Here are the detailed notes on **Uniform Convergence, Continuity, and the Space $\mathcal{C}(K)$**, based on the provided text.

1. Specific Sequence Problems (Introductory Examples)

Problem 17: M-Test Failure

Consider functions defined on $[0, 1]$:

$$f_n(x) = \begin{cases} \frac{1}{n}, & \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \\ 0, & \text{elsewhere} \end{cases}$$

- **Result:** The series $\sum f_n(x)$ converges uniformly on $[0, 1]$.
- **Note:** The Weierstrass M-test fails here, demonstrating that the M-test is a sufficient but not necessary condition for uniform convergence.

2. Uniform Convergence and Continuity

This section addresses whether the limit of a sequence of continuous functions is continuous.

Theorem 8.3.1: Interchange of Limits

Let $\{f_n\}$ be a sequence of real-valued functions converging uniformly to f on a subset E of a metric space. Let p be a limit point of E.
If $\lim_{x \to p} f_n(x) = A_n$ for each n, then:

1. The sequence $\{A_n\}$ converges.
2. $\lim_{x \to p} f(x) = \lim_{n \to \infty} A_n$.

Formulaic Representation:

$$\lim_{x \to p}(\lim_{n \to \infty} f_n(x)) = \lim_{n \to \infty}(\lim_{x \to p} f_n(x))$$

Proof (Key Ideas):

- **Step 1 (A_n is Cauchy):** Since f_n converges uniformly, $|f_n(x) - f_m(x)| < \epsilon$. Letting $x \to p$, we get $|A_n - A_m| \le \epsilon$. Thus $\{A_n\}$ converges to some limit A.
- **Step 2 (Convergence to A):** Use the $\epsilon/3$ argument via the Triangle Inequality:

$$|f(x) - A| \le |f(x) - f_m(x)| + |f_m(x) - A_m| + |A_m - A|$$

- Term 1 is small due to uniform convergence.
- Term 2 is small because $\lim_{x \to p} f_m(x) = A_m$.
- Term 3 is small because $A_m \to A$.

Corollary 8.3.2: Preservation of Continuity

- **(a) Sequences:** If $\{f_n\}$ are continuous on E and converge uniformly to f, then f is continuous on E.
- **(b) Series:** If $\{f_n\}$ are continuous and $\sum f_n$ converges uniformly to S, then S is continuous.

Proof Idea:
If p is a limit point, apply Theorem 8.3.1. Since f_n is continuous, $\lim_{x \to p} f_n(x) = f_n(p)$. The theorem implies $\lim_{x \to p} f(x) = \lim f_n(p) = f(p)$.

Counter-Examples (Why Uniformity Matters)

- **Example 8.3.3:** The sequence $f_n(x) = x^n$ on $[0, 1]$ converges pointwise to a discontinuous function (0 on $[0, 1)$, 1 at $x = 1$). Thus, convergence is **not** uniform.
- The series $\sum_{k=0}^{\infty} x^2 (\frac{1}{1+x^2})^k$ converges to a function discontinuous at $x = 0$, implying convergence is not uniform near 0.

3. Dini's Theorem

Does pointwise convergence ever imply uniform convergence? Generally no, but yes under specific conditions (monotonicity and compactness).

Example 8.3.4 (Counter-example):

$$S_n(x) = nxe^{-nx^2}, \quad x \in [0, 1]$$

- $S_n \to 0$ pointwise.
- Functions are continuous.
- **Failure:** The maximum value is $\sqrt{n/2e}$, which tends to ∞. The "hump" moves toward 0 but gets infinitely tall. Convergence is **not** uniform.

Theorem 8.3.5: Dini's Theorem

Let K be a **compact** subset. Let $\{f_n\}$ be a sequence of continuous functions on K such that:

1. $\{f_n\}$ converges **pointwise** to a continuous function f.
2. The sequence is **monotone**: $f_n(x) \geq f_{n+1}(x)$ for all x, n (or increasing).

Result: $\{f_n\}$ converges **uniformly** to f on K.

Proof (Key Ideas):

1. Define $g_n = f_n - f$. Then g_n is continuous, $g_n \geq 0$, $g_n \geq g_{n+1}$, and $g_n \to 0$ pointwise.
2. Let $\epsilon > 0$. Define sets $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
3. Since g_n is continuous, K_n is closed. Since K is compact, K_n is **compact**.
4. By monotonicity ($g_n \geq g_{n+1}$), the sets are nested: $K_{n+1} \subset K_n$.
5. Since $g_n(x) \to 0$ for every x, the intersection $\bigcap K_n = \emptyset$.
6. **Finite Intersection Property:** For compact sets, if the infinite intersection is empty, a finite intersection must be empty. Thus, $K_N = \emptyset$ for some N.
7. This implies $0 \leq g_n(x) < \epsilon$ for all $n \geq N$ and all $x \in K$, proving uniform convergence.

Example 8.3.6 (Necessity of Compactness):
$f_n(x) = \frac{1}{nx+1}$ on $(0, 1)$. Monotonically decreases to 0. Convergence is **not** uniform (near $x = 0$, $f_n \to 1$) because the domain $(0, 1)$ is not compact.

4. The Space $\mathcal{C}(K)$

This section formalizes the set of continuous functions as a normed linear space. Let K be a compact set. $\mathcal{C}(K)$ is the vector space of all continuous real-valued functions on K.

Uniform Norm

Definition 8.3.7: For $f \in \mathcal{C}(K)$, the uniform norm is:

$$||f||_u = \max\{|f(x)| : x \in K\}$$

(Note: Maximum exists because K is compact and f is continuous).

Theorem 8.3.8 (Equivalence):
A sequence f_n converges uniformly to f in $\mathcal{C}(K)$ **if and only if** it converges in the uniform norm:

$$\lim_{n \to \infty} ||f_n - f||_u = 0$$

Completeness

Definition 8.3.10:

- A sequence $\{x_n\}$ in a normed space is **Cauchy** if $||x_n - x_m|| < \epsilon$ for large n, m.
- A space is **Complete** if every Cauchy sequence converges to an element within the space.

Theorem 8.3.11: The space $(\mathcal{C}(K), || \cdot ||_u)$ is **complete**.
Proof (Key Ideas):

1. Take a Cauchy sequence $\{f_n\}$ in $\mathcal{C}(K)$.
2. $|f_n(x) - f_m(x)| \leq ||f_n - f_m||_u < \epsilon$. This means $\{f_n(x)\}$ is a uniformly Cauchy sequence of real numbers.
3. By previous theorems (Cauchy criterion for uniform convergence), $\{f_n\}$ converges uniformly to some function f.
4. Since f_n are continuous and convergence is uniform, limit f is continuous (Corollary 8.3.2). Thus $f \in \mathcal{C}(K)$.
5. Therefore, the sequence converges to f in the norm.

5. Contraction Mappings

Extension of contraction functions to normed linear spaces.

Definition 8.3.12:
Let $(X, || \cdot ||)$ be a normed linear space. A mapping $T : X \to X$ is a **contraction** if there exists a constant c ($0 < c < 1$) such that for all $x, y \in X$:

$$||T(x) - T(y)|| \leq c||x - y||$$

Theorem 8.3.13: Banach Fixed Point Theorem

Let X be a **complete** normed linear space. If T is a contraction mapping, there exists a **unique** fixed point $x \in X$ such that $T(x) = x$.

Proof (Key Ideas):

1. **Existence (Iterative Sequence):**
   - Pick arbitrary x_0. Define $x_n = T(x_{n-1})$.
   - Show $\{x_n\}$ is Cauchy: $||x_{n+1} - x_n|| \leq c^n||x_1 - x_0||$.
   - Using geometric series sum, $||x_{n+m} - x_n|| \leq \frac{c^n}{1-c}||x_1 - x_0||$.

- Since $0 < c < 1$, $c^n \to 0$, so sequence is Cauchy.
- Since X is complete, $x_n \to x$ for some $x \in X$.
- By continuity of T: $x = \lim x_{n+1} = \lim T(x_n) = T(x)$.

2. **Uniqueness:**
   - Assume $T(x) = x$ and $T(y) = y$.
   - $||x - y|| = ||T(x) - T(y)|| \leq c||x - y||$.
   - Since $c < 1$, this implies $||x - y|| = 0$, so $x = y$.