The Borel-Cantelli Lemma

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Abstract

The Borel-Cantelli lemma is a fundamental result in probability theory that describes the behavior of events that occur infinitely often. In this project, we explore the Borel-Cantelli lemma and its applications in various fields.

Background

1. A σ-field F is a purposeful collection of subsets of a sample space Ω that satisfy the following criteria:

   a) The entire sample space Ω is contained within F
   b) F is closed under complement, i.e. if a set B is in F, the complement of B must also be in F,
   c) F is closed under countable unions, i.e., if B1, B2, B3, … are in F, the union of all of them must also be in F.

The largest σ-field is the power class 2^Ω, which contains all possible subsets of Ω. The smallest σ-field contains just the empty set and Ω itself.

2. A probability measure is a real-valued function (denoted by P) that assigns a probability to every event in a σ-field F. It must satisfy the following axioms:

   a) 0 <= P(A) <= 1 for any event A
   b) The probability that nothing in the sample space happens is 0 and the probability that something in the sample space happens is 1
   c) For any collection of disjoint events A_i from the σ-field A_i:

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k).$$

This is known as countable additivity.

Borel-Cantelli

Suppose you flip a biased coin. Then you flip it again. And again. Now flip it infinitely many times. Let A_i be the event that the i-th flip results in heads. Suppose that the coins bias changes over time such that $P(A_i)$ = 1/i for all i.

Now consider the event that infinitely many flips result in heads, denoted by $\limsup_{i\to\infty} A_i$.

The Borel Cantelli Lemma tells us the following:
If each A_i is independent, then

$$\sum_{i=1}^{\infty} P(A_i) = \infty \quad \Rightarrow \quad P\left(\limsup_{i\to\infty} A_i\right) = 1.$$

Despite the fact that the probability of getting heads on the i-th flip approaches 0 and thus becomes increasingly unlikely, the sum of the probabilities of each A_i is infinite (The series 1/n diverges). Consequently, because each coin flip is independent, the probability that we get infinitely many heads is 1. That is to say it will most certainly happen.

On the other hand, if the sum of the probabilities for each A_i is finite, than the probability that the event happens infinitely many times is 0:

$$\sum_{i=1}^{\infty} P(A_i) < \infty \quad \Rightarrow \quad P\left(\limsup_{i\to\infty} A_i\right) = 0.$$

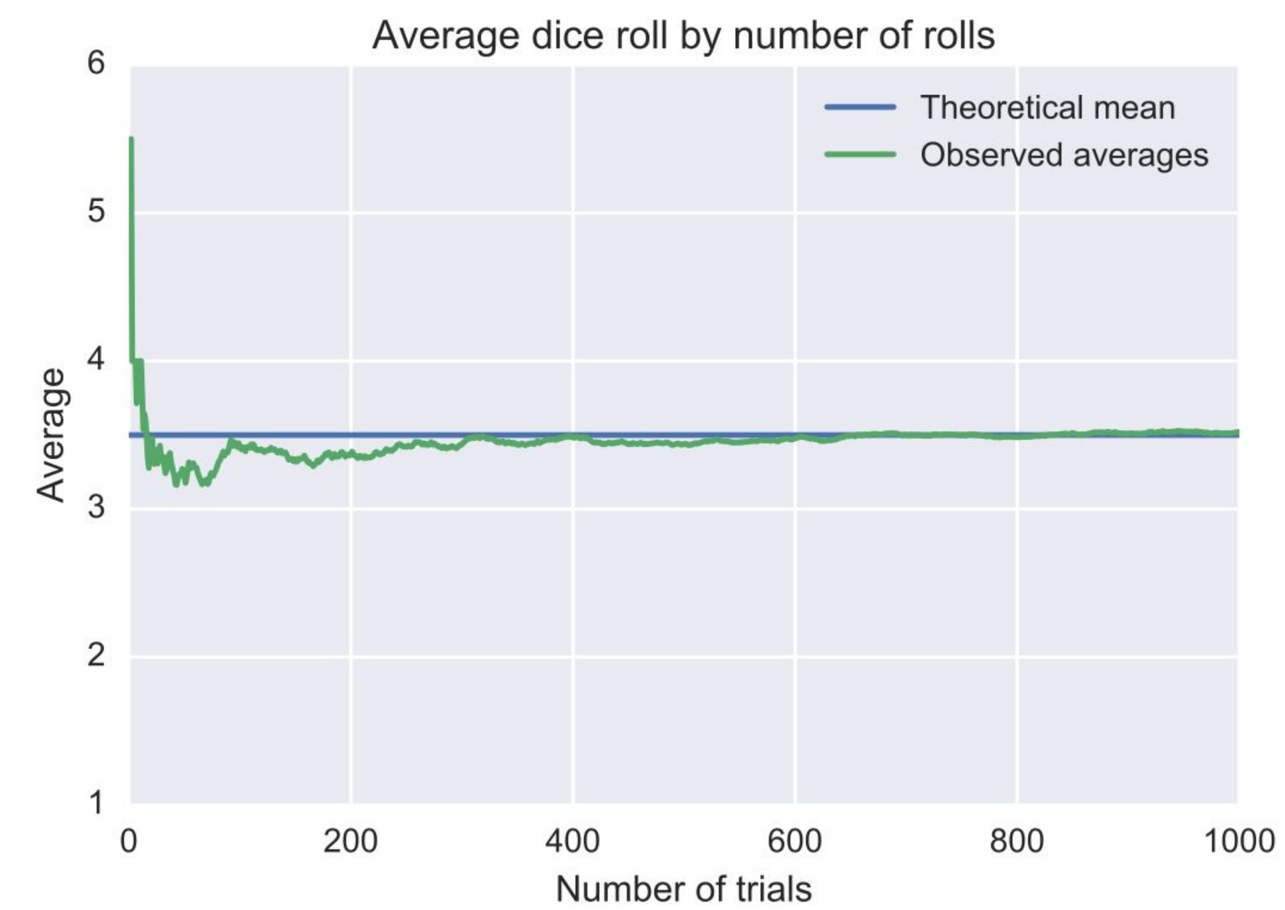
Applications

The Borel-Cantelli Lemma is particularly useful in establishing the Strong Law of Large Numbers:

$$P\left(\omega \in \Omega\colon \lim_{n\to\infty} [X_n(\omega)] = \mu\right) = 1$$

This tells us that the sample average will almost surely converge to the theoretical mean as the number of samples approaches infinity.

Ever hear the saying ‘The house always wins’? This is why! Despite only having a minute advantage in odds, casinos have a positive expected net return. By the law of large numbers, a casino will almost surely profit in the long run. This is known as the Gambler’s Ruin, a concept which dates back to Pascal in 1656.



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References

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