Here are concise, comprehensive study notes based on the provided text, focusing on definitions, key theorems, and the logical structure of proofs.

Study Notes: Series of Real Numbers

1. Absolute and Conditional Convergence

Definitions

- **Absolute Convergence:** A series $\sum a_k$ is absolutely convergent if the series of absolute values $\sum |a_k|$ converges.
- **Conditional Convergence:** A series is conditionally convergent if $\sum a_k$ converges, but $\sum |a_k|$ diverges.

Examples

1. **Alternating Harmonic Series:** $\sum \frac{(-1)^{k+1}}{k}$
   - Converges (by Alternating Series Test).
   - However, $\sum |\frac{(-1)^{k+1}}{k}| = \sum \frac{1}{k} = \infty$ (Harmonic series diverges).
   - $\therefore$ **Conditionally Convergent**.
2. **Inverse Square Alternating Series:** $\sum \frac{(-1)^{k+1}}{k^2}$
   - $\sum \frac{1}{k^2} < \infty$.
   - $\therefore$ **Absolutely Convergent**.

Theorem 7.3.3: Absolute Implies Convergence

Statement: Every absolutely convergent series converges.
Proof Idea:

- Assume $\sum |a_k| < \infty$.
- By the **Triangle Inequality**: $|\sum_{k=p}^{q} a_k| \leq \sum_{k=p}^{q} |a_k|$.
- Since $\sum |a_k|$ converges, it satisfies the Cauchy Criterion (the tail $\sum_{k=p}^{q} |a_k|$ becomes arbitrarily small).
- Therefore, $|\sum_{k=p}^{q} a_k|$ also becomes arbitrarily small.
- By the Cauchy Criterion for series, $\sum a_k$ converges.

2. Testing for Absolute Convergence

To test $\sum a_k$ for absolute convergence, apply standard tests (Section 7.1) to $\sum |a_k|$.

Theorem 7.3.4: Root and Ratio Tests

Let $\alpha = \limsup_{k\to\infty} \sqrt[k]{|a_k|}$ and $R = \limsup_{k\to\infty} |\frac{a_{k+1}}{a_k}|$, $r = \liminf_{k\to\infty} |\frac{a_{k+1}}{a_k}|$.

1. **Convergence:** If $\alpha < 1$ or $R < 1$, the series is **absolutely convergent**.
2. **Divergence:** If $\alpha > 1$ or $r > 1$, the series **diverges** (not just fails absolute convergence, but fails to converge entirely).
   - *Reasoning:* If limit > 1, the terms $|a_k|$ grow or do not approach 0. If $\lim a_k \neq 0$, the series diverges.
3. **Inconclusive:** If $\alpha = 1$ or $r \leq 1 \leq R$, the test gives no information.

3. Rearrangements of Series

Definition

A series $\sum a'_k$ is a **rearrangement** of $\sum a_k$ if there exists a one-to-one bijection $j : \mathbb{N} \to \mathbb{N}$ such that $a'_k = a_{j(k)}$.

- *Essence:* All terms of the original series appear exactly once, just in a different order.

The Critical Question

Does rearranging the terms change the sum?

- **Absolutely Convergent Series:** No.
- **Conditionally Convergent Series: Yes.**

Example of Failure (Conditional Convergence)

Consider $\sum \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$ (Sums to s).

- Rearrange to take two positive terms for every one negative term:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \dots$$

- The positive terms accumulate faster than the negative terms reduce the sum.
- The new sum s' can be shown to be strictly greater than the original sum ($s' > s$).

4. Convergence Theorems regarding Rearrangement

Theorem 7.3.8: Rearrangement of Absolute Series

Statement: If $\sum a_k$ converges absolutely, then *every* rearrangement converges to the same sum.

Proof Idea (Internal Logic):

1. **Tail Control:** Since $\sum |a_k|$ converges, for any ϵ, we can choose N such that the sum of absolute values of the "tail" (terms after index N) is $< \epsilon$.
2. **Matching Terms:** Let s_n be the partial sum of the original and s'_n be the partial sum of the rearrangement. Choose an index p large enough so that the first p terms of the rearrangement include all the first N terms of the original series.
3. **Cancellation:** In the difference $|s_n - s'_n|$ (for large n), the first N terms cancel out perfectly.
4. **Bound Remainder:** The remaining terms belong to the "tail" (indices $> N$). Since the sum of the absolute values of the tail is bounded by ϵ, the difference $|s_n - s'_n|$ is negligible. The limits must be identical.

Theorem 7.3.9: Riemann's Rearrangement Theorem

Statement: If $\sum a_k$ is **conditionally convergent**, then for any real number α, there exists a rearrangement $\sum a'_k$ that converges to α.

Proof Construction (The "Greedy" Algorithm):

1. **Decomposition:** Separate the series into positive terms (P_k) and negative terms (Q_k).
   - Because the series is *conditionally* convergent:
     - $\sum P_k = \infty$ (Positive part diverges)
     - $\sum Q_k = \infty$ (Negative part diverges)
     - $P_k \to 0$ and $Q_k \to 0$ (Terms vanish)
     - *Note:* If these didn't diverge, the series would be absolutely convergent or divergent.

2. **Algorithm to target α:**
   - **Step 1:** Add enough positive terms P_k until the partial sum just exceeds α. (Possible because $\sum P_k = \infty$).
   - **Step 2:** Subtract enough negative terms Q_k until the partial sum just drops below α. (Possible because $\sum Q_k = \infty$).
   - **Step 3:** Repeat indefinitely.
3. **Convergence:**
   - At each step, we overshoot or undershoot α by at most the value of the single term we just added/subtracted.
   - Since $a_k \to 0$ (the individual terms approach zero), the size of these overshoots/undershoots approaches 0.
   - Therefore, the partial sums converge to α.

Remark: This theorem implies conditionally convergent series are unstable regarding order; one can even rearrange them to diverge to $\pm\infty$.