Here is a concise, comprehensive cheat sheet covering Chapters 5.1, 5.2, and 5.3 based on the provided documents.

Calculus Cheat Sheet: Derivatives & Mean Value Theorems

5.1 The Derivative

1. Definitions

- **The Derivative:** For $f : I \to \mathbb{R}$ and $p \in I$:

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

- **One-Sided Derivatives:**
  - **Right:** $f'_+(p) = \lim_{h\to 0^+} \frac{f(p+h)-f(p)}{h}$
  - **Left:** $f'_-(p) = \lim_{h\to 0^-} \frac{f(p+h)-f(p)}{h}$
- **Existence:** $f'(p)$ exists $\iff$ $f'_+(p)$ and $f'_-(p)$ exist and are **equal**.

2. Analyzed Functions (Examples)

- **Power ($f(x) = x^n$):** $f'(x) = nx^{n-1}$.
- **Square Root ($f(x) = \sqrt{x}, x > 0$):** Uses rationalization. $f'(x) = \frac{1}{2\sqrt{x}}$.
- **Sine ($f(x) = \sin x$):** Uses $\sin(A + B)$. $f'(x) = \cos x$.
- **Absolute Value ($f(x) = |x|$ at $x = 0$):**
  - $f'_+(0) = 1$, $f'_-(0) = -1$.
  - **Result:** Not differentiable at 0.
- **Cusp ($g(x) = x^{3/2}$ at $x = 0$):**
  - $g'(0) = \lim_{h\to 0^+} \sqrt{h} = 0$. Differentiable.
- **Oscillating Discontinuity ($f(x) = x \sin(1/x)$ at $x = 0$):**
  - Limit oscillates. **Result:** Not differentiable at 0.
- **Diff. with Discontinuous Derivative ($g(x) = x^2 \sin(1/x)$ at $x = 0$):**
  - $g'(0) = 0$ (via Squeeze Thm).
  - For $x \neq 0$, $g'(x) = 2x \sin(1/x) - \cos(1/x)$.

- **Result:** $\lim_{x\to0} g'(x)$ DNE. g is differentiable, but g' is **discontinuous** at 0.

3. Theorems & Arithmetic

- **Diff $\implies$ Cont:** If $f'(p)$ exists, f is continuous at p. (Converse is **False**).
- **Product Rule:** $(fg)' = f'g + fg'$. (Proof uses "add/subtract $f(x+h)g(x)$" trick).
- **Reciprocal Rule:** $(1/g)' = -g'/g^2$.
- **Quotient Rule:** $(f/g)' = (f'g - fg')/g^2$.
- **Chain Rule:** If $h = g \circ f$, then $h'(x) = g'(f(x)) \cdot f'(x)$.
    - *Example:* $h(x) = \sin(1/x^2) \implies h'(x) = \cos(1/x^2) \cdot (-2x^{-3})$.

5.2 The Mean Value Theorem (MVT)

1. Local Extrema

- **Theorem:** If f has a local extremum at interior point p and $f'(p)$ exists, then $f'(p) = 0$.
- **Corollary:** On $[a,b]$, extrema occur where $f' = 0$, f' DNE, or at endpoints.

2. Core Theorems

- **Rolle's Theorem:**
    - **Hypotheses:** Continuous on $[a,b]$, Differentiable on (a,b), $f(a) = f(b)$.
    - **Conclusion:** $\exists c \in (a,b)$ such that $f'(c) = 0$.
- **Mean Value Theorem (Lagrange):**
    - **Hypotheses:** Continuous on $[a,b]$, Differentiable on (a,b).
    - **Conclusion:** $\exists c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.
    - *Example Application:* Prove $\frac{x}{1+x} \leq \ln(1+x) \leq x$ for $x > -1$.
        - Apply MVT to $\ln(1+t)$ on $[0,x]$. $f'(c) = \frac{1}{1+c}$. Analyze bounds of c.
- **Cauchy MVT:**
    - $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ (assuming $g' \neq 0$).

3. Monotonicity & Subtle Pathologies

- **Standard Rule:**
    - $f' \geq 0$ on $I \implies$ Increasing.
    - $f' = 0$ on $I \implies$ Constant.
- **Pointwise vs. Neighborhood (Crucial):**

    - $f'(c) > 0$ **does NOT** imply f is increasing on a neighborhood $(c - \delta, c + \delta)$.
    - **Counter-Example:** $f(x) = x + 2x^2 \sin(1/x)$ at $x = 0$.
        - $f'(0) = 1 > 0$, but f' oscillates signs near 0.
    - **Continuous Derivative:** If $f'(c) > 0$ **and** f' is continuous at c, *then* f is increasing near c.
- **First Derivative Test Converse Error:**
    - Relative min at c **does NOT** imply f is decreasing to the left and increasing to the right.
    - **Counter-Example:** $f(x) = x^4(2 + \sin(1/x))$. Absolute min at 0, but oscillates (not monotone) near 0.

4. Derivative Properties

- **Limits of Derivatives:** If $\lim_{x \to a^+} f'(x) = L$, then $f'_+(a) = L$. (Derivatives cannot have simple jump discontinuities).
- **Darboux's Theorem (IVT for Derivatives):** If f is differentiable, f' satisfies the Intermediate Value Property (even if discontinuous).
- **Inverse Function Thm:** If $f' \neq 0$, $(f^{-1})'(y) = \frac{1}{f'(x)}$.

5.3 L'Hospital's Rule

1. Theorem Statement

Given f, g differentiable on (a, b), $g' \neq 0$:
If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ AND one of the following holds:

1. **Case $0/0$:** $\lim f(x) = 0$ and $\lim g(x) = 0$.
2. **Case ∞/∞:** $\lim g(x) = \pm\infty$.

Then: $\lim_{x \to a} \frac{f(x)}{g(x)} = L$

2. Proof Concepts

- $0/0$ **Case:** Uses **Cauchy MVT** on $[a, x]$. Define $f(a) = g(a) = 0$.
- ∞/∞ **Case:** Uses MVT + Bounding argument. Does not require continuity at a.

3. Worked Examples

- **A. Basic ($0/0$):**

$$\lim_{x\to 0^+} \frac{\ln(1+x)}{x} \xrightarrow{L'H} \lim_{x\to 0^+} \frac{1/(1+x)}{1} = 1$$

- **B. Repeated Use ($0/0$):**

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \xrightarrow{L'H} \lim_{x\to 0} \frac{\sin x}{2x} \xrightarrow{L'H} \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

- **C. Substitution Required:**
  - **Problem:** $\lim_{x\to 0^+} \frac{e^{-1/x}}{x}$. Direct differentiation fails (gets messier).
  - **Fix:** Let $t = 1/x$. As $x \to 0^+, t \to \infty$.
  - **New Limit:** $\lim_{t\to\infty} \frac{t}{e^t}$ (Case ∞/∞).
  - **Apply L'H:** $\lim \frac{1}{e^t} = 0$.