Exploring the Structure and Smoothness of Manifolds

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1 Multilinear maps

A tensor $\mathcal{T}$ of order r can be expressed as the *tensor product* of r vectors:

$$\mathcal{T} = u_1 \otimes u_2 \otimes \ldots \otimes u_r \tag{1}$$

We herein fix $r = 3$ whenever it eases exposition. Recall that a vector $u \in U$ can be expressed as the combination of the basis vectors of U. Transform these basis vectors with a matrix A, and if the resulting vector u' is equivalent to uA, then the components of u are said to be *covariant*. If $u' = A^{-1}u$, i.e., the vector changes inversely with the change of basis, then the components of u are *contravariant*. By *Einstein notation*, we index the covariant components of a tensor in subscript and the contravariant components in superscript.

Just as the components of a vector u can be indexed by an integer i (as in u_i), tensor components can be indexed as $\mathcal{T}_{ijk}$. Additionally, as we can view a matrix to be a linear map $M : U \to V$ from one finite-dimensional vector space to another, we can consider a tensor to be a *multilinear* map $\mathcal{T} : V^{*r} \times V^s \to \mathbb{R}$, where V^s denotes the s-th-order Cartesian product of vector space V with itself and likewise for its algebraic dual space V^*. In this sense, a tensor maps an ordered sequence of vectors to one of its (scalar) components. Just as a linear map satisfies $M(a_1u_1 + a_2u_2) = a_1M(u_1) + a_2M(u_2)$, we call an r-th-order tensor multilinear if it satisfies

$$\mathcal{T}(u_1, \ldots, a_1v_1 + a_2v_2, \ldots, u_r) = a_1\mathcal{T}(u_1, \ldots, v_1, \ldots, u_r) + a_2\mathcal{T}(u_1, \ldots, v_2, \ldots, u_r), \tag{2}$$

for scalars a_1 and a_2.

Let $\{\mathcal{T}\}$ be the set of all tensors. Endow $\{\mathcal{T}\}$ with the binary operator $+$ that maps two tensors, $\mathcal{T}_1$ and $\mathcal{T}_2$, to a tensor whose ijk-th component is the scalar addition of the ijk-th components of $\mathcal{T}_1$ and $\mathcal{T}_2$. Let scalar multiplication be defined on $\{\mathcal{T}\}$ such that the multiplication of $\mathcal{T}$ by a scalar a results in a multiplication of its components by a. Further endow the set with the tensor product operator as defined above. Then with these operators, $\{\mathcal{T}\}$ forms an algebra over the field $\{a\}$, the set of all a for which the above properties hold.

2 Differentiation on smooth manifolds

Let X be any set and T be a collection of subsets of X. Call the members of T *open sets*. T forms a topology on X if its members satisfy:

- T contains X and the empty set $\varnothing$
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open

Then (X, T) is a *topological space*. For simplicity, we often omit T and refer to a topological space by its underlying set X.

An open set containing $x \in X$ is a *neighborhood* of x. T is *Hausdorff* if there exist disjoint sets $A, B \in T$ such that $a \in A$ and $b \in B$, $\forall a, b \in X$. That is, on a Hausdorff topological space, any two points lie in disjoint neighborhoods.

Proposition 2.1. *A subspace of a Hausdorff space is Hausdorff.*

Proof. Let H and $G \subseteq H$ be topological spaces. (We omit their topologies for simplicity.) Suppose G is not Hausdorff. Then for some $a, b \in G$, there exist $A, B \subseteq G$ such that $a \in A$, $b \in B$, and $A \cap B \neq \varnothing$. Yet, $a, b \in H$ and $A, B \subseteq H$, so the same can be said for H. Thus, G not Hausdorff implies H not Hausdorff. $\square$

Recall that a map $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $c \in \mathbb{R}$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon, \forall x \in \mathbb{R}$. Intuitively, this definition of continuity seems to highlight a relationship between "neighborhoods" of size ϵ and δ. We can analogously understand continuity of maps between topological spaces.

Let X, Y be two topological spaces and $A \in X, B \in Y$ be open sets. A map $f : X \to Y$ is continuous if $\forall B \in Y$ and $\forall x \in X$, there exists a neighborhood A of x such that $f(A) \in B$. Equivalently, we can say that the inverse image of an open set in Y is open.

If both f and its inverse are continous, then f is a *homeomorphism*. Then there exists a continuous map $g : Y \to X$ such that $f \circ g = g \circ f = 1$. Then X and Y are *homeomorphic* or *topologically equivalent*.

Recall that a map $f : \mathbb{R} \to \mathbb{R}$ is smooth if $\partial^k f / \partial x^k$ exists and is continuous $\forall x \in \mathbb{R}$ and $k = 1, 2, 3, \ldots$. Equivalently, we say that f is $C^\infty(\mathbb{R})$. We can similarly define smoothness for maps between open sets.

Let $X, Y \subseteq \mathbb{R}^n$ be open sets. The map $f : X \to Y$ is smooth if every component of the Jacobian matrix

$$Df(x) := \left[\frac{\partial^i f}{\partial x^j}\right] \tag{3}$$

exists and is continuous $\forall i, j = 1, 2, 3, \ldots$. Here, we note that, for a single component of f (i.e., a map $f^i : \mathbb{R}^n \to \mathbb{R}$),

$$\frac{\partial f^i}{\partial x} = \frac{\partial^n f^i}{\partial x^1 \ldots \partial x^n}. \tag{4}$$

Let $f : X \to Y$ be a homeomorphism, where $X, Y \subseteq \mathbb{R}$ are open sets. If both f and its inverse are smooth, then f is a *diffeomorphism*.