Application of the second order Gaussian kinematic formula to CMB data analysis

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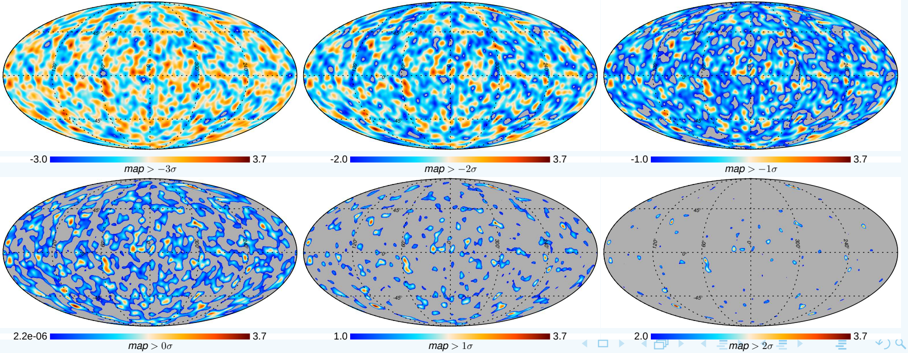
December 10, 2018

Outline

- Let M be a general Riemannian manifold. In particular, for CMB we can think of M as a sphere S^2.
- The basic set of random geometrical objects are $\mathcal{R}$ valued random field $f(x)$ defined on M and its excursion sets A

$$A_u(f, M) = \{x \in M : f(x) \geq u\}$$



Lifschitz-Killing Curvatures (Minkowski Functionals)

- Lifschitz-Killing Curvatures (LKCs), a.k.a Minkowski Functionals (MFs), can be defined using a tube formula:

$$\mu(\mathrm{Tube}(\mathrm{M},\rho)) = \sum_{\mathrm{j}=0}^{\mathrm{n}=\dim(\mathrm{M})} \omega_{\mathrm{j}} \mathcal{L}_{\mathrm{n}-\mathrm{j}}(\mathrm{M})\rho^{\mathrm{j}}$$

  where $\mathrm{Tube}(\mathrm{M},\rho) = \{\mathrm{t} \in \mathcal{R}^{\mathrm{N}} : \mathrm{dist}(\mathrm{M},\mathrm{x}) \leq \rho\}$ is a tube of radius ρ bounding M; μ is Lebesgue measure; and w_{j} is the volume of a unit ball in $\mathcal{R}^{\mathrm{j}}$.

- LKCs depend on the Riemannian metric, and are a measure of the k-dimensional size of the Riemannian manifold M.

Lifschitz-Killing Curvatures (Minkowski Functionals) II

- $\mathcal{L}_0(\mathrm{A_u(f)})$ is the genus or the Euler-Poincarè characteristic (minima+maxima-saddles) of the excursion regions. The third Minkowski functional.
- $\mathcal{L}_1(\mathrm{A_u(f)})$ is half the boundary length of the excursion regions, e.g. the second Minkowski functional.
- $\mathcal{L}_2(\mathrm{A_u(f)})$ is the area of the excursion regions, e.g. the first Minkowski functional.

The Gaussian Kinematic Formula (GKF)

- Developed by Adler & Taylor, it is about expected values of Lifshitz-Killing curvatures (LKCs)/Minkowski Functionals (MFs) for excursion regions.

$$\mathbb{E}\mathcal{L}_i(A_u(f, M)) = \sum_{k=0}^{\dim M - i} \begin{bmatrix} i+k \\ k \end{bmatrix} \mathcal{L}_{i+k}(M)\mathcal{M}_k([u, \infty))$$

where

$$\begin{bmatrix} i+k \\ k \end{bmatrix} = \binom{i+k}{k} \frac{\omega_{i+k}}{\omega_k \omega_i}$$

- $\mathcal{M}$ is given by

$$\mathcal{M}_j^{\gamma_k}([u, \infty)) = (2\pi)^{-1/2} H_{j-1}(u) e^{-u^2/2}.$$

where H_j is the Hermite polynomials: $H_0(u) = 1$, $H_1(u) = 2u$, $H_2(u) = 4u^2 - 1$, $H_3(u) = 8u^3 - 12u$

The power of GKF

- Splits the metric dependence of the field from that of excursion set behaviour.
- The $\mathcal{L}_k(M)$ part contains all the metric property. If the metric is scaled by λ, $\mathcal{L}_k(M)$ scales by λ^k.
- On a sphere the scaling λ required to go to harmonic and needlet spaces are given by:

$$\lambda_s = \begin{cases} \sqrt{\frac{s(s+1)}{2}}, & \text{if } f(x) = T_\ell(x) \\ \sqrt{\frac{\sum_\ell b^2(\frac{\ell}{2^s}) C_\ell \frac{2\ell+1}{4\pi} \frac{\ell(\ell+1)}{2}}{\sum_\ell b^2(\frac{\ell}{2^s}) C_\ell \frac{2\ell+1}{4\pi}}}, & \text{if } f(x) = \beta_j(x) \end{cases}$$

- The $\mathcal{M}_k([u,\infty))$ part dependence only on the excursion threshold u. It absorbs all non-linear transformations:

$$H_{ns}(x) := H_n(f(x)) = \frac{f^n(x)}{\sum_l b^2(\frac{l}{B^s}) \frac{2l+1}{4\pi} C_l} - 1 \text{ .where } n = 2, 3, ..$$

Applications of GKF in cosmology

- Our interest here is to compute the expected values of the LKCs (MFs) in harmonic and needlet space.
- The advantages of implementing LKCs on needlet space are:
  - Needlets enjoy very good localization in pixel space - are minimally affected by masked regions, especially at high-frequency j.
  - The double-localization properties of needlets (in real and harmonic space) allow a precise interpretation of any possible anomalies - offer a scale-by-scale probe of asymmetries and relevant features e.g. Cold Spot.

LKCs for a Gaussian field

- The expected value of the first Lipschitz-Killing curvature (e.g. Euler-Poincarè characteristic)

$$\mathbb{E}\mathcal{L}_0(\mathrm{A_u}(\mathrm{f}(\mathrm{x}),\mathrm{S}^2)) = 2\left\{1-\Phi(\mathrm{u})\right\} + \lambda_{\mathrm{s}}^2 \frac{\mathrm{u}e^{-\mathrm{u}^2/2}}{\sqrt{(2\pi)^3}} 4\pi \ ;$$

- The second Lipschitz-Killing curvature (e.g., half the boundary length)

$$\mathbb{E}\mathcal{L}_1(\mathrm{A_u}(\mathrm{f}(\mathrm{x}),\mathrm{S}^2)) = \pi \times \lambda_{\mathrm{s}} \mathrm{e}^{-\mathrm{u}^2/2} \ ;$$

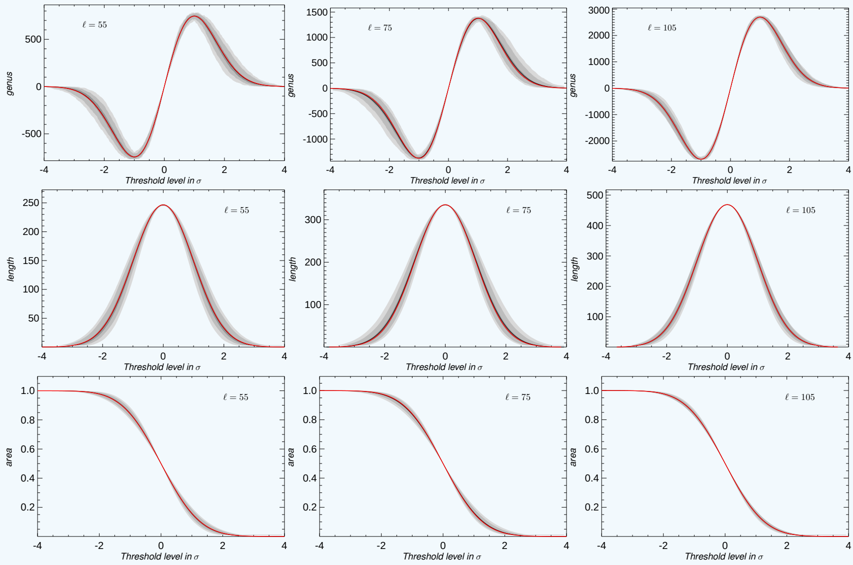
- The third Lipschitz-Killing curvature (e.g., the area of the excursion region)

$$\mathbb{E}\mathcal{L}_2(\mathrm{A_u}(\mathrm{f}(\mathrm{x}),\mathrm{S}^2)) = 4\pi \times \left\{1-\Phi(\mathrm{u})\right\} \ .$$

Computing LKCs from a (CMB) map

- Harmonic space - obtain $T_\ell(x)$ maps; normalize each map by the expected RMS; power transform normalized T_ℓ maps to obtain NG maps.
- Needlet space - apply the standard needlet filter to the spherical harmonic coefficients; obtain needlet maps $\beta_j(x)$; normalize each map by the expected RMS; power transform normalized $\beta_j(x)$ maps to obtain NG maps. The j^{th} needlet map has a compact support for ℓ-range between B^{j-1} and B^{j+1} where we used B=1.5.
- The area functional is computed by finding the ratio of Healpix pixels above a certain temperature threshold.
- The length and genus functionals are computed by using the method described in Eriksen et. al. 2004 paper.

Multipole space - Gaussian case



Analytical (blue curve) vs Simulation (black and grey. Grey Shades are 68, 95 and

Summary

- Exact formula on full sky - no flat sky approximation!
- Analytical results agrees very well with simulations.
- Evaluation in both multipole and needlet space:
  - separates scale/frequency
  - mitigates contamination of high-frequency modes by the large low-ℓ cosmic variances.
  - allows a unified measure of deviation from Gaussianity (work in progress)
- Handles effect of mask analytically!!
- Handles non-Gaussianity analytically.
- Analytical predications of the LKCs variances (work in progress).