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KU-MATH

FinKont2: Hand-in Exercise #3

Answers (individually composed – but feel free to discuss in groups) must be handed in via Absalon no later than 23:59 CET on Sunday April 6, 2025.

Portfolio insurance, static hedging, stochastic volatility

Remark: A longer story about optimal dynamics portfolio choice, fixed fraction strategies, and who would (or would not) like to have portfolio insurance can be told (see for instance https://tinyurl.com/67bdb7fn) — but that's not our concern in this Hand-in.

We consider first the Black-Scholes model, i.e. there is a constant interest rate r and a stock whose price follows,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^P(t)$$

where W^P is a Brownian motion under the real-world probability measure P. As usual, Q denotes the martingale measure in this model. When numerical values are needed, use $r = 0.02$, $\mu = 0.07$, and $\sigma = 0.20$.

<u>1a</u>
Consider a self-financing strategy that has the fixed fraction a invested in the stock and let A_t^a denote its value process. Use results from Björk's Section 6.4 to find dA_t^a, thus proving that A_t^a is geometric Brownian motion under both P and Q. What are the drift rates and what is the volatility? Show that we can write

$$A_t^a = g(t)(S_t)^a,$$

where g is a function that depends on time (and model parameters) but not on S_t. (And where the last factor on the right-hand side is S_t raised to the power a.)

A (highly stylized, arguably) pension company invests its clients savings (over some time interval $[0;T]$) according to the fixed fraction strategy described above. The pension company is considering offering portfolio insurance to its clients, more specifically a contract that pays $(K - A_T^a)^+$ at time T for some K.

<u>1b</u>
What would it cost for the pension company to replicate a portfolio insurance contract (give a formula) and how can the contract be replicated by trading dynamically in the stock (careful; S, not A^a – use last part of question 1a) and the risk-free asset (devise and run a discrete hedge simulation experiment)? Assume $S_0 = A_0^a = 1$, $T = 30$, $K = e^{rT}$, $a = 0.5$.

<u>1c</u>
Compute (give formulas) for the following quantities

- $e^{-rT}\mathrm{E}^P((K - A_T^a)^+)$
- $e^{-rT}\mathrm{E}^Q((K - (S_T)^a)^+)$
- $ae^{-rT}\mathrm{E}^Q((K - S_T)^+)$

How do these quantities compare to the replication price you found in question 1b?

1d
Prove equation (1) – the so-called spanning formula – in Carr & Madan (2001).

1e
Use the spanning formula (in a suitably discretized version) to calculate the composition of a (static) portfolio of (expiry T) put options on the stock that replicates the portfolio insurance contract. (Use the result from the last part of question 1a.) What happens to the price of the put portfolio as you make the discretization finer and finer?

We now – and for the rest the exercise – change to a market where S has Heston-dynamics, which we for simplicty give directly under a martingale measure Q,

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{V(t)}S(t)dW_1^Q(t) \\ dV(t) &= \kappa(\theta - V(t))dt + \epsilon\sqrt{V(t)}dW_2^Q(t), \end{aligned}$$

where $dW_1^Q(t)dW_2^Q(t) = \rho dt$.
When numerical values are needed, use $V_0 = \theta = 0.20^2$, $\kappa = 2.0$, $\epsilon = 1.0$, $\rho = -0.5$.

1f
Find dA_t^a and explain how $e^{-rT}\mathrm{E}^Q((K - A_T^a)^+)$ can be calculated using the Heston (call price) formula (along with a put/call-parity type argument) and that this gives an arbitrage-free price of the portfolio insurance contract.

1g
Calculate $e^{-rT}\mathrm{E}^Q((K - A_T^a)^+)$ using $S_0 = A_0^a = 1$, $T = 30$, $K = e^{rT}$, $a = 0.5$ and other parameters as given above. (For this you will need an implementation of the Heston formula. There is a variety to choose from here.)

1h
Consider the static put option hedge portfolio from question 1e. Calculate its time 0-price assuming that put options are priced with the Q-parameters given above. Is this a perfectly replicating strategy? What happens if $a = 1$?

1i
Assume now that $a = 1$ (but all other things as above; and we're still in the Heston model)– so the portfolio insurance contract is just a put option on the stock. What is the Δ of the put price (i.e. the derivative of the price wrt. the stock price)? Demonstrate by simulation (feel free to use the R-code for simulation from the March 18 lectures) that a strategy that holds $\Delta(t)$ in the stock at time t and is kept self-financing vith the risk-free asset *does not* perfectly replicate the put option payoff.

1j

Explain how to perfectly replicate an expiry T, strike K_1-put option in the Heston model by trading dynamically in the stock, the risk-free asset and a put option with strike K_0 (and expiry T). (Follow the argumentation on page 5 in Gatheral's book. You must be as explicit as possible about how to compute things – but implementing and testing things experimentally is not required.) What if rather than the strike K_0 put, what you can trade is an expiry T variance swap on the stock?