# Chapter 9: Fourier Series

## 9.1 Orthogonal Functions

**Core Concept:** Approximating functions using a linear combination of orthogonal functions, analogous to vector decomposition in $\mathbb{R}^n$.

### 1. Inner Product and Orthogonality

For Riemann integrable functions on $[a, b]$, denoted $\mathcal{R}[a, b]$, the **inner product** is defined as:

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$$

- **Norm:** $||f||_2 = \sqrt{\langle f, f \rangle} = \left[ \int_a^b f^2(x)\, dx \right]^{1/2}$.
- **Orthogonality:** Two functions $\phi, \psi$ are orthogonal if $\langle \phi, \psi \rangle = 0$.
- **Orthonormality:** A sequence $\{\phi_n\}$ is orthonormal if $\langle \phi_n, \phi_m \rangle = 0$ for $n \neq m$ and $||\phi_n||^2 = 1$.

### 2. Key Examples of Orthogonal Systems

- **Example 9.1.2(a):** $\{1, x\}$ on $[-1, 1]$.

$$\int_{-1}^1 1 \cdot x\, dx = \left[ \frac{x^2}{2} \right]_{-1}^1 = 0$$

- **Example 9.1.2(b):** $\{\sin nx\}_{n=1}^\infty$ on $[-\pi, \pi]$.

$$\int_{-\pi}^\pi \sin nx \sin mx\, dx = 0 \quad (\text{for } n \neq m)$$

  *Norm squared:* $\int_{-\pi}^\pi \sin^2 nx\, dx = \pi$.
- **Example 9.1.2(c):** $\{1, \sin \frac{n\pi x}{L}, \cos \frac{n\pi x}{L}\}$ on $[-L, L]$.
  This is the general trigonometric system.

# 3. Approximation in the Mean

We seek constants $c_n$ to minimize the mean square error between $f$ and the partial sum $S_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$:

$$E_N = ||f - S_N||_2^2 = \int_a^b [f(x) - S_N(x)]^2 \, dx$$

**Theorem 9.1.4 (Best Approximation):**

The error is minimized if and only if $c_n$ are the **Fourier Coefficients**:

$$c_n = \frac{\langle f, \phi_n \rangle}{||\phi_n||^2} = \frac{\int_a^b f(x)\phi_n(x) \, dx}{\int_a^b \phi_n^2(x) \, dx}$$

With these coefficients, the error identity is:

$$||f - S_N||^2 = ||f||^2 - \sum_{n=1}^{N} c_n^2 ||\phi_n||^2$$

**Example 9.1.6:**

Approximating $f(x) = x^3 + 1$ on $[-1, 1]$ using orthogonal system $\{1, x\}$.

- $c_1 = \frac{\int (x^3 + 1)(1)}{\int 1^2} = 1$
- $c_2 = \frac{\int (x^3 + 1)(x)}{\int x^2} = \frac{3}{5}$
- Result: $S_2(x) = 1 + \frac{3}{5}x$.

# 4. Bessel's Inequality

Since the error $||f - S_N||^2 \geq 0$, it follows that:

$$\sum_{n=1}^{\infty} c_n^2 ||\phi_n||^2 \leq ||f||^2$$

**Corollary:** For orthogonal systems, Fourier coefficients $c_n \to 0$ as $n \to \infty$ (normalized by norm).

# 9.2 Completeness and Parseval's Equality

## 1. Convergence in the Mean

A sequence $f_n$ converges to $f$ **in the mean** if:

$$\lim_{n \to \infty} \int_a^b [f(x) - f_n(x)]^2 \, dx = 0 \quad \Longleftrightarrow \quad \lim_{n \to \infty} ||f - f_n||_2 = 0$$

- **Theorem:** Uniform convergence $\implies$ Convergence in the mean.
- **Counter-Example 9.2.3 (Mean $\not\Rightarrow$ Pointwise):**
  A sequence of "moving bump" functions $f_n$ on $[0, 1]$.
  - Constructed by dividing $[0, 1]$ into intervals of size $1/2^k$. $f_n$ is 1 on a shrinking sub-interval and 0 elsewhere.
  - $\int f_n^2 \to 0$ (Converges in mean to 0).
  - For any $x$, $f_n(x)$ oscillates between 0 and 1 infinitely often (Diverges pointwise).

## 2. Completeness

An orthogonal system $\{\phi_n\}$ is **complete** if Parseval's Equality holds for *every $f \in \mathcal{R}[a, b]$*.
**Parseval's Equality:**

$$\sum_{n=1}^{\infty} c_n^2 ||\phi_n||^2 = ||f||^2$$

- Completeness is equivalent to the partial sums $S_N$ converging to $f$ in the mean.
- **Uniqueness Theorem:** If $\{\phi_n\}$ is complete and $f$ is continuous, and all Fourier coefficients are zero ($\int f\phi_n = 0$), then $f(x) \equiv 0$.

# 9.3 Trigonometric and Fourier Series

## 1. The Trigonometric System

On $[-\pi, \pi]$, the system is $\{1, \cos nx, \sin nx\}$.

- $||1||^2 = 2\pi$
- $||\cos nx||^2 = \pi$, $||\sin nx||^2 = \pi$

## 2. Definitions

For $f \in \mathcal{R}[-\pi, \pi]$:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

**Fourier Series:**

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

## 3. Examples of Calculation

- **Example 9.3.3(a): Step Function**
  $f(x) = 0$ for $x \in [-\pi, 0)$, $f(x) = 1$ for $x \in [0, \pi)$.
  - $a_0 = 1$.
  - $a_n = 0$ (calculation yields $\sin n\pi = 0$).
  - $b_n = \frac{1}{n\pi}[1 - (-1)^n]$. (Non-zero only for odd $n$).
  - Series: $\frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$.
- **Example 9.3.3(b): $f(x) = x$**
  - $f$ is odd $\implies a_n = 0$.
  - $b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$. Integration by parts yields $b_n = \frac{2(-1)^{n+1}}{n}$.
  - Series: $2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

## 4. Riemann-Lebesgue Lemma

For $f \in \mathcal{R}[-\pi, \pi]$:

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

- **Example 9.3.5:** If $f$ is not integrable (e.g., $f(x) = 1/x$), the limit may not be zero ($\lim \to \pi/2$). This confirms integrability is required.

## 5. Sine and Cosine Series

- If $f$ is defined on $[0, \pi]$:
    - **Odd Extension:** Yields **Sine Series** ($a_n = 0$).
    - **Even Extension:** Yields **Cosine Series** ($b_n = 0$).

# 9.4 Convergence in the Mean

**Goal:** Prove that for $f \in \mathcal{R}[-\pi, \pi]$, the Fourier series converges to $f$ in the mean.

## 1. The Dirichlet Kernel ($D_n$)

The partial sum $S_n(x)$ can be written as an integral convolution:

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(x - t) \, dt$$

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin((n + 1/2)t)}{2 \sin(t/2)}$$

- **Problem:** $D_n$ is *not* a "good" kernel (approximate identity) because $\int |D_n| \to \infty$. This makes pointwise convergence difficult to prove directly.

## 2. The Fejér Kernel ($F_n$)

To solve the convergence problem, we look at **Cesàro means** (averages of partial sums):

$$\sigma_n(x) = \frac{S_0(x) + \cdots + S_n(x)}{n + 1}$$

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) F_n(x - t) \, dt$$

$$F_n(t) = \frac{1}{2(n+1)} \left[ \frac{\sin\left(\frac{n+1}{2}t\right)}{\sin(t/2)} \right]^2$$

**Properties of $F_n$ (Approximate Identity):**

1. $F_n(t) \geq 0$.
2. $\frac{1}{\pi} \int F_n = 1$.
3. $F_n \to 0$ uniformly outside any neighborhood of $t = 0$.

# 3. Fejér's Theorem

**Theorem 9.4.5:** If $f$ is continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$, then $\sigma_n(x)$ converges uniformly to $f(x)$.

**Proof Logic:**
Using the properties of the Fejér kernel (Approximate Identity), the convolution $\sigma_n$ approximates any continuous periodic function uniformly.

**Corollary 9.4.6:** Since $\sigma_n \to f$ uniformly, $S_n$ converges to $f$ **in the mean** for continuous functions. (The mean square error of $S_n$ is always $\leq$ the mean square error of $\sigma_n$).

# 4. General Convergence and Parseval's

**Theorem 9.4.7:** For *any $f \in \mathcal{R}[-\pi, \pi]$*, $S_n$ converges to $f$ in the mean.
**Proof Idea:** Any Riemann integrable function can be approximated in the mean by a continuous function (Step function approximation lemma). Combined with Fejér's theorem, the result follows.

**Parseval's Equality (Corollary 9.4.9):**

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx$$

**Example 9.4.10:**

Using $f(x) = x$ (from 9.3.3b), $b_n = \frac{2(-1)^{n+1}}{n}$.
Apply Parseval's: $\sum \frac{4}{n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^2}{3}$.
Result: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

# 9.5 Pointwise Convergence

**Question:** When does $S_n(x) \to f(x)$ at a specific point $x$?

## 1. Dirichlet's Theorem

**Theorem 9.5.3:** Let $f$ be periodic $(2\pi)$. If at a point $x_0$:

1. Left and right limits $f(x_0-)$, $f(x_0+)$ exist.
2. Left and right derivatives (or Lipschitz conditions) exist.
   - $|f(x_0 \pm t) - f(x_0\pm)| \le Mt$.

Then:

$$\lim_{n \to \infty} S_n(x_0) = \frac{f(x_0+) + f(x_0-)}{2}$$

**Proof Logic:**
Using the Dirichlet integral representation:

$$S_n(x) - A = \frac{2}{\pi} \int_0^\pi \left[ \frac{f(x+t) + f(x-t)}{2} - A \right] D_n(t)\, dt$$

Set $A = \text{avg limit}$. The term in the bracket divided by $2\sin(t/2)$ behaves like a Riemann integrable function $g(t)$ due to the derivative condition (boundedness near 0).
By Riemann-Lebesgue Lemma, $\int g(t) \sin((n + 1/2)t)dt \to 0$.

## 2. Piecewise Continuity

- **Piecewise Continuous:** Finite number of simple discontinuities (finite jumps).
- **Piecewise Smooth:** $f$ and $f'$ are piecewise continuous.
- **Corollary 9.5.6:** If $f$ is periodic and piecewise smooth, Fourier series converges to the average of limits everywhere.

## 3. Examples

- **Example 9.5.7(a):** Square wave (0, 3, 0).
  - At discontinuity (e.g., $x = \pi/2$), series converges to $3/2$.
  - Evaluating at $x = 0$ yields Leibniz series: $\sum \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$.
- **Example 9.5.7(b):** Triangular wave ($f(x) = |x|$ on $[-\pi, \pi]$).

- Continuous everywhere. Series converges to $|x|$.
- Evaluating at $x = 0$ yields $\sum \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$.

# 4. Differentiation of Fourier Series

**Theorem 9.5.8:**

Differentiation term-by-term is valid if:

1. $f$ is continuous on $[-\pi, \pi]$ AND $f(-\pi) = f(\pi)$ (Periodic continuity).
2. $f'$ is piecewise continuous.

Then at points where $f''(x)$ exists:

$$f'(x) = \sum_{n=1}^{\infty}(-na_n \sin nx + nb_n \cos nx)$$

*Note: If $f$ is discontinuous, the differentiated series generally diverges.*

**Example 9.5.9:**

- $f(x) = x^2$ is continuous and periodic.
- Differentiating the series for $x^2$ leads to the series for $2x$ (which matches the calculated series for $x$).
- Conversely, integrating the series for $2x$ term-by-term yields the series for $x^2$. Evaluating at $x = 0$ often yields sums of series like $\sum \frac{1}{(2k-1)^2}$.

# 4. Differentiation of Fourier Series

Simply differentiating a Fourier series term-by-term is not always valid (the resulting series might diverge).

**Theorem:** If $f$ is **continuous** everywhere (including $f(-\pi) = f(\pi)$) and $f'$ is **piecewise continuous**, then the Fourier series of $f'$ is obtained by differentiating the series of $f$ term-by-term.

$$f'(x) \sim \sum(-na_n \sin nx + nb_n \cos nx)$$