Section 4.2: Continuous Functions

1. Definition of Continuity

- **Metric Space Definition:** Let (X, d) be a metric space and $E \subset X$. A function $f : E \to \mathbb{R}$ is continuous at $p \in E$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$|f(x) - f(p)| < \epsilon \quad \text{for all } x \in E \text{ with } d(x, p) < \delta$$

- **Topological Phrasing:** f is continuous at p if and only if f maps the δ-neighborhood of p into the ϵ-neighborhood of $f(p)$:

$$x \in N_\delta(p) \cap E \implies f(x) \in N_\epsilon(f(p))$$

- **Sequential Criterion:** f is continuous at p if and only if for every sequence $\{p_n\}$ in E with $p_n \to p$, we have $\lim_{n\to\infty} f(p_n) = f(p)$.
- **Isolated Points:** If p is an isolated point of E, every function f is continuous at p because there exists a δ such that $N_\delta(p) \cap E = \{p\}$.

2. Specific Examples of Continuity (4.2.2)

- **(a) Removable Discontinuity:**

$$g(x) = \frac{x^2 - 4}{x - 2}, \quad x \neq 2; \quad g(2) = 2$$

  - $\lim_{x\to 2} g(x) = 4 \neq g(2)$. Thus, discontinuous at $x = 2$.
  - *Correction:* Redefining $g(2) = 4$ makes it continuous.
- **(b) Rational Indicator:**

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ x, & x \notin \mathbb{Q} \end{cases}$$

  - Continuous at $p = 0$ since $\lim_{x\to 0} f(x) = 0 = f(0)$.
  - Discontinuous at every $p \neq 0$.
- **(c) Dirichlet Function:**

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

  - Discontinuous at every $p \in \mathbb{R}$ because rationals and irrationals are dense in $\mathbb{R}$.
- **(d) Reciprocal:** $f(x) = 1/x$ is continuous on $(0, 1)$.
- **(e) Oscillating Function:**

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

  - Continuous at $x = 0$ because $|x\sin(1/x)| \leq |x|$, so the limit is 0.
- **(f) Sine Function:** $f(x) = \sin x$ is continuous on $\mathbb{R}$.
  - *Proof:* Using $|\sin y - \sin x| = 2|\cos\frac{y+x}{2}\sin\frac{y-x}{2}| \leq |y - x|$, we can choose $\delta = \epsilon$.
- **(g) Thomae's Function (Popcorn Function):**

$$f(x) = \begin{cases} 1/n, & x = m/n \in \mathbb{Q} \cap (0, 1) \text{ (lowest terms)} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

  - **Discontinuous** at every rational p (since $f(p) \neq 0$).
  - **Continuous** at every irrational p (limit is 0).
  - *Proof Sketch:* For $\epsilon > 0$, only finitely many rationals have a denominator n such that $1/n \geq \epsilon$. We can choose δ to exclude these specific rationals from the neighborhood of irrational p.

3. Algebra and Composition

- **Algebraic Operations:** If f, g are continuous at p, then $f + g$, $f - g$, fg, and f/g (provided $g(p) \neq 0$) are continuous at p.
- **Composition (Theorem 4.2.4):**
  If f is continuous at p and g is continuous at $f(p)$, then $h = g \circ f$ is continuous at p.
  - *Examples:* Polynomials, Rational functions (on domain), and $\sin(p(x))$ are continuous.

4. Topological Characterization (Theorem 4.2.6)

- **Theorem:** A function $f : E \to \mathbb{R}$ is continuous on E **if and only if** $f^{-1}(V)$ is open in E for every open subset V of $\mathbb{R}$.
- **Proof:**
  1. Continuity $\implies$ Open Pre-images

- Let $p \in f^{-1}(V)$, which means $f(p) \in V$.

$$\underbrace{f(N_\delta(p) \cap E) \subseteq N_\epsilon(f(p))}_{\text{Since } f \text{ is continuous}} \subseteq \underbrace{V}_{\text{Since } V \text{ is open}}$$

- By definition of inverse image ($A \subseteq B \iff f^{-1}(A) \subseteq f^{-1}(B)$):

$$N_\delta(p) \cap E \subseteq f^{-1}(N_\epsilon(f(p))) \subseteq f^{-1}(V)$$

Since $N_\delta(p) \cap E \subseteq f^{-1}(V)$, the set $f^{-1}(V)$ **is open**.

2. Open Pre-images $\implies$ Continuity
- Let $\epsilon > 0$. Set $V = N_\epsilon(f(p))$ (which is open).

$$\underbrace{N_\delta(p) \cap E \subseteq f^{-1}(V)}_{\text{Since } f^{-1}(V) \text{ is open}}$$

- Applying f to both sides gives the continuity definition directly:

$$f(N_\delta(p) \cap E) \subseteq V = N_\epsilon(f(p))$$

- **Example:** For $f(x) = \sqrt{x}$ on $[0, \infty)$, let $V = (a, b)$.
  - If $a \leq 0 < b$, then $f^{-1}(V) = [0, b^2)$.
  - While $[0, b^2)$ is not open in $\mathbb{R}$, it **is** open in $[0, \infty)$ because it can be written as $(-b^2, b^2) \cap [0, \infty)$.
- **Warning:** The forward image of an open set is **not** necessarily open.
  - *Example*: $f(x) = x^2$ for $x \leq 2$ and $6 - x$ for $x > 2$. $f((-1, 1)) = [0, 1)$, which is not open.

5. Continuity and Compactness

- **Theorem 4.2.8 (Preservation of Compactness):**
  If K is a compact subset of a metric space X and $f : K \to \mathbb{R}$ is continuous on K, then the image $f(K)$ is compact.
  - **Proof:**
    a. Let $\{V_\alpha\}$ be an arbitrary open cover of $f(K)$.
    b. Since f is continuous, each $f^{-1}(V_\alpha)$ is open in K, forming an open cover of K.
    c. Since K is compact, there exists a finite subcover corresponding to indices $\alpha_1, \ldots, \alpha_n$.
    d. The collection $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$ covers $f(K)$. Thus, $f(K)$ is compact.

- **Corollary 4.2.9 (Extreme Value Theorem):**
  If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous, then f attains its maximum and minimum on K.

$$\exists p, q \in K \text{ s.t. } f(q) \leq f(x) \leq f(p) \quad \forall x \in K$$

  - **Concise Proof:**
    By Theorem 4.2.8, the image $f(K)$ is compact in $\mathbb{R}$.
    - **Bounded:** Since compact sets in $\mathbb{R}$ are bounded, $M = \sup f(K)$ and $m = \inf f(K)$ exist.
    - **Closed:** Since compact sets in $\mathbb{R}$ are closed, $f(K)$ contains its limit points, so $M, m \in f(K)$.
    - **Conclusion:** Therefore, there exist $p, q \in K$ such that $f(p) = M$ and $f(q) = m$.
- **Counter-examples (4.2.10):**
  The theorem fails if K is not compact (i.e., not closed or not bounded).
  - **Unbounded Domain:** $f(x) = \frac{x^2}{1+x^2}$ on $[0, \infty)$. The supremum is 1, but $f(x) < 1$ for all x, so the maximum is never attained.
  - **Not Closed Domain:** $g(x) = x$ on $(0, 1)$. The supremum is 1 and infimum is 0, but neither is contained in the range $(0, 1)$.

6. Intermediate Value Theorem (IVT)

- **Theorem 4.2.11 (Intermediate Value Theorem):**
  Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < \gamma < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = \gamma$.
  - **Key Intuition:** Continuity implies **local persistence of inequalities**. If $f(c) \neq \gamma$, the function forces values to stay strictly above or below γ in a small neighborhood, contradicting the precise boundary nature of the supremum.
  - **Rigorous Proof Sketch:**
    a. **Construct Set:** Let $A = \{x \in [a, b] : f(x) \leq \gamma\}$. Since $a \in A$, $A \neq \emptyset$. Since A is bounded by b, let $c = \sup A$.
    b. **Contradiction (Case 1: $f(c) < \gamma$):**
       Let $\epsilon = \frac{1}{2}(\gamma - f(c)) > 0$. By continuity, $\exists \delta > 0$ such that $f(x) < f(c) + \epsilon < \gamma$ for $x \in N_\delta(c)$.
       This implies there exist points $x > c$ where $f(x) < \gamma$, meaning $x \in A$. This contradicts that c is the upper bound of A.

c. **Contradiction (Case 2: $f(c) > \gamma$):**
Similar logic shows that for some neighborhood, $f(x) > \gamma$. This implies c is not a limit point of A (or $c \notin A$), contradicting the properties of the supremum.

d. **Conclusion:** Therefore, $f(c) = \gamma$.

- **Corollary 4.2.12 (Topological Characterization):**
If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous on I, then $f(I)$ is an interval.
  - **Proof:**
  To show $f(I)$ is an interval, let $s, t \in f(I)$ with $s < t$ and pick any γ such that $s < \gamma < t$. There exist $a, b \in I$ with $f(a) = s, f(b) = t$. Since I is an interval, the closed segment between a and b lies in I. By the **Intermediate Value Theorem**, there exists c between a and b such that $f(c) = \gamma$. Thus $\gamma \in f(I)$, implying $f(I)$ is an interval.
- **Corollary 4.2.13 (Existence of Roots):**
For every $\gamma > 0$ and $n \in \mathbb{N}$, there exists a unique $y > 0$ such that $y^n = \gamma$.
- **Corollary 4.2.14 (Fixed Point Theorem):**
If $f : [0, 1] \to [0, 1]$ is continuous, there exists $y \in [0, 1]$ such that $f(y) = y$.
  - **Proof Technique (Auxiliary Function):**
    a. Define $g(x) = f(x) - x$.
    b. **Evaluate Endpoints:**
       - $g(0) = f(0) - 0 \geq 0$ (since $f(0) \in [0, 1]$).
       - $g(1) = f(1) - 1 \leq 0$ (since $f(1) \in [0, 1]$).
    c. **Apply IVT:** Since g is continuous and 0 lies between $g(1)$ and $g(0)$, there exists y such that $g(y) = 0$.
    d. **Result:** $f(y) - y = 0 \implies f(y) = y$.

Caveats & Counter-examples (4.2.15)

- **Converse is False:** A function can satisfy the intermediate value property (Darboux property) but be discontinuous.
  - *Example:* $f(x) = \sin(1/x)$ for $x > 0$ and $f(0) = 0$. This function takes every value between -1 and 1 in any neighborhood of 0, satisfying the property, but is discontinuous at 0.
- **Requires Completeness of $\mathbb{R}$:** The IVT relies on the Least Upper Bound Property.
  - *Example:* Let $f(x) = x^2$ on the rational interval $E = [0, 2] \cap \mathbb{Q}$.
  - We have $f(0) < 2 < f(2)$, but there is **no rational number** $c \in E$ such that $c^2 = 2$. Thus, IVT fails in $\mathbb{Q}$.