Section 4.1: Limit of a Function

1. Definition of a Limit

Context: Let (X, d) be a metric space, $E \subset X$, and $f : E \to \mathbb{R}$. Let p be a **limit point** of E.

Definition 4.1.1 ($\epsilon - \delta$ Definition):
$\lim_{x \to p} f(x) = L$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$|f(x) - L| < \epsilon$$

for all $x \in E$ satisfying $0 < d(x, p) < \delta$.

- **Metric Space Extension:** If $f : E \to (Y, \rho)$, replace $|f(x) - L|$ with $\rho(f(x), L)$.
- **Neighborhood Notation:** $f(x) \in N_\epsilon(L)$ for all $x \in E \cap (N_\delta(p) \setminus \{p\})$.

Key Remarks:

1. **Dependence:** δ depends on ϵ, the function f, and often the point p.
2. **Isolated Points:** If p is not a limit point (isolated), the limit is meaningless.
3. **Value at p:** It is **not** required that $p \in E$. Even if $p \in E$, $\lim_{x \to p} f(x)$ need not equal $f(p)$.

2. $\epsilon - \delta$ Proof Strategies (Examples)

A. Rational Functions (Algebraic Manipulation)

Example 4.1.2(c): $h(x) = \frac{\sqrt{x+1}-1}{x}$ for $x \in (-1, 0) \cup (0, \infty)$.

- **Claim:** $\lim_{x \to 0} h(x) = 1/2$.
- **Analysis:** Rationalize the numerator:

$$|h(x) - \frac{1}{2}| = \left| \frac{1}{\sqrt{x+1}+1} - \frac{1}{2} \right| = \frac{|x|}{2(\sqrt{x+1}+1)^2}$$

  Since $(\sqrt{x+1}+1)^2 > 1$, we have $|h(x) - 1/2| < |x|/2$.
- **Choice:** Set $\delta = \min\{1, 2\epsilon\}$. Then $|h(x) - 1/2| < \epsilon$.

B. The Dirichlet Function (Nowhere Continuous)

Example 4.1.2(d): $f(x) = 1$ if $x \in \mathbb{Q}$, 0 if $x \notin \mathbb{Q}$.

- **Claim:** $\lim_{x \to p} f(x)$ does not exist for any $p \in \mathbb{R}$.
- **Proof:** Fix L. Let $\epsilon = \max\{|L-1|, |L|\}$.
  - Density of $\mathbb{Q}$ implies $\exists x \in \mathbb{Q}$ near $p \implies |1 - L| = \epsilon$.
  - Density of irrationals implies $\exists x \notin \mathbb{Q}$ near $p \implies |0 - L| = \epsilon$.
  - Regardless of δ, $|f(x) - L| \geq \epsilon$ is always possible. Thus, limit fails.

C. Modified Dirichlet Function (Limit exists at one point)

Example 4.1.2(e): $f(x) = 0$ if $x \in \mathbb{Q}$, x if $x \notin \mathbb{Q}$.

- **At $x = 0$:** Since $|f(x)| \leq |x|$, choosing $\delta = \epsilon$ proves $\lim_{x \to 0} f(x) = 0$.
- **At $p \neq 0$:** Limit does not exist (similar argument to Dirichlet function).

D. Dependence of δ on p (Uniformity issue)

Example 4.1.2(f): $f(x) = 1/x$ on $(0, 1)$. Show $\lim_{x \to p}(1/x) = 1/p$.

- **Inequality:** $|\frac{1}{x} - \frac{1}{p}| = \frac{|x-p|}{xp}$.
  Restricting $x > p/2$ implies $\frac{1}{xp} < \frac{2}{p^2}$.
- **Choice:** $\delta = \min\{p/2, p^2\epsilon/2\}$.
- **Key Insight:** As $p \to 0$, $p^2\epsilon/2 \to 0$. δ *must* shrink as p approaches 0; it cannot be independent of p for this domain.

E. Multivariable Limit

Example 4.1.2(g): $f(x, y) = \frac{xy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus (0, 0)$. Show limit at $(1, 2)$ is $2/5$.

- **Technique:** Algebraic factorization and Triangle Inequality.
  i. **Common Denominator:**

  $$|f(x, y) - \frac{2}{5}| = \left|\frac{5xy - 2x^2 - 2y^2}{5(x^2 + y^2)}\right|$$

  ii. **Numerator Decomposition:**
  The numerator is rewritten to isolate terms approaching zero, $(x - 1)$ and $(y - 2)$:

  $$5xy - 2x^2 - 2y^2 = (x - 2y)(y - 2) + (4y - 2x)(x - 1)$$

  iii. **Triangle Inequality ($|a + b| \leq |a| + |b|$):**

$$\leq \frac{|x-2y||y-2|}{5(x^2+y^2)} + \frac{|4y-2x||x-1|}{5(x^2+y^2)} \leq \frac{(|x|+2|y|)|y-2|+(4|y|+2|x|)|x-1|}{5(x^2+y^2)}$$

- **Bounding:** Restrict (x, y) to the neighborhood $N_{1/2}(1, 2)$.
    - This implies $1/2 < |x| < 3/2$ and $3/2 < |y| < 5/2$.
    - Using these values, we bound the coefficients (e.g., $5(x^2+y^2) > 25/2$) to find a constant $K = 26/25$.
    - $$|f(x,y) - \frac{2}{5}| < \frac{26}{25}(|y-2| + |x-1|)$$
- **Result:** Given ϵ, choose $\delta < \min\{1/2, \frac{25}{52}\epsilon\}$.

3. Sequential Criterion for Limits

Theorem 4.1.3: $\lim_{x\to p} f(x) = L$ **if and only if** for every sequence $\{p_n\}$ in E ($p_n \neq p$) with $p_n \to p$, the sequence $\{f(p_n)\} \to L$.

Proof Technique: Constructing a Sequence ($\delta = 1/n$)
The "If" direction ($\Leftarrow$) is often proven by contradiction (Contrapositive).

- **Assumption:** Suppose $\lim_{x\to p} f(x) \neq L$.
- **Negation of Limit:** There exists an $\epsilon_0 > 0$ such that for any $\delta > 0$, there is an x with $0 < |x - p| < \delta$ but $|f(x) - L| \geq \epsilon_0$.
- **Construction:** For each $n \in \mathbb{N}$, choose $\delta = 1/n$.
    - We can find a point p_n such that $0 < |p_n - p| < 1/n$ and $|f(p_n) - L| \geq \epsilon_0$.
- **Conclusion:** The sequence $\{p_n\}$ converges to p (since $|p_n - p| < 1/n \to 0$), but $\{f(p_n)\}$ does not converge to L. This contradicts the hypothesis.

Corollary 4.1.4 (Uniqueness): If a limit exists, it is unique.

Application: Disproving Existence of Limits
To show a limit **does not exist**:

1. Find a sequence $p_n \to p$ where $\{f(p_n)\}$ diverges.
2. Find two sequences $p_n \to p, r_n \to p$ where $\lim f(p_n) \neq \lim f(r_n)$.

Example 4.1.5(a): $f(x) = \sin(1/x)$ at $x \to 0$.

- Choose $p_n = \frac{2}{(2n+1)\pi}$.
- $f(p_n) = \sin((2n+1)\pi/2) = (-1)^n$.

- The sequence $(-1)^n$ does not converge. Thus, $\lim_{x\to 0} \sin(1/x)$ does not exist.

4. Limit Theorems

Let $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$.

1. **Algebra:**
   - $\lim(f+g) = A+B$
   - $\lim(fg) = AB$
   - $\lim(f/g) = A/B$ (provided $B \neq 0$).

   Proof Detail for Quotient (Bounding away from 0):
   To prove $\lim \frac{1}{g(x)} = \frac{1}{B}$, we must ensure $g(x)$ does not vanish near p.
   - **Specific Choice of ϵ:** Set $\epsilon = \frac{|B|}{2}$.
   - **Logic:** Since $\lim g(x) = B$, $\exists \delta_1 > 0$ such that whenever $0 < |x-p| < \delta_1$, we have $|g(x) - B| < \frac{|B|}{2}$.
   - **Triangle Inequality:**

   $$|g(x)| = |B - (B - g(x))| \geq |B| - |g(x) - B| > |B| - \frac{|B|}{2} = \frac{|B|}{2}$$

   - **Result:** $|g(x)| > \frac{|B|}{2} > 0$. The denominator is strictly bounded away from zero by a specific positive constant in this neighborhood.
2. **Boundedness Theorem (Thm 4.1.8):**
   If g is bounded on E ($|g(x)| \leq M$) and $\lim_{x\to p} f(x) = 0$, then $\lim_{x\to p} f(x)g(x) = 0$.
   - *Example 4.1.10(c):* $\lim_{x\to 0} x\sin(1/x) = 0$. Since $|\sin(1/x)| \leq 1$ is bounded and $x \to 0$, the product goes to 0.
3. **Squeeze Theorem (Thm 4.1.9):**
   If $g(x) \leq f(x) \leq h(x)$ and $\lim g(x) = \lim h(x) = L$, then $\lim f(x) = L$.

5. Essential Trigonometric Limit

Claim: $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

- **Geometric Proof:** Using unit circle areas (Triangle OPQ, Sector OPR, Triangle ORS).

$$\text{Area}(\triangle OPQ) < \text{Area}(\text{Sector}) < \text{Area}(\triangle ORS)$$

$$\frac{1}{2}\sin t\cos t < \frac{1}{2}t < \frac{1}{2}\tan t$$

- **Inequality:** $\cos t < \frac{\sin t}{t} < \frac{1}{\cos t}$.
- **Conclusion:** Since $\lim_{t\to 0}\cos t = 1$, by Squeeze Theorem, limit is 1.

6. Limits at Infinity

Definition 4.1.11:
Let domain of f be unbounded above. $\lim_{x\to\infty} f(x) = L$ if $\forall\epsilon > 0, \exists M \in \mathbb{R}$ such that:

$$|f(x) - L| < \epsilon$$

for all $x \in \mathrm{Dom}(f) \cap (M, \infty)$.
(Analogous definition exists for $x \to -\infty$ using $x < M$).

Examples:

1. **Damped Sine:** $f(x) = \frac{\sin x}{x}$ on $(0, \infty)$.
   Since $|f(x)| \leq 1/x$, choosing $M = 1/\epsilon$ proves limit is 0.
2. **Oscillation at Infinity:** $f(x) = x\sin(\pi x)$.
   Choosing sequence $p_n = n + 1/2$ gives $f(p_n) = (-1)^n(n + 1/2)$. This is unbounded; limit does not exist.

Section 4.2: Continuous Functions

1. Definition of Continuity

- **Metric Space Definition:** Let (X, d) be a metric space and $E \subset X$. A function $f : E \to \mathbb{R}$ is continuous at $p \in E$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$|f(x) - f(p)| < \epsilon \quad \text{for all } x \in E \text{ with } d(x, p) < \delta$$

- **Topological Phrasing:** f is continuous at p if and only if f maps the δ-neighborhood of p into the ϵ-neighborhood of $f(p)$:

$$x \in N_\delta(p) \cap E \implies f(x) \in N_\epsilon(f(p))$$

- **Sequential Criterion:** f is continuous at p if and only if for every sequence $\{p_n\}$ in E with $p_n \to p$, we have $\lim_{n\to\infty} f(p_n) = f(p)$.
- **Isolated Points:** If p is an isolated point of E, every function f is continuous at p because there exists a δ such that $N_\delta(p) \cap E = \{p\}$.

2. Specific Examples of Continuity (4.2.2)

- **(a) Removable Discontinuity:**

$$g(x) = \frac{x^2 - 4}{x - 2}, \quad x \neq 2; \quad g(2) = 2$$

  - $\lim_{x\to 2} g(x) = 4 \neq g(2)$. Thus, discontinuous at $x = 2$.
  - *Correction:* Redefining $g(2) = 4$ makes it continuous.
- **(b) Rational Indicator:**

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ x, & x \notin \mathbb{Q} \end{cases}$$

  - Continuous at $p = 0$ since $\lim_{x\to 0} f(x) = 0 = f(0)$.
  - Discontinuous at every $p \neq 0$.
- **(c) Dirichlet Function:**

$$
f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}
$$

  - Discontinuous at every $p \in \mathbb{R}$ because rationals and irrationals are dense in $\mathbb{R}$.
- **(d) Reciprocal:** $f(x) = 1/x$ is continuous on $(0, 1)$.
- **(e) Oscillating Function:**

$$
f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}
$$

  - Continuous at $x = 0$ because $|x \sin(1/x)| \leq |x|$, so the limit is 0.
- **(f) Sine Function:** $f(x) = \sin x$ is continuous on $\mathbb{R}$.
  - *Proof:* Using $|\sin y - \sin x| = 2|\cos \frac{y+x}{2} \sin \frac{y-x}{2}| \leq |y - x|$, we can choose $\delta = \epsilon$.
- **(g) Thomae's Function (Popcorn Function):**

$$
f(x) = \begin{cases} 1/n, & x = m/n \in \mathbb{Q} \cap (0, 1) \text{ (lowest terms)} \\ 0, & x \notin \mathbb{Q} \end{cases}
$$

  - **Discontinuous** at every rational p (since $f(p) \neq 0$).
  - **Continuous** at every irrational p (limit is 0).
  - *Proof Sketch:* For $\epsilon > 0$, only finitely many rationals have a denominator n such that $1/n \geq \epsilon$. We can choose δ to exclude these specific rationals from the neighborhood of irrational p.

3. Algebra and Composition

- **Algebraic Operations:** If f, g are continuous at p, then $f + g$, $f - g$, fg, and f/g (provided $g(p) \neq 0$) are continuous at p.
- **Composition (Theorem 4.2.4):**
  If f is continuous at p and g is continuous at $f(p)$, then $h = g \circ f$ is continuous at p.
  - *Examples:* Polynomials, Rational functions (on domain), and $\sin(p(x))$ are continuous.

4. Topological Characterization (Theorem 4.2.6)

- **Theorem:** A function $f : E \to \mathbb{R}$ is continuous on E **if and only if** $f^{-1}(V)$ is open in E for every open subset V of $\mathbb{R}$.
- **Proof:**
  1. Continuity $\implies$ Open Pre-images

- Let $p \in f^{-1}(V)$, which means $f(p) \in V$.

$$\underbrace{f(N_\delta(p) \cap E) \subseteq N_\epsilon(f(p))}_{\text{Since } f \text{ is continuous}} \subseteq \underbrace{V}_{\text{Since } V \text{ is open}}$$

- By definition of inverse image ($A \subseteq B \iff f^{-1}(A) \subseteq f^{-1}(B)$):

$$N_\delta(p) \cap E \subseteq f^{-1}(N_\epsilon(f(p))) \subseteq f^{-1}(V)$$

Since $N_\delta(p) \cap E \subseteq f^{-1}(V)$, the set $f^{-1}(V)$ **is open**.

2. Open Pre-images $\implies$ Continuity

- Let $\epsilon > 0$. Set $V = N_\epsilon(f(p))$ (which is open).

$$\underbrace{N_\delta(p) \cap E \subseteq f^{-1}(V)}_{\text{Since } f^{-1}(V) \text{ is open}}$$

- Applying f to both sides gives the continuity definition directly:

$$f(N_\delta(p) \cap E) \subseteq V = N_\epsilon(f(p))$$

- **Example:** For $f(x) = \sqrt{x}$ on $[0, \infty)$, let $V = (a, b)$.
  - If $a \le 0 < b$, then $f^{-1}(V) = [0, b^2)$.
  - While $[0, b^2)$ is not open in $\mathbb{R}$, it **is** open in $[0, \infty)$ because it can be written as $(-b^2, b^2) \cap [0, \infty)$.
- **Warning:** The forward image of an open set is **not** necessarily open.
  - *Example*: $f(x) = x^2$ for $x \le 2$ and $6 - x$ for $x > 2$. $f((-1, 1)) = [0, 1)$, which is not open.

5. Continuity and Compactness

- **Theorem 4.2.8 (Preservation of Compactness):**
  If K is a compact subset of a metric space X and $f : K \to \mathbb{R}$ is continuous on K, then the image $f(K)$ is compact.
  - **Proof:**
    a. Let $\{V_\alpha\}$ be an arbitrary open cover of $f(K)$.
    b. Since f is continuous, each $f^{-1}(V_\alpha)$ is open in K, forming an open cover of K.
    c. Since K is compact, there exists a finite subcover corresponding to indices $\alpha_1, \ldots, \alpha_n$.
    d. The collection $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$ covers $f(K)$. Thus, $f(K)$ is compact.

- **Corollary 4.2.9 (Extreme Value Theorem):**
  If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous, then f attains its maximum and minimum on K.

$$\exists p, q \in K \text{ s.t. } f(q) \leq f(x) \leq f(p) \quad \forall x \in K$$

  - **Concise Proof:**
    By Theorem 4.2.8, the image $f(K)$ is compact in $\mathbb{R}$.
    - **Bounded:** Since compact sets in $\mathbb{R}$ are bounded, $M = \sup f(K)$ and $m = \inf f(K)$ exist.
    - **Closed:** Since compact sets in $\mathbb{R}$ are closed, $f(K)$ contains its limit points, so $M, m \in f(K)$.
    - **Conclusion:** Therefore, there exist $p, q \in K$ such that $f(p) = M$ and $f(q) = m$.
- **Counter-examples (4.2.10):**
  The theorem fails if K is not compact (i.e., not closed or not bounded).
  - **Unbounded Domain:** $f(x) = \frac{x^2}{1+x^2}$ on $[0, \infty)$. The supremum is 1, but $f(x) < 1$ for all x, so the maximum is never attained.
  - **Not Closed Domain:** $g(x) = x$ on $(0, 1)$. The supremum is 1 and infimum is 0, but neither is contained in the range $(0, 1)$.

6. Intermediate Value Theorem (IVT)

- **Theorem 4.2.11 (Intermediate Value Theorem):**
  Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < \gamma < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = \gamma$.
  - **Key Intuition:** Continuity implies **local persistence of inequalities**. If $f(c) \neq \gamma$, the function forces values to stay strictly above or below γ in a small neighborhood, contradicting the precise boundary nature of the supremum.
  - **Rigorous Proof Sketch:**
    a. **Construct Set:** Let $A = \{x \in [a, b] : f(x) \leq \gamma\}$. Since $a \in A$, $A \neq \emptyset$. Since A is bounded by b, let $c = \sup A$.
    b. **Contradiction (Case 1: $f(c) < \gamma$):**
    Let $\epsilon = \frac{1}{2}(\gamma - f(c)) > 0$. By continuity, $\exists \delta > 0$ such that $f(x) < f(c) + \epsilon < \gamma$ for $x \in N_\delta(c)$.
    This implies there exist points $x > c$ where $f(x) < \gamma$, meaning $x \in A$. This contradicts that c is the upper bound of A.

c. **Contradiction (Case 2: $f(c) > \gamma$):**
Similar logic shows that for some neighborhood, $f(x) > \gamma$. This implies c is not a limit point of A (or $c \notin A$), contradicting the properties of the supremum.

d. **Conclusion:** Therefore, $f(c) = \gamma$.

- **Corollary 4.2.12 (Topological Characterization):**
If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous on I, then $f(I)$ is an interval.
  - **Proof:**
To show $f(I)$ is an interval, let $s, t \in f(I)$ with $s < t$ and pick any γ such that $s < \gamma < t$. There exist $a, b \in I$ with $f(a) = s$, $f(b) = t$. Since I is an interval, the closed segment between a and b lies in I. By the **Intermediate Value Theorem**, there exists c between a and b such that $f(c) = \gamma$. Thus $\gamma \in f(I)$, implying $f(I)$ is an interval.
- **Corollary 4.2.13 (Existence of Roots):**
For every $\gamma > 0$ and $n \in \mathbb{N}$, there exists a unique $y > 0$ such that $y^n = \gamma$.
- **Corollary 4.2.14 (Fixed Point Theorem):**
If $f : [0, 1] \to [0, 1]$ is continuous, there exists $y \in [0, 1]$ such that $f(y) = y$.
  - **Proof Technique (Auxiliary Function):**
    a. Define $g(x) = f(x) - x$.
    b. **Evaluate Endpoints:**
      - $g(0) = f(0) - 0 \geq 0$ (since $f(0) \in [0, 1]$).
      - $g(1) = f(1) - 1 \leq 0$ (since $f(1) \in [0, 1]$).
    c. **Apply IVT:** Since g is continuous and 0 lies between $g(1)$ and $g(0)$, there exists y such that $g(y) = 0$.
    d. **Result:** $f(y) - y = 0 \implies f(y) = y$.

Caveats & Counter-examples (4.2.15)

- **Converse is False:** A function can satisfy the intermediate value property (Darboux property) but be discontinuous.
  - *Example:* $f(x) = \sin(1/x)$ for $x > 0$ and $f(0) = 0$. This function takes every value between -1 and 1 in any neighborhood of 0, satisfying the property, but is discontinuous at 0.
- **Requires Completeness of $\mathbb{R}$:** The IVT relies on the Least Upper Bound Property.
  - *Example:* Let $f(x) = x^2$ on the rational interval $E = [0, 2] \cap \mathbb{Q}$.
  - We have $f(0) < 2 < f(2)$, but there is **no rational number** $c \in E$ such that $c^2 = 2$. Thus, IVT fails in $\mathbb{Q}$.

Section 4.3: Uniform Continuity

1. Concept: Pointwise vs. Uniform Continuity

- **Standard Continuity (Pointwise):**
  - A function $f : E \to \mathbb{R}$ is continuous on E if for each $p \in E$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in E \cap N_\delta(p)$.
  - **Dependency:** Here, δ depends on **both** ϵ and the specific point p. (i.e., $\delta = \delta(\epsilon, p)$).
  - *Problem:* As p changes, the required δ might get infinitely small (e.g., $f(x) = 1/x$ near 0).
- **Uniform Continuity (Definition 4.3.1):**
  - Let $E \subset (X, d)$ and $f : E \to \mathbb{R}$.
  - f is **uniformly continuous** on E if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in E, \text{ if } d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

  - **Dependency:** Here, δ depends **only** on ϵ, f, and the set E. It is **independent** of the point x.

2. Examples of Uniform Continuity

Case A: $f(x) = x^2$ on a Bounded Subset E

- **Claim:** f is uniformly continuous on E if E is bounded.
- **Proof:**
  - Since E is bounded, $\exists C > 0$ such that $|x| \le C$ for all $x \in E$.
  - Consider $|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y|$.
  - Using the bound: $|x + y| \le |x| + |y| \le 2C$.
  - Thus, $|f(x) - f(y)| \le 2C|x - y|$.
  - Choose $\delta = \epsilon/2C$. If $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Case B: $f(x) = \sin x$ on $\mathbb{R}$

- **Claim:** f is uniformly continuous on $\mathbb{R}$.
- **Proof:**
  - Standard trigonometric inequality: $|\sin x - \sin y| \le |x - y|$.
  - Choose $\delta = \epsilon$.

- If $|x-y|<\delta$, then $|f(x)-f(y)| \leq |x-y| < \epsilon$.

Case C: Non-Example ($f(x) = 1/x$ on $(0,1)$)

- **Claim:** f is **not** uniformly continuous on $(0,1)$.
- **Proof (Contradiction):**
    - Assume f is uniformly continuous. Let $\epsilon = 1$.
    - There must exist $\delta > 0$ (assume $\delta < 1$) such that $|x-y| < \delta \implies |1/x - 1/y| < 1$.
    - Choose $x \in (0, 1/2)$ such that x is very small, and let $y = x + \delta/2$.
    - Then $|x-y| = \delta/2 < \delta$.
    - However, analysis shows that as $x \to 0$, the difference $|1/x - 1/(x+\delta/2)|$ becomes arbitrarily large, eventually exceeding 1.
    - Specifically, if x is small enough ($x < \delta/2$), the inequality fails, leading to a contradiction.

3. Lipschitz Functions

- **Definition:** A function $f : E \to \mathbb{R}$ satisfies a **Lipschitz condition** on E if there exists a constant $M > 0$ such that:

$$|f(x) - f(y)| \leq Md(x,y) \quad \forall x, y \in E$$

    - These are called Lipschitz functions.
    - Functions with bounded derivatives are Lipschitz.
- **Theorem 4.3.3:**
    - If f satisfies a Lipschitz condition on E, then f is **uniformly continuous** on E.
    - *Proof Logic:* Simply choose $\delta = \epsilon/M$.
- **Important Note (Converse is False):**
    - Not every uniformly continuous function is Lipschitz.
    - *Example:* $f(x) = \sqrt{x}$ on $[0, \infty)$ is uniformly continuous, but **not** Lipschitz (the derivative is unbounded near 0).

4. The Uniform Continuity Theorem

Determining uniform continuity for non-Lipschitz functions is difficult. Compactness provides a sufficient condition.

- **Theorem 4.3.4:**

- If K is a **compact** metric space and $f : K \to \mathbb{R}$ is **continuous** on K, then f is **uniformly continuous** on K.

- **Proof Sketch:**
  i. Fix $\epsilon > 0$. Since f is continuous, for every $p \in K$, there is a δ_p valid locally.
  ii. The neighborhoods $N_{\delta_p/2}(p)$ form an open cover of K.
  iii. Since K is compact, there is a **finite subcover** corresponding to points $p_1, \ldots, p_n$.
  iv. Define $\delta = \frac{1}{2}\min\{\delta_{p_1}, \ldots, \delta_{p_n}\}$. This δ works globally for the whole set.
- **Corollary 4.3.5 (Heine's Theorem):**
  - A continuous real-valued function on a **closed and bounded interval** $[a, b]$ is uniformly continuous.

5. Necessity of Compactness (Example 4.3.6)

To guarantee uniform continuity via Theorem 4.3.4, the domain must be **both** closed and bounded (Compact).

- **Closed but Not Bounded:**
  - Set: $[0, \infty)$.
  - Function: $f(x) = x^2$.
  - Result: Continuous, but **not** uniformly continuous (values grow too fast as $x \to \infty$).
- **Bounded but Not Closed:**
  - Set: $(0, 1)$.
  - Function: $f(x) = 1/x$.
  - Result: Continuous, but **not** uniformly continuous (values grow too fast as $x \to 0$).

Section 4.4: Monotone Functions and Discontinuities

I. Right and Left Limits

1. Definitions

Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$.

- **Right Limit ($f(p+)$):** Suppose p is a limit point of $E \cap (p, \infty)$.

$$\lim_{x \to p^+} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } p < x < p + \delta.$$

- **Left Limit ($f(p-)$):** Suppose p is a limit point of $E \cap (-\infty, p)$.

$$\lim_{x \to p^-} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } p - \delta < x < p.$$

2. Relationship to General Limits

For $p \in \text{Int}(I)$, $\lim_{x \to p} f(x)$ exists if and only if:

1. Both $f(p+)$ and $f(p-)$ exist.
2. $f(p+) = f(p-)$.

3. One-Sided Continuity

- **Right Continuous at p:** $\lim_{x \to p^+} f(x) = f(p)$.
- **Left Continuous at p:** $\lim_{x \to p^-} f(x) = f(p)$.
- **Theorem 4.4.3:** f is continuous at p iff $f(p+) = f(p-) = f(p)$.

II. Classification of Discontinuities

If f is discontinuous at p, it falls into one of two main categories:

1. Simple Discontinuities (First Kind)

Both $f(p+)$ and $f(p-)$ exist.

- **Removable Discontinuity:** $f(p+)$ and $f(p-)$ exist and are equal, but differ from $f(p)$ (or $f(p)$ is undefined).

- *Example:* $g(x) = \frac{x^2-4}{x-2}$. Limit is 4, but undefined at $x = 2$. Can be "removed" by defining $g(2) = 4$.
- **Jump Discontinuity:** $f(p+) \neq f(p-)$.
  - *Example:* The greatest integer function $f(x) = [x]$. At integer n:
    - $f(n-) = n - 1$
    - $f(n+) = n$
    - Jump size is 1.

2. Discontinuities of the Second Kind

Either $f(p+)$ or $f(p-)$ (or both) does not exist.

- *Example:* $f(x) = \sin(\frac{1}{x})$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.
  - $f(0-) = 0$.
  - $f(0+)$ does not exist (oscillates between -1 and 1).

3. Specific Examples Analyzed

- **Example 4.4.5(a):** Piecewise function defined as x for $x \leq 1$ and $3 - x^2$ for $x > 1$.
  - $f(1-) = 1$, $f(1+) = 2$.
  - Left continuous, but jump discontinuity at $x = 1$.
- **Example 4.4.5(d):** $g(x) = \sin(2\pi x[x])$.
  - Continuous on $(n, n+1)$ as $[x]$ is constant n.
  - At integers n, limits from both sides go to 0, so it is continuous at integers.
  - However, it is not *uniformly* continuous on $\mathbb{R}$.

III. Monotone Functions

1. Definitions

Let f be defined on an interval I.

- **Monotone Increasing:** $x < y \implies f(x) \leq f(y)$.
- **Monotone Decreasing:** $x < y \implies f(x) \geq f(y)$.
- **Strictly Increasing:** $x < y \implies f(x) < f(y)$.

2. Theorem 4.4.7 (Existence of Limits)

If f is monotone increasing on an open interval I, then for every $p \in I$, both $f(p+)$ and $f(p-)$ exist. The following inequality holds:

$$\sup_{x<p} f(x) = f(p-) \le f(p) \le f(p+) = \inf_{x>p} f(x)$$

Furthermore, if $p < q$, then $f(p+) \le f(q-)$.

Proof of Theorem 4.4.7

- **Part 1: Existence of Left Limit $f(p-)$**
  Let $S = \{f(x) : x < p\}$. Since f is monotonic increasing, $f(p)$ serves as an upper bound for S.
  - By the **Completeness Property**, the supremum $A = \sup S$ exists, and clearly $A \le f(p)$.
  - **Convergence:** For any $\epsilon > 0$, since A is the least upper bound, $A - \epsilon$ is not an upper bound. Thus, there exists $x_0 < p$ such that $A - \epsilon < f(x_0) \le A$.
  - By monotonicity, for all $x \in (x_0, p)$, we have $f(x_0) \le f(x) \le A$.
  - Consequently, $|f(x) - A| < \epsilon$, which implies $f(p-) = A \le f(p)$.
- **Part 2: Existence of Right Limit $f(p+)$**
  By an analogous argument using the **infimum** of the set $\{f(x) : x > p\}$, we establish that the right limit exists and satisfies:

$$f(p) \le f(p+) = \inf_{x>p} f(x)$$

- **Part 3: Relation for distinct points ($p < q$)**
  Let $p < q$. Choose any x such that $p < x < q$.
  From the definitions of the one-sided limits established above:

$$f(p+) = \inf_{z>p} f(z) \le f(x) \le \sup_{z<q} f(z) = f(q-)$$

3. Corollary 4.4.8 (Countability of Discontinuities)

The set of discontinuities of a monotone function is **at most countable**.

- *Proof Logic:* Each discontinuity is a jump $(f(p-), f(p+))$. Since intervals are disjoint for distinct points, we can map each jump to a rational number within that interval. Since $\mathbb{Q}$ is countable, the set of jumps is countable.

IV. Construction of Functions with Prescribed Discontinuities

We can construct a monotone function that is discontinuous exactly at a specific countable set of points, with controlled jump sizes.

1. The Unit Jump Function $I(x)$

$$I(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

- The function $I(x)$ is **right continuous** at 0 with a unit jump: $I(0+) = 1$ and $I(0-) = 0$.
- **Shifted Function:** $I_k(x) = I(x - a_k)$ represents a unit step occurring strictly at $x = a_k$.

2. Theorem 4.4.10 (General Construction)

Let $\{x_n\}_{n=1}^{\infty}$ be a countable subset of (a, b). Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} c_n$ converges.

Define the function f on $[a, b]$ by:

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

Detailed Analysis of f:

- **Well-Defined and Monotone:**
  Since $0 \leq c_n I(x - x_n) \leq c_n$, the partial sums $s_n(x)$ are bounded above by $\sum c_n$. Thus, $f(x)$ converges for all x. Furthermore, because each term $I(x - x_n)$ is non-decreasing, f is **monotone increasing** on $[a, b]$.
  - Boundary values: $f(a) = 0$ (since $x_n > a$) and $f(b) = \sum_{n=1}^{\infty} c_n$.
- **Continuity on $[a, b] \setminus \{x_n\}$:**
  The function is continuous at any point p where $p \neq x_n$ for any n. The proof relies on splitting the set of points $E = \{x_n\}$:
  - **Isolated Points:** If p is not a limit point of E, there exists a neighborhood $(p - \delta, p + \delta)$ containing no x_n. In this interval, f is constant, and therefore continuous.
  - **Limit Points:** If p is a limit point of E, continuity is proved using the Cauchy criterion. For any $\epsilon > 0$, we can choose an integer N such that the "tail" of the series is small ($\sum_{k=N+1}^{\infty} c_k < \epsilon$). By choosing a neighborhood δ small enough to exclude the first N points, the variation of f near p is bounded by ϵ, proving continuity.
- **Right Continuity Everywhere:**
  For any point x_n (or any p), f is right continuous:

$$f(x_n+) = f(x_n)$$

  This is because as $x \to x_n$ from the right, the terms $I(x - x_k)$ do not change state (they remain

1 for $x_k \leq x_n$ and 0 for $x_k > x$ locally). The variation is again controlled by the tail of the convergent series $\sum c_n$.

- **Discontinuity at $\{x_n\}$:**
  At each prescribed point x_n, the function exhibits a jump discontinuity exactly equal to the weight c_n:

$$f(x_n) - f(x_n-) = c_n$$

  - **Left Limit:** As $y \to x_n$ from the left ($y < x_n$), the term $c_n I(y - x_n)$ is 0.
  - **Value at x_n:** Exactly at x_n, the term becomes $c_n I(0) = c_n$.
  - This confirms that f has a countable number of discontinuities strictly located at the set $\{x_n\}$.

3. Examples of Construction

- **Step Function:** If $\{x_n\}$ is a finite set, f is a standard step function with finitely many jumps.
- **Rational Discontinuities:** Let $\{x_n\}$ be an enumeration of the rational numbers $\mathbb{Q} \cap (0, 1)$ and $c_n = 2^{-n}$. The resulting function is strictly increasing, discontinuous at every rational number in $(0, 1)$, and continuous at every irrational number.
- **Distribution Functions:** If the weights are normalized such that $\sum c_n = 1$, the function behaves like a cumulative distribution function (CDF) used in probability theory.

V. Inverse Functions

1. Logic of Invertibility

- **Strict Monotonicity:** Let f be a strictly increasing real-valued function on an interval I. If $x, y \in I$ with $x < y$, then $f(x) < f(y)$. The same logic applies if f is strictly decreasing.
- **Injectivity (One-to-One):** Strictly monotone functions imply that $f(x) \neq f(y)$ for any distinct $x, y \in I$. Therefore, f is one-to-one.
- **Existence:** Because f is one-to-one, it possesses an inverse function f^{-1} defined on the range $f(I)$.

2. Theorem 4.4.12: Continuity of the Inverse Function

- **Statement:** Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then the inverse function f^{-1} is strictly monotone and continuous on the interval $J = f(I)$.
- **Proof Summary:**
  i. **Image is an Interval:** Since f is continuous on the interval I, the image $J = f(I)$ must also be an interval (by Corollary 4.2.12).

ii. **Monotonicity of Inverse:** If f is strictly increasing, f^{-1} is also strictly increasing. For $y_1, y_2 \in J$ with $y_1 < y_2$, their pre-images must satisfy $x_1 < x_2$.
iii. **Continuity of Inverse:** To prove f^{-1} is continuous at $y_0 \in J$:
   - Consider an ϵ-neighborhood around $x_0 = f^{-1}(y_0)$.
   - Due to the strict monotonicity and continuity of f, we can map the interval $(x_0 - \epsilon, x_0]$ to $(y_0 - \delta, y_0]$ where $\delta = f(x_0) - f(x_0 - \epsilon)$.
   - This ensures that for y within δ of y_0, $|f^{-1}(y_0) - f^{-1}(y)| < \epsilon$, proving continuity.

3. Example 4.4.13 (n-th Roots)

- $f(x) = x^n$ is strictly increasing and continuous on $I = [0, \infty)$.
- Therefore, the inverse $g(x) = \sqrt[n]{x}$ is strictly increasing and continuous on $J = [0, \infty)$.

4. Remark (Conditions for Monotonicity)

- **The Converse:** If f is one-to-one and continuous on an interval I, then f is necessarily strictly monotone (increasing or decreasing). This results from the Intermediate Value Theorem.
- **Essential Conditions:** This converse holds **only** if:
  i. f is continuous.
  ii. The domain of f is an interval.

  If either condition is unmet, a one-to-one function might not be strictly monotone.

Chapter 5.1: The Derivative

1. Definition of the Derivative

Historical Context: Formulated rigorously by Cauchy (1821) using limits, moving away from vague notions of tangent lines and velocity.

Definition 5.1.1 (The Derivative)
Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. For a fixed $p \in I$, the derivative $f'(p)$ is:

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

Alternatively, letting $x = p + h$:

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

- **Geometric interpretation:** Slope of the tangent line at $(p, f(p))$.
- **Physical interpretation:** Instantaneous velocity.

2. One-Sided Derivatives

If p is an endpoint or we need to analyze corner points, we use one-sided limits (Definition 5.1.2).

- **Right Derivative ($f'_+(p)$):** $\lim_{h \to 0^+} \frac{f(p+h) - f(p)}{h}$
- **Left Derivative ($f'_-(p)$):** $\lim_{h \to 0^-} \frac{f(p+h) - f(p)}{h}$

Key Property:
For an interior point $p \in I$, $f'(p)$ exists **if and only if** both $f'_+(p)$ and $f'_-(p)$ exist and are **equal**.

3. Worked Examples (Specific Functions)

The text analyzes the differentiability of several specific functions to illustrate the definition.

A. Power Function ($f(x) = x^2$)

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0}(2x + h) = 2x$$

(Note: Generalizes to $f(x) = x^n \implies f'(x) = nx^{n-1}$).

B. Square Root ($f(x) = \sqrt{x}, x > 0$)

Uses rationalization:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \xrightarrow{h \to 0} \frac{1}{2\sqrt{x}}$$

C. Sine Function ($f(x) = \sin x$)

Uses the identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right)$$

Since $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$, $f'(x) = \cos x$.

D. Absolute Value ($f(x) = |x|$) at $x = 0$

- $f'_+(0) = \lim_{h \to 0^+} \frac{h}{h} = 1$
- $f'_-(0) = \lim_{h \to 0^-} \frac{-h}{h} = -1$
- **Result:** $f'_+(0) \neq f'_-(0)$, so f is **not differentiable** at 0.

E. Cusp ($g(x) = x^{3/2}$) at $x = 0$

$$g'(0) = \lim_{h \to 0^+} \frac{h^{3/2}}{h} = \lim_{h \to 0^+} \sqrt{h} = 0$$

Differentiable at 0.

F. Oscillating Discontinuity ($f(x) = x \sin(1/x)$ for $x \neq 0$, $f(0) = 0$)

$$f'(0) = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

Result: Limit does not exist. Not differentiable at 0.

G. Differentiable with Discontinuous Derivative ($g(x) = x^2 \sin(1/x)$ for $x \neq 0$, $g(0) = 0$)

- **At $x = 0$:**

$$g'(0) = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

  (By Squeeze Theorem).
- **For $x \neq 0$:** $g'(x) = 2x \sin(1/x) - \cos(1/x)$.
- **Observation:** $\lim_{x \to 0} g'(x)$ does not exist (due to $\cos(1/x)$).
- **Conclusion:** g is differentiable everywhere, but g' is **not continuous** at 0.

4. Relationship between Differentiability and Continuity

Theorem 5.1.4:
If f is differentiable at p, then f is continuous at p.

Proof Sketch:

$$\lim_{t \to p}(f(t) - f(p)) = \lim_{t \to p} \left[\frac{f(t) - f(p)}{t - p} \cdot (t - p)\right] = f'(p) \cdot 0 = 0$$

Thus $\lim_{t \to p} f(t) = f(p)$.

- **Converse:** False. Continuity does **not** imply differentiability (e.g., $f(x) = |x|$).
- **Weierstrass Function:** An example of a function continuous everywhere but differentiable nowhere.

5. Arithmetic of Derivatives

Theorem 5.1.5 (Algebraic Rules)
Let f, g be differentiable at x.

A. Sum Rule

$$(f + g)'(x) = f'(x) + g'(x)$$

B. Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Key Proof Idea: The "Add and Subtract" Trick
Direct substitution creates a mixed term $f(x+h)g(x+h)$ that cannot be factored. To fix this, we insert a "middle term" into the numerator.

Proof Sketch:

1. **Setup:** Start with the difference quotient:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

2. **The Trick:** Add and subtract $f(x+h)g(x)$:

$$\frac{f(x+h)g(x+h) - \mathbf{f(x+h)g(x)} + \mathbf{f(x+h)g(x)} - f(x)g(x)}{h}$$

3. **Group & Limit:** Separate into two parts:

$$f(x+h)\underbrace{\left[\frac{g(x+h)-g(x)}{h}\right]}_{\to g'(x)} + g(x)\underbrace{\left[\frac{f(x+h)-f(x)}{h}\right]}_{\to f'(x)}$$

Note: $f(x+h) \to f(x)$ because differentiability implies continuity.

C. Quotient Rule (Reciprocal Case)

To prove the quotient rule, we first focus on the derivative of the reciprocal.

The Reciprocal Rule:

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{[g(x)]^2}$$

Proof Sketch:

1. **Common Denominator:**

$$\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \frac{g(x) - g(x+h)}{h \cdot g(x+h)g(x)}$$

2. **Extract Derivative Definition:**
   Recognize that $g(x) - g(x+h) = -[g(x+h) - g(x)]$:

$$-\underbrace{\left[\frac{g(x+h) - g(x)}{h}\right]}_{\to g'(x)} \cdot \frac{1}{g(x+h)g(x)}$$

3. **Take Limit ($h \to 0$):**
   Since g is continuous, $g(x+h) \to g(x)$, giving the denominator $[g(x)]^2$.

(Note: The full Quotient Rule is simply the Product Rule applied to $f(x) \cdot [1/g(x)]$.)

6. The Chain Rule

Theorem 5.1.6 (Composition)

If f is differentiable at x and g is differentiable at $y = f(x)$, then $h = g \circ f$ is differentiable at x:

$$h'(x) = g'(f(x)) \cdot f'(x)$$

Proof: Linear Approximation (to avoid division by zero).

Standard limits fail if $f(t) - f(x) = 0$. Instead, use "error terms" (u, v) that go to 0:

- **f:** $f(t) - f(x) = (t - x)[f'(x) + u(t)]$
- **g:** $g(s) - g(y) = (s - y)[g'(y) + v(s)]$
- Set $s = f(t)$ and $y = f(x)$. The difference quotient becomes:

$$\frac{g(f(t)) - g(f(x))}{t - x} = [f'(x) + u(t)] \cdot [g'(y) + v(f(t))]$$

- As $t \to x$, the error terms vanish ($u, v \to 0$), leaving $f'(x)g'(y)$.

Examples:

- **Composite Trig:** $h(x) = \sin(1/x^2)$.

$$h'(x) = \cos\left(\frac{1}{x^2}\right) \cdot \frac{d}{dx}(x^{-2}) = \cos\left(\frac{1}{x^2}\right) \cdot (-2x^{-3})$$

- **Power Rule Extension:** $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$.

Chapter 5.2: The Mean Value Thm.

1. Local Maxima and Minima

Definition 5.2.1:
A function $f : E \to \mathbb{R}$ has a:

- **Local maximum** at $p \in E$ if $\exists \delta > 0$ such that $f(x) \leq f(p)$ for all $x \in E \cap N_\delta(p)$.
- **Absolute maximum** if $f(x) \leq f(p)$ for all $x \in E$.
  (Analogous definitions apply for minimums).

Theorem 5.2.2 (Relationship to Derivative):
Let f be defined on interval I. If f has a local extremum at an interior point $p \in \text{Int}(I)$ and f is differentiable at p, then:

$$f'(p) = 0$$

- **Proof Idea:** Analyze the difference quotient. If p is a max, $\frac{f(t)-f(p)}{t-p} \leq 0$ for $t > p$ (implies $f'_+(p) \leq 0$) and ≥ 0 for $t < p$ (implies $f'_-(p) \geq 0$). Since $f'(p)$ exists, limits must be equal, thus 0.

Corollary 5.2.3:
For continuous f on $[a, b]$, relative extrema at $p \in (a, b)$ imply either $f'(p)$ does not exist or $f'(p) = 0$.

- *Note:* This does not apply to endpoints. At endpoints, we can only conclude inequality (e.g., if max at a, $f'(a) \leq 0$).

2. Rolle's Theorem

Theorem 5.2.5 (Rolle's Theorem):
Suppose f is:

1. Continuous on $[a, b]$
2. Differentiable on (a, b)
3. $f(a) = f(b)$

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

- **Geometric Interpretation:** There is at least one point where the tangent line is horizontal.
- **Proof Idea:**
  - Since $[a, b]$ is compact, f attains max and min (EVT).
  - If f is constant, $f'(x) = 0$ everywhere.
  - If not constant, extremum occurs at an interior point c. By Theorem 5.2.2, $f'(c) = 0$.
- **Example:** $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$. Derivative undefined at endpoints, but theorem holds ($c = 0$).

3. The Mean Value Theorem (MVT)

Theorem 5.2.6 (Lagrange):
Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that:

$$f(b) - f(a) = f'(c)(b - a)$$

- **Geometric Interpretation:** There is a point c where the tangent slope equals the secant slope connecting $(a, f(a))$ and $(b, f(b))$.
- **Proof Idea:**
  Construct an auxiliary function $g(x)$ representing the vertical distance between the curve and the secant line:

  $$g(x) = f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

  Since $g(a) = g(b) = 0$, applying Rolle's Theorem to g yields $g'(c) = 0$, which rearranges to the MVT equation.

Example 5.2.7 (Inequalities):
Using MVT to prove $\frac{x}{1+x} \leq \ln(1 + x) \leq x$ for $x > -1$.

- Let $f(x) = \ln(1 + x)$. $f(0) = 0$.
- By MVT, $\ln(1 + x) = f(x) - f(0) = f'(c)x = \frac{x}{1+c}$ for some c between 0 and x.
- Analyze bounds of $\frac{1}{1+c}$ based on $0 < c < x$ or $x < c < 0$ to derive the inequality.

4. Cauchy Mean Value Theorem

Theorem 5.2.8:
If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If $g'(x) \neq 0$, this can be written as:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

- **Proof Idea:** Define $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. Since $h(a) = h(b)$, apply Rolle's Theorem to find $h'(c) = 0$.
- **Geometric Interpretation:** For a parametric curve defined by $(g(t), f(t))$, there is a point where the tangent slope equals the slope of the chord connecting the endpoints.

5. Applications: Monotonicity

1. Monotonicity on Intervals (Theorem 5.2.9)

Let f be differentiable on an interval I. The sign of the derivative determines the monotonicity of the function on that interval:

- $f'(x) \geq 0, \forall x \in I \implies f$ is **monotone increasing**.
- $f'(x) > 0, \forall x \in I \implies f$ is **strictly increasing**.
- $f'(x) \leq 0, \forall x \in I \implies f$ is **monotone decreasing**.
- $f'(x) = 0, \forall x \in I \implies f$ is **constant**.

Proof Idea: Apply the Mean Value Theorem (MVT) to arbitrary $x_1 < x_2$. The sign of $f(x_2) - f(x_1)$ is determined entirely by $f'(c)(x_2 - x_1)$.

2. Pointwise vs. Neighborhood Behavior (Crucial Distinction)

Observation:
The condition $f'(c) > 0$ at a **single point** c behaves differently than $f'(x) > 0$ on an **interval**.

A. What $f'(c) > 0$ implies:
If $f'(c) > 0$, there exists a $\delta > 0$ such that:

- $f(x) < f(c)$ for all $x \in (c - \delta, c)$
- $f(x) > f(c)$ for all $x \in (c, c + \delta)$
  (See Exercise 17).

B. What $f'(c) > 0$ does NOT imply:
It does **not** imply that f is increasing on the interval $(c - \delta, c + \delta)$.

- **Reason:** $f'(x)$ may assume both positive and negative values in every neighborhood of c.
- **Counter-Example (Exercise 20):**

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  Here, $f'(0) = 1 > 0$, yet f is **not monotone** on any interval containing 0 due to rapid oscillation.

C. Sufficient Condition for Monotonicity:
If we require **continuity** of the derivative:

- If $f'(c) > 0$ **AND** f' is continuous at c, then there exists a $\delta > 0$ such that $f'(x) > 0$ for all $x \in (c - \delta, c + \delta)$.
- $\therefore f$ is increasing on $(c - \delta, c + \delta)$.

3. Relative Extrema & The First Derivative Test

The First Derivative Test:
Used to classify critical points where $f'(c) = 0$ or $f'(c)$ does not exist.
Suppose f is continuous on (a, b).

1. If $f'(x) < 0$ on (a, c) AND $f'(x) > 0$ on (c, b):
   - f is decreasing to the left and increasing to the right.
   - $\implies f$ has a **relative minimum** at c.
2. (Similarly for relative maximum if signs switch from $+$ to $-$).

The False Converse:
One naturally assumes: *"If f has a relative minimum at c, then f must be decreasing to the left and increasing to the right."*

- **This is FALSE.**
- A function can have a relative minimum at c without being monotone on the immediate left or right sides.

Counter-Example (Example 5.2.10):

$$f(x) = \begin{cases} x^4(2 + \sin(1/x)) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- f has an absolute minimum at $x = 0$ (since $f(x) > 0$ for $x \neq 0$).
- However, $f'(x)$ oscillates between positive and negative values in every neighborhood of 0.
- Thus, f is **not** "decreasing then increasing" in the standard monotonic sense near 0.

6. Limits of Derivatives

Theorem 5.2.11:
If f is continuous on $[a, b)$ and differentiable on (a, b), and $\lim_{x \to a^+} f'(x) = L$, then the right-hand derivative exists and:

$$f'_+(a) = \lim_{x \to a^+} f'(x)$$

- **Proof Idea:** Use MVT on $[a, a + h]$. $f(a + h) - f(a) = f'(\zeta_h)h$. As $h \to 0$, $\zeta_h \to a$, so the difference quotient converges to L.
- **Implication:** Derivatives cannot have simple jump discontinuities; discontinuities must be of the second kind (oscillatory).

7. Intermediate Value Theorem for Derivatives

Theorem 5.2.13 (Darboux's Theorem):
If f is differentiable on I and $a, b \in I$ with $a < b$, then for any λ between $f'(a)$ and $f'(b)$, there exists $c \in (a, b)$ such that:

$$f'(c) = \lambda$$

- **Significance:** Derivatives possess the Intermediate Value Property even if they are **not continuous**.

- **Proof Idea:** Construct $g(x) = f(x) - \lambda x$. Depending on signs of $g'(a)$ and $g'(b)$, g attains a local extremum interior to the interval. At that extremum c, $g'(c) = f'(c) - \lambda = 0$.

8. Inverse Function Theorem

Theorem 5.2.14:
If f is differentiable on interval I and $f'(x) \neq 0$ for all $x \in I$:

1. f is one-to-one.
2. f^{-1} is continuous and differentiable on $J = f(I)$.
3. The derivative is given by:

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

- **Proof Idea:** $f' \neq 0$ implies f' maintains a single sign (by Darboux's Theorem), so f is strictly monotone (one-to-one). Differentiability follows from limits.

Section 5.3 L'Hospital's Rule

1. Infinite Limits (Definitions)

Before introducing L'Hospital's rule, formal definitions for infinite limits are established to handle the indeterminate form ∞/∞.

- **Definition 5.3.1:** Let f be defined on a subset $E \subset \mathbb{R}$ and p be a limit point of E.
  - $\lim_{x\to p} f(x) = \infty$ if for every $M \in \mathbb{R}$, there exists $\delta > 0$ such that $f(x) > M$ for all $x \in E$ with $0 < |x-p| < \delta$.
  - $\lim_{x\to p} f(x) = -\infty$ is defined similarly (where $f(x) < M$).
- **Note:** These definitions extend to limits at infinity ($\lim_{x\to\infty}$) and one-sided limits ($\lim_{x\to p^+}$).

2. L'Hospital's Rule (Theorem 5.3.2)

This rule evaluates limits of indeterminate forms $0/0$ or ∞/∞.

Hypotheses:

1. f, g are real-valued differentiable functions on (a, b).
2. $g'(x) \neq 0$ for all $x \in (a, b)$.
3. $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where $L \in \mathbb{R} \cup \{-\infty, \infty\}$.

Conditions (Indeterminate Forms):

- **(a) Case $0/0$:** $\lim_{x\to a^+} f(x) = 0$ and $\lim_{x\to a^+} g(x) = 0$.
- **(b) Case ∞/∞:** $\lim_{x\to a^+} g(x) = \pm\infty$ (Note: $f(x)$ does not strictly need to tend to ∞, but usually does).

Conclusion:

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

(Note: The rule applies equally to $x \to b^-$, $x \to p$, or $x \to \pm\infty$).

3. Proofs: Core Ideas

Case A: Indeterminate Form $0/0$ (Finite a)

- **Key Tool:** Generalized Mean Value Theorem (GMVT).
- **Method:**
  i. Since we are dealing with limits approaching a, we define $f(a) = g(a) = 0$ to make the functions continuous at a.
  ii. Consider a sequence $\{x_n\} \to a^+$. Apply GMVT on the interval $[a, x_n]$.
  iii. There exists c_n between a and x_n such that:

$$\frac{f(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(c_n)}$$

  iv. Since $f(a) = g(a) = 0$, this simplifies to $\frac{f(x_n)}{g(x_n)} = \frac{f'(c_n)}{g'(c_n)}$.
  v. As $n \to \infty$, $x_n \to a$ implies $c_n \to a$. Therefore, the limit of the ratio of functions equals the limit of the ratio of derivatives.

Case B: Limits at Infinity ($x \to -\infty$)

- **Key Tool:** Substitution.
- **Method:** Let $x = -1/t$. As $t \to 0^+$, $x \to -\infty$.
  - Define $\phi(t) = f(-1/t)$ and $\psi(t) = g(-1/t)$.
  - Using chain rule differentiation, the problem reduces to a limit at 0^+, which allows the use of the previous proof logic.

Case C: Indeterminate Form ∞/∞

- **Key Tool:** GMVT + Bounding Argument (No need for f, g to be continuous at a).
- **Method:**
  i. Assume $\lim \frac{f'(x)}{g'(x)} = L$.
  ii. Fix y and let x vary. Apply GMVT on interval (x, y) to get $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(\zeta)}{g'(\zeta)}$.
  iii. **Algebraically rearrange** the GMVT equation to isolate $\frac{f(x)}{g(x)}$:

$$\frac{f(x)}{g(x)} = \frac{f'(\zeta)}{g'(\zeta)} \left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)}$$

  iv. Since $g(x) \to \infty$ as $x \to a^+$, the term $\frac{g(y)}{g(x)} \to 0$ and $\frac{f(y)}{g(x)} \to 0$.
  v. This implies that for x sufficiently close to a, $\frac{f(x)}{g(x)}$ becomes arbitrarily close to $\frac{f'(\zeta)}{g'(\zeta)} \cdot (1 - 0) + 0$, which converges to L.

4. Examples

(a) Basic Application ($0/0$)

Problem: Compute $\lim_{x \to 0^+} \frac{\ln(1+x)}{x}$.

- **Form:** $0/0$ (since $\ln(1) = 0$).
- **Apply L'Hospital's:**

$$\lim_{x \to 0^+} \frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = \lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1} = 1$$

- **Note:** This can also be proven using inequalities derived from the Taylor expansion or Mean Value Theorem (i.e., $\frac{x}{1+x} \le \ln(1+x) \le x$), but L'Hospital's is more direct.

(b) Repeated Application

Problem: Compute $\lim_{x \to 0} \frac{1-\cos x}{x^2}$.

- **Form:** $0/0$.
- **First Application:**

$$\lim_{x \to 0} \frac{\sin x}{2x}$$

  This is *still* form $0/0$.
- **Second Application:**

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

- **Result:** The limit is $1/2$.

(c) Importance of Substitution (Avoiding Complexity)

Problem: Compute $\lim_{x \to 0^+} \frac{e^{-1/x}}{x}$.

- **Form:** $0/0$ (since $e^{-\infty} \to 0$).
- **Direct L'Hospital's Failure:** Differentiating directly gives $\frac{e^{-1/x} \cdot (1/x^2)}{1}$, which simplifies to $\frac{e^{-1/x}}{x^2}$. This is *more* complicated than the original.
- **Correct Approach:** Use substitution $t = 1/x$. As $x \to 0^+$, $t \to \infty$.

$$\lim_{t \to \infty} \frac{t}{e^t}$$

- **New Form:** ∞/∞.
- **Apply L'Hospital's:**

$$\lim_{t \to \infty} \frac{1}{e^t} = 0$$

- **Result:** The original limit is 0.

Section 6.1 The Riemann Integral

1. Upper and Lower Sums

Definition (Partition):
A partition $\mathcal{P}$ of a closed interval $[a, b]$ is a finite set of points $\{x_0, x_1, \dots, x_n\}$ such that:

$$a = x_0 < x_1 < \cdots < x_n = b$$

Let $\Delta x_i = x_i - x_{i-1}$.

Definition (Bounds):
For a bounded function f on $[a, b]$, on each subinterval $[x_{i-1}, x_i]$, define:

- $m_i = \inf\{f(t) : x_{i-1} \le t \le x_i\}$
- $M_i = \sup\{f(t) : x_{i-1} \le t \le x_i\}$

Definition (Sums):

- **Lower Sum:** $\mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i$ (Inscribed rectangles)
- **Upper Sum:** $\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i$ (Circumscribed rectangles)

Basic Inequality:
For any partition $\mathcal{P}$:

$$\mathcal{L}(\mathcal{P}, f) \le \mathcal{U}(\mathcal{P}, f)$$

2. Upper and Lower Integrals

Boundedness:
If $m \le f(t) \le M$ for all t, then for any partition $\mathcal{P}$:

$$m(b - a) \le \mathcal{L}(\mathcal{P}, f) \le \mathcal{U}(\mathcal{P}, f) \le M(b - a)$$

Refinement Lemma:
A partition $\mathcal{P}^*$ is a **refinement** of $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}^*$. Adding points improves the approximation:

$$\mathcal{L}(\mathcal{P}, f) \leq \mathcal{L}(\mathcal{P}^*, f) \leq \mathcal{U}(\mathcal{P}^*, f) \leq \mathcal{U}(\mathcal{P}, f)$$

Key Proof Idea: Adding a point x^* in $[x_{k-1}, x_k]$ splits the interval. Since the infimum over a subset is larger (or equal) and the supremum is smaller (or equal), the lower sum increases and the upper sum decreases.

Comparison of Any Two Partitions:
For *any* partitions $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{Q}, f)$.
(Proof uses the common refinement $\mathcal{P} \cup \mathcal{Q}$).

Definition (Integrals):

- **Lower Integral:** $\underline{\int_a^b} f = \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f)$
- **Upper Integral:** $\overline{\int_a^b} f = \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f)$

Theorem 6.1.4:
Always, $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

3. Definition of the Riemann Integral

Definition 6.1.5:
A bounded function f is **Riemann integrable** on $[a, b]$ (denoted $f \in \mathcal{R}[a, b]$) if:

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

The common value is denoted $\int_a^b f(x)dx$.

4. Examples

(a) Dirichlet Function (Not Integrable):
$f(x) = 1$ if $x \in \mathbb{Q}$, 0 if $x \notin \mathbb{Q}$ on $[a, b]$.

- For any interval, density of rationals/irrationals implies $m_i = 0$ and $M_i = 1$.

- $\mathcal{L}(\mathcal{P}, f) = 0 \implies \underline{\int} f = 0$.
- $\mathcal{U}(\mathcal{P}, f) = b - a \implies \overline{\int} f = b - a$.
- Since $0 \neq b - a$, $f \notin \mathcal{R}[a, b]$.

(b) Step Function:
$f(x) = 0$ for $x < 1/2$, $f(x) = 1$ for $x \geq 1/2$ on $[0, 1]$.

- Proof involves isolating the discontinuity at $1/2$ within a small interval of the partition. The error term $\mathcal{U} - \mathcal{L}$ can be made arbitrarily small by shrinking the interval covering $1/2$.
- Result: $\int_0^1 f = 1/2$.

(c) $f(x) = x$ on $[a, b]$:

- f is increasing $\implies m_i = x_{i-1}, M_i = x_i$.
- $\mathcal{U} - \mathcal{L} = \sum (x_i - x_{i-1}) \Delta x_i$.
- By choosing equal width $\Delta x, \mathcal{U} - \mathcal{L} = \Delta x(b - a)$. As $\Delta x \to 0$, difference goes to 0.
- $\int_a^b x dx = \frac{1}{2}(b^2 - a^2)$.

(d) $f(x) = x^2$ on $[0, 1]$:

- Uses equal partitions. $m_i = (\frac{i-1}{n})^2, M_i = (\frac{i}{n})^2$.
- Calculations use sum of squares formula $\sum i^2 = \frac{1}{6} m(m+1)(2m+1)$.
- Taking limits as $n \to \infty$, both sums converge to $1/3$.

5. Riemann's Criterion for Integrability

Theorem 6.1.7:
$f \in \mathcal{R}[a, b]$ **if and only if** for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that:

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon$$

- *Significance:* Allows checking integrability without knowing the value of the integral.
- *Proof Key Idea:*
  - $\Rightarrow$: Use definitions of supremum and infimum to find partitions close to the integrals, then combine them.
  - $\Leftarrow$: If $\mathcal{U} - \mathcal{L} < \epsilon$, then $0 \leq \overline{\int} - \underline{\int} < \epsilon$ for all ϵ, forcing equality.

6. Integrability of Specific Classes of Functions

Theorem 6.1.8:

1. **Continuous Functions:** If f is continuous on $[a, b]$, it is integrable.
   - *Proof Idea:* Use **Uniform Continuity**. Given ϵ, choose δ such that $|x - t| < \delta \implies |f(x) - f(t)| < \frac{\epsilon}{b-a}$. Then $M_i - m_i < \frac{\epsilon}{b-a}$. Summing gives $\mathcal{U} - \mathcal{L} < \epsilon$.
2. **Monotone Functions:** If f is monotone on $[a, b]$, it is integrable.
   - *Proof Idea:* Telescoping sum. For uniform partition with width h, $\sum(M_i - m_i)\Delta x_i = h\sum(f(x_i) - f(x_{i-1})) = h(f(b) - f(a))$. We can make h small enough to satisfy the ϵ condition.

7. The Composition Theorem

Theorem 6.1.9
Let $f \in \mathcal{R}[a, b]$ with range in $[c, d]$ and let $\varphi : [c, d] \to \mathbb{R}$ be **continuous**. Then the composition $\varphi \circ f$ is Riemann integrable on $[a, b]$ (i.e., $\varphi \circ f \in \mathcal{R}[a, b]$).

Proof Strategy

Since φ is continuous on a closed interval, it is **uniformly continuous** and **bounded**. We use these properties to control the upper and lower sums of the composition.

1. Setup & Uniform Continuity

- Let $K = \sup\{|\varphi(t)| : t \in [c, d]\}$.
- Let $\epsilon > 0$ be given. Define a strict tolerance $\epsilon' = \frac{\epsilon}{b-a+2K}$.
- By **uniform continuity** of φ, there exists $\delta \in (0, \epsilon')$ such that:

$$|s - t| < \delta \implies |\varphi(s) - \varphi(t)| < \epsilon'$$

2. Partitioning & The "Small Oscillation" Trick
Since f is integrable, choose a partition $P = \{x_0, \dots, x_n\}$ such that:

$$U(P, f) - L(P, f) < \delta^2$$

Note: We use δ^2 to force the total width of "bad" intervals to be small later.

3. Splitting the Indices (The Crucial Step)

Let M_k, m_k be bounds for f, and M_k^*, m_k^* be bounds for $\varphi \circ f$ on interval k.
We split the partition indices $k \in \{1, \dots, n\}$ into two disjoint sets:

- **Set A (Small Oscillation):** k where $M_k - m_k < \delta$.
  - Here, $f(t)$ varies little, so $\varphi(f(t))$ varies by less than ϵ'.
  - $\implies M_k^* - m_k^* \le \epsilon'$.
- **Set B (Large Oscillation):** k where $M_k - m_k \ge \delta$.
  - Here, we only know φ is bounded by K.
  - $\implies M_k^* - m_k^* \le 2K$.

4. The Estimates

We split the difference between Upper and Lower sums:

$$U(P, \varphi \circ f) - L(P, \varphi \circ f) = \sum_{k \in A} (M_k^* - m_k^*)\Delta x_k + \sum_{k \in B} (M_k^* - m_k^*)\Delta x_k$$

- **Estimate for A (Good behavior):**

$$\sum_A (M_k^* - m_k^*)\Delta x_k \le \epsilon' \sum_A \Delta x_k \le \epsilon'(b-a)$$

- **Estimate for B (Bad behavior, but small width):**
  First, bound the total width of set B:

$$\sum_B \delta \Delta x_k \le \sum_B (M_k - m_k)\Delta x_k \le U(P, f) - L(P, f) < \delta^2$$

  Dividing by δ, we get $\sum_B \Delta x_k < \delta < \epsilon'$.
  Thus:

$$\sum_B (M_k^* - m_k^*)\Delta x_k \le 2K \sum_B \Delta x_k < 2K\epsilon'$$

5. Conclusion

Combining the sums:

$$U - L < \epsilon'(b-a) + 2K\epsilon' = \epsilon'(b - a + 2K) = \epsilon$$

Since the difference is arbitrarily small, $\varphi \circ f \in \mathcal{R}[a, b]$. ■

Corollary 6.1.10
If $f \in \mathcal{R}[a, b]$, then:

1. $|f| \in \mathcal{R}[a, b]$ (using $\varphi(t) = |t|$)
2. $f^2 \in \mathcal{R}[a, b]$ (using $\varphi(t) = t^2$)

Important Warning:
The composition of two Riemann integrable functions is **not** necessarily integrable.

- *Counter-example (6.1.14b):* If f is the Riemann function and g is an indicator function, $g \circ f$ may fail integrability.

8. Lebesgue's Theorem

Definition (Measure Zero):
A set $E \subset \mathbb{R}$ has measure zero if for any $\epsilon > 0$, E can be covered by a countable union of open intervals $\{I_n\}$ such that $\sum length(I_n) < \epsilon$.

- *Examples:* Finite sets, countable sets (like $\mathbb{Q}$), and even the Cantor set (uncountable) have measure zero.

Theorem 6.1.13 (Lebesgue):
A bounded function f on $[a, b]$ is Riemann integrable **if and only if** the set of its discontinuities has **measure zero**.

Applications:

- Continuous functions: Discontinuity set is empty (measure 0) $\implies$ Integrable.
- Monotone functions: Discontinuities are countable (measure 0) $\implies$ Integrable.
- Function with finite discontinuities $\implies$ Integrable.

9. Advanced Examples (Lebesgue Application)

(a) Thomae's Function (Popcorn Function):

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ (lowest terms)} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

- Continuous at all irrationals (measure of rationals is 0).
- Discontinuous at all rationals.
- By Lebesgue's Theorem, it is **integrable** and $\int_0^1 f = 0$.

(b) Counter-example for Composition:

Let f be Thomae's function (integrable). Let $g(y) = 1$ if $y \in (0, 1]$, $g(0) = 0$.

- g is integrable (only discontinuous at 0).
- $g \circ f$ results in the Dirichlet function (1 at rationals, 0 at irrationals).
- $g \circ f$ is **not** integrable.
- Reason: f maps rationals to non-zero values (where $g = 1$) and irrationals to 0 (where $g = 0$).

6.2 Properties of the Riemann Integral

This section derives fundamental algebraic and order properties of the Riemann integral and establishes the equivalence between the Darboux definition (upper/lower sums) and the original Riemann definition (limit of sums).

1. Linearity and Algebra of Integrable Functions

Theorem 6.2.1: Let $f, g \in \mathcal{R}[a, b]$.

1. **Sum:** $f + g \in \mathcal{R}[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
2. **Scalar Multiplication:** $cf \in \mathcal{R}[a, b]$ for any $c \in \mathbb{R}$, and $\int_a^b cf = c \int_a^b f$.
3. **Product:** $fg \in \mathcal{R}[a, b]$.

Proof of (1) - Sum:

- **Key Idea:** Relate the supremum of the sum to the sum of supremums.
- Let $\mathcal{P} = \{x_0, \dots, x_n\}$ be a partition.
- Inequality: $\sup_{[x_{i-1}, x_i]} (f + g) \le \sup_{[x_{i-1}, x_i]} f + \sup_{[x_{i-1}, x_i]} g$.
- This implies $\mathcal{U}(\mathcal{P}, f + g) \le \mathcal{U}(\mathcal{P}, f) + \mathcal{U}(\mathcal{P}, g)$.
- Using refinements $\mathcal{P}_f$ and $\mathcal{P}_g$ where upper sums are within $\epsilon/2$ of the integral, let $\mathcal{Q} = \mathcal{P}_f \cup \mathcal{P}_g$.
- $\overline{\int}(f + g) \le \mathcal{U}(\mathcal{Q}, f + g) < \int f + \int g + \epsilon$.
- A similar argument with lower sums shows $\underline{\int}(f + g) \ge \int f + \int g$.
- Since $\underline{\int} \le \overline{\int}$, equality holds.

Proof of (3) - Product:

- **Key Idea:** Use the algebraic identity $fg = \frac{1}{4}[(f + g)^2 - (f - g)^2]$.
- Since $f, g \in \mathcal{R}$, $f \pm g \in \mathcal{R}$.
- Square functions of integrable functions are integrable (Corollary 6.1.10).
- By linearity (proven in parts 1 and 2), the linear combination of these squares is integrable.

2. Absolute Value Property

Theorem 6.2.2: If $f \in \mathcal{R}[a, b]$, then $|f| \in \mathcal{R}[a, b]$ and:

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Proof:

- **Integrability:** $|f| \in \mathcal{R}$ follows from composition properties (Corollary 6.1.10).
- **Inequality:** Choose $c = \pm 1$ such that $\left|\int f\right| = c \int f$.
- Note that $cf(x) \leq |f(x)|$ for all x.
- Therefore, $\int cf \leq \int |f|$.
- Substituting back: $\left|\int f\right| = c \int f = \int cf \leq \int |f|$.

3. Additivity Over Intervals

Theorem 6.2.3: Let f be bounded on $[a, b]$ and let $a < c < b$. Then $f \in \mathcal{R}[a, b]$ if and only if $f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$.
If integrable, the identity holds:

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof:

- **Step 1: Upper Integrals:**
  - Let $\mathcal{P}$ be a partition of $[a, b]$ containing c. It splits into partitions $\mathcal{P}_1$ of $[a, c]$ and $\mathcal{P}_2$ of $[c, b]$.
  - $\mathcal{U}(\mathcal{P}, f) = \mathcal{U}(\mathcal{P}_1, f) + \mathcal{U}(\mathcal{P}_2, f)$.
  - Taking infimums yields: $\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f$.
  - (Note: If a partition doesn't contain c, taking the refinement $\mathcal{P} \cup \{c\}$ only decreases the sum, preserving the inequality for the infimum).
- **Step 2: Lower Integrals:**
  - Similarly, $\underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f$.
- **Conclusion:**
  - If f is integrable on the sub-intervals, the upper and lower sums match on the right-hand side, forcing them to match on the left (whole interval).
  - Conversely, if $f \in \mathcal{R}[a, b]$, equality of total upper/lower sums forces equality on the sub-intervals (since upper $\geq$ lower always).

4. Riemann's Definition of the Integral

While Darboux used sup/inf sums, Riemann used arbitrary tags.

Definitions:

- **Riemann Sum:** Given a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ and tags $t_i \in [x_{i-1}, x_i]$:

$$\mathcal{S}(\mathcal{P}, f) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

- **Norm (Mesh):** $||\mathcal{P}|| = \max\{\Delta x_i : i = 1, \ldots, n\}$.
- **Limit Definition:** $\lim_{||\mathcal{P}|| \to 0} \mathcal{S}(\mathcal{P}, f) = I$ means:
  Given $\epsilon > 0$, there exists $\delta > 0$ such that for *any* partition $\mathcal{P}$ with $||\mathcal{P}|| < \delta$ and *any* choice of tags t_i:

$$|\mathcal{S}(\mathcal{P}, f) - I| < \epsilon$$

5. Equivalence of Definitions

Theorem 6.2.6: $f \in \mathcal{R}[a, b]$ (Darboux sense) if and only if the Riemann limit exists. If so, the limit equals the integral.

Proof Highlights:

Part A: Limit Exists $\implies$ Darboux Integrable

- Assume limit is I. Given ϵ, choose δ per definition.
- Pick a partition $\mathcal{P}$ with $||\mathcal{P}|| < \delta$.
- Because $M_i = \sup f$, we can choose tags t_i such that $f(t_i)$ is arbitrarily close to M_i.
- This implies $\mathcal{U}(\mathcal{P}, f) < \mathcal{S}(\mathcal{P}, f) + \epsilon < I + 2\epsilon$.
- Similarly for lower sums. Thus, $\mathcal{U} - \mathcal{L}$ can be made arbitrarily small.

Part B: Darboux Integrable $\implies$ Limit Exists

- Given $f \in \mathcal{R}[a, b]$, there exists a partition $\mathcal{Q} = \{x_0, \ldots, x_N\}$ such that $\mathcal{U}(\mathcal{Q}) - \mathcal{L}(\mathcal{Q}) < \epsilon$.
- Let $|f(x)| \leq M$. Choose $\delta = \epsilon/(NM)$.
- Let $\mathcal{P} = \{y_0, \ldots, y_n\}$ be *any* partition with $||\mathcal{P}|| < \delta$.
- **Classification of Intervals:** Split intervals of $\mathcal{P}$ into two types:
  i. **Good intervals:** Subsets of intervals in $\mathcal{Q}$ (contain no points of $\mathcal{Q}$ in their interior).
  ii. **Bad intervals:** Contain points of $\mathcal{Q}$.

- **Bounding the Sum:**
  - There are at most $N-1$ "bad" intervals. Their total length is small (controlled by δ). The contribution to the sum is bounded by $2M(N-1)\delta < 2\epsilon$.
  - On "good" intervals, the difference between $f(t_k)$ and the sup/inf of $\mathcal{Q}$ is controlled by the original ϵ.
- Result: $|\mathcal{S}(\mathcal{P}, f) - \int f| < 3\epsilon$.

6. Application Example

Example 6.2.7: Calculating $\int_a^b x\,dx$

- Since $f(x) = x$ is continuous, it is integrable.
- We can compute the limit using a specific choice of tags (since the limit is unique).
- **Choice:** Midpoint rule. $t_i = \frac{x_{i-1}+x_i}{2}$.
- **Riemann Sum:**

$$\mathcal{S}(\mathcal{P}, f) = \sum_{i=1}^{n} \left(\frac{x_{i-1} + x_i}{2} \right) (x_i - x_{i-1})$$

- Using difference of squares $(x+y)(x-y) = x^2 - y^2$:

$$\mathcal{S}(\mathcal{P}, f) = \frac{1}{2} \sum_{i=1}^{n} (x_i^2 - x_{i-1}^2)$$

- **Telescoping Sum:**

$$= \frac{1}{2}[(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + \cdots + (x_n^2 - x_{n-1}^2)]$$

$$= \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}(b^2 - a^2)$$

6.3 Fundamental Theorem of Calculus

This section establishes the connection between differentiation and integration (inverse operations) using Riemann or Darboux sums.

I. The First Fundamental Theorem (Evaluation)

Definition 6.3.1 (Antiderivative)

A function F on an interval I is an antiderivative of f if $F'(x) = f(x)$ for all $x \in I$.

- **Note:** Antiderivatives are not unique ($F(x) + C$).

Theorem 6.3.2 (Fundamental Theorem of Calculus - Part 1)

If $f \in \mathcal{R}[a, b]$ (Riemann integrable) and F is an antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

- **Proof (Key Idea):**
    i. Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be any partition of $[a, b]$.
    ii. Apply the **Mean Value Theorem** to F on each subinterval $[x_{i-1}, x_i]$:

    $$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i = f(t_i)\Delta x_i$$

    for some $t_i \in (x_{i-1}, x_i)$.
    iii. Summing over i yields a telescoping sum:

    $$\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = F(b) - F(a)$$

    iv. Since this Riemann sum equals $F(b) - F(a)$ for *any* partition, and f is integrable:

    $$\mathcal{L}(\mathcal{P}, f) \leq F(b) - F(a) \leq \mathcal{U}(\mathcal{P}, f) \implies \int_a^b f = F(b) - F(a)$$

Examples 6.3.3 (Notable Cases)

- **(b) Discontinuous Derivative:**
  Let $F(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $F(0) = 0$.
  Then $f(x) = F'(x)$ exists everywhere but is discontinuous at $x = 0$. Since f is bounded and

continuous almost everywhere, $f \in \mathcal{R}[0,1]$.
Result: $\int_0^1 f = F(1) - F(0) = \sin 1$.

- **(c) Piecewise Function:**
Let $f(x) = 1$ for $x \in [0,1)$ and $f(x) = x - 1$ for $x \in [1,2]$.
Define $F(x) = \int_0^x f(t)dt$.
Result: Even though f is discontinuous at $x = 1$, the accumulation function $F(x)$ is **continuous everywhere**.

II. The Second Fundamental Theorem (Differentiation)

Theorem 6.3.4 (Fundamental Theorem of Calculus - Part 2)
Let $f \in \mathcal{R}[a,b]$ and define F on $[a,b]$ by:

$$F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on $[a,b]$. Furthermore, if f is **continuous** at $c \in [a,b]$, then F is differentiable at c and:

$$F'(c) = f(c)$$

- **Proof (Key Idea):**
To show $F'(c) = f(c)$, consider the difference quotient:

$$\frac{F(c+h) - F(c)}{h} - f(c) = \frac{1}{h}\int_c^{c+h} (f(t) - f(c))\,dt$$

Since f is continuous at c, for any $\epsilon > 0$, $|f(t) - f(c)| < \epsilon$ for sufficiently small h.

$$\left|\frac{1}{h}\int_c^{c+h} (f(t) - f(c))\,dt\right| < \frac{1}{h}(h \cdot \epsilon) = \epsilon$$

Thus, limit as $h \to 0$ is 0, implying $F'(c) = f(c)$.
- **Remarks:**
    - Integrability of f implies continuity of F.
    - Continuity of f implies differentiability of F.

III. Application: The Natural Logarithm

Example 6.3.5 (Definition of $\ln x$)
For $x > 0$, define the natural logarithm as:

$$L(x) = \int_1^x \frac{1}{t}\,dt$$

Since $1/t$ is continuous, $L'(x) = 1/x$.

Properties:

1. **Log Rules:** $L(ab) = L(a) + L(b)$.
   - *Proof Idea:* Differentiate $L(ax)$ to get $1/x$, implying $L(ax) = L(x) + C$. Use $L(1) = 0$ to find $C = L(a)$.
2. **Power Rule:** $L(b^r) = rL(b)$ for $r \in \mathbb{R}$.
3. **Euler's Number:** $L(e) = 1$.
   - *Derivation:* $1 = L'(1) = \lim_{n\to\infty} nL(1 + 1/n) = L(\lim(1 + 1/n)^n) = L(e)$.

IV. Consequences of FTC

Theorem 6.3.6 (Mean Value Theorem for Integrals)
If f is continuous on $[a, b]$, there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

- **Proof:** Let $F(x) = \int_a^x f$. By standard MVT, $F(b) - F(a) = F'(c)(b-a)$. Since $F'(c) = f(c)$, the result follows.

Theorem 6.3.7 (Integration by Parts)
If f, g are differentiable and $f', g' \in \mathcal{R}[a, b]$:

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'$$

- **Proof:** Integrate the product rule $(fg)' = fg' + gf'$ and apply FTC.

Theorem 6.3.8 (Change of Variable)
Let φ be differentiable on $[a, b]$ with $\varphi' \in \mathcal{R}$. If f is continuous on the range of φ:

$$\int_a^b f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$$

- **Proof:** Let $F(x) = \int f(s)ds$. By Chain Rule, $\frac{d}{dt}F(\varphi(t)) = f(\varphi(t))\varphi'(t)$. Apply FTC.

Examples 6.3.9

- **(a) Substitution:** Evaluating $\int_0^2 t/(1+t^2)dt$ using $\varphi(t) = 1 + t^2$.
- **(b) Trig Substitution:** Evaluating $\int_0^a \sqrt{a^2 - x^2}dx$ using $x = a\sin t$. Converts to $\int \cos^2 t$, solved via half-angle identity.

6.4 Improper Riemann Integrals

Motivation: Standard Riemann integration requires a function f to be **bounded** on a **closed and bounded** interval $[a, b]$. If f is unbounded or the interval is infinite, we extend the definition.

I. Unbounded Functions on Finite Intervals

Definition 6.4.1 (Singularity at Endpoint)

Let f be defined on (a, b) such that $f \in \mathcal{R}[c, b]$ for every $c \in (a, b)$. The **improper Riemann integral** is defined as:

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f$$

- **Convergent:** The limit exists.
- **Divergent:** The limit does not exist.

Note: A similar definition applies if f is unbounded at b (limit as $c \to b^-$).

Interior Singularities

If f is unbounded at p where $a < p < b$, split the integral:

$$\int_a^b f = \int_a^p f + \int_p^b f$$

The integral converges only if **both** parts converge independently.

Examples on Finite Intervals

1. Example (Divergence): $f(x) = 1/x$ on $(0, 1]$

- f is unbounded at 0.
- $\int_c^1 \frac{1}{x}\, dx = \ln(1) - \ln(c) = -\ln c$.
- Limit: $\lim_{c \to 0^+} (-\ln c) = \infty$.
- **Conclusion:** The integral diverges.

2. Example (Convergence): $f(x) = \ln x$ on $(0, 1]$

- f is continuous on $(0, 1]$. Use Integration by Parts with $u = \ln x, dv = dx$:

$$\int_c^1 \ln x \, dx = [x \ln x]_c^1 - \int_c^1 dx = -c \ln c - (1 - c)$$

- Using L'Hospital's Rule ($c = 1/t$): $\lim_{c \to 0^+} c \ln c = 0$.
- Limit: $\lim_{c \to 0^+} (-c \ln c - 1 + c) = -1$.
- **Conclusion:** Converges to -1.

3. Example (Piecewise):
Let $f(x) = 0$ on $[-1, 0]$ and $f(x) = 1/x$ on $(0, 1]$.

- The integral over $[-1, 1]$ fails to exist because the sub-interval integral $\int_0^1 (1/x)$ diverges.

Comparison with Standard Riemann Integral

Properties of standard Riemann integrals do **not** always hold for improper integrals.

- **Property failure: $f \in \mathcal{R} \implies f^2 \in \mathcal{R}$ is FALSE.**
  - *Counter-example:* $f(x) = 1/\sqrt{x}$ on $(0, 1]$.
  - $\int_0^1 x^{-1/2} \, dx = \lim_{c \to 0^+} [2\sqrt{x}]_c^1 = 2$ (Converges).
  - However, $f^2(x) = 1/x$, which diverges on $(0, 1]$.
- **Property failure: $f \in \mathcal{R} \implies |f| \in \mathcal{R}$ is FALSE.**
  - Convergence of $\int f$ does not imply convergence of $\int |f|$.

II. Infinite Intervals

Definition 6.4.3
Let f be defined on $[a, \infty)$ and $f \in \mathcal{R}[a, c]$ for all $c > a$.

$$\int_a^\infty f = \lim_{c \to \infty} \int_a^c f$$

- For $(-\infty, b]$: Take limit as $c \to -\infty$.
- For $(-\infty, \infty)$: Split at any fixed $p \in \mathbb{R}$.

$$\int_{-\infty}^\infty f = \int_{-\infty}^p f + \int_p^\infty f$$

Condition: Both separate integrals must converge.

Important Warning:
One cannot compute $\int_{-\infty}^{\infty} f$ as $\lim_{c\to\infty} \int_{-c}^{c} f$.

- *Counter-example:* $f(x) = x$.
  - $\lim_{c\to\infty} \int_{-c}^{c} x\,dx = 0$.
  - But $\int_0^{\infty} x\,dx = \infty$. Thus, the improper integral on $(-\infty, \infty)$ **diverges**.

Examples on Infinite Intervals

1. Example: $f(x) = 1/x^2$ **on** $[1, \infty)$

- $\int_1^c x^{-2}\,dx = [-1/x]_1^c = 1 - \frac{1}{c}$.
- $\lim_{c\to\infty}(1 - 1/c) = 1$.
- **Conclusion:** Converges to 1.

2. Example: $f(x) = \frac{\sin x}{x}$ **on** $[\pi, \infty)$

- *Fact:* The integral of f converges (conditional convergence).
- *Proof focus:* The integral of the absolute value **diverges** ($\int |f| = \infty$).

Proof of Absolute Divergence for $\frac{\sin x}{x}$**:**
We estimate the sum of integrals over intervals $[k\pi, (k+1)\pi]$.

1. Consider $\int_{\pi}^{\infty} \frac{|\sin x|}{x}\,dx$.
2. On the sub-interval $[(k+\frac{1}{4})\pi, (k+\frac{3}{4})\pi]$, we know $|\sin x| \geq \frac{\sqrt{2}}{2}$.
3. Also, $\frac{1}{x} \geq \frac{1}{(k+1)\pi}$.
4. Bounding the integral for the k-th segment:

$$\text{Area} \geq \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{1}{(k+1)\pi}\right) \cdot \left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{4(k+1)}$$

5. Summing from $k = 1$ to n:

$$\int_{\pi}^{(n+1)\pi} |f| \geq \frac{\sqrt{2}}{4} \sum_{k=1}^{n} \frac{1}{k+1}$$

6. Since the Harmonic series $\sum \frac{1}{k}$ diverges, the integral diverges to ∞.

III. Absolute Integrability & Comparison Test

Definition: f is **absolutely integrable** on $[a, \infty)$ if $f \in \mathcal{R}[a, c]$ for all c and $\int_a^\infty |f|$ converges.

- *Note:* Absolute integrability implies convergence of the improper integral (similar to series).

Theorem 6.4.5: Comparison Test
Let $g \geq 0$ be a function where $\int_a^\infty g(x)\,dx < \infty$ (converges).
If f satisfies:

1. $f \in \mathcal{R}[a, c]$ for every $c > a$, and
2. $|f(x)| \leq g(x)$ for all $x \in [a, \infty)$,

Then:

- $\int_a^\infty f(x)\,dx$ **converges**.
- $|\int_a^\infty f(x)\,dx| \leq \int_a^\infty g(x)\,dx$.

6.5 The Riemann-Stieltjes Integral

1. Motivation & Concept

The Riemann-Stieltjes integral unifies discrete and continuous summation into a single formula.

- **Physical Example (Moment of Inertia I):**
  - *Discrete:* For point masses m_i at r_i: $I = \sum r_i^2 m_i$.
  - *Continuous:* For a wire with density $\rho(x)$: $I = \int x^2 \rho(x) dx$.
  - *Unified:* Using a mass distribution function $m(x)$, both can be written as:

$$I = \int_0^l x^2 dm(x)$$

2. Definition of the Integral

Let α be a **monotone increasing** function on $[a, b]$ and f be a bounded real-valued function.

Partition and Sums

For a partition $\mathcal{P} = \{x_0, \dots, x_n\}$ of $[a, b]$:

- Define $\Delta\alpha_i$ instead of Δx_i:

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$$

- **Upper Sum:** $\mathcal{U}(\mathcal{P}, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$, where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$.
- **Lower Sum:** $\mathcal{L}(\mathcal{P}, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$, where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.

Integrability

f is Riemann-Stieltjes integrable with respect to α, denoted $f \in \mathcal{R}(\alpha)$, if the upper and lower integrals meet:

$$\underline{\int_a^b} f d\alpha = \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f, \alpha) = \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f, \alpha) = \overline{\int_a^b} f d\alpha$$

The common value is denoted $\int_a^b f d\alpha$.

> **Note:** If $\alpha(x) = x$, this reduces to the standard Riemann integral.

3. Key Examples (Existence & Non-Existence)

A. The Unit Jump Function (Discrete behavior)

Let $I_c(x)$ be the unit jump at c ($a < c \leq b$):

$$I_c(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

- **Result:** If f is continuous at c, then:

$$\int_a^b f dI_c = f(c)$$

- **Proof Idea:** For any partition where $c \in (x_{k-1}, x_k]$, only the k-th term has $\Delta\alpha_k \neq 0$ (specifically $\Delta\alpha_k = 1$). Thus sums collapse to M_k and m_k. Since f is continuous, as $\Delta x \to 0$, $M_k, m_k \to f(c)$.

B. The Dirichlet Function

Let $f(x) = 1$ if $x \in \mathbb{Q}$, and 0 if $x \notin \mathbb{Q}$.

- **Result:** Not integrable for any non-constant α.
- **Reason:** In every interval, density of rationals/irrationals implies $M_i = 1$ and $m_i = 0$. Thus $\mathcal{U} = \alpha(b) - \alpha(a)$ and $\mathcal{L} = 0$.

4. Conditions for Integrability

Theorem: Cauchy Criterion

$f \in \mathcal{R}(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that:

$$\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) < \epsilon$$

Theorem: Sufficient Classes for Integrability

1. **Continuous f:** If f is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha)$.
2. **Monotone f / Continuous α:** If f is monotone and α is **continuous**, then $f \in \mathcal{R}(\alpha)$.
   - *Proof Idea:* Since α is continuous, use the Intermediate Value Theorem to choose a partition where $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$. The telescoping sum of f then bounds the difference $U - L$.

5. Properties of the Integral

If $f, g \in \mathcal{R}(\alpha)$ and $c \in \mathbb{R}$:

1. **Linearity:** $\int(f+g)d\alpha = \int fd\alpha + \int gd\alpha$ and $\int cfd\alpha = c\int fd\alpha$.
2. **Additivity of α:** $\int fd(\alpha_1 + \alpha_2) = \int fd\alpha_1 + \int fd\alpha_2$.
3. **Domain Splitting:** $\int_a^b = \int_a^c + \int_c^b$.
4. **Comparison:** If $f(x) \leq g(x)$, then $\int fd\alpha \leq \int gd\alpha$.
5. **Boundedness:**

$$\left|\int_a^b fd\alpha\right| \leq \int_a^b |f|d\alpha \leq M[\alpha(b) - \alpha(a)]$$

   (where $|f(x)| \leq M$).

6. Fundamental Theorems

Mean Value Theorem for Integrals

If f is continuous and α is monotone increasing, there exists $c \in [a, b]$ such that:

$$\int_a^b fd\alpha = f(c)[\alpha(b) - \alpha(a)]$$

- *Proof Idea:* Use the Intermediate Value Theorem on the range of values bounded by $\inf f$ and $\sup f$.

Integration by Parts

If α and β are monotone increasing, then $\alpha \in \mathcal{R}(\beta) \iff \beta \in \mathcal{R}(\alpha)$.
Formula:

$$\int_a^b \alpha d\beta = \alpha(b)\beta(b) - \alpha(a)\beta(a) - \int_a^b \beta d\alpha$$

- *Proof Idea:* Relies on the summation identity for partitions:
  $\mathcal{U}(\mathcal{P}, \alpha, \beta) - \mathcal{L}(\mathcal{P}, \alpha, \beta) = \mathcal{U}(\mathcal{P}, \beta, \alpha) - \mathcal{L}(\mathcal{P}, \beta, \alpha)$.
  If the difference for one integral approaches 0, it must for the other.

7. Evaluating the Integral (Computational Theorems)

The Riemann-Stieltjes integral usually reduces to one of two forms for calculation:

Case A: α is a Step Function (Summation)

If $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$ where $\sum c_n$ converges and f is continuous:

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

- *Proof Idea:* Use the Unit Jump Example result and linearity. The infinite series case is handled by bounding the tail of the series.
- **Example:** $\int_0^2 e^x d[x]$ where $[x]$ is the floor function.
  - $[x]$ jumps at $x = 1$ and $x = 2$ within (0, 2].
  - Result: $e^1 + e^2$.

Case B: α is Differentiable (Reduction to Riemann)

If α is differentiable and $\alpha' \in \mathcal{R}[a, b]$ (Riemann integrable), then:

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$$

- *Proof Idea:* By MVT, $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$. The Riemann-Stieltjes sum $\sum f(s_i)\Delta\alpha_i$ approximates the Riemann sum $\sum f(s_i)\alpha'(t_i)\Delta x_i$.

- **Example:** $\int_0^1 \sin(\pi x) d(x^2) = \int_0^1 \sin(\pi x) \cdot (2x) dx$.

8. Riemann-Stieltjes Sums and Limits

A Riemann-Stieltjes sum is defined as $\mathcal{S}(\mathcal{P}, f, \alpha) = \sum f(t_i)\Delta\alpha_i$ where t_i are arbitrary tags in the intervals.

- **Warning:** Unlike the standard Riemann integral, the existence of the integral $\int f d\alpha$ does **not** guarantee that $\lim_{||\mathcal{P}|| \to 0} S(\mathcal{P}, f, \alpha)$ exists.
- **Counter-Example:**
  - $f(x) = 1$ for $x > 1$, 0 else.
  - $\alpha(x) = 1$ for $x \geq 1$, 0 else.
  - Integral exists (value 0 via continuity properties).
  - Sum limit does not exist: Depending on whether the tag t_k is chosen at 1 or slightly right of 1, the sum oscillates between 0 and 1.
- **Convergence Condition:** However, if f continuous and α monotone, the limit of sums *does* equal the integral.

7.1 Series of Real Numbers

1. Fundamentals of Series

Definitions

Let $\{a_k\}$ be a sequence of real numbers.

- **Infinite Series:** Denoted as $\sum_{k=1}^{\infty} a_k$.
- **Partial Sums:** The sequence of partial sums $\{s_n\}$ is defined as $s_n = \sum_{k=1}^{n} a_k$.
- **Convergence:** The series $\sum a_k$ converges to a sum s if and only if the sequence of partial sums $\{s_n\}$ converges to s (i.e., $\lim_{n\to\infty} s_n = s$).
- **Divergence:** If $\{s_n\}$ diverges, the series diverges. If $\lim s_n = \infty$, we write $\sum a_k = \infty$.

Linearity of Convergent Series (Theorem 7.1.1)

If $\sum a_k = \alpha$ and $\sum b_k = \beta$, then:

1. $\sum(c \cdot a_k) = c\alpha$ for any constant $c \in \mathbb{R}$.
2. $\sum(a_k + b_k) = \alpha + \beta$.

Proof Idea:
This follows directly from the limit laws for sequences. Since $s_n \to \alpha$ and $t_n \to \beta$, the partial sum of the combined series is $(s_n + t_n)$, which converges to $\alpha + \beta$.

2. Comparison Tests

These tests are primarily used for series with **non-negative terms** ($a_k \geq 0$).

Comparison Test (Theorem 7.1.2)

Suppose $0 \leq a_k \leq Mb_k$ for some constant $M > 0$ and all $k \geq k_0$.

1. If $\sum b_k$ **converges**, then $\sum a_k$ **converges**.
2. If $\sum a_k$ **diverges**, then $\sum b_k$ **diverges**.

Proof:

- **Convergence Case:** By the Cauchy Criterion, if $\sum b_k$ converges, the tail sum can be made arbitrarily small ($\sum_{k=m+1}^{n} b_k < \epsilon/M$). Since $a_k \leq M b_k$, the tail sum of a_k is bounded by ϵ. Thus, $\sum a_k$ converges.
- **Divergence Case:** Logical contrapositive of the above.

Limit Comparison Test (Corollary 7.1.3)

Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

1. If $0 < L < \infty$: $\sum a_k$ converges $\iff$ $\sum b_k$ converges.
2. If $L = 0$ and $\sum b_k$ converges: $\sum a_k$ converges.

Note: If $L = 0$ and $\sum a_k$ converges, nothing can be concluded about $\sum b_k$.

Examples (7.1.4)

1. **Series:** $\sum \frac{k}{3^k}$.
   - **Compare with:** $\sum (1/2)^k$ (Geometric series).
   - **Logic:** Since $\lim k(2/3)^k = 0$, eventually $\frac{k}{3^k} \leq \frac{1}{2^k}$. Since $\sum (1/2)^k$ converges, the original series **converges**.
2. **Series:** $\sum \sqrt{\frac{k+1}{2k^3+1}}$.
   - **Compare with:** $\sum \frac{1}{k}$ (Harmonic series, divergent).
   - **Logic:** As $k \to \infty$, terms behave like $\sqrt{\frac{k}{2k^3}} \approx \frac{1}{k\sqrt{2}}$. Using the Limit Comparison Test, the limit of the ratio is $\frac{\sqrt{2}}{2} > 0$. Since $\sum \frac{1}{k}$ diverges, the original series **diverges**.
3. **Counter-example ($L = 0$):**
   - Let $a_k = 2^{-k}$ and $b_k = 1/k$.
   - $\lim(a_k/b_k) = 0$. $\sum a_k$ converges, but $\sum b_k$ diverges.

3. Integral Test

Theorem 7.1.5

Let f be a function on $[1, \infty)$ that is **non-negative** and **monotone decreasing**, such that $f(k) = a_k$. Then:

$$\sum_{k=1}^{\infty} a_k < \infty \iff \int_1^{\infty} f(x)\,dx < \infty$$

Proof Idea (Geometric intuition):
We approximate the area under $f(x)$ using rectangles of width 1.

- **Upper Bound:** The sum $\sum_{k=2}^{n} a_k$ represents the "lower" rectangular approximation and is less than the integral $\int_1^n f(x)dx$.
- **Lower Bound:** The sum $\sum_{k=1}^{n-1} a_k$ represents the "upper" rectangular approximation and is greater than the integral.
- Thus, the series and the integral bound each other; if one is finite, so is the other.

Examples (7.1.6)

1. **p-series:** $\sum_{k=1}^{\infty} \frac{1}{k^p}$
   - Evaluate $\int_1^{\infty} x^{-p} dx$.
   - **If $p > 1$:** Integral converges $\implies$ Series **Converges**.
   - **If $p \leq 1$:** Integral diverges ($\ln x$ or x^{1-p}) $\implies$ Series **Diverges**.
2. **Logarithmic Series:** $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$
   - Function $f(x) = \frac{1}{x \ln x}$ is decreasing.
   - $\int_2^c \frac{1}{x \ln x} dx = \ln(\ln c) - \ln(\ln 2)$.
   - As $c \to \infty$, integral $\to \infty$. Series **Diverges**.
3. **Mixed p-log series:** $\sum \frac{\ln k}{k^p}$
   - **$p = 1$:** Integral of $\frac{\ln x}{x}$ is $\frac{1}{2}(\ln x)^2 \to \infty$. **Diverges**.
   - **$p > 1$:** Write $p = q + r$ with $q > 1, r > 0$. Comparing with convergent $\sum \frac{1}{k^q}$ shows **Convergence**.
   - **$p < 1$:** Compares with divergent $\sum \frac{1}{k^p}$. **Diverges**.

4. Root and Ratio Tests

These tests are crucial for power series. We use Limit Superior ($\limsup$) and Limit Inferior ($\liminf$) for broader applicability.

Ratio Test (Theorem 7.1.7)

Let $\sum a_k$ be a series of positive terms. Define:

$$R = \limsup_{k \to \infty} \frac{a_{k+1}}{a_k}, \quad r = \liminf_{k \to \infty} \frac{a_{k+1}}{a_k}$$

1. If $R < 1$: Series **Converges**.
2. If $r > 1$: Series **Diverges**.
3. If $r \le 1 \le R$: **Inconclusive**.

Proof Idea (Convergence case):
If $R < 1$, choose a constant c such that $R < c < 1$. Eventually, the ratio $\frac{a_{n+1}}{a_n} < c$. This implies $a_{n+m} < a_n c^m$. The series behaves like a geometric series $\sum c^n$ (which converges because $c < 1$). By Comparison Test, $\sum a_n$ converges.

Root Test (Theorem 7.1.8)

Let $\sum a_k$ be a series of non-negative terms. Define:

$$\alpha = \limsup_{k \to \infty} \sqrt[k]{a_k}$$

1. If $\alpha < 1$: Series **Converges**.
2. If $\alpha > 1$: Series **Diverges**.
3. If $\alpha = 1$: **Inconclusive**.

Proof Idea:
Similar to Ratio Test. If $\alpha < 1$, choose c such that $\alpha < c < 1$. Eventually $\sqrt[n]{a_n} < c$, meaning $a_n < c^n$. Compare with convergent geometric series $\sum c^n$.
If $\alpha > 1$, then $a_n > 1$ for infinitely many n, so terms do not approach 0. Diverges.

Hierarchy of Tests (Theorem 7.1.10)

For any sequence of positive numbers:

$$\liminf \frac{a_{n+1}}{a_n} \le \liminf \sqrt[n]{a_n} \le \limsup \sqrt[n]{a_n} \le \limsup \frac{a_{n+1}}{a_n}$$

Significance: The Root Test is strictly stronger than the Ratio Test. If the Ratio Test works, the Root Test works. However, there are cases where the Ratio Test fails but the Root Test succeeds.

Examples (7.1.9)

1. **p-series ($\sum 1/k^p$):**

- Ratio and Root limits are both 1. Both tests are **Inconclusive**. (We know result depends on p, but these tests cannot detect it).

2. **Factorial Series ($\sum p^k/k!$):**
   - Ratio: $\frac{a_{k+1}}{a_k} = \frac{p}{k+1} \to 0$.
   - $R = 0 < 1$. Series **Converges** for all p. (Root test is hard to apply here due to factorial).
3. **Mixed Sequence (Root > Ratio):**
   - $a_n = 1/2^k$ (if $n = 2k$) and $a_n = 1/3^k$ (if $n = 2k + 1$).
   - **Ratio Test:** Oscillates. $\liminf \frac{a_{n+1}}{a_n} = 0$, $\limsup \frac{a_{n+1}}{a_n} = \infty$. **Inconclusive**.
   - **Root Test:** Subsequential limits of $\sqrt[n]{a_n}$ are $1/\sqrt{2}$ and $1/\sqrt{3}$.
   - $\alpha = \limsup \sqrt[n]{a_n} = 1/\sqrt{2} \approx 0.707 < 1$.
   - **Result:** Series **Converges** by Root Test.

7.2 The Dirichlet Test and Applications

1. Abel Partial Summation Formula

This formula is the discrete analogue of **Integration by Parts**. It is the fundamental tool required to prove the Dirichlet Test.

Theorem 7.2.1
Let $\{a_k\}$ and $\{b_k\}$ be sequences of real numbers.
Define partial sums $A_0 = 0$ and $A_n = \sum_{k=1}^{n} a_k$ for $n \geq 1$.
If $1 \leq p \leq q$, then:

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p$$

Proof (Key Idea):
We substitute a_k using the difference of partial sums: $a_k = A_k - A_{k-1}$.

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} (A_k - A_{k-1}) b_k = \sum_{k=p}^{q} A_k b_k - \sum_{k=p}^{q} A_{k-1} b_k$$

By shifting the index of the second summation (letting $j = k - 1$), we align the terms to factor out A_k. The middle terms collapse into the form $A_k(b_k - b_{k+1})$, leaving the boundary terms $A_q b_q$ and $-A_{p-1} b_p$.

2. The Dirichlet Test

This test provides convergence criteria for series of the form $\sum a_k b_k$, usually where one part oscillates and the other decays.

Theorem 7.2.2
The series $\sum_{k=1}^{\infty} a_k b_k$ converges if the sequences satisfy three conditions:

1. **Bounded Partial Sums:** The sequence $A_n = \sum_{k=1}^{n} a_k$ is bounded (i.e., $|A_n| \leq M$ for some $M > 0$).
2. **Monotonicity:** $\{b_k\}$ is decreasing ($b_1 \geq b_2 \geq \cdots \geq 0$).
3. **Limit Zero:** $\lim_{k\to\infty} b_k = 0$.

Proof (Key Idea):
We use the **Cauchy Criterion**. We need to show that for large enough p, q, the tail sum $|\sum_{k=p}^{q} a_k b_k|$ is arbitrarily small.
Using Abel's Formula and the triangle inequality:

$$\left|\sum_{k=p}^{q} a_k b_k\right| \leq \sum_{k=p}^{q-1} |A_k|(b_k - b_{k+1}) + |A_q|b_q + |A_{p-1}|b_p$$

Since $|A_k| \leq M$ and terms $(b_k - b_{k+1})$ are non-negative (because b_k is decreasing):

$$\leq M \left[\sum_{k=p}^{q-1}(b_k - b_{k+1}) + b_q + b_p\right]$$

The summation inside is a telescoping sum that simplifies to $b_p - b_q$. Thus, the expression simplifies to $2Mb_p$. Since $b_p \to 0$, this value becomes smaller than any ϵ, proving convergence.

3. Application I: Alternating Series

The most common application of the Dirichlet Test is for alternating series.

Theorem 7.2.3 (Alternating Series Test)
If $\{b_k\}$ satisfies $b_1 \geq b_2 \geq \cdots \geq 0$ and $\lim_{k\to\infty} b_k = 0$, then:

$$\sum_{k=1}^{\infty} (-1)^{k+1} b_k \quad \text{converges.}$$

Proof:
Set $a_k = (-1)^{k+1}$. The partial sums A_n alternate between 1 and 0. Thus, $|A_n| \leq 1$ for all n. Since partial sums are bounded and $\{b_k\}$ decreases to zero, the Dirichlet Test applies directly.

Error Estimation

For alternating series, we can easily bound the error between the limit s and the partial sum s_n.

Theorem 7.2.4
Let $s = \sum_{k=1}^{\infty} (-1)^{k+1} b_k$ and s_n be the n-th partial sum. Then:

$$|s - s_n| \leq b_{n+1}$$

Proof (Key Idea):

- Consider the even partial sums $\{s_{2n}\}$. Because $s_{2n} = (b_1 - b_2) + \cdots + (b_{2n-1} - b_{2n})$ and terms are decreasing, $\{s_{2n}\}$ is **increasing**.
- Similarly, $\{s_{2n+1}\}$ is **decreasing**.
- Since the series converges to s, the limit is "trapped" between consecutive partial sums: $s_{2n} \leq s \leq s_{2n+1}$.
- The distance between consecutive sums is $|s_{k+1} - s_k| = b_{k+1}$. Therefore, the distance from s_k to s cannot exceed b_{k+1}.

Example 7.2.5
Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$.

- Since $\left\{\frac{1}{2k-1}\right\}$ decreases to 0, the series converges.
- (Note: It converges to $\pi/4$).
- **Error:** By Theorem 7.2.4, $\left|\frac{\pi}{4} - s_n\right| \leq \frac{1}{2n+1}$.
- **Result:** Convergence is very slow. To get 2 decimal places of accuracy (< 0.01), we need $2n + 1 > 100$, implying $n \approx 50$.

4. Application II: Trigonometric Series

We examine series of the form $\sum b_k \sin(kt)$ and $\sum b_k \cos(kt)$.

Theorem 7.2.6
Let $\{b_k\}$ be a sequence where $b_1 \geq b_2 \geq \cdots \geq 0$ and $\lim_{k\to\infty} b_k = 0$.

1. **Sine Series:** $\sum_{k=1}^{\infty} b_k \sin(kt)$ converges for all $t \in \mathbb{R}$.
2. **Cosine Series:** $\sum_{k=1}^{\infty} b_k \cos(kt)$ converges for all $t \in \mathbb{R}$, **except** possibly where $t = 2p\pi$ for integers p.

Proof (Key Idea):
We must prove that the partial sums of the trigonometric parts ($a_k = \sin kt$ or $\cos kt$) are bounded.
Key Identity for partial sums of sine (for $t \neq 2p\pi$):

$$A_n = \sum_{k=1}^{n} \sin kt = \frac{\cos \frac{1}{2}t - \cos(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t}$$

Bounding A_n:

$$|A_n| \leq \frac{|\cos \frac{1}{2}t| + |\cos(n + \frac{1}{2})t|}{2|\sin \frac{1}{2}t|} \leq \frac{2}{2|\sin \frac{1}{2}t|} = \frac{1}{|\sin \frac{1}{2}t|}$$

This is a finite constant for any fixed $t \neq 2p\pi$.

- Since A_n is bounded and $b_k \downarrow 0$, the **Dirichlet Test** proves convergence.
- **Case $t = 2p\pi$:**
  - For sine: $\sin(k \cdot 2p\pi) = 0$. The sum is $\sum 0$, which converges.
  - For cosine: $\cos(k \cdot 2p\pi) = 1$. The sum becomes $\sum b_k$. Since we only know $b_k \to 0$, this might diverge (e.g., if $b_k = 1/k$).

Example 7.2.7

1. **Series:** $\sum_{k=1}^{\infty} \frac{1}{k} \cos kt$
   - Converges for all $t \neq 2p\pi$ (by Theorem 7.2.6).
   - If $t = 2p\pi$, series becomes $\sum \frac{1}{k}$ (Harmonic Series), which **diverges**.
2. **Series:** $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos kt$
   - Converges for all $t \in \mathbb{R}$.
   - Even at $t = 2p\pi$, it becomes $\sum \frac{1}{k^2}$, which converges (p-series, $p = 2$).

Here are concise, comprehensive study notes based on the provided text, focusing on definitions, key theorems, and the logical structure of proofs.

Study Notes: Series of Real Numbers

1. Absolute and Conditional Convergence

Definitions

- **Absolute Convergence:** A series $\sum a_k$ is absolutely convergent if the series of absolute values $\sum |a_k|$ converges.
- **Conditional Convergence:** A series is conditionally convergent if $\sum a_k$ converges, but $\sum |a_k|$ diverges.

Examples

1. **Alternating Harmonic Series:** $\sum \frac{(-1)^{k+1}}{k}$
    - Converges (by Alternating Series Test).
    - However, $\sum \left|\frac{(-1)^{k+1}}{k}\right| = \sum \frac{1}{k} = \infty$ (Harmonic series diverges).
    - $\therefore$ **Conditionally Convergent**.
2. **Inverse Square Alternating Series:** $\sum \frac{(-1)^{k+1}}{k^2}$
    - $\sum \frac{1}{k^2} < \infty$.
    - $\therefore$ **Absolutely Convergent**.

Theorem 7.3.3: Absolute Implies Convergence

Statement: Every absolutely convergent series converges.
Proof Idea:

- Assume $\sum |a_k| < \infty$.
- By the **Triangle Inequality**: $\left|\sum_{k=p}^{q} a_k\right| \leq \sum_{k=p}^{q} |a_k|$.
- Since $\sum |a_k|$ converges, it satisfies the Cauchy Criterion (the tail $\sum_{k=p}^{q} |a_k|$ becomes arbitrarily small).
- Therefore, $\left|\sum_{k=p}^{q} a_k\right|$ also becomes arbitrarily small.
- By the Cauchy Criterion for series, $\sum a_k$ converges.

2. Testing for Absolute Convergence

To test $\sum a_k$ for absolute convergence, apply standard tests (Section 7.1) to $\sum |a_k|$.

Theorem 7.3.4: Root and Ratio Tests

Let $\alpha = \limsup_{k\to\infty} \sqrt[k]{|a_k|}$ and $R = \limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$, $r = \liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$.

1. **Convergence:** If $\alpha < 1$ or $R < 1$, the series is **absolutely convergent**.
2. **Divergence:** If $\alpha > 1$ or $r > 1$, the series **diverges** (not just fails absolute convergence, but fails to converge entirely).
    - *Reasoning:* If limit > 1, the terms $|a_k|$ grow or do not approach 0. If $\lim a_k \neq 0$, the series diverges.
3. **Inconclusive:** If $\alpha = 1$ or $r \leq 1 \leq R$, the test gives no information.

3. Rearrangements of Series

Definition

A series $\sum a'_k$ is a **rearrangement** of $\sum a_k$ if there exists a one-to-one bijection $j : \mathbb{N} \to \mathbb{N}$ such that $a'_k = a_{j(k)}$.

- *Essence:* All terms of the original series appear exactly once, just in a different order.

The Critical Question

Does rearranging the terms change the sum?

- **Absolutely Convergent Series:** No.
- **Conditionally Convergent Series: Yes.**

Example of Failure (Conditional Convergence)

Consider $\sum \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$ (Sums to s).

- Rearrange to take two positive terms for every one negative term:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \dots$$

- The positive terms accumulate faster than the negative terms reduce the sum.
- The new sum s' can be shown to be strictly greater than the original sum ($s' > s$).

4. Convergence Theorems regarding Rearrangement

Theorem 7.3.8: Rearrangement of Absolute Series

Statement: If $\sum a_k$ converges absolutely, then *every* rearrangement converges to the same sum.

Proof Idea (Internal Logic):

1. **Tail Control:** Since $\sum |a_k|$ converges, for any ϵ, we can choose N such that the sum of absolute values of the "tail" (terms after index N) is $< \epsilon$.
2. **Matching Terms:** Let s_n be the partial sum of the original and s'_n be the partial sum of the rearrangement. Choose an index p large enough so that the first p terms of the rearrangement include all the first N terms of the original series.
3. **Cancellation:** In the difference $|s_n - s'_n|$ (for large n), the first N terms cancel out perfectly.
4. **Bound Remainder:** The remaining terms belong to the "tail" (indices $> N$). Since the sum of the absolute values of the tail is bounded by ϵ, the difference $|s_n - s'_n|$ is negligible. The limits must be identical.

Theorem 7.3.9: Riemann’s Rearrangement Theorem

Statement: If $\sum a_k$ is **conditionally convergent**, then for any real number α, there exists a rearrangement $\sum a'_k$ that converges to α.

Proof Construction (The "Greedy" Algorithm):

1. **Decomposition:** Separate the series into positive terms (P_k) and negative terms (Q_k).
   - Because the series is *conditionally* convergent:
     - $\sum P_k = \infty$ (Positive part diverges)
     - $\sum Q_k = \infty$ (Negative part diverges)
     - $P_k \to 0$ and $Q_k \to 0$ (Terms vanish)
     - *Note:* If these didn't diverge, the series would be absolutely convergent or divergent.

2. **Algorithm to target α:**
   - **Step 1:** Add enough positive terms P_k until the partial sum just exceeds α. (Possible because $\sum P_k = \infty$).
   - **Step 2:** Subtract enough negative terms Q_k until the partial sum just drops below α. (Possible because $\sum Q_k = \infty$).
   - **Step 3:** Repeat indefinitely.
3. **Convergence:**
   - At each step, we overshoot or undershoot α by at most the value of the single term we just added/subtracted.
   - Since $a_k \to 0$ (the individual terms approach zero), the size of these overshoots/undershoots approaches 0.
   - Therefore, the partial sums converge to α.

Remark: This theorem implies conditionally convergent series are unstable regarding order; one can even rearrange them to diverge to $\pm\infty$.

Here are the structured study notes based on the provided text, focusing on **Square Summable Sequences** and **Normed Linear Spaces**.

1. The Space l^2 (Square Summable Sequences)

Definition

The set l^2 consists of all sequences of real numbers $\{a_k\}_{k=1}^{\infty}$ such that the sum of their squares converges.

$$l^2 = \left\{ \{a_k\} : \sum_{k=1}^{\infty} a_k^2 < \infty \right\}$$

The Norm

For a sequence $\{a_k\} \in l^2$, the norm (magnitude) is defined as:

$$||\{a_k\}||_2 = \sqrt{\sum_{k=1}^{\infty} a_k^2}$$

Examples of Convergence in l^2

- **Case 1:** $\{1/k\}$.
  The norm squared is $\sum (1/k)^2 = \sum 1/k^2$. Since this is a p-series with $p = 2$, it converges.
  - $\therefore \{1/k\} \in l^2$.
- **Case 2:** $\{1/\sqrt{k}\}$.
  The norm squared is $\sum (1/\sqrt{k})^2 = \sum 1/k$. This is the harmonic series (diverges).
  - $\therefore \{1/\sqrt{k}\} \notin l^2$.
- **General Case:** $\{1/k^q\}$ for fixed q.
  The series is $\sum 1/k^{2q}$. By p-series test, this converges if $2q > 1$.
  - $\therefore \{1/k^q\} \in l^2 \iff q > 1/2$.

2. The Cauchy-Schwarz Inequality

This is the fundamental inequality allowing us to define angles and geometry in infinite-dimensional spaces.

Finite Version (Theorem 7.4.3)

For real numbers $a_1, ..., a_n$ and $b_1, ..., b_n$:

$$\sum_{k=1}^{n} |a_k b_k| \le \sqrt{\sum_{k=1}^{n} a_k^2} \sqrt{\sum_{k=1}^{n} b_k^2}$$

Proof (Key Idea):

We construct a non-negative quadratic function.

1. Let $\lambda \in \mathbb{R}$. Consider the square of a linear combination:

$$0 \le \sum_{k=1}^{n} (|a_k| - \lambda |b_k|)^2$$

2. Expand the square:

$$0 \le \sum a_k^2 - 2\lambda \sum |a_k b_k| + \lambda^2 \sum b_k^2$$

   Let $A = \sum a_k^2$, $B = \sum b_k^2$, and $C = \sum |a_k b_k|$.

$$0 \le A - 2\lambda C + \lambda^2 B$$

3. **Optimization:** To minimize the expression, choose $\lambda = C/B$ (assuming $B \neq 0$; if $B = 0$, the inequality is trivial).

$$0 \le A - 2\left(\frac{C}{B}\right) C + \left(\frac{C}{B}\right)^2 B \implies 0 \le A - \frac{C^2}{B}$$

$$C^2 \le AB \implies C \le \sqrt{A}\sqrt{B}$$

   This yields the desired inequality.

Infinite Version (Corollary 7.4.4)

If $\{a_k\}, \{b_k\} \in l^2$, then the series $\sum a_k b_k$ converges absolutely and:

$$\sum_{k=1}^{\infty} |a_k b_k| \le ||\{a_k\}||_2 \cdot ||\{b_k\}||_2$$

Proof: Apply the finite theorem to partial sums S_n and let $n \to \infty$.

Inner Product

Based on this, we define the inner product of two sequences in l^2 as:

$$\langle a, b \rangle = \sum_{k=1}^{\infty} a_k b_k$$

Consequently: $|\langle a, b \rangle| \leq ||a||_2 ||b||_2$.

3. Minkowski's Inequality (Triangle Inequality)

This theorem proves that the "length" of a sum is less than the sum of the "lengths."

Theorem 7.4.5

If $\{a_k\}, \{b_k\} \in l^2$, then their sum $\{a_k + b_k\} \in l^2$ and:

$$||\{a_k + b_k\}||_2 \leq ||\{a_k\}||_2 + ||\{b_k\}||_2$$

Proof (Key Idea):

1. Start with the square of the sum term: $(a_k + b_k)^2$.
2. Expand and apply absolute values:

$$(a_k + b_k)^2 = a_k^2 + 2a_k b_k + b_k^2 \leq a_k^2 + 2|a_k b_k| + b_k^2$$

3. Sum over all k:

$$||\{a + b\}||_2^2 \leq ||a||_2^2 + 2\sum |a_k b_k| + ||b||_2^2$$

4. **Apply Cauchy-Schwarz** to the middle term:

$$||\{a + b\}||_2^2 \leq ||a||_2^2 + 2||a||_2||b||_2 + ||b||_2^2$$

5. Recognize the perfect square on the right side:

$$||\{a + b\}||_2^2 \leq (||a||_2 + ||b||_2)^2$$

6. Taking the square root proves the theorem.

4. Normed Linear Spaces

The space l^2 is an example of a broader algebraic structure called a Normed Linear Space.

Vector Space Definition
A set X is a vector space over $\mathbb{R}$ if it satisfies standard axioms (commutativity, associativity, existence of zero element and additive inverses, distributivity of scalar multiplication).

- l^2 is a vector space where addition is component-wise: $\{a_k\} + \{b_k\} = \{a_k + b_k\}$.
- The zero element is the sequence of all zeros.

Norm Definition (Def 7.4.8)
Let X be a vector space. A function $||\cdot|| : X \to \mathbb{R}$ is a **norm** if it satisfies four properties for all $x, y \in X$ and $c \in \mathbb{R}$:

1. **Non-negativity:** $||x|| \geq 0$.
2. **Zero property:** $||x|| = 0 \iff x = 0$.
3. **Homogeneity:** $||cx|| = |c| \cdot ||x||$.
4. **Triangle Inequality:** $||x + y|| \leq ||x|| + ||y||$.

Verification for l^2 (Theorem 7.4.6)

- Properties 1, 2 are obvious from the definition of the sum of squares.
- Property 3 is easily proven by factoring out constants from the series.
- Property 4 is exactly Minkowski's Inequality.
- Therefore, $(l^2, ||\cdot||_2)$ is a normed linear space.

Distance and Convergence

- **Distance (Metric):** Defined as $d(x, y) = ||x - y||$. This satisfies metric space properties (positivity, symmetry, triangle inequality).
- **Norm Convergence:** A sequence of vectors $\{x_n\}$ converges to x if:

$$\lim_{n \to \infty} ||x_n - x|| = 0$$

  This is analogous to standard convergence in $\mathbb{R}$, but care must be taken (e.g., the Bolzano-Weierstrass theorem fails in infinite-dimensional spaces like l^2).

Here are the organized notes on **Section 8.1: Pointwise Convergence and Interchange of Limits**, based on the provided text.

1. Definitions: Sequences and Series of Functions

Pointwise Convergence of Sequences

Let $E \subset X$ be a metric space (e.g., $E \subseteq \mathbb{R}$). A sequence of real-valued functions $\{f_n\}$ defined on E **converges pointwise** to a function f on E if the sequence of real numbers $\{f_n(x)\}$ converges for every $x \in E$.

- **Notation:** $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$.
- **Formal Definition ($\epsilon - n_0$):**
  The sequence converges pointwise to f if for every $x \in E$ and given $\epsilon > 0$, there exists a positive integer n_0 such that:

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } n \geq n_0$$

  - *Crucial Note:* The integer n_0 depends on both ϵ and x. Notation: $n_0 = n_0(x, \epsilon)$.

Pointwise Convergence of Series

For a sequence $\{f_n\}$, we define the sequence of partial sums $\{S_n\}$ by:

$$S_n(x) = \sum_{k=1}^{n} f_k(x)$$

The series $\sum_{k=1}^{\infty} f_k$ converges pointwise on E if the sequence $\{S_n(x)\}$ converges for each $x \in E$. The sum is denoted as:

$$S(x) = \sum_{k=1}^{\infty} f_k(x)$$

2. The Interchange of Limits Questions

Historically, mathematicians (including Cauchy) incorrectly believed that properties of f_n (like continuity) automatically transferred to the limit function f. The text investigates three major questions regarding the validity of interchanging limits:

1. **Continuity:** If each f_n is continuous at p, is $f = \lim f_n$ continuous at p?
    - *Mathematically:* Does $\lim_{t \to p}(\lim_{n \to \infty} f_n(t)) = \lim_{n \to \infty}(\lim_{t \to p} f_n(t))$?
2. **Differentiability:** If each f_n is differentiable, is f differentiable?
    - *Mathematically:* Does $f'(p) = \lim_{n \to \infty} f_n'(p)$?
3. **Integrability:** If each f_n is Riemann integrable, is f integrable? And do the integrals converge?
    - *Mathematically:* Does $\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$?

General Answer: No. Pointwise convergence is insufficient to preserve these properties.

3. Counter-Examples (Proofs by Example)

The following examples demonstrate why the answer to the above questions is "No."

(a) Continuity is NOT preserved

- **Function:** $f_n(x) = x^n$ on domain $E = [0, 1]$.
- **Properties of f_n:** Each f_n is continuous on $[0, 1]$.
- **Pointwise Limit (f):**
    - If $0 \leq x < 1$, $x^n \to 0$.
    - If $x = 1$, $1^n \to 1$.

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

- **Conclusion:** The limit function f is **discontinuous** at $x = 1$, despite all f_n being continuous.

(b) Continuity is NOT preserved (Series Example)

- **Function:** Series defined by partial sums of $f_k(x) = \frac{x^2}{(1+x^2)^k}$ on $\mathbb{R}$.
- **Sum Calculation:**
    - If $x = 0$: All terms are 0, so Sum $= 0$.
    - If $x \neq 0$: This is a geometric series with ratio $r = \frac{1}{1+x^2} < 1$.

$$\sum_{k=0}^{\infty} x^2 \left(\frac{1}{1+x^2} \right)^k = x^2 \left[\frac{1}{1 - \frac{1}{1+x^2}} \right] = x^2 \left[\frac{1+x^2}{x^2} \right] = 1 + x^2$$

- **Limit Function:**

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 + x^2, & x \neq 0 \end{cases}$$

- **Conclusion:** The sum function f is **discontinuous** at $x = 0$.

(c) Riemann Integrability is NOT preserved

- **Function:** Let $\{x_k\}$ be an enumeration of rationals in $[0, 1]$.

$$f_n(x) = \begin{cases} 1, & \text{if } x = x_k \text{ for } 1 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

- **Properties of f_n:** Each f_n is continuous except at finite points ($x_1 ... x_n$), so f_n is Riemann integrable with $\int_0^1 f_n = 0$.
- **Pointwise Limit (f):** As $n \to \infty$, f becomes the Dirichlet function:

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \text{ (rational)} \\ 0, & \text{if } x \notin \mathbb{Q} \text{ (irrational)} \end{cases}$$

- **Conclusion:** The limit function f is **not Riemann integrable** (it is nowhere continuous).

(d) Integral Value is NOT preserved ($\lim \int \neq \int \lim$)

- **Function:** $f_n(x) = nx(1 - x^2)^n$ on $[0, 1]$.
- **Properties of f_n:** Continuous and integrable.
- **Pointwise Limit (f):**
  - For $0 < x < 1$: limit is 0 (exponential decay dominates linear growth).
  - For $x = 0$ or $x = 1$: $f_n(x) = 0$.
  - Thus, $f(x) = 0$ for all x, and $\int_0^1 f(x)dx = 0$.
- **Sequence of Integrals:**
  Using substitution ($u = 1 - x^2$):

$$\int_0^1 f_n(x)dx = \int_0^1 nx(1 - x^2)^n dx = \frac{n}{2} \int_0^1 u^n du = \frac{n}{2(n+1)}$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{n}{2n + 2} = \frac{1}{2}$$

- **Conclusion:**

$$\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 (\lim_{n\to\infty} f_n) \quad \left(\frac{1}{2} \neq 0\right)$$

(e) Differentiability is NOT preserved

- **Function:** $f_n(x) = \frac{\sin(nx)}{n}$ on $\mathbb{R}$.
- **Pointwise Limit (f):** Since $|\sin(nx)| \leq 1$, $|f_n(x)| \leq 1/n$. Thus $f(x) = 0$.
  - The derivative of the limit is $f'(x) = 0$.
- **Sequence of Derivatives:**

$$f_n'(x) = \cos(nx)$$

At $x = 0$, $f_n'(0) = \cos(0) = 1$.

$$\lim_{n\to\infty} f_n'(0) = 1$$

- **Conclusion:**

$$\frac{d}{dx}(\lim_{n\to\infty} f_n(x)) \neq \lim_{n\to\infty} f_n'(x) \quad (0 \neq 1)$$

Here are detailed study notes on **Uniform Convergence**, **The Cauchy Criterion**, and the **Weierstrass M-Test**, based on the provided text.

1. Pointwise vs. Uniform Convergence

To preserve properties like continuity or to interchange limit operations (integrals, derivatives), pointwise convergence is often insufficient. Stronger conditions are required.

Pointwise Convergence (Recap)

A sequence $\{f_n\}$ converges pointwise to f on a set E if for every $x \in E$ and $\epsilon > 0$, there exists an integer n_0 such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq n_0$.

- **Crucial Limitation:** The integer n_0 depends on both ϵ **and** x.
- Notation: $n_0 = n_0(x, \epsilon)$.

Uniform Convergence (Definition 8.2.1)

A sequence $\{f_n\}$ converges **uniformly** to f on E if for every $\epsilon > 0$, there exists an integer n_0 such that:

$$|f_n(x) - f(x)| < \epsilon$$

for **all** $x \in E$ and all $n \geq n_0$.

- **Key Distinction:** The integer n_0 depends **only** on ϵ, not on x. The dependence on x is removed.
- **Geometric Interpretation:** For $n \geq n_0$, the entire graph of $f_n(x)$ lies within a "tube" of width 2ϵ centered around $f(x)$.

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon$$

2. Analysis of Specific Examples

Example A: $f_n(x) = x^n$ on $[0, 1]$

- **Pointwise Limit:**

$$f(x) = \lim_{n \to \infty} x^n = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

The limit function f is discontinuous, even though each f_n is continuous.

- **Uniform Convergence Test:**
  If convergence were uniform, there would exist n_0 such that $|x^{n_0} - 0| < \epsilon$ for all $x \in [0, 1)$.
  However, if we pick $\epsilon < 1$, the inequality $x^{n_0} < \epsilon$ fails as x approaches 1.
  - **Conclusion:** Convergence is **not uniform** on $[0, 1]$.
  - **Note:** Convergence *is* uniform on $[0, a]$ for any fixed $0 < a < 1$, because $|x^n| \le a^n$, and $a^n \to 0$ independent of x.

Example B: The "Sliding Hump" ($S_n(x) = nxe^{-nx^2}$)

Consider the partial sums $S_n(x) = nxe^{-nx^2}$ on $[0, 1]$.

- **Pointwise Limit:** For any fixed x, $\lim_{n \to \infty} S_n(x) = 0$. The limit function is $S(x) = 0$.
- **Uniform Convergence Test:**
  We examine the maximum value of the function to see if it stays within ϵ distance of 0.
  Using the derivative, S_n has a maximum at $x_n = \sqrt{1/2n}$.

$$M_n = \max_{x \in [0,1]} S_n(x) = S_n\left(\sqrt{\frac{1}{2n}}\right) = \sqrt{\frac{n}{2e}}$$

* **Conclusion:** Since $M_n \to \infty$ as $n \to \infty$, the graph does not flatten out into an ϵ-strip. The convergence is **not uniform**.

- **Terminology:** These are called "sliding-hump" functions because the peak moves toward 0 (slides) but grows infinitely high (hump) as n increases.

3. Criteria for Uniform Convergence

The Supremum Test (Theorem 8.2.5)

This is the most practical method for determining uniform convergence.
Let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$.
The sequence $\{f_n\}$ converges uniformly to f on E if and only if:

$$\lim_{n\to\infty} M_n = 0$$

- **Application to Example A (x^n):** On $[0,1]$, $\sup |x^n - f(x)| = 1$ (near $x = 1$). Since limit is 1, not 0, it is not uniform.
- **Application to Example B (nxe^{-nx^2}):** On $[0,\infty)$, $M_n = \sqrt{n/2e} \to \infty$. Not uniform. However, on $[a,\infty)$ for fixed $a > 0$, the maximum eventually occurs at the endpoint a, and decreases to 0. Thus, it is uniform on $[a,\infty)$.

The Cauchy Criterion (Theorem 8.2.3)

A sequence $\{f_n\}$ converges uniformly on E if and only if for every $\epsilon > 0$, there exists n_0 such that:

$$|f_n(x) - f_m(x)| < \epsilon$$

for all $x \in E$ and all $n, m \geq n_0$.

Proof Idea:

1. **Forward ($\Rightarrow$):** Follows from the triangle inequality, similar to standard Cauchy sequences.
2. **Reverse ($\Leftarrow$):**
   - Fix x. The sequence $\{f_n(x)\}$ is a Cauchy sequence of real numbers, so it converges to some value y. Define $f(x) = y$.
   - To prove uniformity, take the limit as $m \to \infty$ in the Cauchy inequality $|f_n(x) - f_m(x)| < \epsilon$. This yields $|f_n(x) - f(x)| \leq \epsilon$, valid for all x, proving uniform convergence.

Corollary for Series (8.2.4):
A series $\sum f_k$ converges uniformly on E if and only if for large enough n, m:

$$|\sum_{k=n+1}^{m} f_k(x)| < \epsilon \quad \forall x \in E$$

4. The Weierstrass M-Test

A powerful tool for proving the uniform convergence of series.

Theorem 8.2.7

Suppose $\{f_k\}$ is defined on E and there exists a sequence of constants $\{M_k\}$ such that:

1. $|f_k(x)| \le M_k$ for all $x \in E$ and $k \in \mathbb{N}$.
2. $\sum_{k=1}^{\infty} M_k < \infty$ (The series of constants converges).

Then, $\sum_{k=1}^{\infty} f_k(x)$ converges **uniformly** and **absolutely** on E.

Proof Idea:
Let $S_n(x)$ be the partial sums. For $n > m$:

$$|S_n(x) - S_m(x)| = \left| \sum_{k=m+1}^{n} f_k(x) \right| \le \sum_{k=m+1}^{n} |f_k(x)| \le \sum_{k=m+1}^{n} M_k$$

Since $\sum M_k$ converges, the tail of the series approaches 0. By the Cauchy Criterion, $\sum f_k$ converges uniformly.

Examples of M-Test

1. **Trigonometric Series:** $\sum \frac{\cos kx}{k^p}$ for $p > 1$.
   - Bound: $\left|\frac{\cos kx}{k^p}\right| \le \frac{1}{k^p}$.
   - Since $\sum \frac{1}{k^p}$ converges (p-series), the trig series converges uniformly on $\mathbb{R}$.
2. **Geometric Series:** $\sum (x/2)^k$ on $[-a, a]$ where $0 < a < 2$.
   - Bound: $|(x/2)^k| \le (a/2)^k$.
   - Since $a/2 < 1$, $\sum (a/2)^k$ converges. The series converges uniformly on $[-a, a]$.
   - *Note:* It does **not** converge uniformly on the open interval $(-2, 2)$ because the terms are not bounded by a single convergent geometric series near the endpoints.

5. Uniform vs. Absolute Convergence

Uniform convergence and absolute convergence are distinct concepts. One does not necessarily imply the other.

Case A: Uniform $\not\Rightarrow$ Absolute

Example: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$ on $[0, 1]$.

- **Uniform:** Let $a_k(x) = x^k/k$. The sequence $a_k(x)$ is decreasing and goes to 0 uniformly. By the Alternating Series Estimation Theorem, the error $|S(x) - S_n(x)| \le a_{n+1}(x) \le \frac{1}{n+1}$. Since the error bound $\frac{1}{n+1}$ is independent of x, convergence is uniform.

- **Not Absolute:** At $x = 1$, the series of absolute values is $\sum \frac{1}{k}$ (Harmonic Series), which diverges.

Case B: Absolute $\not\Rightarrow$ Uniform

Example: $\sum x^2(1 + x^2)^{-k}$ (from Example 8.1.2b).

- **Absolute:** The series consists of non-negative terms and converges pointwise to a discontinuous function (0 at $x = 0$, $1 + x^2$ elsewhere).
- **Not Uniform:** A series of continuous functions converging to a discontinuous function **cannot** be uniformly convergent (a key theorem to be discussed in later sections). Thus, absolute convergence does not guarantee uniformity.

Here are the detailed notes on **Uniform Convergence, Continuity, and the Space $\mathcal{C}(K)$**, based on the provided text.

1. Specific Sequence Problems (Introductory Examples)

Problem 17: M-Test Failure

Consider functions defined on $[0, 1]$:

$$f_n(x) = \begin{cases} \frac{1}{n}, & \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \\ 0, & \text{elsewhere} \end{cases}$$

- **Result:** The series $\sum f_n(x)$ converges uniformly on $[0, 1]$.
- **Note:** The Weierstrass M-test fails here, demonstrating that the M-test is a sufficient but not necessary condition for uniform convergence.

2. Uniform Convergence and Continuity

This section addresses whether the limit of a sequence of continuous functions is continuous.

Theorem 8.3.1: Interchange of Limits

Let $\{f_n\}$ be a sequence of real-valued functions converging uniformly to f on a subset E of a metric space. Let p be a limit point of E.
If $\lim_{x \to p} f_n(x) = A_n$ for each n, then:

1. The sequence $\{A_n\}$ converges.
2. $\lim_{x \to p} f(x) = \lim_{n \to \infty} A_n$.

Formulaic Representation:

$$\lim_{x \to p}(\lim_{n \to \infty} f_n(x)) = \lim_{n \to \infty}(\lim_{x \to p} f_n(x))$$

Proof (Key Ideas):

- **Step 1 (A_n is Cauchy):** Since f_n converges uniformly, $|f_n(x) - f_m(x)| < \epsilon$. Letting $x \to p$, we get $|A_n - A_m| \leq \epsilon$. Thus $\{A_n\}$ converges to some limit A.
- **Step 2 (Convergence to A):** Use the $\epsilon/3$ argument via the Triangle Inequality:

$$|f(x) - A| \le |f(x) - f_m(x)| + |f_m(x) - A_m| + |A_m - A|$$

- Term 1 is small due to uniform convergence.
- Term 2 is small because $\lim_{x \to p} f_m(x) = A_m$.
- Term 3 is small because $A_m \to A$.

Corollary 8.3.2: Preservation of Continuity

- **(a) Sequences:** If $\{f_n\}$ are continuous on E and converge uniformly to f, then f is continuous on E.
- **(b) Series:** If $\{f_n\}$ are continuous and $\sum f_n$ converges uniformly to S, then S is continuous.

Proof Idea:
If p is a limit point, apply Theorem 8.3.1. Since f_n is continuous, $\lim_{x \to p} f_n(x) = f_n(p)$. The theorem implies $\lim_{x \to p} f(x) = \lim f_n(p) = f(p)$.

Counter-Examples (Why Uniformity Matters)

- **Example 8.3.3:** The sequence $f_n(x) = x^n$ on $[0, 1]$ converges pointwise to a discontinuous function (0 on $[0, 1)$, 1 at $x = 1$). Thus, convergence is **not** uniform.
- The series $\sum_{k=0}^{\infty} x^2 (\frac{1}{1+x^2})^k$ converges to a function discontinuous at $x = 0$, implying convergence is not uniform near 0.

3. Dini's Theorem

Does pointwise convergence ever imply uniform convergence? Generally no, but yes under specific conditions (monotonicity and compactness).

Example 8.3.4 (Counter-example):

$$S_n(x) = nxe^{-nx^2}, \quad x \in [0, 1]$$

- $S_n \to 0$ pointwise.
- Functions are continuous.
- **Failure:** The maximum value is $\sqrt{n/2e}$, which tends to ∞. The "hump" moves toward 0 but gets infinitely tall. Convergence is **not** uniform.

Theorem 8.3.5: Dini's Theorem

Let K be a **compact** subset. Let $\{f_n\}$ be a sequence of continuous functions on K such that:

1. $\{f_n\}$ converges **pointwise** to a continuous function f.
2. The sequence is **monotone**: $f_n(x) \geq f_{n+1}(x)$ for all x, n (or increasing).

Result: $\{f_n\}$ converges **uniformly** to f on K.

Proof (Key Ideas):

1. Define $g_n = f_n - f$. Then g_n is continuous, $g_n \geq 0$, $g_n \geq g_{n+1}$, and $g_n \to 0$ pointwise.
2. Let $\epsilon > 0$. Define sets $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
3. Since g_n is continuous, K_n is closed. Since K is compact, K_n is **compact**.
4. By monotonicity ($g_n \geq g_{n+1}$), the sets are nested: $K_{n+1} \subset K_n$.
5. Since $g_n(x) \to 0$ for every x, the intersection $\bigcap K_n = \emptyset$.
6. **Finite Intersection Property:** For compact sets, if the infinite intersection is empty, a finite intersection must be empty. Thus, $K_N = \emptyset$ for some N.
7. This implies $0 \leq g_n(x) < \epsilon$ for all $n \geq N$ and all $x \in K$, proving uniform convergence.

Example 8.3.6 (Necessity of Compactness):
$f_n(x) = \frac{1}{nx+1}$ on $(0, 1)$. Monotonically decreases to 0. Convergence is **not** uniform (near $x = 0$, $f_n \to 1$) because the domain $(0, 1)$ is not compact.

4. The Space $\mathcal{C}(K)$

This section formalizes the set of continuous functions as a normed linear space. Let K be a compact set. $\mathcal{C}(K)$ is the vector space of all continuous real-valued functions on K.

Uniform Norm

Definition 8.3.7: For $f \in \mathcal{C}(K)$, the uniform norm is:

$$||f||_u = \max\{|f(x)| : x \in K\}$$

(Note: Maximum exists because K is compact and f is continuous).

Theorem 8.3.8 (Equivalence):
A sequence f_n converges uniformly to f in $\mathcal{C}(K)$ **if and only if** it converges in the uniform norm:

$$\lim_{n\to\infty} ||f_n - f||_u = 0$$

Completeness

Definition 8.3.10:

- A sequence $\{x_n\}$ in a normed space is **Cauchy** if $||x_n - x_m|| < \epsilon$ for large n, m.
- A space is **Complete** if every Cauchy sequence converges to an element within the space.

Theorem 8.3.11: The space $(\mathcal{C}(K), ||\cdot||_u)$ is **complete**.
Proof (Key Ideas):

1. Take a Cauchy sequence $\{f_n\}$ in $\mathcal{C}(K)$.
2. $|f_n(x) - f_m(x)| \leq ||f_n - f_m||_u < \epsilon$. This means $\{f_n(x)\}$ is a uniformly Cauchy sequence of real numbers.
3. By previous theorems (Cauchy criterion for uniform convergence), $\{f_n\}$ converges uniformly to some function f.
4. Since f_n are continuous and convergence is uniform, limit f is continuous (Corollary 8.3.2). Thus $f \in \mathcal{C}(K)$.
5. Therefore, the sequence converges to f in the norm.

5. Contraction Mappings

Extension of contraction functions to normed linear spaces.

Definition 8.3.12:
Let $(X, ||\cdot||)$ be a normed linear space. A mapping $T : X \to X$ is a **contraction** if there exists a constant c ($0 < c < 1$) such that for all $x, y \in X$:

$$||T(x) - T(y)|| \leq c||x - y||$$

Theorem 8.3.13: Banach Fixed Point Theorem

Let X be a **complete** normed linear space. If T is a contraction mapping, there exists a **unique** fixed point $x \in X$ such that $T(x) = x$.

Proof (Key Ideas):

1. **Existence (Iterative Sequence):**
   - Pick arbitrary x_0. Define $x_n = T(x_{n-1})$.
   - Show $\{x_n\}$ is Cauchy: $||x_{n+1} - x_n|| \leq c^n||x_1 - x_0||$.
   - Using geometric series sum, $||x_{n+m} - x_n|| \leq \frac{c^n}{1-c}||x_1 - x_0||$.

- Since $0 < c < 1$, $c^n \to 0$, so sequence is Cauchy.
- Since X is complete, $x_n \to x$ for some $x \in X$.
- By continuity of T: $x = \lim x_{n+1} = \lim T(x_n) = T(x)$.

2. **Uniqueness:**
   - Assume $T(x) = x$ and $T(y) = y$.
   - $||x - y|| = ||T(x) - T(y)|| \leq c||x - y||$.
   - Since $c < 1$, this implies $||x - y|| = 0$, so $x = y$.

limit $\Longleftrightarrow$ seq criterion $\Longleftrightarrow$ $f(p+) = f(p-)$

$\forall \varepsilon > 0 \quad \exists \delta > 0$

s.t $|f(x) - L| < \varepsilon$

$0 < d(x,p) < \delta$

if $\forall \{p_n\} \to p$

then $\{f(p_n)\} \to L$

limit is unique

---

continuous $\Longleftrightarrow$ topological $\Longleftrightarrow$ $f(p) = f(p+) = f(p-)$

$\forall \varepsilon > 0 \quad \exists \delta > 0$

s.t $|f(x) - f(p)| < \varepsilon$

$d(x,p) < \delta$

$f^{-1}(U) \xrightarrow{\text{Conti}} V$

Compact $\xrightarrow{\text{conti}}$ Compact

m, M EVT

Uniform Conti

IVT

$[a \quad c \quad b]$

$\xrightarrow{\text{conti}}$ $[m \quad f(a) \quad \gamma \quad f(b) \quad M]$

Uniform Conti

Uniform Conti

$\forall \varepsilon > 0 \quad \exists \delta > 0$

s.t $|f(x) - f(y)| < \varepsilon$

if $d(x,y) < \delta$

Lipschitz Func

$|f(x) - f(y)|$

$\leq M \cdot d(x,y)$

---

mono increasing

有 有

$f(p-) \leq f(p) \leq f(p+)$

$\shortparallel$

$\sup_{x<p} f(x) \leq f(y-)$

countable

discon ti $\precsim \mathbb{N}$

jump $\longmapsto \mathbb{Q}$

injection

f interval / continuous $\Leftarrow$를 위해 필요함,

then strict mono $\Longleftrightarrow$ one-to-one

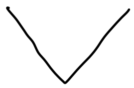
f	then	f^{-1}
interval		interval
strict mono		strict mono
continue		continue

derivatives

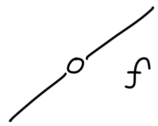
$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

$f_-'(p)$ $x \to p-$

$f_+'(p)$ $x \to p+$



$f_-'(0) = -1$ $f_+'(0) = +1$



$f'(0+) = 1$

$f'(0-) = 1$

difable

$\Rightarrow$ conti

chain rule

$\because$ $f(t) - f(x)$

$= (t - x)\left(f'(x) + \cancel{u(t)}\right)$

---

f on (~~interval~~) $[a,b]$

$p \in$ ~~interior point~~ (a,b) is local extremum

either $\begin{cases} f'(p) = 0 \\ \text{not difable at } p \end{cases}$

Rolle's Thm

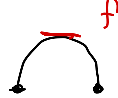
f conti $[a,b]$

difable (a,b)

$f(a) = f(b)$

then $\exists f'(c) = 0$

$f'(a) = 0$

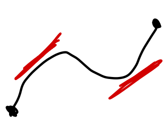


Mean Value Thm

f conti $[a,b]$

difable (a,b)

then $\exists f'(c) = \dfrac{f(a) - f(b)}{a - b}$ ✓



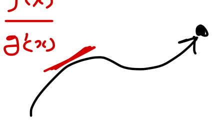
Cauchy MVT

f, g conti $[a,b]$

difable (a,b)

then $\exists \underbrace{\dfrac{f'(c)}{g'(c)}}_{\text{순간속도의 비율}} = \underbrace{\dfrac{f(a) - f(b)}{g(a) - g(b)}}_{\text{평균속도의 비율}}$ by 철수, 영희

$\dfrac{f'(x)}{g'(x)}$



$(f(x), g(x))$

---

f on (interval)

$f'(x) \geq 0 \Rightarrow$ increasing

$f'(x) > 0 \Rightarrow$ mono increasing

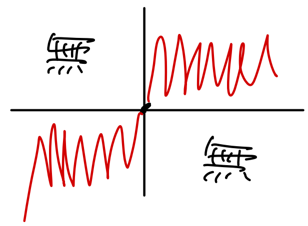
$f'(c) > 0$ (and f' conti) implies

$f(c-\delta, c) < f(c) < f(c, c+\delta)$ (and increasing)

$f'(0) = 1$

$$\begin{cases} f(x) = x + 2x^2 \sin\left(\frac{1}{x}\right) \\ f'(x) = 1 + 4x\sin\left(\frac{1}{x}\right) \\ \qquad - 2\cos\left(\frac{1}{x}\right) \end{cases}$$

oscilate,



not increasing

---

f is conti on (a, b)

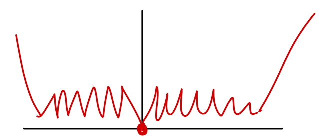
f decreasing at $c-$, increasing at $c+$ $\Longrightarrow$ c is local min ($\not\Longleftarrow$)

$f'(0) = 0$

$f(x) = x^4\left(2 + \sin\left(\frac{1}{x}\right)\right)$

$f'(x) = 4x^3\left(2 + \sin\left(\frac{1}{x}\right)\right) - x^2 \cdot \cos\left(\frac{1}{x}\right)$

oscilate,



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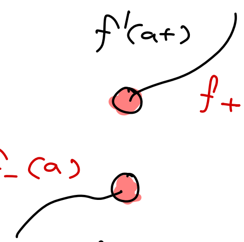
f conti $[a, b)$

diffable (a, b)

if $f'(a+)$ exists

then $f_+(a) = f'(a+)$



$\because$ MVT

$$\frac{f(a+h) - f(a)}{h} = f'(c)$$

$h \to 0+$ then $c \to a+$

$(LHS) \to f_+(a)$

Darboux

f difable on I

$f'(a) < \lambda < f'(b)$

then $\exists\, f'(c) = \lambda$

$\because$ $g(x) = f(x) - \lambda x$

$g(a) \searrow$ $\nearrow g(b)$

$\times$

local min

$f'(a)$가 존재

$\Rightarrow$ f'는 a에서 $\begin{cases} \text{연속} \\ \text{oscilate} \\ \text{~~jump~~ disconti} \\ \text{~~removable~~ disconti} \end{cases}$

---

f difable on I

$\forall x\ f'(x) \neq 0$ $\Rightarrow$ f strict mono $\Rightarrow$ f one-to-one, f^{-1} conti $\Rightarrow$ f^{-1} diffable.

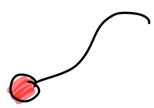
f' do not jump

① $f'(c) > 0 \not\Longrightarrow (c-\delta, c+\delta)$ 에서 증가

f'가 c에서 연속이면 성립

② $\lim_{x\to a+} f'(x)$가 존재한다면,

$$f'_+(a) = \lim_{x\to a+} f'(x)$$



③ f'가 불연속이더라도 IVP가 성립한다.

$f'(a) < \lambda < f'(b)$ 이면

$\exists\ f'(c) = \lambda$

④ $f'(a)$ well defined $\Longrightarrow$ continuous (O)

oscillate (O)

jump disconti (X)

⑤ $f'(x) \neq 0$ in I then f strict mono

hence f^{-1} diffable.

로피탈의 정리

① f, g real valued, diffable on (a, b)

② $g'(x) \neq 0$ in (a, b)

③ $\displaystyle\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in [-\infty, \infty]$

④ $\begin{cases} \displaystyle\lim_{x\to a+} f(x) = 0 \text{ and } \lim_{x\to a+} g(x) = 0 \\ \displaystyle\lim_{x\to a+} g(x) = \pm\infty \end{cases}$

then $\displaystyle\lim_{x\to a+} \frac{f(x)}{g(x)} = L$

$\because$ define $f(a) = g(a) = 0$

more fixed

GMVT $\displaystyle\frac{f(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(c_n)}$

$x_n \to a+$ then $\displaystyle\lim_{x\to a+} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a+} \frac{f'(x)}{g'(x)}$

$\because$

$\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$,

more fixed

GMVT $\displaystyle\frac{f(x) - f(a+)}{g(x) - g(a+)} = \frac{f'(\xi)}{g'(\xi)}$

then $(\cdots)$ $\displaystyle\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}\left(1 - \frac{g(a+)}{g(x)}\right) + \frac{f(a+)}{g(x)}$