Final Model?

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1 Static Model with multivariate Pareto distribution

Economy is made up of infinite workers who provide one unit of labor inelastically. Each worker is endowed with two productivity profiles, one for urban labor and one for rural labor. Workers draw their urban and rural productivity from two independent Pareto distributions:

$$z_u \sim Pareto(1, \theta_u)$$
$$z_r \sim Pareto(1, \theta_r)$$

Each has a scale factor of 1, and theta is the shape parameter. Since the distributions are independent the joint distribution is just the two density functions multiplied together:

$$f(z_r, z_u) = f_{z_r}(z_r) f_{z_u}(z_u) = \frac{\theta_r}{z_r^{\theta_r+1}} \frac{\theta_u}{z_u^{\theta_u+1}}$$
$$\theta_r, \theta_u > 3$$

The inequality condition is important since it ensures the second moment exists which allows the variance to be calculated [1]. The variance is used to match moments in the data.

The production functions are:

$$Y_r = A_r N_r^{\phi}$$
$$Y_u = A_u N_u - T N_u^p$$
$$N_r = \int_1^{\infty} \int_{z_r^*(z_u)}^{\infty} z_r f(z_r) dz_r f(z_u) dz_u$$
$$N_u = \int_1^{\infty} \int_1^{z_r^*(z_u)} f(z_r) dz_r z_u f(z_u) dz_u$$
$$N_u^p = \int_1^{\infty} \int_1^{z_r^*(z_u)} f(z_r) dz_r f(z_u) dz_u$$
$$0 < \phi < 1$$

[1] I think this is true, but I'm not sure

N_r and N_u are the efficiency units of labor in each sector of the economy. The bound of $z_r^*(z_u)$ is derived later, but represents the threshold of indifference for workers choosing between the two economies. N_u^p is the population working in the urban economy. T is a fixed cost to working in the urban economy. This is the reason we multiply T by N_u^p rather than N_u since the cost each worker pays to migrate is not dependent on the worker's efficiency. ϕ is a scale parameter than ensures diminishing returns to labor in the rural economy.It also ensures there are at least a few workers in this sector since as we derive below $\lim_{N_r \to 0} w_r = \infty$. [2]

The wages per efficiency unit in the rural and urban economies are determined by differentiating the total production functions by efficiency units of labor. This gives:

$$w_r = \frac{\partial Y_r}{\partial N_r} = \phi A_r N_r^{\phi - 1}$$
$$w_u = \frac{\partial Y_u}{\partial N_u} = A_u$$

Below is the equation representing the indifference condition for a worker in this economy. On left hand side is their urban income. We subtract T because they must pay this amount to migrate from the country. The right hand side is their rural income which is achieved by multiplying their rural productivity by the rural wage.

$$z_u A_u - T = z_r^* \phi A_r N_r^{\phi - 1}$$
$$z_r^*(z_u) = \frac{z_u A_u - T}{\phi A_r N_r^{\phi - 1}}$$

A worker with a given rural and urban productivity profile chooses their sector according to:

$$z_r > z_r^*(z_u) \implies rural$$
$$z_r < z_r^*(z_u \implies urban$$

1.1 Finding N_r

There is only one unknown in this model which is N_r. To make the numerical integration easier and to accommodate the non linear solver we can take the

[2]Since $N_r \in \mathbb{R}^+$ we can write $w_r = \phi A_r N_r^{\phi-1}$. Since $\phi - 1 < 0$ we get our desired result: $\lim_{N_r \to 0} w_r = \infty$

first integral for N_r [3] which gives us:

$$N_r = \frac{\theta_r}{\theta_r - 1} N_r^{-(\phi-1)(1-\theta_r)} \int_1^\infty \frac{\theta_u}{z_u^{\theta_u+1}} \left(\frac{z_u A_u - T}{A_r \phi} \right)^{1-\theta_r} dz_u$$
$$C = \frac{\theta_r}{\theta_r - 1} \int_1^\infty \frac{\theta_u}{z_u^{\theta_u+1}} \left(\frac{z_u A_u - T}{A_r \phi} \right)^{1-\theta_r} dz_u$$
$$N_r = C N_r^{-(\phi-1)(1-\theta_r)}$$

This form can be plugged into the non-linear solver. However, there are some values for C that cause N_r to exceed the maximum rural productivity units possible in the model. Thus we derive a maximum for N_r

$$N_r^{max} = \int_1^\infty \int_1^\infty z_r f_{z_r}(z_r) dz_r f_{z_u}(z_u) dz_u$$
$$N_r^{max} = \frac{\theta_r}{\theta_r - 1} \int_1^\infty f_{z_u}(z_u) dz_u$$
$$N_r^{max} = \frac{\theta_r}{\theta_r - 1}$$

Thus the final formula for N_r is:

$$N_r = \max(C N_r^{-(\phi-1)(1-\theta_r)}, \frac{\theta_r}{\theta_r - 1}) \tag{1}$$

1.2 Calculating Moments of the Distribution

To match wage moments in the data we must calculate the first and second moments of the urban and rural wages.

$$E(z_r | z_r > z_r^*(z_u)) = \frac{\int_1^\infty \int_{z_r^*(z_u)}^\infty z_r f_{z_r}(z_r) dz_r f_{z_u}(z_u) dz_u}{P(z_r > z_r^*(z_u))}$$
$$E(z_r^2 | z_r > z_r^*(z_u)) = \frac{\int_1^\infty \int_{z_r^*(z_u)}^\infty z_r^2 f_{z_r}(z_r) dz_r f_{z_u}(z_u) dz_u}{P(z_r > z_r^*(z_u))}$$
$$E(z_u | z_r < z_r^*(z_u)) = \frac{\int_1^\infty \int_1^{z_r^*(z_u)} f_{z_r}(z_r) dz_r z_u f_{z_u}(z_u) dz_u}{P(z_r < z_r^*(z_u))}$$
$$E(z_u^2 | z_r < z_r^*(z_u)) = \frac{\int_1^\infty \int_1^{z_r^*(z_u)} f_{z_r}(z_r) dz_r z_u^2 f_{z_u}(z_u) dz_u}{P(z_r < z_r^*(z_u))}$$

The first moments are fairly simple as we've already written formulas for the numerators:

$$E(z_r | z_r > z_r^*(z_u)) = \frac{N_r}{P(z_r > z_r^*(z_u))}$$
$$E(z_u | z_r < z_r^*(z_u)) = \frac{N_u}{P(z_r < z_r^*(z_u))}$$

[3] See Appendix Note 1

The second moments are more complicated. To make the numerical integration easier, we take the first integrals by hand to get [4]:

$$E(z_r^2|z_r > z_r^*(z_u)) = \left(\frac{\theta_r}{\theta_r - 2}\left(\int_1^\infty (z_r^*(z_u))^{2-\theta_r}\frac{\theta_u}{z_u^{\theta_u+1}}dz_u\right)\right)P(z_r > z_r^*(z_u))^{-1}$$
$$E(z_u^2|z_r < z_r^*(z_u)) = \left(\frac{\theta_r}{2 - \theta_r} - \int_1^\infty z_r^*(z_u)^{-\theta_r}\frac{\theta_u}{z_u^{\theta_u-1}}dz_u\right)P(z_r < z_r^*(z_u))^{-1}$$

1.3 Calculating Rural and Urban income variance

To match the variance of income in the data we have one more step to go. We define $I(z_r, z_u)$, the income of an individual with a given urban and rural productivity profile as:

$$I(z_u, z_r) = \max\{A_r\phi N_r^{\phi-1}z_r, A_u z_u - T\}$$

Individuals make their choice on where to work rationally and choose the sector that will net them the larger income. Notice that the line of indifference between the two sectors for an individual with a given z_r and z_u is once again $z_r^*(z_u)$ where:

$$A_r\phi N_r^{\phi-1}z_r > A_u z_u - T \implies z_r > z_r^*(z_u)$$
$$A_r\phi N_r^{\phi-1}z_r < A_u z_u - T \implies z_r < z_r^*(z_u)$$

We use this definition to compute the first and second moments of the incomes in the rural sector using the calculations done earlier for the first and second moments of wages in the distribution. **This is the section I am confused on. I know I need to divide by the area I'm integrating over, but the previous expectations I wrote out already divided by the area. But now when I take the expectation over I do I need to divide by the area again? And also, do I divide by the area squared for the second moment?** [5]:

$$E(I(z_u, z_r)|z_r > z_r^*(z_u)) = \frac{A_r\phi N_r^\phi}{P(z_r > z_r^*(z_u))}$$
$$E(I(z_u, z_r)^2|z_r > z_r^*(z_u)) = \frac{(A_r\phi N_r^{\phi-1})^2 E(z_r^2|z_r > z_r^*(z_u))}{P(z_r < z_r^*(z_u))}$$

And in the urban economy this gives:

$$E(I(z_u, z_r)|z_r < z_r^*(z_u))) = \frac{A_u N_u - T}{P(z_r < z_r^*(z_u))}$$
$$E(I(z_u, z_r)^2|z_r < z_r^*(z_u))) = \frac{A_u^2 E(z_u^2|z_r < z_r^*(z_u)) - 2TA_u E(z_u|z_r < z_r^*(z_u)) + T^2}{P(z_r < z_r^*(z_u))}$$

[4] See appendix 2 and 3

[5] See Appendix 4 for details on calculations for both rural and urban incomes

Finally we can compute the variance of incomes in each sector of the economy:

$$Var(I(z_u, z_r)|z_r > z_r^*(z_u))) = E(I(z_u, z_r)^2|z_r > z_r^*(z_u))) - \{E(I(z_u, z_r)|z_r > z_r^*(z_u)))\}^2$$
$$Var(I(z_u, z_r)|z_r < z_r^*(z_u))) = E(I(z_u, z_r)^2|z_r < z_r^*(z_u))) - \{E(I(z_u, z_r)|z_r < z_r^*(z_u)))\}^2$$

2 Appendix

1. Taking inner integral of N_r

$$\begin{aligned}
N_r &= \int_1^\infty \int_{z_r^*(z_u)}^\infty z_r \frac{\theta_r}{z_r^{\theta_r+1}} dz_r f_{z_u}(z_u) dz_u \\
&= \int_1^\infty \int_{z_r^*(z_u)}^\infty \theta_r z_r^{-\theta_r} dz_r f_{z_u}(z_u) dz_u \\
&= \int_1^\infty \frac{\theta_r}{\theta_r - 1} z_r^{1-\theta_r}\Big|_{z_r^*(z_u)}^\infty f_{z_u}(z_u) dz_u \\
&= \int_1^\infty \frac{\theta_r}{\theta_r - 1} z_r^*(z_u)^{1-\theta_r} f_{z_u}(z_u) dz_u \\
&= \frac{\theta_r}{\theta_r - 1} \int_1^\infty \left(\frac{z_u A_u - T}{\phi A_r N_r^{\phi-1}}\right)^{1-\theta_r} \frac{\theta_u}{z_u^{\theta_u+1}} dz_u \\
&= \frac{\theta_r}{\theta_r - 1} N_r^{-(\phi-1)(1-\theta_r)} \int_1^\infty \frac{\theta_u}{z_u^{\theta_u+1}} \left(\frac{z_u A_u - T}{A_r \phi}\right)^{1-\theta_r} dz_u
\end{aligned}$$

2. Taking the inner integral of $E(z_r^2|z_r > z_r^*(z_u))$

$$\begin{aligned}
E(z_r^2|z_r > z_r^*(z_u)) &= \left(\int_1^\infty \int_{z_r^*(z_u)}^\infty z_r^2 f_{z_r}(z_r) dz_r f_{z_u}(z_u) dz_u\right) P(z_r > z_r^*(z_u))^{-1} \\
&= \left(\int_1^\infty \int_{z_r^*(z_u)}^\infty \frac{\theta_r}{z_r^{\theta_r-1}} dz_r f_{z_u}(z_u) dz_u\right) P(z_r > z_r^*(z_u))^{-1} \\
&= \left(\int_1^\infty \frac{\theta_r}{\theta_r - 2} z_r^*(z_u))^{2-\theta_r} f_{z_u}(z_u) dz_u\right) P(z_r > z_r^*(z_u))^{-1} \\
&= \left(\frac{\theta_r}{\theta_r - 2}\left(\int_1^\infty (z_r^*(z_u))^{2-\theta_r} \frac{\theta_u}{z_u^{\theta_u+1}} dz_u\right)\right) P(z_r > z_r^*(z_u))^{-1}
\end{aligned}$$

3. Taking the inner integral of $E(z_u^2|z_r < z_r^*(z_u))$

$$
\begin{aligned}
E(z_u^2|z_r < z_r^*(z_u)) &= \left(\int_1^\infty \int_1^{z_r^*(z_u)} f_{z_r}(z_r)dz_r z_u^2 f_{z_u}(z_u)dz_u\right) P(z_r < z_r^*(z_u))^{-1} \\
&= \left(\int_1^\infty \left(1 - z_r^*(z_u)^{-\theta_r}\right) z_u^2 f_{z_u}(z_u)dz_u\right) P(z_r < z_r^*(z_u))^{-1} \\
&= \left(\int_1^\infty \frac{\theta_u}{z_u^{\theta_u - 1}} dz_u - \int_1^\infty z_r^*(z_u)^{-\theta_r} \frac{\theta_u}{z_u^{\theta_u - 1}} dz_u\right) P(z_r < z_r^*(z_u))^{-1} \\
&= \left(\frac{\theta_r}{2 - \theta_r} - \int_1^\infty z_r^*(z_u)^{-\theta_r} \frac{\theta_u}{z_u^{\theta_u - 1}} dz_u\right) P(z_r < z_r^*(z_u))^{-1}
\end{aligned}
$$

4. Deriving moments for rural and urban incomes:
First moment of rural income:

$$
\begin{aligned}
E(I(z_u, z_r)|z_r > z_r^*(z_u)) &= A_r \phi N_r^{\phi-1} E(z_r|z_r > z_r^*(z_u)) \\
&= \frac{A_r \phi N_r^{\phi}}{P(z_r > z_r^*(z_u))}
\end{aligned}
$$

Second moment of rural income:

$$
\begin{aligned}
E(I(z_u, z_r)^2|z_r > z_r^*(z_u)) &= \{A_r \phi N_r^{\phi-1} E(z_r|z_r > z_r^*(z_u))\}^2 \\
&= \frac{(A_r \phi N_r^{\phi-1})^2 E(z_r^2|z_r > z_r^*(z_u))}{P(z_r > z_r^*(z_u))}
\end{aligned}
$$

First moment of urban income:

$$
\begin{aligned}
E(I(z_u, z_r)|z_r < z_r^*(z_u)) &= A_u E(z_u|z_r < z_r^*(z_u)) - T \\
&= \frac{A_u N_u - T}{P(z_r < z_r^*(z_u))}
\end{aligned}
$$

Second moment of urban income:

$$
\begin{aligned}
E(I(z_u, z_r)^2|z_r < z_r^*(z_u)) &= E(\{A_u z_u - T\}^2|z_r < z_r^*(z_u)) \\
&= E(A_u^2 z_u^2 - 2TA_u z_u + T^2|z_r < z_r^*(z_u)) \\
&= \frac{A_u^2 E(z_u^2|z_r < z_r^*(z_u)) - 2TA_u E(z_u|z_r < z_r^*(z_u)) + T^2}{P(z_r < z_r^*(z_u))}
\end{aligned}
$$