7.1 Series of Real Numbers

1. Fundamentals of Series

Definitions

Let $\{a_k\}$ be a sequence of real numbers.

- **Infinite Series:** Denoted as $\sum_{k=1}^{\infty} a_k$.
- **Partial Sums:** The sequence of partial sums $\{s_n\}$ is defined as $s_n = \sum_{k=1}^{n} a_k$.
- **Convergence:** The series $\sum a_k$ converges to a sum s if and only if the sequence of partial sums $\{s_n\}$ converges to s (i.e., $\lim_{n\to\infty} s_n = s$).
- **Divergence:** If $\{s_n\}$ diverges, the series diverges. If $\lim s_n = \infty$, we write $\sum a_k = \infty$.

Linearity of Convergent Series (Theorem 7.1.1)

If $\sum a_k = \alpha$ and $\sum b_k = \beta$, then:

1. $\sum(c \cdot a_k) = c\alpha$ for any constant $c \in \mathbb{R}$.
2. $\sum(a_k + b_k) = \alpha + \beta$.

Proof Idea:
This follows directly from the limit laws for sequences. Since $s_n \to \alpha$ and $t_n \to \beta$, the partial sum of the combined series is $(s_n + t_n)$, which converges to $\alpha + \beta$.

2. Comparison Tests

These tests are primarily used for series with **non-negative terms** ($a_k \geq 0$).

Comparison Test (Theorem 7.1.2)

Suppose $0 \leq a_k \leq M b_k$ for some constant $M > 0$ and all $k \geq k_0$.

1. If $\sum b_k$ **converges**, then $\sum a_k$ **converges**.
2. If $\sum a_k$ **diverges**, then $\sum b_k$ **diverges**.

Proof:

- **Convergence Case:** By the Cauchy Criterion, if $\sum b_k$ converges, the tail sum can be made arbitrarily small ($\sum_{k=m+1}^{n} b_k < \epsilon/M$). Since $a_k \le Mb_k$, the tail sum of a_k is bounded by ϵ. Thus, $\sum a_k$ converges.
- **Divergence Case:** Logical contrapositive of the above.

Limit Comparison Test (Corollary 7.1.3)

Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

1. If $0 < L < \infty$: $\sum a_k$ converges $\iff$ $\sum b_k$ converges.
2. If $L = 0$ and $\sum b_k$ converges: $\sum a_k$ converges.

Note: If $L = 0$ and $\sum a_k$ converges, nothing can be concluded about $\sum b_k$.

Examples (7.1.4)

1. **Series:** $\sum \frac{k}{3^k}$.
   - **Compare with:** $\sum (1/2)^k$ (Geometric series).
   - **Logic:** Since $\lim k(2/3)^k = 0$, eventually $\frac{k}{3^k} \le \frac{1}{2^k}$. Since $\sum (1/2)^k$ converges, the original series **converges**.
2. **Series:** $\sum \sqrt{\frac{k+1}{2k^3+1}}$.
   - **Compare with:** $\sum \frac{1}{k}$ (Harmonic series, divergent).
   - **Logic:** As $k \to \infty$, terms behave like $\sqrt{\frac{k}{2k^3}} \approx \frac{1}{k\sqrt{2}}$. Using the Limit Comparison Test, the limit of the ratio is $\frac{\sqrt{2}}{2} > 0$. Since $\sum \frac{1}{k}$ diverges, the original series **diverges**.
3. **Counter-example ($L = 0$):**
   - Let $a_k = 2^{-k}$ and $b_k = 1/k$.
   - $\lim(a_k/b_k) = 0$. $\sum a_k$ converges, but $\sum b_k$ diverges.

3. Integral Test

Theorem 7.1.5

Let f be a function on $[1, \infty)$ that is **non-negative** and **monotone decreasing**, such that $f(k) = a_k$. Then:

$$\sum_{k=1}^{\infty} a_k < \infty \iff \int_1^{\infty} f(x)\,dx < \infty$$

Proof Idea (Geometric intuition):
We approximate the area under $f(x)$ using rectangles of width 1.

- **Upper Bound:** The sum $\sum_{k=2}^{n} a_k$ represents the "lower" rectangular approximation and is less than the integral $\int_1^n f(x)dx$.
- **Lower Bound:** The sum $\sum_{k=1}^{n-1} a_k$ represents the "upper" rectangular approximation and is greater than the integral.
- Thus, the series and the integral bound each other; if one is finite, so is the other.

Examples (7.1.6)

1. **p-series:** $\sum_{k=1}^{\infty} \frac{1}{k^p}$
   - Evaluate $\int_1^{\infty} x^{-p} dx$.
   - **If $p > 1$:** Integral converges $\implies$ Series **Converges**.
   - **If $p \leq 1$:** Integral diverges ($\ln x$ or x^{1-p}) $\implies$ Series **Diverges**.
2. **Logarithmic Series:** $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$
   - Function $f(x) = \frac{1}{x \ln x}$ is decreasing.
   - $\int_2^c \frac{1}{x \ln x} dx = \ln(\ln c) - \ln(\ln 2)$.
   - As $c \to \infty$, integral $\to \infty$. Series **Diverges**.
3. **Mixed p-log series:** $\sum \frac{\ln k}{k^p}$
   - $p = 1$**:** Integral of $\frac{\ln x}{x}$ is $\frac{1}{2}(\ln x)^2 \to \infty$. **Diverges**.
   - $p > 1$**:** Write $p = q + r$ with $q > 1, r > 0$. Comparing with convergent $\sum \frac{1}{k^q}$ shows **Convergence**.
   - $p < 1$**:** Compares with divergent $\sum \frac{1}{k^p}$. **Diverges**.

4. Root and Ratio Tests

These tests are crucial for power series. We use Limit Superior ($\limsup$) and Limit Inferior ($\liminf$) for broader applicability.

Ratio Test (Theorem 7.1.7)

Let $\sum a_k$ be a series of positive terms. Define:

$$R = \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}, \quad r = \liminf_{k\to\infty} \frac{a_{k+1}}{a_k}$$

1. If $R < 1$: Series **Converges**.
2. If $r > 1$: Series **Diverges**.
3. If $r \le 1 \le R$: **Inconclusive**.

Proof Idea (Convergence case):
If $R < 1$, choose a constant c such that $R < c < 1$. Eventually, the ratio $\frac{a_{n+1}}{a_n} < c$. This implies $a_{n+m} < a_n c^m$. The series behaves like a geometric series $\sum c^n$ (which converges because $c < 1$). By Comparison Test, $\sum a_n$ converges.

Root Test (Theorem 7.1.8)

Let $\sum a_k$ be a series of non-negative terms. Define:

$$\alpha = \limsup_{k\to\infty} \sqrt[k]{a_k}$$

1. If $\alpha < 1$: Series **Converges**.
2. If $\alpha > 1$: Series **Diverges**.
3. If $\alpha = 1$: **Inconclusive**.

Proof Idea:
Similar to Ratio Test. If $\alpha < 1$, choose c such that $\alpha < c < 1$. Eventually $\sqrt[n]{a_n} < c$, meaning $a_n < c^n$. Compare with convergent geometric series $\sum c^n$.
If $\alpha > 1$, then $a_n > 1$ for infinitely many n, so terms do not approach 0. Diverges.

Hierarchy of Tests (Theorem 7.1.10)

For any sequence of positive numbers:

$$\liminf \frac{a_{n+1}}{a_n} \le \liminf \sqrt[n]{a_n} \le \limsup \sqrt[n]{a_n} \le \limsup \frac{a_{n+1}}{a_n}$$

Significance: The Root Test is strictly stronger than the Ratio Test. If the Ratio Test works, the Root Test works. However, there are cases where the Ratio Test fails but the Root Test succeeds.

Examples (7.1.9)

1. **p-series ($\sum 1/k^p$):**

- Ratio and Root limits are both 1. Both tests are **Inconclusive**. (We know result depends on p, but these tests cannot detect it).

2. **Factorial Series ($\sum p^k/k!$):**
    - Ratio: $\frac{a_{k+1}}{a_k} = \frac{p}{k+1} \to 0$.
    - $R = 0 < 1$. Series **Converges** for all p. (Root test is hard to apply here due to factorial).
3. **Mixed Sequence (Root > Ratio):**
    - $a_n = 1/2^k$ (if $n = 2k$) and $a_n = 1/3^k$ (if $n = 2k + 1$).
    - **Ratio Test:** Oscillates. $\liminf \frac{a_{n+1}}{a_n} = 0$, $\limsup \frac{a_{n+1}}{a_n} = \infty$. **Inconclusive**.
    - **Root Test:** Subsequential limits of $\sqrt[n]{a_n}$ are $1/\sqrt{2}$ and $1/\sqrt{3}$.
    - $\alpha = \limsup \sqrt[n]{a_n} = 1/\sqrt{2} \approx 0.707 < 1$.
    - **Result:** Series **Converges** by Root Test.