February 18, 2025
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KU-MATH

# FinKont2: Hand-In Exercise #1

Answers (individually composed – but feel free to discuss in groups) must be handed in via Absalon no later than 23:59 CET on Tuesday February 25, 2025. There are 10 equal-weighted (number.letter)-named questions.

## Moments Etc. (30%)

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1.a
Consider the Vasicek (or Ornstein/Uhlenbeck) stochastic differential equation (SDE)

$$dX(t) = \kappa(\theta - X(t))dt + \sigma dW(t).$$

Show that

$$X(t+u)|\mathcal{F}_t \sim N\left(X(t)e^{-\kappa u} + \theta(1 - e^{-\kappa u}), \frac{\sigma^2(1 - e^{-2\kappa u})}{2\kappa}\right)$$

Hint: Look at the dynamics of the process defined by $Z(t) = e^{\kappa t}X(t)$ and use Björk's Lemma 4.18.

1.b
Consider the Cox/Ingersoll/Ross (or $\sqrt{\ }$) SDE

$$dX(t) = \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dW(t), \quad X(0) = x.$$

Put $m(t) = \mathbf{E}(X(t))$. Show that

$$m(t) = X(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}).$$

Hint: Write the SDE on integral form, take mean, interchange, and observe that the $m$-function solves an ordinary differential equation (ODE).

1.c – still considering the Cox/Ingersoll/Ross (or $\sqrt{}$) SDE
Derive an ODE for the second moment, $h(t) := \mathbf{E}(X^2(t))$. Hint: Use Ito on $X^2$ and do as in 1.b.
Prove (e.g. by direct verification) that

$$\mathrm{var}(X(t)) = X(0)\frac{\sigma^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \theta\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa t})^2.$$

Remark: Because $X$ is a time-homogeneous Markov process conditional moments are found by shifting the formulas/equations.

# The Bachelier Model (30%)

The Bachelier model – named after the French finance (and stochastic processes) pioneer Louis Bachelier who used it in his 1900 dissertation – models the price of a stock, $S$, by a so-called arithmetic (as oppsosed to geometric) process, which has dynamics of the form

$$dS(t) = \ldots dt + \sigma dW(t),$$

the important point being that the term in front of $dW(t)$ is constant, i.e. it does not have $S(t)$ itself in it.

2.a
Suppose the interest rate is 0 and consider a strike-$K$ expiry-$T$ call-option. Show that its arbitrage-free time-$t$ price is

$$\pi^{\text{call, Bach}}(t) = (S(t) - K)\Phi\left(\frac{S(t) - K}{\sigma\sqrt{T - t}}\right) + \sigma\sqrt{T - t}\phi\left(\frac{S(t) - K}{\sigma\sqrt{T - t}}\right),$$

where $\Phi$ and $\phi$ denote, respectively, the standard normal distribution and density function. Show that the time-$t$ $\Delta$-hedge-ratio of the call-option is

$$\Delta^{\text{call, Bach}}(t) = \Phi\left(\frac{S(t) - K}{\sigma\sqrt{T - t}}\right).$$

Hint: You may without proof use that if $X \sim N(\mu, \sigma^2)$ then

$$\mathrm{E}(X\mathbf{1}_{l \le X \le h}) = \mu\left(\Phi\left(\frac{h - \mu}{\sigma}\right) - \Phi\left(\frac{l - \mu}{\sigma}\right)\right) + \sigma\left(\phi\left(\frac{l - \mu}{\sigma}\right) - \phi\left(\frac{h - \mu}{\sigma}\right)\right),$$

where as usual $\mathbf{1}$ denotes the indicator function.

2.b (where we still assume that the interest rate is 0)
Assume that $S(0) = 100$, $T = 0.25$, and $\sigma = 15$.
What do the implied volatilities across strikes look like in the Bachelier model? Comment on the results. What happens if instead we assume that $S(0) = 50$?

2.c
What does the call-price formula look like in a Bachelier model with a non-zero (but constant) interest rate $r$? Hint: Use the trick from 1.a and be careful about time-dependent parameters.

# Quanto Hedging and The Kingdom of Denmark Put (40%)

In this exercise we consider an arbitrage-free currency model of Black/Scholes-type. More specifically, we think of and refer to "domestic" as "US" and "foreign" as "Japan". With Björk's Proposition 18.7 (with a sign typo corrected) in mind we write bank-accounts, exchange-rate and Japanese stock dynamics under the US-martingale measure as

$$\begin{aligned} dB_{US}(t) &= r_{US}B_{US}(t)dt \\ dX(t) &= X(t)(r_{US}-r_J)dt + X(t)\sigma_X^\top dW(t) \\ dB_J(t) &= r_J B_J(t)dt \\ dS_J(t) &= S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)\sigma_J^\top dW(t), \end{aligned}$$

where the $\sigma$'s are 2-dimensional (constant column) vectors. (We don't need a US stock here; we'll get enough fun as it is.)

For numerical experiments, you can/may/should assume that $r_{US} = 0.03$ $r_J = 0.00$, $\sigma_X^\top = (0.1, 0.02)$ a, and $\sigma_J^\top = (0, 0.25)$.

Consider a *quanto put*. This is an option that at time $T$ pays off

$$Y_0(K - S_J(T))^+,$$

where (the constant) $Y_0$ is some agreed-upon-in-advance exchange-rate; it could be the time-0 exchange rate or a forward exchange rate.

<u>3.a</u>
Show that the arbitrage-free time-$t$ price of the quanto put is $F^{QP}(t, S_J(t))$, where the function $F^{QP}$ is defined by

$$F^{QP}(t,s) = Y_0 e^{-r_{US}(T-t)}\left(K\Phi(-d_2(t,s)) - e^{(r_J-\sigma_X^\top\sigma_J)(T-t)}s\Phi(-d_1(t,s))\right),$$

$\Phi$ is the standard normal distribution function as usual,

$$d_{1/2}(t,s) = \frac{\ln(s/K) + (r_J - \sigma_X^\top\sigma_J \pm ||\sigma_J||^2/2)(T-t)}{\sqrt{T-t}||\sigma_J||},$$

and $||\cdot||$ denotes the Euclidian norm of a vector.
Show that

$$\frac{\partial F^{QP}(t,s)}{\partial s} = Y_0 e^{(r_J-\sigma_X^\top\sigma_J-r_{US})(T-t)}\left(\Phi(d_1(t,s)) - 1\right) =: g(t,s).$$

("=:" and ":=" mean "defined as", with the term closest to ":" being defined.)
Hint: Careful "pattern recognition" brings you back to Black-Scholes, thus requiring no new calculations.

<u>3.b</u>
Consider now the usual set-up for a discrete hedge experiment, i.e. the interval $[0;T]$ is split into $n$ pieces at the equidistant time-points $t_i$, where we adjust our portfolio.
Illustrate experimentally (say, in the plausible case where $Y_0 = X(0) = 1/100$ and $K = S_J(0) = 30,000$ and $T = 2$) that a strategy that

- at time $t_i$ holds
$$\Delta^{QP}(t_i, S_J(t_i), X(t_i)) := \frac{g(t_i, S_J(t_i))}{X(t_i)}$$
units of the foreign stock, and
- is then made self-financing with the domestic bank-account,

*does not* replicate the pay-off of the quanto put (even in the '$n \to \infty$'-limit).

3.c

Suppose now that the strategy from 3.b in amended/extended by holding (at time $t_i$)

$$-\Delta^{QP}(t_i, S_J(t_i), X(t_i))S_J(t_i)$$

units of currency, which are instantly deposited in the foreign bank (giving you $-\Delta^{QP}S_J/B_J$ units of it), where they earn interest; it is still kept self-financing via domestic bank-account. Illustrate experimentally that this strategy ("suitably initiated" and "in the limit") *does* replicate the pay-off of the quanto put.

3.d

Explain why (i.e.: prove mathematically that) the strategy from part 3.c works (replication-wise in continuous time) while the strategy from 3.b does not.