Power Set

For a set S, define $\mathcal{P}(S) = \{T \mid T \subseteq S\}$.
(set of all subsets of S.)

E.g. if $S = \{1,2\}$, $\mathcal{P}(S) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$.

Note: there is a 1-1 correspondence between

$$\mathcal{P}(S) \longleftrightarrow \{f : S \longrightarrow \{0,1\}\}$$

(set of all boolean functions from S)

How? Define, for any subset $T \subseteq S$, $\chi_T : S \to \{0,1\}$
as follows:

$$\chi_T(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{else} \end{cases}$$

Then the map $\chi : \mathcal{P}(S) \longrightarrow \{f : S \to \{0,1\}\}$
is 1-1, onto as desired.

(χ_T is often called the "characteristic function" for T)

Say $S = \{1,2,3\}$. Any function from $S \to \{0,1\}$
could be described by a table:

S	$\{0,1\}$
$1 \longrightarrow$	$0/1$
$2 \longrightarrow$	$0/1$
$3 \longrightarrow$	$0/1$

E.g. if $T = \{2,3\} \subseteq S$,

$\chi_T(1) = 0$

$\chi_T(2) = 1$

$\chi_T(3) = 1$.

Goal: write a <u>recursive</u> function that gives $\mathcal{P}(S)$ on input S.

- Assuming we had a working solution for smaller sets, how could we build a soln for S?

(Also what would be a good base case? How about $S = \{\}$? Then $\mathcal{P}(S) = \{\{\}\}$.)

Say $S = \{1,2,3\}$. Define $\bar{S} = \{1,2\}$ $(= S \setminus \{3\}.)$

$$\mathcal{P}(\bar{S}) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$$

add a 3!

what's missing? $\longrightarrow$ $\underline{\{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}}$

call this $\mathcal{P}(\bar{S}) + 3$

Then $\mathcal{P}(S) = \mathcal{P}(\bar{S}) \cup (\mathcal{P}(\bar{S}) + 3)$

In general, let $x \in S$, and set $\bar{S} = \overline{S \setminus \{x\}}$.

Then $\mathcal{P}(S) = \mathcal{P}(\bar{S}) \cup (\mathcal{P}(\bar{S}) + x)$.

(Note on notation: $A \setminus B$ = "set difference"
$$= \{x \in A \mid x \notin B\}$$
$$= A \cap B^c.)$$