Section 4.3: Uniform Continuity

1. Concept: Pointwise vs. Uniform Continuity

- **Standard Continuity (Pointwise):**
  - A function $f : E \to \mathbb{R}$ is continuous on E if for each $p \in E$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in E \cap N_\delta(p)$.
  - **Dependency:** Here, δ depends on **both** ϵ and the specific point p. (i.e., $\delta = \delta(\epsilon, p)$).
  - *Problem:* As p changes, the required δ might get infinitely small (e.g., $f(x) = 1/x$ near 0).
- **Uniform Continuity (Definition 4.3.1):**
  - Let $E \subset (X, d)$ and $f : E \to \mathbb{R}$.
  - f is **uniformly continuous** on E if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in E, \text{ if } d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

  - **Dependency:** Here, δ depends **only** on ϵ, f, and the set E. It is **independent** of the point x.

2. Examples of Uniform Continuity

Case A: $f(x) = x^2$ on a Bounded Subset E

- **Claim:** f is uniformly continuous on E if E is bounded.
- **Proof:**
  - Since E is bounded, $\exists C > 0$ such that $|x| \le C$ for all $x \in E$.
  - Consider $|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y|$.
  - Using the bound: $|x + y| \le |x| + |y| \le 2C$.
  - Thus, $|f(x) - f(y)| \le 2C|x - y|$.
  - Choose $\delta = \epsilon/2C$. If $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Case B: $f(x) = \sin x$ on $\mathbb{R}$

- **Claim:** f is uniformly continuous on $\mathbb{R}$.
- **Proof:**
  - Standard trigonometric inequality: $|\sin x - \sin y| \le |x - y|$.
  - Choose $\delta = \epsilon$.

- If $|x-y|<\delta$, then $|f(x)-f(y)| \leq |x-y| < \epsilon$.

Case C: Non-Example ($f(x) = 1/x$ on $(0, 1)$)

- **Claim:** f is **not** uniformly continuous on $(0, 1)$.
- **Proof (Contradiction):**
  - Assume f is uniformly continuous. Let $\epsilon = 1$.
  - There must exist $\delta > 0$ (assume $\delta < 1$) such that $|x-y| < \delta \implies |1/x - 1/y| < 1$.
  - Choose $x \in (0, 1/2)$ such that x is very small, and let $y = x + \delta/2$.
  - Then $|x-y| = \delta/2 < \delta$.
  - However, analysis shows that as $x \to 0$, the difference $|1/x - 1/(x+\delta/2)|$ becomes arbitrarily large, eventually exceeding 1.
  - Specifically, if x is small enough ($x < \delta/2$), the inequality fails, leading to a contradiction.

3. Lipschitz Functions

- **Definition:** A function $f : E \to \mathbb{R}$ satisfies a **Lipschitz condition** on E if there exists a constant $M > 0$ such that:

$$|f(x) - f(y)| \leq Md(x, y) \quad \forall x, y \in E$$

  - These are called Lipschitz functions.
  - Functions with bounded derivatives are Lipschitz.
- **Theorem 4.3.3:**
  - If f satisfies a Lipschitz condition on E, then f is **uniformly continuous** on E.
  - *Proof Logic:* Simply choose $\delta = \epsilon/M$.
- **Important Note (Converse is False):**
  - Not every uniformly continuous function is Lipschitz.
  - *Example:* $f(x) = \sqrt{x}$ on $[0, \infty)$ is uniformly continuous, but **not** Lipschitz (the derivative is unbounded near 0).

4. The Uniform Continuity Theorem

Determining uniform continuity for non-Lipschitz functions is difficult. Compactness provides a sufficient condition.

- **Theorem 4.3.4:**

    - If K is a **compact** metric space and $f : K \to \mathbb{R}$ is **continuous** on K, then f is **uniformly continuous** on K.
- **Proof Sketch:**
    i. Fix $\epsilon > 0$. Since f is continuous, for every $p \in K$, there is a δ_p valid locally.
    ii. The neighborhoods $N_{\delta_p/2}(p)$ form an open cover of K.
    iii. Since K is compact, there is a **finite subcover** corresponding to points $p_1, \dots, p_n$.
    iv. Define $\delta = \frac{1}{2} \min\{\delta_{p_1}, \dots, \delta_{p_n}\}$. This δ works globally for the whole set.
- **Corollary 4.3.5 (Heine's Theorem):**
    - A continuous real-valued function on a **closed and bounded interval** $[a, b]$ is uniformly continuous.

5. Necessity of Compactness (Example 4.3.6)

To guarantee uniform continuity via Theorem 4.3.4, the domain must be **both** closed and bounded (Compact).

- **Closed but Not Bounded:**
    - Set: $[0, \infty)$.
    - Function: $f(x) = x^2$.
    - Result: Continuous, but **not** uniformly continuous (values grow too fast as $x \to \infty$).
- **Bounded but Not Closed:**
    - Set: $(0, 1)$.
    - Function: $f(x) = 1/x$.
    - Result: Continuous, but **not** uniformly continuous (values grow too fast as $x \to 0$).