Hand-In Exercise #1

Continuous Time Finance 2 (FinKont2)

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Part 1: Moments Etc.

Question 1.a

We consider the Vasicek (or Ornstein/Uhlenbeck) stochastic differential equation (SDE) given by

$$dX(t) = \kappa(\theta - X(t))dt + \sigma dW(t).$$

To show that

$$X(t+u) \mid \mathcal{F}_t \sim \mathcal{N}\left(X(t)e^{-\kappa u} + \theta(1 - e^{-\kappa u}), \frac{\sigma^2(1 - e^{-2\kappa u})}{2\kappa}\right),$$

we use the hint and look at the dynamics of the process defined by $Z(t) = e^{\kappa t}X(t)$. We first get

$$\begin{aligned} e^{\kappa t}X(t) &= e^{\kappa t}\kappa(\theta - X(t))dt + e^{\kappa t}\sigma dW(t) \\ &= e^{\kappa t}\kappa\theta dt - e^{\kappa t}\kappa X(t)dt + e^{\kappa t}\sigma dW(t) \\ \Leftrightarrow \quad e^{\kappa t}X(t) + e^{\kappa t}\kappa X(t)dt &= e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma dW(t). \end{aligned}$$

Observing that $\frac{d}{dt}(e^{\kappa t}X(t)) = e^{\kappa t}dX(t) + \kappa e^{\kappa t}X(t)dt$, the dynamics of $Z(t)$ are then given by

$$dZ(t) = d(e^{\kappa t}X(t)) = e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma dW(t).$$

Since we are interested in the expectation of $X(t+u)$ conditional on $\mathcal{F}_t$, we integrate both sides to write the dynamics in integral form over the interval $[t, t+u]$. We obtain that

$$\begin{aligned} \int_t^{t+u} dZ(s) &= \int_t^{t+u} d(e^{\kappa s}X(s)) \\ &= \int_t^{t+u} e^{\kappa s}\kappa\theta ds + \int_t^{t+u} e^{\kappa s}\sigma dW(s) \\ &= \kappa\theta \int_t^{t+u} e^{\kappa s}ds + \sigma \int_t^{t+u} e^{\kappa s}dW(s) \\ \Leftrightarrow \quad e^{\kappa(t+u)}X(t+u) - e^{\kappa t}X(t) &= \kappa\theta \int_t^{t+u} e^{\kappa s}ds + \sigma \int_t^{t+u} e^{\kappa s}dW(s) \\ \Leftrightarrow \quad e^{\kappa(t+u)}X(t+u) &= e^{\kappa t}X(t) + \kappa\theta \int_t^{t+u} e^{\kappa s}ds + \sigma \int_t^{t+u} e^{\kappa s}dW(s). \end{aligned}$$

The first integral on the right-hand side can easily be computed by noting that

$$\int_t^{t+u} e^{\kappa s}ds = \left[\frac{1}{\kappa}\left(e^{\kappa s}\right)\right]_{s=t}^{s=t+u} = \frac{1}{\kappa}\left(e^{\kappa(t+u)} - e^{\kappa t}\right) \Rightarrow \kappa\theta \int_t^{t+u} e^{\kappa s}ds = \theta\left(e^{\kappa(t+u)} - e^{\kappa t}\right).$$

To then isolate $X(t+u)$, we multiply the entire equation by $e^{-\kappa(t+u)}$, which gives

$$\begin{aligned} e^{\kappa(t+u)}X(t+u) &= e^{\kappa t}X(t) + \kappa\theta \int_t^{t+u} e^{\kappa s} ds + \sigma \int_t^{t+u} e^{\kappa s} dW(s) \\ \Leftrightarrow X(t+u) &= e^{-\kappa u}X(t) + \theta(1-e^{-\kappa u}) + \sigma \int_t^{t+u} e^{-\kappa((t+u)-s)} dW(s) \\ &= e^{-\kappa u}X(t) + \theta(1-e^{-\kappa u}) + \sigma \int_t^{t+u} e^{\kappa(s-t-u)} dW(s). \end{aligned}$$

Conditioned on $\mathcal{F}_t$, the first two terms are known (i.e., $\mathcal{F}_t$-measurable). The only "new" randomness arises from the Brownian increment $\int_t^{t+u} e^{\kappa(s-t-u)} dW(s)$ independent of $\mathcal{F}_t$. For each fixed t and u, the integrand $e^{\kappa(s-t-u)}$ is deterministic in s. Hence, by Lemma 4.18 in Björk (2019), the integral is normally distributed with mean 0 (conditional on $\mathcal{F}_t$), and its variance is found via Itô's isometry. Since the integral is independent of $\mathcal{F}_t$, the right-hand side is a sum of an $\mathcal{F}_t$-measurable term plus an independent normal increment. So, $X(t+u)|\mathcal{F}_t \sim \mathcal{N}(\mathbb{E}[X(t+u)|\mathcal{F}_t], \mathbb{V}[X(t+u)|\mathcal{F}_t])$, where

$$\begin{aligned} \mathbb{E}[X(t+u)|\mathcal{F}_t] &= \mathbb{E}\left[e^{-\kappa u}X(t) + \theta(1-e^{-\kappa u}) + \sigma \int_t^{t+u} e^{\kappa(s-t-u)} dW(s) \middle| \mathcal{F}_t \right] \\ &= e^{-\kappa u}X(t) + \theta(1-e^{-\kappa u}), \end{aligned}$$

since $\mathbb{E}\left[\int_t^{t+u} e^{\kappa(s-t-u)} dW(s)\right] = 0$. Also, the variance is given by

$$\begin{aligned} \mathbb{V}[X(t+u)|\mathcal{F}_t] &= \mathbb{V}\left[\sigma \int_t^{t+u} e^{\kappa(s-t-u)} dW(s) \,\middle|\, \mathcal{F}_t \right] \\ &= \sigma^2\, \mathbb{V}\left[\int_t^{t+u} e^{\kappa(s-t-u)} dW(s) \,\middle|\, \mathcal{F}_t \right] \\ &= \sigma^2 \int_t^{t+u} \left(e^{\kappa(s-t-u)} \right)^2 ds \\ &= \sigma^2 \int_t^{t+u} e^{2\kappa(s-t-u)} ds \\ &= \sigma^2 \left[\frac{e^{2\kappa(s-t-u)}}{2\kappa} \right]_{s=t}^{s=t+u} \\ &= \frac{\sigma^2\left(1-e^{-2\kappa u}\right)}{2\kappa} \end{aligned}$$

which is exactly the parameters for the normal distribution of $X(t+u) \mid \mathcal{F}_t$ that we had to show.

Question 1.b

We consider the Cox/Ingersoll/Ross stochastic differential equation (SDE) given by

$$dX(t) = \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dW(t), \quad X(0) = x$$

and set $m(t) = \mathbb{E}(X(t))$. To show that

$$m(t) = X(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}),$$

we start by writing the SDE on integral form, such that

$$\begin{aligned} X(t) &= X(0) + \int_0^t \kappa(\theta - X(s))ds + \int_0^t \sigma\sqrt{X(s)}dW(s) \\ &= x + \kappa\int_0^t (\theta - X(s))ds + \sigma\int_0^t \sqrt{X(s)}dW(s). \end{aligned}$$

Then, we take the expectation of both sides and use the linearity of expectations, giving

$$\begin{aligned} m(t) = E[X(t)] &= E\left[x + \kappa\int_0^t (\theta - X(s))ds + \sigma\int_0^t \sqrt{X(s)}dW(s)\right] \\ &= E[x] + E\left[\kappa\int_0^t (\theta - X(s))ds\right] + E\left[\sigma\int_0^t \sqrt{X(s)}dW(s)\right] \\ &= x + \kappa\int_0^t (\theta - E[X(s)])ds \\ &= x + \kappa\int_0^t (\theta - m(s))ds \end{aligned}$$

since x is constand and the stochastic integral $\int_0^t \sqrt{X(s)}dW(s)$ is a martingale with mean zero. We now differentiate both sides with respect to t, which gives

$$\frac{d}{dt}m(t) = \kappa(\theta - m(t)).$$

This is a first-order linear ordinary differential equation (ODE) of type $m'(t)+p(t)m(t) = q(t)$ with

$$m'(t) + \kappa m(t) = \kappa\theta, \quad m(0) = X(0) = x.$$

By using the technique of an integrating factor, we now solve this ODE. Here, $p(t) = \kappa$ and $q(t) = \kappa\theta$, so the integrating factor can be computed by

$$\mu(t) = e^{\int p(t)dt} = e^{\int \kappa dt} = e^{\kappa t}.$$

Multiplying both sides of the original ODE by $e^{\kappa t}$ gives

$$m'(t) + \kappa m(t) \Leftrightarrow e^{\kappa t} m'(t) + e^{\kappa t}\kappa m(t) = e^{\kappa t}\kappa\theta,$$

where we notice that the left-hand side is exactly the derivative of $e^{\kappa t}m(t)$, since

$$\frac{d}{dt}\left(e^{\kappa t}m(t)\right) = e^{\kappa t}m'(t) + e^{\kappa t}\kappa m(t),$$

such that the ODE can be rewritten as

$$\frac{d}{dt}\left(e^{\kappa t}m(t)\right) = e^{\kappa t}\kappa\theta.$$

Integrating both sides, using $m(0) = X(0)$ and solving for $m(t)$ then gives us the ODE solution by

$$\begin{aligned}
\int_0^t \frac{d}{dt}\left(e^{\kappa s}m(s)\right) ds = \int_0^t e^{\kappa s}\kappa\theta ds \\
&\Leftrightarrow e^{\kappa t}m(t) - e^{\kappa \cdot 0}m(0) = \kappa\theta \int_0^t e^{\kappa s} ds \\
&\Leftrightarrow e^{\kappa t}m(t) - m(0) = \kappa\theta \left[\frac{1}{\kappa}e^{\kappa s}\right]_{s=0}^{s=t} \\
&\Leftrightarrow e^{\kappa t}m(t) - x = \kappa\theta \cdot \frac{1}{\kappa}\left(e^{\kappa t} - 1\right) \\
&\Leftrightarrow e^{\kappa t}m(t) - x = \theta\left(e^{\kappa t} - 1\right) \\
&\Leftrightarrow e^{\kappa t}m(t) = x + \theta\left(e^{\kappa t} - 1\right) \\
&\Leftrightarrow m(t) = e^{-\kappa t}x + e^{-\kappa t}\theta\left(e^{\kappa t} - 1\right) \\
&\Leftrightarrow m(t) = e^{-\kappa t}X(0) + \theta(1 - e^{-\kappa t}),
\end{aligned}$$

and this is exactly the solution we set out to find.

Question 1.c

We first have to derive an ODE for the second moment defined by $h(t) := \mathbb{E}(X^2(t))$. Using the hint from the exercise, we apply Proposition 4.12 with Itô's formula in Björk (2019) to the stochastic process $X^2(t)$. We define $f(x) = x^2$ and note that its derivatives are $f'(x) = 2x$ and $f''(x) = 2$.

Hence, Itô's formula and the dynamics of $X(t)$ gives

$$\begin{aligned}
d(X^2(t)) &= \frac{\partial f}{\partial x} dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX(t))^2 \\
&= 2X(t)dX(t) + \frac{1}{2}\cdot 2(dX(t))^2 \\
&= 2X(t)\left[\kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dW(t)\right] + \left[\kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dW(t)\right]^2 \\
&= 2X(t)\kappa(\theta - X(t))dt + 2X(t)\sigma\sqrt{X(t)}dW(t) + \sigma^2 X(t)dt \\
&= \left[2\kappa X(t)(\theta - X(t)) + \sigma^2 X(t)\right]dt + 2X(t)\sigma\sqrt{X(t)}dW(t),
\end{aligned}$$

where we have used that $(dt)^2 = 0$, $(dW(t))^2 = dt$ and $dt \cdot dW(t) = 0$. On integral form, this reads

$$\begin{aligned}
X^2(t) &= X^2(0) + \int_0^t \left[2\kappa X(s)(\theta - X(s)) + \sigma^2 X(s)\right]ds + \int_0^t 2X(s)\sigma\sqrt{X(s)}dW(s) \\
&= X^2(0) + 2\kappa\int_0^t \left[\left(\theta + \frac{\sigma^2}{2\kappa}\right)X(s) - X^2(s)\right]ds + 2\sigma\int_0^t X(s)\sqrt{X(s)}dW(s).
\end{aligned}$$

Then, we take the expectation of both sides and use the linearity of expectations, such that

$$\begin{aligned}
\mathbb{E}[X^2(t)] &= \mathbb{E}[X^2(0)] + \mathbb{E}\left[2\kappa\int_0^t \left[\left(\theta + \frac{\sigma^2}{2\kappa}\right)X(s) - X^2(s)\right]ds\right] + \mathbb{E}\left[2\sigma\int_0^t X(s)\sqrt{X(s)}dW(s)\right] \\
&= X^2(0) + 2\kappa\int_0^t \left[\left(\theta + \frac{\sigma^2}{2\kappa}\right)\mathbb{E}[X(s)] - \mathbb{E}[X^2(s)]\right]ds,
\end{aligned}$$

where the expectation of the stochastic integral vanished, see Lemma 4.18 in Björk (2019). Defining $\mathbb{E}[X^2(t)] := h(t)$, $\mathbb{E}[X(s)] := m(s)$ and $\mathbb{E}[X^2(s)] := h(s)$, we differentiate w.r.t t and get an ODE by

$$\frac{d}{dt}h(t) = 2\kappa\left(\theta + \frac{\sigma^2}{2\kappa}\right)m(t) - 2\kappa h(t), \quad h(0) = X^2(0).$$

By using the technique of an integrating factor as in Question 1.b, we start by multiplying both sides of the ODE by the integrating factor $e^{2\kappa t}$ and inserting $m(t)$ from 1.b, which gives

$$\begin{aligned}
e^{2\kappa t}dh(t) + e^{2\kappa t}2\kappa h(t)dt &= e^{2\kappa t}2\kappa\left(\theta + \frac{\sigma^2}{2\kappa}\right)m(t)dt \\
\Leftrightarrow d(e^{2\kappa t}h(t)) &= 2\kappa\left(\theta + \frac{\sigma^2}{2\kappa}\right)e^{2\kappa t}[e^{-\kappa t}X(0) + \theta(1 - e^{-\kappa t})]dt \\
&= 2\kappa\left(\theta + \frac{\sigma^2}{2\kappa}\right)\left(e^{\kappa t}X(0) + \theta(e^{2\kappa t} - e^{\kappa t})\right)dt.
\end{aligned}$$

Integrating both sides and solving for $h(t)$ then gives us the ODE solution by

$$
\begin{aligned}
e^{2\kappa t}h(t) - X(0)^2 &= \int_0^t 2\kappa\left(\theta + \frac{\sigma^2}{2\kappa}\right)\left(e^{\kappa s}X(0) + \theta\big(e^{2\kappa s} - e^{\kappa s}\big)\right)ds \\
&= 2\kappa\left(\theta + \frac{\sigma^2}{2\kappa}\right)\left[X(0)\int_0^t e^{\kappa s}ds + \theta\int_0^t \left(e^{2\kappa s} - e^{\kappa s}\right)ds\right] \\
&= 2\kappa\left(\theta + \frac{\sigma^2}{2\kappa}\right)\left[X(0)\frac{e^{\kappa t}-1}{\kappa} + \theta\left(\frac{e^{2\kappa t}-1}{2\kappa} - \frac{e^{\kappa t}-1}{\kappa}\right)\right] \\
&= 2\left(\theta + \frac{\sigma^2}{2\kappa}\right)\left[X(0)(e^{\kappa t}-1) + \theta\left(\frac{e^{2\kappa t}-1}{2} - (e^{\kappa t}-1)\right)\right] \\
\Leftrightarrow \quad h(t) &= e^{-2\kappa t}X(0)^2 + 2\left(\theta + \frac{\sigma^2}{2\kappa}\right)e^{-2\kappa t}\left[X(0)(e^{\kappa t}-1) + \theta\left(\frac{e^{2\kappa t}-1}{2} - (e^{\kappa t}-1)\right)\right] \\
&= e^{-2\kappa t}X(0)^2 + \left(\theta + \frac{\sigma^2}{2\kappa}\right)\left[2X(0)(e^{-\kappa t} - e^{-2\kappa t}) + \theta(1-e^{-\kappa t})^2\right] \\
&= e^{-2\kappa t}X(0)^2 + 2X(0)\theta\big(e^{-\kappa t} - e^{-2\kappa t}\big) + \theta^2(1-e^{-\kappa t})^2 \\
&\qquad + \frac{\sigma^2 X(0)}{\kappa}\big(e^{-\kappa t} - e^{-2\kappa t}\big) + \theta\frac{\sigma^2}{2\kappa}(1-e^{-\kappa t})^2.
\end{aligned}
$$

Next, we have to prove that

$$
\mathrm{var}(X(t)) = X(0)\frac{\sigma^2}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \theta\frac{\sigma^2}{2\kappa}(1-e^{-\kappa t})^2,
$$

and since we from earlier know that $m(t) = \mathbb{E}(X(t))$ and $h(t) = \mathbb{E}(X^2(t))$, we by definition have

$$
\mathrm{var}(X(t)) = \mathbb{E}(X^2(t)) - (\mathbb{E}(X(t)))^2) = h(t) - m^2(t).
$$

We have just derived an expression for $h(t)$, so we just need to derive an expression for $m^2(t)$. By using the solution for $m(t)$ from Question 1.b, we simply get that

$$
m^2(t) = \left(e^{-\kappa t}X(0) + \theta(1-e^{-\kappa t})\right)^2 = e^{-2\kappa t}X(0)^2 + 2X(0)\theta(e^{-\kappa t} - e^{-2\kappa t}) + \theta^2(1-e^{-\kappa t})^2.
$$

Inserting $h(t)$ and $m^2(t)$ and collecting terms in the equation for $\mathrm{var}(X(t))$, we get the desired by

$$
\begin{aligned}
\mathrm{var}(X(t)) &= h(t) - m^2(t) \\
&= e^{-2\kappa t}X(0)^2 + 2X(0)\theta\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \theta^2(1-e^{-\kappa t})^2 \\
&\qquad + \frac{\sigma^2 X(0)}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \theta\frac{\sigma^2}{2\kappa}(1-e^{-\kappa t})^2 \\
&\qquad - e^{-2\kappa t}X(0)^2 - 2X(0)\theta(e^{-\kappa t} - e^{-2\kappa t}) - \theta^2(1-e^{-\kappa t})^2 \\
&= \frac{\sigma^2 X(0)}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \theta\frac{\sigma^2}{2\kappa}(1-e^{-\kappa t})^2.
\end{aligned}
$$

Part 2: The Bachelier Model

In this part, we consider the Bachelier model, which describes a stock's price S using an arithmetic process rather than a geometric one. The model is given by the stochastic differential equation

$$dS(t) = ...dt + \sigma dW(t),$$

with the important distinction that the coefficient σ in front of the Brownian motion increment $dW(t)$ is constant (i.e., it does not have $S(t)$ itself in it).

Question 2.a

Supposing that the interest rate is 0 and considering a strike-K expiry-T call-option, we first have to show that its arbitrage-free time-t price is given by

$$\pi^{\text{call, Bach}}(t) = (S(t) - K)\Phi\left(\frac{S(t) - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{S(t) - K}{\sigma\sqrt{T-t}}\right),$$

where Φ and ϕ denote, respectively, the standard normal distribution and density function.

Since we suppose that $r = 0$, the (risk-neutral) $\mathbb{Q}$-dynamics of the stock price are

$$dS(t) = 0dt + \sigma dW^{\mathbb{Q}}(t) = \sigma dW^{\mathbb{Q}}(t),$$

which in integral form can be written as

$$S(T) = S(t) + \int_t^T \sigma dW(s) = S(t) + \sigma(W(T) - W(t)).$$

By Theorem 11.19 in Björk (2019) with the Risk Neutral Valuation Formula, the time-t value of a European call-option with strike K at time T with payoff $X = (S(T) - K)^+$ for $r = 0$ is simply

$$\begin{aligned}\pi^{\text{call, Bach}}(t) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} X\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[(S(T) - K)^+\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[(S(T) - K)\mathbf{1}_{0 \le S(T) - K \le \infty}\right],\end{aligned}$$

where $\mathbf{1}$ denotes the indicator function. Since $W(T) - W(t) \sim \mathcal{N}(0, T-t)$ by the definition of a

Brownian motion (see e.g. Definition 4.1 in Björk (2019)), we see that

$$S(T) \sim \mathcal{N}\left(S(t), \sigma^2(T-t)\right) \Rightarrow S(T) - K \sim \mathcal{N}\left(S(t) - K, \sigma^2(T-t)\right),$$

From the hint in the exercise, we know that if $X \sim \mathcal{N}(\mu, \sigma^2)$, it then applies that

$$\mathbb{E}(X\mathbf{1}_{l \leq X \leq h}) = \mu\left(\Phi\left(\frac{h-\mu}{\sigma}\right) - \Phi\left(\frac{l-\mu}{\sigma}\right)\right) + \sigma\left(\phi\left(\frac{l-\mu}{\sigma}\right) - \phi\left(\frac{h-\mu}{\sigma}\right)\right).$$

Using the previous distribution of $S(T)-K$ and the indicator function $\mathbf{1}_{l \leq S(T)-K \leq h} = \mathbf{1}_{0 \leq S(T)-K \leq \infty}$, we obtain the desired arbitrage-free time-t price of the call-option by

$$\begin{aligned}
\pi^{\text{call, Bach}}(t) &= \mathbb{E}_t^{\mathbb{Q}}\left[(S(T)-K)\mathbf{1}_{0 \leq S(T)-K \leq \infty}\right] \\
&= (S(t)-K)\left(\Phi\left(\frac{\infty-(S(t)-K)}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{0-(S(t)-K)}{\sigma\sqrt{T-t}}\right)\right) \\
&\qquad + \sigma\sqrt{T-t}\left(\phi\left(\frac{0-(S(t)-K)}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{\infty-(S(t)-K)}{\sigma\sqrt{T-t}}\right)\right) \\
&\overset{(1)}{=} (S(t)-K)\left(\Phi(\infty) - \Phi\left(-\frac{(S(t)-K)}{\sigma\sqrt{T-t}}\right)\right) + \sigma\sqrt{T-t}\left(\phi\left(-\frac{(S(t)-K)}{\sigma\sqrt{T-t}}\right) - \phi(\infty)\right) \\
&\overset{(2)}{=} (S(t)-K)\left(1 - \left(1 - \Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\right)\right) + \sigma\sqrt{T-t}\left(\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) - 0\right) \\
&= (S(t)-K)\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right),
\end{aligned}$$

where we have used the following (trivial) properties: (1) $\infty \pm k = \infty$ for any $k \in \mathbb{R}$ (2) $\Phi(\infty) = 1$ and $\phi(\infty) = 0$ (limit properties of CDF and PDF for the standard normal distribution), together with $\Phi(-x) = 1 - \Phi(x)$ and $\phi(-x) = \phi(x)$ (symmetry of both the normal CDF and PDF).

Next, we have to show that the time-t Δ-hedge-ratio of the call-option is given by

$$\Delta^{\text{call, Bach}}(t) = \Phi\left(\frac{S(T)-K}{\sigma\sqrt{T-t}}\right).$$

By Definition 10.4 in Björk (2019), the Δ of the call-option is given by

$$\begin{aligned}
\Delta^{\text{call, Bach}}(t) &= \frac{\partial \pi^{\text{call, Bach}}(t)}{\partial S(t)} \\
&= \frac{\partial}{\partial S(t)}\left((S(t)-K)\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\right) \\
&= \frac{\partial}{\partial S(t)}\left((S(t)-K)\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\right) + \frac{\partial}{\partial S(t)}\left(\sigma\sqrt{T-t}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\right).
\end{aligned}$$

To differentiate the first term, we use the product rule and the fact that $\Phi' = \phi$, such that

$$\begin{aligned}\frac{\partial}{\partial S(t)}\left((S(t)-K)\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\right) &= \Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) + (S(T)-K)\Phi'\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\frac{\partial}{\partial S(t)}\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\\ &= \Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) + (S(T)-K)\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\frac{1}{\sigma\sqrt{T-t}}\\ &= \Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\frac{(S(T)-K)}{\sigma\sqrt{T-t}}.\end{aligned}$$

To differentiate the second term, we use the chain rule and the fact that $\phi'(x) = -x\phi(x)$, such that

$$\begin{aligned}\frac{\partial}{\partial S(t)}\left(\sigma\sqrt{T-t}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\right) &= \sigma\sqrt{T-t}\frac{\partial}{\partial S(t)}\left(\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\right)\\ &= \sigma\sqrt{T-t}\phi'\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\frac{1}{\sigma\sqrt{T-t}}\\ &= -\frac{(S(t)-K)}{\sigma\sqrt{T-t}}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right),\end{aligned}$$

and by combining these two derivatives, we obtain the desired time-t Δ-hedge-ratio by

$$\begin{aligned}\Delta^{\text{call, Bach}}(t) &= \Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) + \phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\frac{(S(T)-K)}{\sigma\sqrt{T-t}} - \frac{(S(t)-K)}{\sigma\sqrt{T-t}}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right)\\ &= \Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right).\end{aligned}$$

Question 2.b

We want to investigate what implied volatilities across strikes look like in the Bachelier model. By Section 7.8.2 in Björk (2019), implied volatility, denoted here by σ_{imp}, is obtained by solving the normal (Bachelier) call-price formula for σ_{imp}. Specifically, we set

$$\pi^{\text{call, Bach}}(t) = \pi^{\text{call, BS}}(S(t), t, T, r, \sigma_{\text{imp}}, K)$$

where $\pi^{\text{call, Bach}}(t)$ is the Bachelier call time-t price from Question 2.a used as the benchmark option here, while $\pi^{\text{call, BS}}(\cdot)$ is the Black–Scholes call-price function. We thus extract the implied volatility σ_{imp} by finding the volatility that equates the Black-Scholes call price to the benchmark option's market price. This allows us to express the original option's value in terms of the benchmark. In this context, implied volatilities reflect how the market perceives deviations between the Black-Scholes and Bachelier models, with higher values indicating a relatively expensive benchmark price.

Having implemented the procedure in Python (see Appendix 1 for the code), we therefore solve the equation for σ_{imp} to find the volatility the market implicitly has used for valuing the benchmark option. For our plot, we use the parameter values specified in the exercise text, which are:

Parameter	$S(0)$	T	σ	r
Value	100	0.25	15	0.0

Table 1: Parameter values used for plotting implied volatilities in the Bachelier model.

As seen in Figure 1, implied volatility decreases as the strike price increases, forming a reverse skew pattern where OTM call options exhibit higher implied volatility than other options. This contrasts with the typical volatility smile observed in the Black-Scholes model. The reverse skew arises because the Bachelier model expresses volatility in absolute terms rather than as a percentage of the underlying price. As a result, deep ITM and deep OTM options are more sensitive to price changes, leading to higher option prices under the Bachelier model compared to Black-Scholes. Consequently, implied volatility is elevated at lower strikes and declines as the strike price increases, reflecting the model's assumption of normally distributed price changes rather than log-normal ones. This also implies that the market considers the price of ITM call options to be relatively high under the Bachelier model compared to Black-Scholes, while OTM call prices are relatively lower.

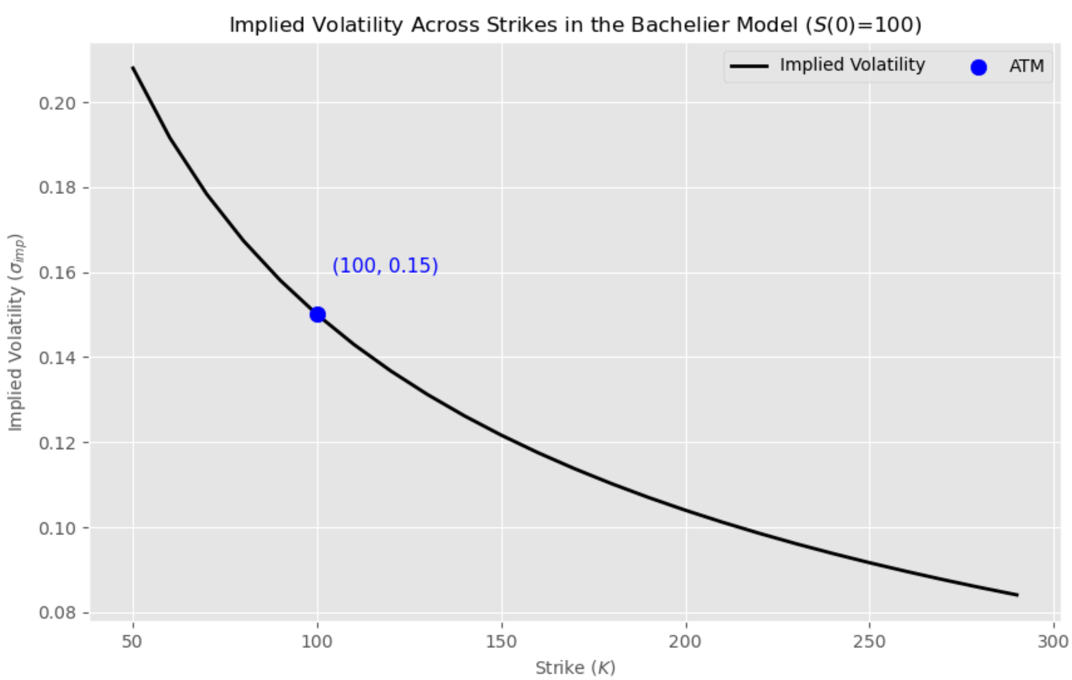

Figure 1: Implied volatilities across different strikes in the Bachelier model for $S(0) = 100$ and $K \in [50, 300]$.

If we instead assume that $S(0) = 50$, the implied volatility curve maintains its reverse skew pattern, but the overall level of implied volatilities increases. As shown in Figure 2, the implied volatility is now higher across all strikes compared to the case where $S(0) = 100$. This happens because the Bachelier model assumes absolute volatility rather than proportional volatility. Since σ remains unchanged, reducing $S(0)$ effectively increases the relative magnitude of price movements, leading to higher implied volatilities. In particular, the ATM implied volatility is now 0.3, doubling from the previous case, highlighting the sensitivity of Bachelier-implied volatilities to changes in the underlying price and its contrast with Black-Scholes, where volatility is expressed in relative terms.

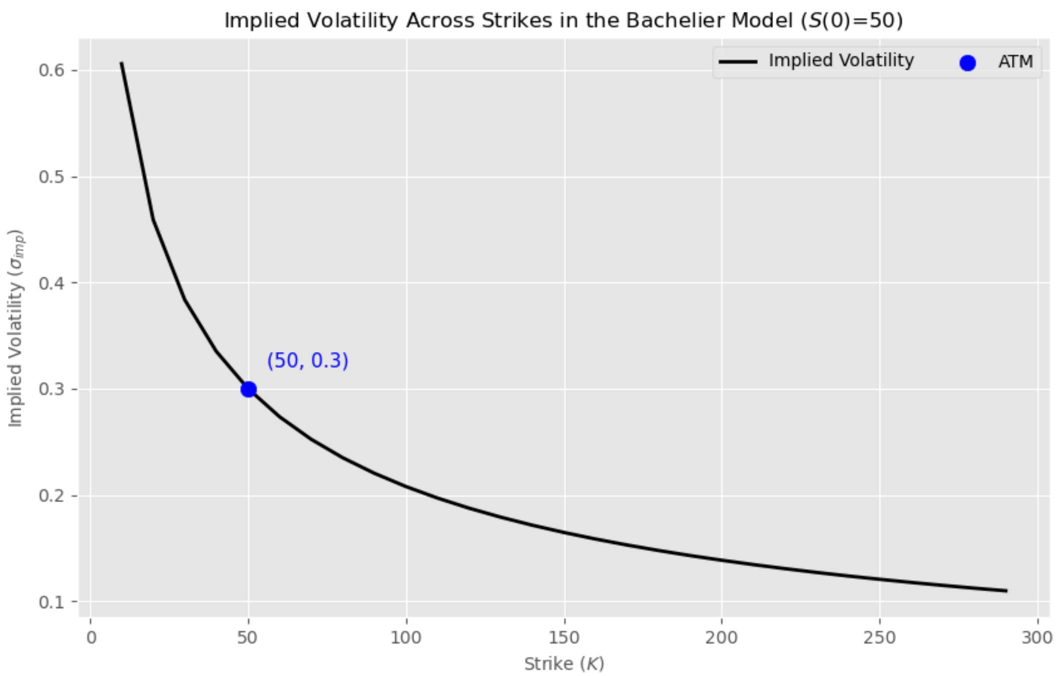

Figure 2: Implied volatilities across different strikes in the Bachelier model for $S(0) = 50$ and $K \in [10, 300]$.

Question 2.c

We want to investigate what the call-price formula looks like in a Bachelier model with a non-zero (but constant) interest rate r. Since r now is non-zero, the $\mathbb{Q}$-dynamics of the stock price are

$$dS(t) = rS(t)dt + \sigma dW^{\mathbb{Q}}(t).$$

To solve this linear SDE, we use the integrating factor method by defining $\tilde{S} = e^{-rt}S(t)$, where e^{-rt} is the integrating factor. The dynamics $d\tilde{S}(t)$ are then computed by Itô's product rule given by

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$$

for two processes $X(t)$ and $Y(t)$. By letting $X(t) = e^{-rt}$ and $Y(t) = S(t)$, we get that

$$\begin{aligned}
d\tilde{S}(t) &= d[e^{-rt}S(t)] \\
&= e^{-rt}dS(t) + S(t)d[e^{-rt}] + d[e^{-rt}]dS(t) \\
&= e^{-rt}(rS(t)dt + \sigma dW^{\mathbb{Q}}(t)) + S(t)(-re^{-rt}dt) + 0 \\
&= re^{-rt}S(t)dt + \sigma e^{-rt}dW^{\mathbb{Q}}(t) - re^{-rt}S(t)dt \\
&= \sigma e^{-rt}dW^{\mathbb{Q}}(t),
\end{aligned}$$

where $d[e^{-rt}]dS(t)$ vanished due to the multiplication rules in Proposition 4.12 in Björk (2019).

Writing this in integral form, using Proposition 5.3 in Björk (2019), we get

$$\begin{aligned}
\int_t^T d\tilde{S}(t) = \int_t^T \sigma e^{-rs}dW^{\mathbb{Q}}(s) \Rightarrow e^{rT}S(T) - e^{rt}S(t) &= \sigma \int_t^T e^{-rs}dW^{\mathbb{Q}}(s) \\
\Leftrightarrow e^{rT}S(T) &= e^{rt}S(t) + \sigma \int_t^T e^{-rs}dW^{\mathbb{Q}}(s) \\
\Leftrightarrow S(T) &= e^{r(T-t)}S(t) + \sigma \int_t^T e^{r(T-s)}dW^{\mathbb{Q}}(s).
\end{aligned}$$

Using Lemma 4.18 in Björk (2019), the stochastic integral

$$\begin{aligned}
\int_t^T e^{r(T-s)}dW^{\mathbb{Q}}(s) &\sim \mathcal{N}\left(0, \int_t^T \left(e^{r(T-s)}\right)^2 ds\right) \\
&= \mathcal{N}\left(0, \int_t^T e^{2r(T-s)}ds\right) \\
&= \mathcal{N}\left(0, \frac{e^{2r(T-t)} - 1}{2r}\right),
\end{aligned}$$

and then it follows that

$$S(T) - K \sim \mathcal{N}\left(e^{r(T-t)}S(t) - K, \sigma^2 \frac{e^{2r(T-t)} - 1}{2r}\right).$$

Now, we use the same trick, Risk Neutral Valuation Formula and properties of the CDF and PDF as in Question 2.a to derive the call-price in a Bachelier model with non-zero (but constant) r by

$$\begin{aligned}
\pi^{\text{call, Bach}}(t) &= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(S(T)-K)^+\right] \\
&= e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(S(T)-K)\mathbf{1}_{0\leq S(T)-K\leq\infty}\right] \\
&= e^{-r(T-t)}(e^{r(T-t)}S(t)-K)\left(\Phi\left(\frac{\infty-(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)-\Phi\left(\frac{0-(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)\right) \\
&\quad + e^{-r(T-t)}\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}\left(\phi\left(\frac{0-(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)-\phi\left(\frac{\infty-(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)\right) \\
&= e^{-r(T-t)}(e^{r(T-t)}S(t)-K)\left(\Phi(\infty)-\Phi\left(-\frac{(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)\right) \\
&\quad + e^{-r(T-t)}\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}\left(\phi\left(-\frac{(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)-\phi(\infty)\right) \\
&= e^{-r(T-t)}(e^{r(T-t)}S(t)-K)\left(\Phi\left(\frac{(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)\right) \\
&\quad + e^{-r(T-t)}\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}\left(\phi\left(-\frac{(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)\right),
\end{aligned}$$

which can be simplified to

$$\pi^{\text{call, Bach}}(t) = (S(t)-e^{-r(T-t)}K)\left(\Phi\left(\frac{(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)\right)+\sigma\sqrt{\frac{1-e^{-2r(T-t)}}{2r}}\left(\phi\left(-\frac{(e^{r(T-t)}S(t)-K)}{\sigma\sqrt{\frac{e^{2r(T-t)}-1}{2r}}}\right)\right).$$

Part 3: Quanto Hedging and The Kingdom of Denmark Put

In this part, we consider an arbitrage-free Black–Scholes-type currency model, with USD as the domestic currency and JPY as the foreign currency. Using Björk's Proposition 18.7, we write the bank-account, exchange-rate, and Japanese stock dynamics under the U.S. martingale measure as

$$\begin{aligned}
dB_{US}(t) &= r_{US}B_{US}(t)dt \\
dX(t) &= X(t)(r_{US}-r_J)dt + X(t)\sigma_X^\top dW(t) \\
dB_J(t) &= r_J B_J(t)dt \\
dS_J(t) &= S_J(t)(r_J-\sigma_X^\top\sigma_J)dt + S_J(t)\sigma_J^\top dW(t),
\end{aligned}$$

where the σ's are two-dimensional (constant column) vectors.

We consider a quanto put option whose payoff at time T is given by

$$Y_0(K - S_J(T))^+,$$

where Y_0 is a constant, pre-agreed exchange rate (e.g., the spot rate at $t = 0$ or a forward rate).

Question 3.a

To show that the arbitrage-free time-t price of the quanto put option is $F^{QP}(t, S_J(t))$ with

$$F^{QP}(t,s) := Y_0 e^{-r_{US}(T-t)} \left(K\Phi(-d_2(t,s)) - e^{(r_J - \sigma_X^\top \sigma_J)(T-t)} s\Phi(-d_1(t,s)) \right),$$

where

$$d_{1/2}(t,s) := \frac{\ln(s/K) + \left(r_J - \sigma_X^\top \sigma_J \pm \|\sigma_J\|^2/2\right)(T-t)}{\sqrt{T-t}\|\sigma_J\|},$$

with Φ being the standard normal distribution function and $||\cdot||$ the Euclidian norm of a vector, we use "pattern recognition" that reduces the problem to a standard Black–Scholes argument.

Under the US martingale measure $\mathbb{Q}$, the foreign stock $S(t)$ (in JPY) satisfies the following SDE

$$dS_j(t) = S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)\sigma_J^\top dW(t)^{\mathbb{Q}^d}.$$

The σ's are two-dimensional vectors, but we instead want the volatility to appear as a one-dimensional vector in front of the single Brownian motion. We can do this by introducing the Euclidian norm of the vector σ_J by $||\sigma_J|| = \sqrt{\sigma_J^\top \sigma_J}$, and then multiplying with $1 = \frac{||\sigma_J||}{||\sigma_J||}$, such that the σ-term now becomes a one-dimensional vector. Now, define a new Brownian motion $\widetilde{W}(t)$[1]. by

$$\widetilde{W}(t) = \frac{\sigma_J}{||\sigma_J||} W^{\mathbb{Q}^d}(t).$$

[1] This is still a Brownian motion by Definition 4.1 in Björk (2019):

1. $\widetilde{W}(0) = 0$: $\widetilde{W}(0) = \frac{\sigma_J}{||\sigma_J||} \cdot W^{\mathbb{Q}^d}(0) = \frac{\sigma_J}{||\sigma_J||} \cdot 0 = 0$.
2. Independent increments: $\frac{\sigma_J}{||\sigma_J||}\left(\widetilde{W}(t) - \widetilde{W}(s)\right) \perp\!\!\!\perp \frac{\sigma_J}{||\sigma_J||}\left(\widetilde{W}(u) - \widetilde{W}(r)\right)$.
3. $\mathcal{N}(0, t-s)$-distributed increments: $\widetilde{W}(t) - \widetilde{W}(s) = \frac{\sigma_J}{\|\sigma_J\|}(W(t) - W(s)) \sim N\left(0, \frac{\|\sigma_J\|^2}{\|\sigma_J\|^2}(t-s)\right) = \mathcal{N}(0, t-s)$.
4. Continuous trajectories: $t \mapsto \frac{\sigma_J}{\|\sigma_J\|} W(t)$ is continuous.

Then, we can write the dynamics of the foreign stock as

$$\begin{aligned}
dS_J(t) &= S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)\sigma_J^\top dW(t)^{\mathbb{Q}^d} \\
&= S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)||\sigma_J|| \left(\frac{\sigma_J}{||\sigma_J||} W^{\mathbb{Q}^d}(t) \right) \\
&= S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)||\sigma_J||\widetilde{W}(t).
\end{aligned}$$

By adding $r_{US} - r_{US} = 0$ inside the drift term, we further get that

$$\begin{aligned}
dS_J(t) &= S_J(t)\Bigg(\underbrace{r_J - \sigma_X^\top \sigma_J}_{\text{original drift}} + \underbrace{(r_{US} - r_{US})}_{=0} \Bigg) dt + S_J(t)\|\sigma_J\| d\widetilde{W}(t) \\
&= S_J(t)\left(r_{US} - \left(r_{US} - (r_J - \sigma_X^\top \sigma_J) \right) \right) dt + S_J(t)\|\sigma_J\| d\widetilde{W}(t) \\
&= S_J(t)\left(r_{US} - \left(r_{US} - r_J + \sigma_X^\top \sigma_J \right) \right) dt + S_J(t)\|\sigma_J\| d\widetilde{W}(t) \\
&= S_J(t)\left(r_{US} - q \right) dt + S_J(t)\|\sigma_J\| d\widetilde{W}(t).
\end{aligned}$$

Hence, $S_J(t)$ “looks like“ a stock with Black-Scholes dynamics with drift r_{US} paying a continuous dividend yield $q = r_{US} - r_J + \sigma_X^\top \sigma_J$. By Proposition 16.7 in Björk (2019) with the risk neutral valuation formula for Black–Scholes models with continuous dividend yields, the price of the quanto put with payoff $Y_0((K - S_J(T))^+$ at time T (in USD) under the US risk-neutral measure $\mathbb{Q}$ is

$$\begin{aligned}
F^{QP}(t, S_J(t)) &= e^{-r_{US}(T-t)} \mathbb{E}_t^{\mathbb{Q}} \left[Y_0(K - S_J(T))^+ \right] \\
&= e^{-r_{US}(T-t)} Y_0 \mathbb{E}_t^{\mathbb{Q}} \left[(K - S_J(T))^+ \right],
\end{aligned}$$

since Y_0 is a constant exchange rate (fixed in advance) that factors out. Recognizing $S_J(t)$ as a dividend-paying asset with yield q, the expectation $\mathbb{E}_t^{\mathbb{Q}} [(K - S_J(T))^+]$ is given by the usual Black–Scholes put formula for spot $S_J(T)$, strike K, rate r_{US}, dividend q and volatility $||\sigma_J||$, i.e. we can rewrite the price of the quanto option, by also using Proposition 16.9 in Björk (2019), as

$$\begin{aligned}
F^{QP}(t, S_J(t)) &= e^{-r_{US}(T-t)} Y_0 \mathbb{E}_t^{\mathbb{Q}} \left[(K - S_J(T))^+ \right] \\
&= Y_0 P_{BS}(t, S_J(t), ||\sigma_J||, r_{US}, r_{US} - r_J + \sigma_X^\top \sigma_J) \\
&= Y_0 P_{BS}(t, S_J(t) e^{-(r_{US} - r_j + \sigma_X^\top \sigma_J)(T-t)}, ||\sigma_J||, r_{US}, 0),
\end{aligned}$$

where we simply replaced $S_J(t)$ by $S_J(t)e^{-q(T-t)} = S_J(t)e^{-(r_{US} - r_j + \sigma_X^\top \sigma_J)(T-t)}$. From Proposition 10.2 in Björk (2019) with the put-call parity, for a put $P_{BS}(t,s)$ and call $C_{BS}(t,s)$ on the same underlying s with strike K and maturity T, we have $P_{BS}(t,s) = Ke^{-r(T-t)} - s + C_{BS}(t,s)$. Also, from Proposition 7.13 in Björk (2019), the call price under the Black–Scholes model is given by

$$C_{BS}(t,s) = s\Phi[d_1(t,s)] - Ke^{-r(T-t)}\Phi[d_2(t,s)],$$

so by first plugging $C_{BS}(t,s)$ into the put–call parity, we obtain that

$$\begin{aligned} P_{BS}(t,s) &= Ke^{-r(T-t)} - s + C_{BS}(t,s) \\ &= Ke^{-r(T-t)} - s + \Big[s\Phi(d_1(t,s)) - Ke^{-r(T-t)}\Phi(d_2(t,s))\Big] \\ &= Ke^{-r(T-t)}[1-\Phi(d_2(t,s))] - s[1-\Phi(d_1(t,s))]. \end{aligned}$$

Recalling that $1-\Phi(x) = \Phi(-x)$ (symmetry of the normal distribution), we get that $1-\Phi(d_2(t,s)) = \Phi(-d_2(t,s))$ and $1-\Phi(d_1(t,s)) = \Phi(-d_1(t,s))$. Applying these identities, we can write the price as

$$P_{BS}(t,s) = Ke^{-r(T-t)}\Phi(-d_2(t,s)) - s\Phi(-d_1(t,s)).$$

Lastly, we substitute this expression for the put price into the equation for $F^{QP}(t, S_J(t))$, such that

$$\begin{aligned} F^{QP}(t,S_J(t)) &= Y_0 P_{BS}(t, S_J(t)e^{-(r_{US}-r_J+\sigma_X^\top\sigma_J)(T-t)}, ||\sigma_J||, r_{US}, 0) \\ &= Y_0 e^{-r_{US}(T-t)}\left(K\Phi(-d_2(t,s)) - e^{-(r_J-\sigma_X^\top\sigma_J)(T-t)}\Phi(-d_1(t,s))\right), \end{aligned}$$

where we used that $r = r_{US}$ and $s = se^{-(r_{US}-r_j+\sigma_X^\top\sigma_J)(T-t)}$ (cf. Proposition 16.9), which was the arbitrage-free time-t price of the quanto put that we had to show. Using the latter expression for s again and Proposition 7.13 in Björk (2019), we also obtain that the d_1 and d_2 terms are

$$\begin{aligned} d_{1/2}(t,s) &= d_{1/2}\left(t, se^{-(r_{US}-r_j+\sigma_X^\top\sigma_J)(T-t)}\right) \\ &= \frac{\log\left(\frac{s}{K}e^{-(r_{US}-r_j+\sigma_X^\top\sigma_J)(T-t)}\right) + \left(r_{US} \pm \frac{1}{2}||\sigma_J||^2\right)(T-t)}{||\sigma_J||\sqrt{T-t}} \\ &= \frac{\log\left(\frac{s}{K}\right) + \left(r_J - \sigma_X^\top\sigma_J \pm \frac{1}{2}||\sigma_J||^2\right)(T-t)}{||\sigma_J||\sqrt{T-t}} \end{aligned}$$

as desired. To then show that

$$\frac{\partial F^{QP}(t,s)}{\partial s} = Y_0 e^{(r_J-\sigma_X^\top\sigma_J-r_{US})(T-t)}(\Phi(d_1(t,s))-1) =: g(t,s),$$

we first note by Section 7.3 in Wilmott (2006) that the Black-Scholes delta of a European put option on a stock paying a continuous dividend yield q is given by

$$\Delta^{\text{put, BS}} = e^{-q(T-t)}(\Phi(d_1)-1),$$

so by using the dividend yield $q = r_{US} - r_J + \sigma_X^\top \sigma_J$ and the constant conversion factor Y_0, we obtain

$$\begin{aligned}\frac{\partial F^{QP}(t,s)}{\partial s} &= Y_0 e^{-q(T-t)}(\Phi(d_1(t,s)) - 1)\\ &= Y_0 e^{-(r_{US} - r_J + \sigma_X^\top \sigma_J)(T-t)}\,(\Phi(d_1(t,s)) - 1)\\ &= Y_0 e^{(r_J - \sigma_X^\top \sigma_J - r_{US})(T-t)}(\Phi(d_1(t,s)) - 1),\end{aligned}$$

as the delta for the quanto put $F^{QP}(t,s)$, which is the desired result.

Question 3.b

We consider a discrete hedging experiment where the interval $[0,T]$ is partitioned into n pieces at the equidistant time points t_i. At each of these t_i's, we rebalance our portfolio so that we hold

$$\Delta^{QP}(t_i, S_J(t_i), X(t_i)) \;=\; \frac{g\big(t_i, S_J(t_i)\big)}{X(t_i)}$$

units of the foreign stock, with the remainder of the portfolio invested in the domestic bank account (i.e., ensuring a self-financing portfolio). Specifically, we implement the following discrete hedging algorithm (see Appendix 2 for the code, including the code for generating the forthcoming plots):

Algorithm 1 Discrete-Time Hedging of the Quanto Put

Require: Time partition $0 = t_0 < t_1 < \cdots < t_n = T$ with $dt = \frac{T}{n}$; initial data $(S(0), X(0))$; option parameters $(K, Y(0))$; interest rates (r_{US}, r_J); volatilities σ_X, σ_J; hedge function $g(t, S_J(t))$.

At $t = 0$**:**

Compute the initial price/outlay $P(0) = F^{QP}(0, S_J(0))$

Set the initial portfolio value $\Pi(0) = Y(0)e^{-r_{US}T}P(0)$.

Define hedge ratio $\Delta^{QP}(0, S_J(0), X(0)) = \frac{g(0, S_J(0))}{X(0)}$, and bank account $B_{US}(0) = \Pi(0) - X(0)S_J(0)\Delta^{QP}(0, S_J(0), X(0))$.

for $t = 1, \ldots, n$: **do**

  Evolve $(S_J(t), X(t))$ over dt (simulating the SDEs under $\mathbb{Q}$).

  $B_{US}(t) \leftarrow B_{US}(t-1)e^{r_{US}dt}$ (growing the old bank account at rate r_{US}).

  Compute the new portfolio value: $\Pi(t) = \Delta^{QP}(t-1, S_J(t-1), X(t-1))\,(X(t)S_J(t)) + B_{US}(t)$.

  Update the hedge ratio: $\Delta^{QP}(t, S_J(t), X(t)) = \frac{g(t, S_J(t))}{X(t)}$.

  Update the bank position: $B_{US}(t) \leftarrow \Pi(t) - X(t)S_J(t)\Delta^{QP}(t, S_J(t), X(t))$.

end for

At $t = n$**:**

Final discounted hedge error: $\tilde{\varepsilon} = e^{-r_{US}t_n}(Y(0)\,(K - S_J(t_n))^+ - \Pi(t_n))$.

For our simulations, we use the parameter values specified in the exercise text, which are:

Parameter	S_0	X_0	T	K	r_{US}	r_J	σ_X	σ_J	N	n
Value	30000	0.01	2	30000	0.03	0.00	$(0.1,\ 0.02)^\top$	$(0,\ 0.25)^\top$	1000	504

Table 2: Parameter values used for the simulations in the discrete hedge experiment.

Since we set $T = 2$ (years) and assume 252 trading days per year, choosing $n = 504$ corresponds to rebalancing daily over the two-year horizon. In the top plot of Figure 3, we plot the terminal value of the hedging portfolio as a function of the stock price at time T, where each dot represents a single simulated path from our experiment, and the black curve is the quanto put payoff for comparison. Although the hedging strategy attempts to track the option payoff's hockey-stick form, the scattered red points illustrate that this discrete hedging strategy does not replicate it.

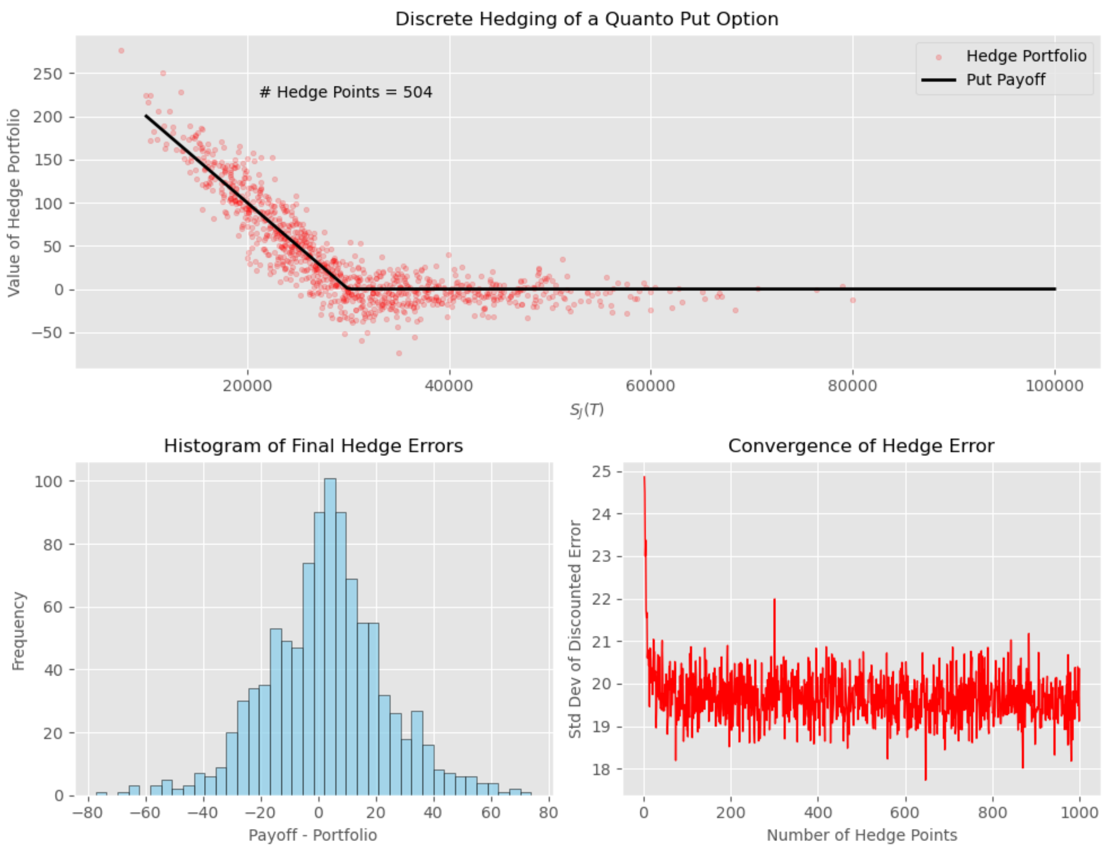

Figure 3: Discrete hedging of a quanto put with 504 hedge points, showing the hedge portfolio vs. put payoff (top), distribution of hedge errors (bottom-left), and convergence of the discounted error (bottom-right).

The bottom-left histogram depicts the distribution of final hedge errors. While these errors center near zero on average, they exhibit non-negligible spread, indicating imperfect replication across individual paths. In addition, the bottom-right plot shows that as the number of hedge points increases, the discounted standard deviation of the hedge error converges to around 19.5, whereas under perfect hedging it would be zero, thus failing to replicate the quanto put payoff for $n \to \infty$.

Question 3.c

We suppose that the strategy from 3.b is extended by holding (at time t_i)

$$-\Delta^{QP}(t_i, S_J(t_i), X(t_i))S_J(t_i)$$

units of currency, which are instantly deposited in the foreign bank (giving $-\Delta^{QP} S_J/B_J$ units of it), where they earn interest; it is still kept self-financing via domestic bank-account.

By extending Algorithm 1 from Question 3.b with the additional step of depositing an amount into the foreign bank account, we obtain a modified hedging strategy that, in the limit, will replicate the quanto put payoff. In particular, we adjust the algorithm as follows: at $t = 0$ we introduce a foreign bank account $B_J(0) = -\Delta^{QP}(0, S_J(0), X(0))S_J(0)$. At each step, we let $B_J(t)$ grow at rate r_J so that the portfolio value becomes $\Pi(t) = \Delta^{QP}(t-1, S_J(t-1), X(t-1))X(t)S_J(t) + X(t)B_J(t) + B_{US}(t)$. We then update the hedge ratio to $\Delta^{QP}(t, S_J(t), X(t)) = \frac{g(t,S_J(t))}{X(t)}$ and reset $B_J(t) = -\Delta^{QP}(t)S_J(t)$, with the domestic bank account determined by the self-financing condition $B_{US}(t) = \Pi(t) - X(t)\left(\Delta^{QP}(t)S_J(t) + B_J(t)\right)$ (see Appendix 2 for the code, including the code for generating the forthcoming plots).

In the top plot of Figure 4, the hedge portfolio (red dots) now closely aligns with the quanto put payoff (black line) across all simulated stock prices $S(T)$. So, in contrast to the previous experiment, this strategy exhibits almost no visible gap between the hedge portfolio and the option payoff. The histogram of final hedge errors (bottom-left plot) is now also tightly clustered around zero, indicating negligible replication error. Finally, the bottom-right plot demonstrates that the standard deviation of the discounted error tends to zero as the number of hedge points increases, confirming that, in the limit for $n \to \infty$, this strategy does indeed replicate the payoff of the quanto put option.

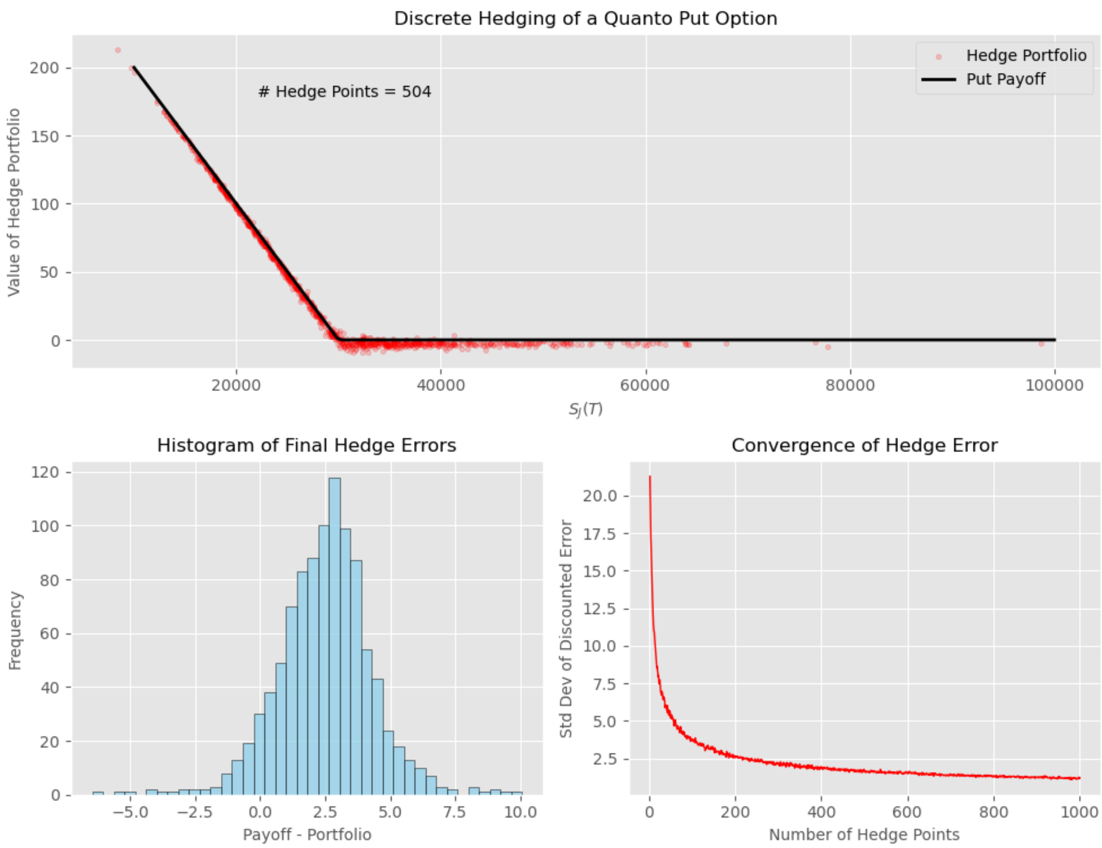

Figure 4: Discrete hedging of a quanto put with 504 hedge points, showing the hedge portfolio vs. put payoff (top), distribution of hedge errors (bottom-left), and convergence of the discounted error (bottom-right).

Question 3.d

We will explain why the strategy from part 3.c works (replication-wise in continuous time), while the strategy from 3.b does not. Let us start by considering a self-financing portfolio h given by

$$V^h(t) = h_0(t)B_{US}(t) + h_1(t)S_J(t)X(t) + h_2(t)B_J(t)X(t),$$

where $h_0(t)$ is the amount held in the domestic bank account (B_{US}), $h_1(t)$ the amount held of the foreign stock (S_J) and $h_2(t)$ the amount held of the foreign bank account (B_J). Since we have assumed that h is self-financing, Definition 6.10 in Björk (2019) gives the dynamics of it by

$$dV^h(t) = h_0(t)dB_{US}(t) + h_1(t)d(S_J(t)X(t)) + h_2(t)d(B_J(t)X(t)),$$

where we already know that $dB_{US}(t) = r_{US}B_{US}(t)dt$. To find the dynamics of $S_J(t)X(t)$ and $B_J(t)X(t)$, we use Itô's product rule given by

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$$

for two processes $X(t)$ and $Y(t)$. Using this and the dynamics in the exercise text, we get that

- $S_J(t)X(t)$:

$$\begin{aligned}
d(S_J(t)X(t)) &= S_J(t)dX(t) + X(t)dS_J(t) + dS_J(t)dX(t) \\
&= S_J(t)(X(t)(r_{US} - r_J)dt + X(t)\sigma_X^\top dW(t)) \\
&\quad + X(t)(S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)\sigma_J^\top dW(t)) \\
&\quad + (S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)\sigma_J^\top dW(t))(X(t)(r_{US} - r_J)dt + X(t)\sigma_X^\top dW(t)) \\
&= S_J(t)X(t)\Big[(r_{US} - r_J)dt + \sigma_X^\top dW(t) + (r_J - \sigma_X^\top \sigma_J)dt + \sigma_J^\top dW(t) + (\sigma_J^\top \sigma_X)dt\Big] \\
&= S_J(t)X(t)\Big[(r_{US} - r_J + r_J - \sigma_X^\top \sigma_J + \sigma_J^\top \sigma_X)dt + (\sigma_X^\top + \sigma_J^\top)dW(t)\Big] \\
&= S_J(t)X(t)\Big[r_{US}dt + (\sigma_X + \sigma_J)^\top dW(t)\Big].
\end{aligned}$$

- $B_J(t)X(t)$:

$$\begin{aligned}
d(B_J(t)X(t)) &= B_J(t)dX(t) + X(t)dB_J(t) + dB_J(t)dX(t) \\
&= B_J(t)\Big(X(t)(r_{US} - r_J)dt + X(t)\sigma_X^\top dW(t)\Big) \\
&\quad + X(t)r_J B_J(t)dt + (r_J B_J(t)dt)\Big(X(t)(r_{US} - r_J)dt + X(t)\sigma_X^\top dW(t)\Big) \\
&= B_J(t)X(t)(r_{US} - r_J)dt + B_J(t)X(t)\sigma_X^\top dW(t) + X(t)r_J B_J(t)dt \\
&= B_J(t)X(t)\Big[(r_{US} - r_J + r_J)dt + \sigma_X^\top dW(t)\Big] \\
&= B_J(t)X(t)\Big[r_{US}dt + \sigma_X^\top dW(t)\Big].
\end{aligned}$$

By substituting these dynamics of $B_{US}(t)$, $S_J(t)X(t)$ and $B_J(t)X(t)$ into $dV^h(t)$, we then obtain

$$\begin{aligned}
dV^\pi(t) &= h_0(t)\left[r_{US}B_{US}(t)dt\right] + h_1(t)\left[S_J(t)X(t)\left(r_{US}dt + (\sigma_X + \sigma_J)^\top dW(t)\right)\right] \\
&\quad + h_2(t)\left[B_J(t)X(t)\left(r_{US}dt + \sigma_X^\top dW(t)\right)\right] \\
&= r_{US}\left[h_0(t)B_{US}(t) + h_1(t)S_J(t)X(t) + h_2(t)B_J(t)X(t)\right]dt \\
&\quad + \left[h_1(t)S_J(t)X(t)(\sigma_X + \sigma_J)^\top + h_2(t)B_J(t)X(t)\sigma_X^\top\right]dW(t).
\end{aligned}$$

We want to compare these dynamics with the dynamics of the quanto put pricing function $F^{QP}(t, S_J(t))$. Using Proposition 4.12 in Björk (2019) with Itô's formula and multiplication rules, we get

$$\begin{aligned}
dF^{QP} &= \frac{\partial F^{QP}}{\partial t}dt + \frac{\partial F^{QP}}{\partial S_J(t)}dS_J(t) + \frac{1}{2}\frac{\partial^2 F^{QP}}{\partial S_J(t)^2}(dS_J(t))^2 \\
&= \frac{\partial F^{QP}}{\partial t}dt + \frac{\partial F^{QP}}{\partial S_J(t)}\left(S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)\sigma_J^\top dW(t)\right) + \frac{1}{2}\frac{\partial^2 F^{QP}}{\partial S_J(t)^2}S_J(t)^2\,\sigma_J^\top \sigma_J\,dt \\
&= \left(\frac{\partial F^{QP}}{\partial t} + \frac{\partial F^{QP}}{\partial S_J(t)}S_J(t)(r_J - \sigma_X^\top \sigma_J) + \frac{1}{2}\frac{\partial^2 F^{QP}}{\partial S_J(t)^2}S_J(t)^2\sigma_J^\top \sigma_J\right)dt + \frac{\partial F^{QP}}{\partial S_J(t)}\,S_J(t)\sigma_J^\top dW(t).
\end{aligned}$$

From Poulsen (2025), we know that hedging works under any probability measure — hence, works for any specific drift rate. Consequently, we must compare only the diffusion terms and verify they coincide, ensuring the same Brownian motion paths are followed. The equation is given by

$$h_1(t)S_J(t)X(t)(\sigma_X + \sigma_J)^\top + h_2(t)\,B_J(t)X(t)\sigma_X^\top = \frac{\partial F^{QP}}{\partial S_J(t)}S_J(t)\sigma_J^\top$$

We then determine the hedging strategy that neutralizes exposure to JPY fluctuations in our USD valuation by matching the terms involving the foreign stock's volatility, ensuring changes in the foreign currency do not affect our USD value:

$$\begin{aligned}
&\frac{\partial F^{QP}}{\partial S_J(t)}S_J(t)\sigma_J^\top(t) = h_1(t)\sigma_J^\top(t)S_J(t)X(t) \\
\Leftrightarrow\quad &\frac{\partial F^{QP}}{\partial S_J(t)} = h_1(t)X(t) \\
\Leftrightarrow\quad &h_1(t) = \frac{\partial F^{QP}}{\partial S_J(t)}\frac{1}{X(t)} = \frac{g\,(t, S_J(t))}{X(t)} = \Delta^{QP}(t, S_J(t), X(t))
\end{aligned}$$

Hereafter, we construct a hedge that neutralizes currency risk in USD by matching the exchange-rate volatility terms. Doing so ensures that the resulting portfolio is locally risk-free with respect to fluctuations in the JPY/USD exchange rate:

$$\begin{aligned}
&h_1(t)S_J(t)X(t)\sigma_X^\top + h_2(t)B_J(t)X(t)\sigma_X^\top = 0 \\
\Leftrightarrow\quad &h_1(t)S_J(t)X(t)\sigma_X^\top = -h_2(t)B_J(t)X(t)\sigma_X^\top \\
\Leftrightarrow\quad &h_2(t) = -\frac{h_1(t)S_J(t)}{B_J(t)} = -\Delta^{QP}(t, S_J(t), X(t))\frac{S_J(t)}{B_J(t)}.
\end{aligned}$$

The second condition shows that to cancel the exposure due to fluctuations in the exchange rate, one must hold a position in the foreign bank account. This extra position — implemented in the modified strategy of question 3.c — is precisely what offsets the residual risk that remains in strategy 3.b. The issue with the strategy from 3.b is that it only rebalances the position in the foreign stock and therefore does not hedge the exchange-rate risk, leaving an unhedged currency exposure that fails to replicate the payoff of the quanto put.

By (14.33) in Björk (2019), the remaining funds are invested in the domestic bank account as

$$h_0(t) = \frac{F^{QP} - h_1(t)S_J(t)X(t) - h_2(t)B_J(t)X(t)}{B_{US}(t)}.$$

If we then substitute the expressions for $h_0(t)$, $h_1(t)$, $h_2(t)$ into V^h, we get that

$$\begin{aligned}
V^h(t) &= h_0(t)B_{US}(t) + h_1(t)S_J(t)X(t) + h_2(t)B_J(t)X(t) \\
&= \frac{F^{QP} - h_1(t)S_J(t)X(t) - h_2(t)B_J(t)X(t)}{B_{US}(t)}B_{US}(t) + \Delta^{QP}(t, S_J(t), X(t))S_J(t)X(t) \\
&\quad - \Delta^{QP}(t, S_J(t), X(t))\frac{S_J(t)}{B_J(t)}B_J(t)X(t) \\
&= F^{QP} - h_1(t)S_J(t)X(t) - h_2(t)B_J(t)X(t) + \Delta^{QP}(t, S_J(t), X(t))S_J(t)X(t) \\
&\quad - \Delta^{QP}(t, S_J(t), X(t))S_J(t)X(t).
\end{aligned}$$

Therefore, while the discrete hedging strategy in 3.b rebalances only the foreign-stock position (via the delta Δ^{QP}), it neglects the exchange-rate fluctuations inherent in the combined dynamics of $S_J(t)X(t)$. By also depositing the corresponding short position of the foreign currency (via $-\Delta^{QP}S_J$), the extended strategy in 3.c aligns both the drift and the diffusion terms with those of the quanto put payoff with the domestic bank account ensuring the portfolio remains self-financing. As a result, replication-wise in continuous time, this extended strategy eliminates the exposure to JPY movements and achieves perfect replication of the quanto put option payoff, while the strategy from 3.b fails to replicate the quanto put payoff since it does not hedge against exchange-rate risk.

Appendix

1. Code for Question 2.b

```
import numpy as np
from scipy.stats import norm
from scipy.optimize import brentq
import matplotlib.pyplot as plt

class BachelierImpliedVolatility:
    """A class to calculate implied volatility using both the Black-Scholes and Bachelier models."""

    def _d(spot, strike, sigma, T):
        """Compute d1 and d2 for the Black-Scholes formula."""
        d1 = (np.log(spot / strike) + 0.5 * sigma**2 * T) / (sigma * np.sqrt(T))
        d2 = d1 - sigma * np.sqrt(T)
        return d1, d2

    def black_scholes_price(spot, strike, T, sigma, option="Call", r=0.0):
        """Calculate the Black-Scholes price for a European option."""
        d1, d2 = BachelierImpliedVolatility._d(spot, strike, sigma, T)
        if option.lower() == "call":
            return spot * norm.cdf(d1) - strike * np.exp(-r * T) * norm.cdf(d2)
        elif option.lower() == "put":
            return strike * np.exp(-r * T) * norm.cdf(-d2) - spot * norm.cdf(-d1)
        else:
            raise ValueError("Option type must be either 'Call' or 'Put'.")

    def black_scholes_implied_vol(obs_price, spot, strike, T, r, sigma_bach, option="Call", low_vol=1e-6,
    ↪   high_vol=10.0):
        """Calculate the Black-Scholes implied volatility.

        It solves for sigma_bs such that:
            BS(spot, strike, T, sigma_bs, option, r) = Bach(spot, strike, T, sigma_bach)
        """
        def difference(sigma_bs):
            return (BachelierImpliedVolatility.black_scholes_price(spot, strike, T, sigma_bs, option, r)
                    - BachelierImpliedVolatility.bachelier_price(spot, strike, T, sigma_bach))
        return brentq(difference, low_vol, high_vol)

    def bachelier_price(spot, strike, T, sigma):
        """Calculate the option price using the Bachelier model."""
        d = (spot - strike) / (sigma * np.sqrt(T))
        return (spot - strike) * norm.cdf(d) + sigma * np.sqrt(T) * norm.pdf(d)

    def plot_implied_volatility(S0, T, sigma_bach, strikes, option="Call", r=0.0, save_path=None):
        """
        Plot the Black-Scholes implied volatility across strikes using the Bachelier model price."""
        vols = np.array([
            BachelierImpliedVolatility.black_scholes_implied_vol(
                BachelierImpliedVolatility.bachelier_price(S0, k, T, sigma_bach),
                S0, k, T, r, sigma_bach, option
```

```
            ) for k in strikes
        ])

        atm_strike = S0
        atm_price = BachelierImpliedVolatility.bachelier_price(S0, atm_strike, T, sigma_bach)
        atm_vol = BachelierImpliedVolatility.black_scholes_implied_vol(atm_price, S0, atm_strike, T, r,
        ↪  sigma_bach, option)

        plt.style.use('ggplot')
        plt.figure(figsize=(10, 6))
        plt.plot(strikes, vols, color="black", linewidth=2, label="Implied Volatility", zorder=1)
        plt.scatter(atm_strike, atm_vol, color="blue", s=80, label="ATM", zorder=2)
        plt.text(atm_strike + 4, atm_vol + 0.01, f"({atm_strike}, {round(atm_vol, 2)})", fontsize=11,
        ↪  color="blue")
        plt.xlabel("Strike ($K$)", fontsize=10)
        plt.ylabel("Implied Volatility ($\\sigma_{imp}$)", fontsize=10)
        plt.title(f"Implied Volatility Across Strikes in the Bachelier Model ($S(0)$={S0})", fontsize=12)
        plt.legend(loc="upper right", fontsize=10, ncol=4)
        plt.xticks(fontsize=10)
        plt.yticks(fontsize=10)

        if save_path:
            plt.savefig(save_path)
        else:
            plt.show()

if __name__ == "__main__":
    S0 = 100
    T = 0.25
    sigma_bach = 15.0
    r = 0.0
    strikes = np.arange(50, 300, 10)
    option_type = "Call"

    BachelierImpliedVolatility.plot_implied_volatility(
        S0=S0,
        T=T,
        sigma_bach=sigma_bach,
        strikes=strikes,
        option=option_type,
        r=r,
        save_path="Assignment_1/question_2b_s0_100.png"
    )
```

2. Code for Question 3.b and 3.c

```
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt
import matplotlib.gridspec as gridspec
```

```python
class QuantoOptionHedgeBase:
    def __init__(self, r_US, r_J, S_J_start, X_0, K, sigma_X, sigma_J, Y_0, T, Nhedges, Nreps):
        self.r_US = r_US
        self.r_J = r_J
        self.S_J_start = S_J_start
        self.X_0 = X_0
        self.K = K
        self.sigma_X = sigma_X
        self.sigma_J = sigma_J
        self.Y_0 = Y_0
        self.T = T
        self.Nhedges = Nhedges
        self.Nreps = Nreps
        self.dt = T / Nhedges
        self.exp_r_dt = np.exp(self.r_US * self.dt)

        self.q_const = self.r_US - self.r_J + np.dot(self.sigma_X.T, self.sigma_J)
        self.norm_sigma_J = np.linalg.norm(self.sigma_J)
        self.norm_sigma_X = np.linalg.norm(self.sigma_X)

        self.drift_SJ = ((self.r_J - np.dot(self.sigma_X.T, self.sigma_J))
                         - 0.5 * self.norm_sigma_J**2) * self.dt
        self.drift_X = ((self.r_US - self.r_J)
                        - 0.5 * self.norm_sigma_X**2) * self.dt
        self.vol_factor = np.sqrt(self.dt)

        self.pf_value = None
        self.S_J = None
        self.X = None

    def _compute_d(self, spot, strike, r, q, sigma, T, d_type=1):
        denom = sigma * np.sqrt(T)
        if d_type == 1:
            return (np.log(spot / strike) + (r - q) * T + 0.5 * sigma**2 * T) / denom
        else:
            return (np.log(spot / strike) + (r - q) * T - 0.5 * sigma**2 * T) / denom

    def black_scholes_price(self, spot, strike, T, r, q, sigma):
        d1 = self._compute_d(spot, strike, r, q, sigma, T, d_type=1)
        d2 = self._compute_d(spot, strike, r, q, sigma, T, d_type=2)
        return (strike * np.exp(-r * T) * norm.cdf(-d2)
                - np.exp(-q * T) * spot * norm.cdf(-d1))

    def _g(self, Y_0, spot, strike, T, r, sigma):
        d1 = self._compute_d(spot, strike, r, self.q_const, sigma, T, d_type=1)
```

```
        return Y_0 * np.exp(-self.q_const * T) * (norm.cdf(d1) - 1)

    def hedge_experiment(self):
        raise NotImplementedError()

    def final_payoff_error(self):
        raise NotImplementedError()

    def convergence_of_hedge_error(cls, ax, r_US, r_J, S_J_start, X_0, K,
                                   sigma_X, sigma_J, Y_0, T, Nreps, max_hedges=1000):
        std_devs = []
        for Nhedges in range(2, max_hedges + 1):
            hedge = cls(r_US, r_J, S_J_start, X_0, K, sigma_X, sigma_J, Y_0, T, Nhedges, Nreps)
            hedge.hedge_experiment()
            error = hedge.final_payoff_error()
            discounted_error = np.exp(-r_US * T) * error
            std_devs.append(np.std(discounted_error))
        ax.plot(range(2, max_hedges + 1), std_devs, linewidth=1, color='red')
        ax.set_xlabel("Number of Hedge Points", fontsize=10)
        ax.set_ylabel("Std Dev of Discounted Error", fontsize=10)
        ax.set_title("Convergence of Hedge Error", fontsize=12)
        ax.grid(True)

    def plot_hedge_experiment(cls, r_US, r_J, S_J_start, X_0, K,
                              sigma_X, sigma_J, Y_0, T, Nreps, Nhedges, save_path=None):
        hedge = cls(r_US, r_J, S_J_start, X_0, K, sigma_X, sigma_J, Y_0, T, Nhedges, Nreps)
        hedge.hedge_experiment()
        final_errors = hedge.final_payoff_error()
        S_final = hedge.S_J
        pf_value_final = hedge.pf_value

        fig = plt.figure(figsize=(10, 8))
        plt.style.use('ggplot')
        gs = gridspec.GridSpec(2, 2, height_ratios=[1, 1])
        ax1 = fig.add_subplot(gs[0, :])
        ax2 = fig.add_subplot(gs[1, 0])
        ax3 = fig.add_subplot(gs[1, 1])

        ax1.scatter(S_final, pf_value_final, s=10, alpha=0.2, label="Hedge Portfolio", color='red')
        S_range = np.linspace(10000, 100000, 200)
        payoff_range = Y_0 * np.maximum(K - S_range, 0)
        ax1.plot(S_range, payoff_range, 'k-', linewidth=2, label="Put Payoff")
        ax1.set_title("Discrete Hedging of a Quanto Put Option", fontsize=12)
        ax1.text(0.35, 0.85, f"# Hedge Points = {Nhedges}",
                 transform=ax1.transAxes, fontsize=10,
```

```
                 verticalalignment='top', horizontalalignment='right')
        ax1.set_xlabel("$S_J(T)$", fontsize=10)
        ax1.set_ylabel("Value of Hedge Portfolio", fontsize=10)
        ax1.legend()

        ax2.hist(final_errors, bins=40, color='skyblue', edgecolor='black', alpha=0.7)
        ax2.set_title("Histogram of Final Hedge Errors", fontsize=12)
        ax2.set_xlabel("Payoff - Portfolio", fontsize=10)
        ax2.set_ylabel("Frequency", fontsize=10)

        cls.convergence_of_hedge_error(ax=ax3, r_US=r_US, r_J=r_J,
                                       S_J_start=S_J_start, X_0=X_0, K=K,
                                       sigma_X=sigma_X, sigma_J=sigma_J,
                                       Y_0=Y_0, T=T, Nreps=Nreps, max_hedges=1000)

        plt.tight_layout(rect=[0, 0, 1, 0.96])
        if save_path:
            plt.savefig(save_path)
        else:
            plt.show()


class QuantoOptionHedge3b(QuantoOptionHedgeBase):
    def hedge_experiment(self):
        option_price = self.black_scholes_price(
            self.S_J_start, self.K, self.T, self.r_US, self.q_const, self.norm_sigma_J
        )
        initial_outlay = self.Y_0 * np.exp(-self.r_US * self.T) * option_price

        self.pf_value = np.full(self.Nreps, initial_outlay)
        self.S_J = np.full(self.Nreps, self.S_J_start, dtype=float)
        self.X = np.full(self.Nreps, self.X_0, dtype=float)

        self.a = self._g(self.Y_0, self.S_J, self.K, self.T, self.r_US, self.norm_sigma_J) /
        ↪  self.X_0
        self.b = self.pf_value - self.X_0 * self.a * self.S_J

        for i in range(1, self.Nhedges):
            Z = np.random.normal(size=(self.Nreps, 2))
            self.S_J *= np.exp(self.drift_SJ + (Z @ self.sigma_J) * self.vol_factor)
            self.X   *= np.exp(self.drift_X  + (Z @ self.sigma_X) * self.vol_factor)

            self.pf_value = self.a * self.X * self.S_J + self.b * self.exp_r_dt
            remaining_T = self.T - (i - 1) * self.dt
            self.a = self._g(self.Y_0, self.S_J, self.K, remaining_T, self.r_US, self.norm_sigma_J)
            ↪  / self.X
```

```
            self.b = self.pf_value - self.X * self.a * self.S_J

    def final_payoff_error(self):
        payoff = self.Y_0 * np.maximum(self.K - self.S_J, 0)
        return payoff - self.pf_value


class QuantoOptionHedge3c(QuantoOptionHedgeBase):
    def hedge_experiment(self):
        option_price = self.black_scholes_price(
            self.S_J_start, self.K, self.T, self.r_US, self.q_const, self.norm_sigma_J
        )
        initial_outlay = self.Y_0 * np.exp(-self.r_US * self.T) * option_price

        self.S_J = np.full(self.Nreps, self.S_J_start, dtype=float)
        self.X   = np.full(self.Nreps, self.X_0, dtype=float)

        self.a = np.zeros(self.Nreps)
        self.c = np.zeros(self.Nreps)
        self.b = np.zeros(self.Nreps)

        a0 = self._g(self.Y_0, self.S_J, self.K, self.T, self.r_US, self.norm_sigma_J) / self.X_0
        c0 = - a0 * self.S_J
        b0 = initial_outlay - (a0 * self.X_0 * self.S_J) - (c0 * self.X_0)

        self.a[:] = a0
        self.c[:] = c0
        self.b[:] = b0

        for i in range(1, self.Nhedges):
            Z = np.random.normal(size=(self.Nreps, 2))
            self.S_J *= np.exp(self.drift_SJ + (Z @ self.sigma_J) * self.vol_factor)
            self.X   *= np.exp(self.drift_X  + (Z @ self.sigma_X) * self.vol_factor)

            old_a, old_c, old_b = self.a, self.c, self.b
            c_grown = old_c * np.exp(self.r_J * self.dt)
            b_grown = old_b * np.exp(self.r_US * self.dt)
            V_before = old_a * self.X * self.S_J + c_grown * self.X + b_grown

            remaining_T = self.T - (i - 1) * self.dt
            new_a = self._g(self.Y_0, self.S_J, self.K, remaining_T, self.r_US, self.norm_sigma_J)
            ↪  / self.X
            new_c = - new_a * self.S_J
            new_b = V_before - (new_a * self.X * self.S_J) - (new_c * self.X)
```

```
            self.a, self.c, self.b = new_a, new_c, new_b

        c_final = self.c * np.exp(self.r_J * self.dt)
        b_final = self.b * np.exp(self.r_US * self.dt)
        self.pf_value = self.a * self.X * self.S_J + c_final * self.X + b_final

    def final_payoff_error(self):
        payoff = self.Y_0 * np.maximum(self.K - self.S_J, 0)
        return payoff - self.pf_value

if __name__ == "__main__":
    r_US = 0.03
    r_J = 0.00
    S_J_start = 30000
    X_0 = 0.01
    K = 30000
    sigma_X = np.array([0.1, 0.02])
    sigma_J = np.array([0.0, 0.25])
    Y_0 = 0.01
    T = 2
    Nreps = 1000
    Nhedges = 504

    # Experiment 3.b
    QuantoOptionHedge3b.plot_hedge_experiment(
        r_US, r_J, S_J_start, X_0, K, sigma_X, sigma_J, Y_0, T, Nreps, Nhedges,
        save_path="Assignment_1/qquestion_3b_quanto_option_hedge_experiment.png"
    )

    # Experiment 3.c
    QuantoOptionHedge3c.plot_hedge_experiment(
        r_US, r_J, S_J_start, X_0, K, sigma_X, sigma_J, Y_0, T, Nreps, Nhedges,
        save_path="Assignment_1/qquestion_3c_quanto_option_hedge_experiment.png"
    )
```

References

Björk, T. (2019). *Arbitrage Theory in Continuous Time.* Oxford Academic, 4th edition.

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