y is regular, there exists an open neighbourhood V_1 of $\widehat{f}(\mathfrak{U})$ such that

$$\widehat{f}(\mathfrak{U}) \in V_1 \subseteq \overline{V_1} \subseteq V.$$

Now $U := f^{-1}(V_1) \subseteq X$ is open and

$$f_*(\mathfrak{U}) \searrow \widehat{f}(\mathfrak{U}) \in V_1$$

which means that $V_1 \in f_*(\mathfrak{U})$ and so $U = f^{-1}(V_1) \in \mathfrak{U}$, i.e. $\mathfrak{U} \in \langle U\rangle$. In order to verify that $\widehat{f}(\langle U\rangle) \subseteq V$, we will prove the stronger containment $\widehat{f}(\langle U\rangle) \subseteq \overline{V_1}$. For suppose there exists $\mathfrak{V} \in \langle U\rangle$ with $\widehat{f}(\mathfrak{V}) \notin \overline{V_1}$. Then

$$f_*(\mathfrak{V}) \searrow \widehat{f}(\mathfrak{V}) \in Y \smallsetminus \overline{V_1}$$

and so $Y \smallsetminus \overline{V_1} \in f_*(\mathfrak{V})$, which means that $f^{-1}(Y \smallsetminus \overline{V_1}) \in \mathfrak{V}$. However, we also have $f^{-1}(V_1) = U \in \mathfrak{V}$. Since $Y \smallsetminus \overline{V_1}$ and V_1 are disjoint, we obtain

$$\varnothing = f^{-1}(\varnothing) = f^{-1}\big((Y \smallsetminus \overline{V_1}) \cap V_1\big) = f^{-1}(Y \smallsetminus \overline{V_1}) \cap f^{-1}(V_1) \in \mathfrak{V},$$

a contradiction. This completes our proof that $\widehat{f}$ is continuous.

Since $\iota(X) \subseteq \beta X$ is dense and Y is Hausdorff, the uniqueness of $\widehat{f}$ follows from Lemma 1.1. $\square$

4. Construction of βX: The General Case

Let X be a completely regular Hausdorff space, and let X_0 be the underlying discrete space (having the same point set as X but the discrete topology). The bijection

$$\iota_0 : X_0 \to X, \quad x \mapsto x$$

is continuous. As before, we denote by $\iota : X_0 \to \beta X_0$ the Stone-Čech compactification of the discrete space X_0. Given an arbitrary compact Hausdorff space Y and a continuous map $f : X \to Y$, we first note that the map $f_0 = f \circ \iota_0^{-1} : X \to Y$ is continuous; so there is a unique map $\widehat{f_0} : \beta X_0 \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, "\iota"] \arrow[d, "\iota_0"'] \arrow[dr, "f_0"] & \beta X_0 \arrow[d, dashed, "\widehat{f_0}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Here you should view X (and therefore also X_0, ι, ι_0 and βX_0) as fixed; but f_0 and $\widehat{f_0}$ vary depending on the choice of f and Y. Now introduce an equivalence relation on βX_0 as follows: Given two ultrafilters $\mathfrak{U}, \mathfrak{V}$ on X_0, we say $\mathfrak{U} \sim \mathfrak{V}$ if $\widehat{f_0}(\mathfrak{U}) = \widehat{f_0}(\mathfrak{V})$ for every continuous map f from X into a compact Hausdorff space Y.

Now consider the quotient space $\beta X := \beta X_0/\!\sim$. Thus by definition,

$$\beta X = \big\{[\mathfrak{U}] \,:\, \mathfrak{U} \in \beta X_0\big\}$$

where each point of βX is an equivalence class of ultrafilters on X_0:

$$[\mathfrak{U}] = \{\mathfrak{V} \in \beta X_0 \,:\, \mathfrak{V} \sim \mathfrak{U}\};$$

and the topology on βX is the finest topology for which the canonical surjection

$$\pi : \beta X_0 \to \beta X, \quad \mathfrak{U} \mapsto [\mathfrak{U}]$$

is continuous, i.e. a subset $U \subseteq \beta X$ is open iff $\pi^{-1}(U)$ is open in βX_0.

4.1 Theorem. $\bar{\iota} : X \to \beta X$ is the Stone-Čech compactification of X, where $\bar{\iota} = \pi \circ \iota \circ \iota_0^{-1}$.

Proof. Since βX is the continuous image of the compact space βX_0 under the canonical surjection π, βX is also compact. Next we show that given $f : X \to Y$, there is a unique continuous map $\widehat{f} : \beta X \to Y$ such that $\widehat{f} \circ \pi = \widehat{f_0}$, giving a commutative diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "\iota"] \arrow[d, "\iota_0"'] \arrow[dr, "f_0"] & \beta X_0 \arrow[r, "\pi"] \arrow[d, dashed, "\widehat{f_0}"] & \beta X \arrow[dl, dashed, "\widehat{f}"] \\
X \arrow[r, "f"] & Y &
\end{tikzcd}$$

Indeed, any such map $\widehat{f}$ must satisfy

$$\widehat{f}([\mathfrak{U}]) = \widehat{f_0}(\mathfrak{U})$$

for all $\mathfrak{U} \in \beta X_0$. The definition of $\sim$-equivalence guarantees that this gives a well-defined function $\widehat{f} : \beta X \to Y$. To see that it is continuous, let $U \subseteq Y$ be open and note that

$$\pi^{-1}(\widehat{f}^{-1}(U)) = \widehat{f_0}^{-1}(U)$$

which is open in βX_0; and by definition of the quotient topology on $\beta X = \beta X_0/{\sim}$, this means that $\widehat{f}^{-1}(U)$ is open in βX. This gives a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\bar{\iota}"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed, "\widehat{f}"] \\
& Y
\end{tikzcd}$$

and we next show that the map

$$\bar{\iota} = \pi \circ \iota \circ \iota_0^{-1} : X \to \beta X$$

is continuous. Let $x \in X$, and let $V \subseteq \beta X$ be an open neighbourhood of $\bar{\iota}(x)$; we must find an open neighbourhood $U \subseteq X$ of x such that $\iota(U) \subseteq \pi^{-1}(V)$. Since $\pi^{-1}(V) \subseteq \beta X_0$ is an open neighbourhood of $\iota(x) = \mathfrak{F}_x$, by Urysohn's Lemma there exists a continuous function $\beta X_0 \to [0,1]$ taking the value 0 at $\mathfrak{F}_x$, and 1 on $\beta X_0 \smallsetminus \pi^{-1}(V)$. By the comments preceding Theorem 2.1, this function has the form $\widehat{f_0}$ for some continuous $f : X \to [0,1]$. Take $U = f^{-1}([0, \frac{1}{2})) \subseteq X$, an open neighbourhood of x; then $\widehat{f_0}(\iota(U)) = f(U) \subseteq [0, \frac{1}{2})$ which gives $\iota(U) \subseteq \pi^{-1}(V)$ as required; thus $\bar{\iota} : X \to \beta X$ is continuous.

To see that βX is Hausdorff, consider distinct points $[\mathfrak{U}] \neq [\mathfrak{U}']$ in βX, so that $\mathfrak{U} \not\sim \mathfrak{U}'$ in βX_0. This means that there exists a compact Hausdorff space Y and a continuous map $\widehat{f_0} : \beta X_0 \to Y$ such that $\widehat{f_0}(\mathfrak{U}) \neq \widehat{f_0}(\mathfrak{U}')$. Take disjoint open sets $U, U' \subset Y$ with $\widehat{f_0}(\mathfrak{U}) \in U$ and $\widehat{f_0}(\mathfrak{U}') \in U'$. As we have seen, there is a continuous map $\widehat{f} : \beta X \to Y$ such that $\widehat{f} \circ \pi = \widehat{f_0}$. It is easily verified that $\widehat{f}^{-1}(U), \widehat{f}^{-1}(U') \subset \beta X$ are disjoint open neighbourhoods of $[\mathfrak{U}]$ and $[\mathfrak{U}']$ respectively. Thus βX is Hausdorff as claimed.

We have verified the uniqueness of $\widehat{f}$ among all continuous maps $\beta X \to Y$ satisfying $\widehat{f} \circ \pi = \widehat{f}_0$. From this we easily deduce the uniqueness of $\widehat{f}$ among all continuous maps $\beta X \to Y$ satisfying $\widehat{f} \circ \bar{\iota} = f$. Indeed, if $\widehat{f}, \widehat{f}' : \beta X \to Y$ are continuous maps satisfying $\widehat{f} \circ \bar{\iota} = \widehat{f}' \circ \bar{\iota}$, then $(\widehat{f} \circ \pi) \circ \iota = (\widehat{f}' \circ \pi) \circ \iota$; and by the universal property of $\iota : X_0 \to \beta X_0$ we obtain $\widehat{f} \circ \pi = \widehat{f}' \circ \pi$ and thence also $\widehat{f} = \widehat{f}'$. □

5. Application: Hindman's Theorem

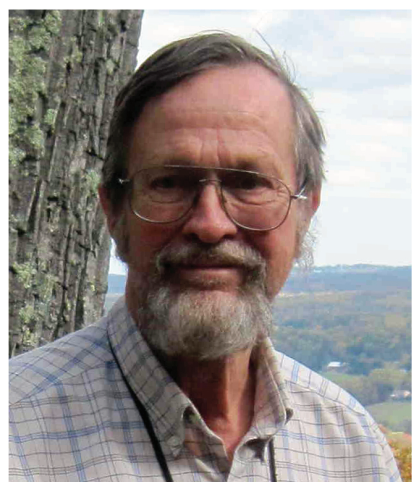
The Stone-Čech compactification of $\mathbb{N} = \{1, 2, 3, \ldots\}$ has far-reaching implications for some areas of additive combinatorics and Ramsey theory. (Although $\mathbb{N} \simeq \omega = \{0, 1, 2, \ldots\}$ and $\beta\mathbb{N} \simeq \beta\omega$ as topological spaces, for algebraic reasons it will be better to use $\mathbb{N}$ which avoids the additive identity.) A cornerstone result of this type is Hindman's Theorem. To introduce the result, for every subset $B \subseteq \mathbb{N}$ we first consider the set of finite sums of distinct elements of B:

$$\Sigma_B = \left\{ \sum S \,:\, S \subseteq B,\ |S| < \infty \right\}.$$

An **IP-set** is a subset $A \subseteq \mathbb{N}$ such that $A \supseteq \Sigma_B$ for some *infinite* subset $B \subseteq \mathbb{N}$. This condition asserts that A is in some sense rather large.

5.1 Theorem (Hindman, 1974). If $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ for some positive integer r, then at least one of the sets C_i is an IP-set.

This result is often paraphrased thus: For any r-colouring of the positive integers, where $r < \infty$, there exists a monochromatic IP-set. The theorem confirms an earlier conjecture of Graham and Rothschild [GR] (and Comfort [C] indicates that this conjecture had been around for some time prior to its publication in 1971). The topological nature of its proof created something of a sensation at the time. This proof, using ultrafilters and the topology of $\beta\mathbb{N}$, is due to Galvin and Glazer (see [C, HS]). This is the most widely read proof of the result, as it is considered the most accessible. Hindman's original proof [H] makes slightly less use of the axiom of choice, so for specialists in reverse mathematics*, it has particular value; but Hindman's proof is considered much more technical.



Neil Hindman
(1943–)

* **Reverse mathematics** is an area of mathematical logic that studies the minimal axioms under which known results can be proved. While most mathematics proceeds from axioms to theorems, this approach starts with a given theorem and generates a list or lists of the required axioms.

Before proceeding with the proof, we require some definitions. For $A \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, define

$$A - k \;=\; \mathbb{N} \cap \{a - k \,:\, a \in A\} = \{n \in \mathbb{N} \,:\, n + k \in A\}.$$

Now given two ultrafilters $\mathfrak{U}, \mathfrak{V}$ on $\mathbb{N}$, define $\mathfrak{U} \oplus \mathfrak{V} \subseteq \mathbb{N}$ to be the collection of all subsets $A \subseteq \mathbb{N}$ such that $A - k \in \mathfrak{U}$ for all k in some $\mathfrak{V}$-set, i.e.

$$\mathfrak{U} \oplus \mathfrak{V} = \big\{A \subseteq \mathbb{N} \,:\, \{k \in \mathbb{N} \,:\, A{-}k \in \mathfrak{U}\} \in \mathfrak{V}\big\}.$$

This operation is in general non-commutative.

5.2 Lemma. If $\mathfrak{U}, \mathfrak{V}$ are ultrafilters on $\mathbb{N}$, then so is $\mathfrak{U} \oplus \mathfrak{V}$. Moreover this operation is associative; and it extends the usual addition in $\mathbb{N}$: for any two principal ultrafilters we have $\mathfrak{F}_{\{a\}} \oplus \mathfrak{F}_{\{b\}} = \mathfrak{F}_{\{a+b\}}$.

Proof. For all $k \in \mathbb{N}$ we have $\varnothing - k = \varnothing \notin \mathfrak{V}$ and $\mathbb{N} - k = \mathbb{N} \in \mathfrak{V}$ so $\varnothing \notin \mathfrak{U} \oplus \mathfrak{V}$, $\mathbb{N} \in \mathfrak{U} \oplus \mathfrak{V}$. If $A, B \in \mathfrak{U} \oplus \mathfrak{V}$ then

$$(A \cap B) - k = (A{-}k) \cap (B{-}k) \in \mathfrak{U}$$

for all k in some $\mathfrak{V}$-set, since this condition holds for both $A{-}k$ and $B{-}k$; thus $A \cap B \in \mathfrak{U} \oplus \mathfrak{V}$. If $A \subseteq B \subseteq \mathbb{N}$ where $A \in \mathfrak{U} \oplus \mathfrak{V}$ then for all k in some $\mathfrak{V}$-set,

$$B{-}k = A{-}k \in \mathfrak{U} \text{ and so also } B{-}k \in \mathfrak{U};$$

thus $B \in \mathfrak{U} \oplus \mathfrak{V}$. If $A \notin \mathfrak{U} \oplus \mathfrak{V}$ and $A' = \mathbb{N} \smallsetminus A$ then for all k in some $\mathfrak{V}$-set,

$$A{-}k \notin \mathfrak{U} \text{ and } \;\mathbb{N} = (A{-}k) \cup (A'{-}k), \text{ so } A{-}k \in \mathfrak{U};$$

thus $A' \in \mathfrak{U} \oplus \mathfrak{V}$. This shows that $\mathfrak{U} \oplus \mathfrak{V} \in \beta\mathbb{N}$, and so '$\oplus$' defines a binary operation on $\beta\mathbb{N}$.

If $a, b \in \mathbb{N}$ then $\{a{+}b\} - b = \{a\} \in \mathfrak{F}_{\{a\}}$ and $\{b\} \in \mathfrak{F}_{\{b\}}$; so $\{a{+}b\} \in \mathfrak{F}_{\{a\}} \oplus \mathfrak{F}_{\{b\}}$. Since the only ultrafilter on $\mathbb{N}$ containing the singleton set $\{a{+}b\}$ is the principal ultrafilter $\mathfrak{F}_{\{a+b\}}$, the final conclusion follows. $\square$

Although '$\oplus$' is not commutative, it is associative:

5.3 Proposition. For all $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \in \beta\mathbb{N}$ we have $(\mathfrak{U} \oplus \mathfrak{V}) \oplus \mathfrak{W} = \mathfrak{U} \oplus (\mathfrak{V} \oplus \mathfrak{W})$.

Proof.

$$\begin{aligned} A \in \mathfrak{U} \oplus (\mathfrak{V} \oplus \mathfrak{W}) \ &\text{iff}\ \{k \in \mathbb{N} \,:\, A{-}k \in \mathfrak{U}\} \in \mathfrak{V} \oplus \mathfrak{W} \\ &\text{iff}\ \big\{\ell \in \mathbb{N} : \{k \in \mathbb{N} \,:\, A{-}k \in \mathfrak{U}\} - \ell \in \mathfrak{V}\big\} \in \mathfrak{W} \\ &\text{iff}\ \big\{\ell \in \mathbb{N} : \{k' \in \mathbb{N} \,:\, (A{-}\ell){-}k' \in \mathfrak{U}\} \in \mathfrak{V}\big\} \in \mathfrak{W} \\ &\qquad\qquad\qquad\qquad (\text{substituting } k = k' + \ell) \\ &\text{iff}\ \big\{\ell \in \mathbb{N} : A{-}\ell \in \mathfrak{U} \oplus \mathfrak{V}\big\} \in \mathfrak{W} \\ &\text{iff}\ A \in (\mathfrak{U} \oplus \mathfrak{V}) \oplus \mathfrak{W}. \end{aligned}$$

□

The map '$\oplus$' is not continuous, i.e. the map $\beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$ is not continuous. However, it satisfies a weaker condition which is sometimes called 'right continuity' (using a different sense of the term than we use for functions $\mathbb{R} \to \mathbb{R}$, however):

5.4 Proposition. For all $\mathfrak{U} \in \beta\mathbb{N}$, the map $\beta\mathbb{N} \to \beta\mathbb{N}$, $\mathfrak{V} \mapsto \mathfrak{U} \oplus \mathfrak{V}$ is continuous.

Proof. It suffices to consider the preimage of a basic open set

$$\langle A \rangle = \{\mathfrak{V} \in \beta\mathbb{N} \,:\, A \in \mathfrak{V}\}$$

where $A \subseteq \mathbb{N}$. We have

$$\begin{aligned} \mathfrak{U} \oplus \mathfrak{V} \in \langle A \rangle \ &\text{iff}\ A \in \mathfrak{U} \oplus \mathfrak{V} \\ &\text{iff}\ \{k \in \mathbb{N} \,:\, A{-}k \in \mathfrak{U}\} \in \mathfrak{V} \\ &\text{iff}\ \mathfrak{V} \in \langle B \rangle \text{ where } B = B_{A,k} = \{k \in \mathbb{N} : A{-}k \in \mathfrak{U}\}. \end{aligned}$$

Since the preimage of basic closed sets are closed (in fact, basic closed sets), the result follows. □

We require the existence of an **idempotent ultrafilter**, i.e. an ultrafilter satisfying $\mathfrak{U} \oplus \mathfrak{U} = \mathfrak{U}$. Since no positive integer a satisfies $a + a = a$, no idempotent ultrafilter can be principal by Lemma 5.2.

5.5 Theorem. There exists $\mathfrak{U} \in \beta\mathbb{N}$ such that $\mathfrak{U} \oplus \mathfrak{U} = \mathfrak{U}$.

Proof. For every $S \subseteq \beta\mathbb{N}$, define

$$S \oplus S = \{\mathfrak{U} \oplus \mathfrak{U}' : \mathfrak{U}, \mathfrak{U}' \in S\}.$$

Consider the collection

$$\mathfrak{S} = \{S \subseteq \beta\mathbb{N} \,:\, S \text{ is nonempty and closed, and } S \oplus S \subseteq S\}.$$

This collection is nonempty since $\beta\mathbb{N} \in \mathfrak{S}$. For every chain $\mathfrak{C} \subseteq \mathfrak{S}$, we claim that $\bigcap \mathfrak{C} \in \mathfrak{S}$ is a lower bound for $\mathfrak{S}$. To see this, note that $\mathfrak{C}$ satisfies the finite intersection property since $\mathfrak{C}$ is totally ordered. Since $\beta\mathbb{N}$ is compact, $\bigcap \mathfrak{C}$ is a nonempty closed set. If $\mathfrak{U} \in S \in \mathfrak{C}$ and $\mathfrak{U}' \in S' \in \mathfrak{C}$, then since $\mathfrak{C}$ is a chain, we may assume $S \subseteq S'$ so that $\mathfrak{U} \oplus \mathfrak{U}' \in S' \subseteq \bigcap \mathfrak{C}$. We have verified all the conditions to show that $\bigcap \mathfrak{C} \in \mathfrak{S}$; and by definition $\bigcap \mathfrak{C} \subseteq S$ for all $S \in \mathfrak{S}$. This shows that every chain in $\mathfrak{S}$ is bounded below. Now by Zorn's Lemma (applied to $\mathfrak{S}$ with *reverse* inclusion), $\mathfrak{S}$ has a minimal member, i.e. there exists $M \in \mathfrak{S}$ satisfying $M \subseteq S$ for all $S \in \mathfrak{S}$. We will show that *every* $\mathfrak{U} \in M$ is idempotent. (Recall that $M \neq \varnothing$ since $M \in \mathfrak{S}$, so *there is* at least one $\mathfrak{U} \in M$.)

Let $\mathfrak{U} \in M$. We show next that

$$\mathfrak{U} \oplus M := \{\mathfrak{U} \oplus \mathfrak{V} \,:\, \mathfrak{V} \in M\} = M.$$

To show this, let $M' = \mathfrak{U} \oplus M \subseteq M$. Then

$$M' \oplus M' \;=\; \mathfrak{U} \oplus M \oplus \mathfrak{U} \oplus M \;\subseteq\; \mathfrak{U} \oplus M \oplus M \oplus M \;\subseteq\; \mathfrak{U} \oplus M = M'.$$

Clearly $M' \neq \varnothing$ since $\mathfrak{U} \oplus \mathfrak{U} \in M'$. Moreover, M' is closed since it is the image of a closed set $M \subseteq \beta\mathbb{N}$ under a continuous map (Proposition 5.4). Thus $M' \in \mathfrak{S}$; and by the minimality of $M \in \mathfrak{S}$, we have $M' = M$ as required.

Again fix $\mathfrak{U} \in M$, and let

$$S = \{\mathfrak{U}' \in M : \mathfrak{U} \oplus \mathfrak{U}' = \mathfrak{U}\}.$$

We will show that $S \in \mathfrak{S}$. First note that $S \neq \varnothing$ since $\mathfrak{U} \in M = \mathfrak{U} \oplus M$. Clearly $S = M \cap \{\mathfrak{U}' \in \beta\mathbb{N} : \mathfrak{U} \oplus \mathfrak{U}' = \mathfrak{U}\}$ is closed since M is closed, and $\{\mathfrak{U}' \in \beta\mathbb{N} : \mathfrak{U} \oplus \mathfrak{U}' = \mathfrak{U}\}$ it is the preimage of a point $\{\mathfrak{U}\}$ under the continuous map of Proposition 5.4. If $\mathfrak{U}', \mathfrak{U}'' \in S$ then

$$\mathfrak{U} \oplus \mathfrak{U}' \oplus \mathfrak{U}'' = \mathfrak{U} \oplus \mathfrak{U}'' = \mathfrak{U}$$

and $\mathfrak{U}' \oplus \mathfrak{U}'' \in M$, so $\mathfrak{U}' \oplus \mathfrak{U}'' \in S$; thus $S \oplus S \subseteq S$. Together this gives $S \in \mathfrak{S}$ as claimed. Again, $S \subseteq M$ so by minimality of M in $\mathfrak{S}$, we have $S = M$. Finally since $\mathfrak{U} \in M$, we obtain $\mathfrak{U} \oplus \mathfrak{U} = \mathfrak{U}$ as desired. $\square$

Under the hypotheses of Theorem 5.1, and assuming $\mathfrak{U}$ is an ultrafilter on $\mathbb{N}$, at least one of the sets C_i is in $\mathfrak{U}$; and so the conclusion of Hindman's Theorem is a consequence of the following:

5.6 Theorem. Let $\mathfrak{U}$ be an idempotent ultrafilter on $\mathbb{N}$. Then every set $A \in \mathfrak{U}$ is an IP-set.

Proof. For each $A \in \mathfrak{U}$, define

$$A^* = \{k \in \mathbb{N} : A - k \in \mathfrak{U}\}.$$

Note that since $\mathfrak{U}$ is idempotent, $A \in \mathfrak{U} = \mathfrak{U} \oplus \mathfrak{U}$ so that $A^* \in \mathfrak{U}$. We will recursively produce

- a chain of subsets $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ with $A_i, A_i^* \in \mathfrak{U}$, and
- an increasing sequence $k_0 < k_1 < k_2 < \cdots$ with each $k_i \in A_i \cap A_i^*$, such that $A_{i+1} = (A_i - k_i) \cap A_i$.

We start with $A_0 = A$. Since $A \cap A^* \in \mathfrak{U}$, there exists $k_0 \in A \cap A^*$. By definition of A^*, $A - k_0 \in \mathfrak{U}$ and so $A_1 := (A - k_0) \cap A \in \mathfrak{U}$. Since $A_1^* \in \mathfrak{U}$ the set $A_1 \cap A_1^* \in \mathfrak{U}$ is infinite, so we may choose $k_1 \in A_1 \cap A_1^*$ such that $k_1 > k_0$. In general having chosen A_i and $k_i \in A_i \cap A_i^*$, by definition the set $A_i - k_i \in \mathfrak{U}$ so $A_{i+1} := (A_i - k_i) \cap A_i \in \mathfrak{U}$. As before, $A_{i+1}^* \in \mathfrak{U}$ so the set $A_{i+1} \cap A_{i+1}^* \in \mathfrak{U}$ is infinite, and we may choose $k_{i+1} \in A_{i+1} \cap A_{i+1}^*$ such that $k_{i+1} > k_i$.

We claim that

$$(*) \qquad k_{n_1} + k_{n_2} + \cdots + k_{n_r} \in A_{n_1} \text{ whenever } n_1 < n_2 < \cdots < n_r,\ r \geqslant 1,$$

which we prove by induction on $r \geqslant 1$. Since $k_n \in A_n$, the initial case holds. Suppose that $r \geqslant 2$ and that the inductive hypothesis holds for the $(r-1)$-term sum

$$k := k_{n_2} + k_{n_3} + \cdots + k_{n_r} \in A_{n_2}\,;$$

then since $k \in A_{n_2} \subseteq A_{n_1+1} = (A_{n_1} - k_{n_1}) \cap A_{n_1}$ we obtain the r-term sum $k_{n_1} + k \in A_{n_1}$ and so (*) holds.

Thus the set of finite sums of $B = \{k_0, k_1, k_2, \ldots\}$ satisfies $\Sigma_B \subseteq A$ as required. □

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