Here is a comprehensive study guide for **Chapter 9: Fourier Series**, organized by logical flow and key concepts. It covers definitions, calculation examples, convergence types, and the logic behind major theorems.

9.1 Orthogonal Functions

1. Inner Product and Orthogonality

To discuss "approximation," we treat functions as vectors.

- **Inner Product:** For $f, g \in \mathcal{R}[a, b]$ (Riemann integrable functions):

$$\langle f, g\rangle = \int_a^b f(x)g(x)\, dx$$

- **Norm:** The "length" of a function is:

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \left[\int_a^b f^2(x)\, dx\right]^{1/2}$$

- **Orthogonality:** Two functions are orthogonal if $\langle f, g\rangle = 0$.
- **Orthonormal System:** A sequence $\{\phi_n\}$ is orthonormal if:

$$\langle \phi_n, \phi_m\rangle = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

2. Examples of Orthogonal Systems

- **Example A:** $\{1, x\}$ on $[-1, 1]$.

$$\int_{-1}^{1} 1 \cdot x\, dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

- **Example B (Trigonometric System):** $\{\sin nx\}_{n=1}^{\infty}$ on $[-\pi, \pi]$.
  For $n \neq m$, using product-to-sum identities:

$$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = 0$$

For $n = m$:

$$\int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi$$

- **Example C:** $\{1, \cos \frac{n\pi x}{L}, \sin \frac{n\pi x}{L}\}$ on $[-L, L]$.

3. Approximation in the Mean (Least Squares)

Given an orthogonal system $\{\phi_n\}$, we want to approximate f using a partial sum $S_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$.
We want to minimize the **mean square error**:

$$E_N = \|f - S_N\|_2^2 = \int_a^b [f(x) - S_N(x)]^2 \, dx$$

Theorem: The error is minimized if and only if c_n are the **Fourier Coefficients**:

$$c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} = \frac{\int_a^b f(x)\phi_n(x) \, dx}{\int_a^b \phi_n^2(x) \, dx}$$

Proof Idea:
Expand the integral $\int (f - \sum c_n \phi_n)^2$. By completing the square with respect to c_n, one finds that the minimum occurs exactly when c_n takes the form above.

4. Bessel's Inequality

Since the error $E_N \geq 0$:

$$\sum_{n=1}^{\infty} c_n^2 \|\phi_n\|^2 \leq \|f\|_2^2$$

Corollary: The coefficients c_n must decay such that $\lim_{n\to\infty} c_n = 0$.

9.2 Completeness and Parseval's Equality

1. Convergence in the Mean

A sequence $\{f_n\}$ converges to f **in the mean** if:

$$\lim_{n\to\infty} \int_a^b [f(x) - f_n(x)]^2 \, dx = 0 \quad \iff \quad \lim_{n\to\infty} \|f - f_n\|_2 = 0$$

Note: Uniform convergence $\implies$ Mean convergence. However, Mean convergence ParseError: KaTeX parse error: Undefined control sequence: \does at position 1: \does NOT imply Pointwise convergence.

2. Completeness

An orthogonal system $\{\phi_n\}$ is **complete** if the Fourier series partial sums S_N converge to f in the mean for every $f \in \mathcal{R}[a, b]$.

3. Parseval's Equality

The system is complete if and only if **Parseval's Equality** holds for all f:

$$\sum_{n=1}^{\infty} c_n^2 \|\phi_n\|^2 = \int_a^b f^2(x) \, dx$$

Concept: This is the infinite-dimensional version of the Pythagorean theorem ($\|v\|^2 = \sum |components|^2$).

9.3 Trigonometric Fourier Series

1. The Definitions

For the system $\{1, \cos nx, \sin nx\}$ on $[-\pi, \pi]$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Formulas:

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$

2. Calculation Examples

Example A: Step Function

$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi \end{cases}$$

- **a_0:** $\frac{1}{\pi} \int_0^{\pi} 1\, dx = 1$.
- **a_n:** $\frac{1}{\pi} \int_0^{\pi} \cos nx\, dx = \frac{1}{n\pi} [\sin nx]_0^{\pi} = 0$.
- **b_n:** $\frac{1}{\pi} \int_0^{\pi} \sin nx\, dx = \frac{-1}{n\pi} [\cos nx]_0^{\pi} = \frac{1}{n\pi}(1 - (-1)^n)$.
  - If n is even, $b_n = 0$.
  - If n is odd $(2k+1)$, $b_n = \frac{2}{(2k+1)\pi}$.
- **Series:** $f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$.

Example B: $f(x) = x$ on $[-\pi, \pi]$

- **Symmetry:** $f(x)$ is **Odd**.
  - $a_n = 0$ (integral of odd $\times$ even is odd, symmetric integral is 0).
- **b_n:** Use integration by parts.

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx\, dx = \frac{2}{\pi} \left[-\frac{x \cos nx}{n} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx\, dx \right]$$

$$b_n = \frac{2}{\pi} \left(-\frac{\pi(-1)^n}{n} \right) = \frac{2(-1)^{n+1}}{n}$$

- **Series:** $x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

3. Riemann-Lebesgue Lemma

For any $f \in \mathcal{R}[-\pi, \pi]$:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

Implication: Fourier coefficients $a_n, b_n \to 0$ as $n \to \infty$.

4. Sine and Cosine Series (Half-Range Expansions)

If f is defined only on $[0, \pi]$:

- **Even Extension (f_e):** Extends f to $[-\pi, \pi]$ as an even function. Generates a **Cosine Series**.
- **Odd Extension (f_o):** Extends f to $[-\pi, \pi]$ as an odd function. Generates a **Sine Series**.

9.4 Convergence in the Mean

1. The Dirichlet Kernel (D_n)

The partial sum $S_n(x)$ can be written as an integral convolution:

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) \, dt$$

where $D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin((n+1/2)t)}{2\sin(t/2)}$.
Properties: $\frac{1}{\pi} \int D_n = 1$, but $\int |D_n| \to \infty$. This makes D_n "bad" for proving uniform convergence directly.

2. The Fejér Kernel (F_n)

To fix the behavior of D_n, we take the arithmetic mean (Cesàro mean) of the partial sums:

$$\sigma_n(x) = \frac{S_0(x) + \cdots + S_n(x)}{n+1}$$

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) F_n(x-t)\, dt$$

where $F_n(t) = \frac{1}{2(n+1)} \left[\frac{\sin((n+1)t/2)}{\sin(t/2)} \right]^2$.

Properties of F_n (Approximate Identity):

1. $F_n(t) \geq 0$.
2. $\frac{1}{\pi} \int_{-\pi}^{\pi} F_n(t)\, dt = 1$.
3. For any $\delta > 0$, $F_n(t) \to 0$ uniformly outside $[-\delta, \delta]$.

3. Fejér's Theorem

If f is continuous on $[-\pi, \pi]$ and periodic ($f(-\pi) = f(\pi)$), then:

$$\sigma_n(x) \to f(x) \quad \text{uniformly on } [-\pi, \pi].$$

Proof Idea: Use the properties of the Approximate Identity. Split the integral into a small region near 0 (where $f(x-t) \approx f(x)$) and the tail (where $F_n \to 0$).

4. Main Result: Mean Convergence

Theorem: For any $f \in \mathcal{R}[-\pi, \pi]$, the Fourier series converges to f **in the mean**.

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2\, dx = 0$$

Corollary (Parseval's Equality for Trig Series):

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx$$

Calculation Example using Parseval:

Using $f(x) = x$ and its coefficients $b_n = \frac{2(-1)^{n+1}}{n}$:

$$\sum_{n=1}^{\infty} \left(\frac{2}{n} \right)^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^2}{3}$$

$$4 \sum \frac{1}{n^2} = \frac{2\pi^2}{3} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

9.5 Pointwise Convergence

1. The Problem

Convergence in the mean does not guarantee that $S_n(x) \to f(x)$ at every specific point x. We need stronger conditions.

2. Dirichlet's Theorem (Sufficient Conditions)

If f is periodic (2π) and piecewise continuous on $[-\pi, \pi]$, and x_0 is a point where:

1. One-sided limits $f(x_0^+)$ and $f(x_0^-)$ exist.
2. One-sided derivatives (or Lipschitz conditions) exist.

$$|f(x_0 \pm t) - f(x_0^\pm)| \leq Mt$$

Then the Fourier series converges to the average of the jump:

$$\lim_{n\to\infty} S_n(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}$$

Note: If f is continuous at x_0, it converges to $f(x_0)$.

3. Application Examples

- **Example A (Discontinuous):**
  $f(x) = 0$ on $[-\pi, -\pi/2)$, 3 on $[-\pi/2, \pi/2]$, 0 on $(\pi/2, \pi)$.
  At $x = \pi/2$, the series converges to $\frac{3+0}{2} = 1.5$.
  At $x = 0$, the series converges to 3.
- **Example B (Continuous):**
  $f(x) = |x|$ on $[-\pi, \pi]$. Continuous everywhere and derivative is piecewise continuous (± 1).
  Series converges to $|x|$ everywhere. Evaluating at $x = 0$ often yields sums of series like $\sum \frac{1}{(2k-1)^2}$.

4. Differentiation of Fourier Series

Simply differentiating a Fourier series term-by-term is not always valid (the resulting series might diverge).

Theorem: If f is **continuous** everywhere (including $f(-\pi) = f(\pi)$) and f' is **piecewise continuous**, then the Fourier series of f' is obtained by differentiating the series of f term-by-term.

$$f'(x) \sim \sum(-na_n \sin nx + nb_n \cos nx)$$