Here is a comprehensive, concise cheat sheet covering Sections 4.1 through 4.4.

Real Analysis Cheat Sheet: Limits & Continuity

4.1 Limit of a Function

1. Definitions

- $\epsilon - \delta$ **Definition:** Let (X, d) be a metric space, $E \subset X$, $f : E \to \mathbb{R}$, and p a limit point of E.

$$\lim_{x \to p} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < d(x,p) < \delta \implies |f(x) - L| < \epsilon$$

  - **Note:** δ depends on ϵ and p. p need not be in E.
- **Sequential Criterion:** $\lim_{x \to p} f(x) = L \iff$ for every sequence $\{p_n\}$ in E ($p_n \neq p$) with $p_n \to p$, the sequence $f(p_n) \to L$.
- **Limit at Infinity:** Let domain be unbounded. $\lim_{x \to \infty} f(x) = L$ if $\forall \epsilon > 0, \exists M \in \mathbb{R}$ such that $x > M \implies |f(x) - L| < \epsilon$.

2. Key Examples & Techniques

- **Rational Functions:** $h(x) = \frac{\sqrt{x+1}-1}{x}$. Rationalize numerator to bound $|h(x) - 1/2|$.
- **Dirichlet Function:** $f(x) = 1$ if $x \in \mathbb{Q}$, 0 if $x \notin \mathbb{Q}$. Limit exists **nowhere** (density of $\mathbb{Q}$ and $\mathbb{Q}^c$).
- **Modified Dirichlet:** $f(x) = x$ if $x \notin \mathbb{Q}$, 0 if $x \in \mathbb{Q}$. Continuous **only at** $x = 0$ (since $|f(x)| \leq |x|$).
- $1/x$ **on** $(0, 1)$**:** To show $\lim_{x \to p} 1/x = 1/p$, δ must shrink as $p \to 0$ ($\delta \approx p^2 \epsilon$).
- **Multivariable:** $f(x, y) = \frac{xy}{x^2+y^2}$. Use Triangle Inequality and restrict to neighborhood (e.g., $N_{1/2}(p)$) to bound denominator.
- **Oscillation:** $\lim_{x \to 0} \sin(1/x)$ DNE. Sequence $p_n = \frac{2}{(2n+1)\pi}$ yields $(-1)^n$ (divergent).

3. Theorems

- **Uniqueness:** If a limit exists, it is unique.

- **Algebra:** $\lim(f \pm g) = A \pm B$, $\lim(fg) = AB$, $\lim(f/g) = A/B$ ($B \neq 0$).
- **Boundedness Thm:** If $|g(x)| \leq M$ and $\lim f(x) = 0 \implies \lim f(x)g(x) = 0$.
  - *Ex:* $\lim_{x \to 0} x \sin(1/x) = 0$.
- **Squeeze Thm:** $g \leq f \leq h$ and $\lim g = \lim h = L \implies \lim f = L$.
  - *App:* $\lim_{t \to 0} \frac{\sin t}{t} = 1$ (via geometry and squeeze: $\cos t < \frac{\sin t}{t} < \frac{1}{\cos t}$).

4.2 Continuous Functions

1. Definitions

- **Metric:** f is continuous at $p \in E$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $d(x,p) < \delta \implies |f(x) - f(p)| < \epsilon$.
- **Topological:** f is continuous on $E \iff f^{-1}(V)$ is open in E for every open set $V \subset \mathbb{R}$.
  - *Note:* Forward image of open set need not be open.
- **Sequential:** $p_n \to p \implies f(p_n) \to f(p)$.
- **Isolated Points:** Functions are always continuous at isolated points of E.

2. Classification of Examples

- **Removable Discontinuity:** $\lim_{x \to p} f(x)$ exists but $\neq f(p)$.
  - *Ex:* $g(x) = \frac{x^2-4}{x-2}$ at $x = 2$. Set $g(2) = 4$ to fix.
- **Thomae's Function (Popcorn):** $f(x) = 1/n$ if $x = m/n \in \mathbb{Q}$, 0 if irrational.
  - Continuous at all **irrationals**.
  - Discontinuous at all **rationals**.
- **Sine:** $f(x) = \sin x$ continuous on $\mathbb{R}$ (Lipschitz with $K = 1$).

3. Continuity & Compactness

- **Preservation:** If f is continuous and K is compact, then $f(K)$ is compact.
- **Extreme Value Theorem:** If K is compact (closed & bounded in $\mathbb{R}$), f attains its **Maximum** and **Minimum** on K.
  - *Counter-ex:* $f(x) = x$ on $(0,1)$ (bounded but not closed) has no max/min.
  - *Counter-ex:* $f(x) = \frac{x^2}{1+x^2}$ on $[0, \infty)$ (closed but not bounded) has no max.

4. Intermediate Value Theorem (IVT)

- **Theorem:** If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) < \gamma < f(b)$, then $\exists c \in (a,b)$ s.t. $f(c) = \gamma$.

- **Corollary:** Continuous image of an interval is an interval.
- **Fixed Point Thm:** Continuous $f : [0, 1] \to [0, 1]$ has y s.t. $f(y) = y$. (Proof via $g(x) = f(x) - x$).
- **Roots:** $y^n = \gamma$ exists for $\gamma > 0$.

4.3 Uniform Continuity

1. Definition

- **Uniform Continuity:** $f : E \to \mathbb{R}$ is uniformly continuous if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, y \in E, d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

- **Key Difference:** In pointwise, δ depends on p ($\delta(\epsilon, p)$). In uniform, δ is independent of p ($\delta(\epsilon)$).

2. Uniform Continuity Theorem

- **Theorem 4.3.4:** If K is **compact** and f is continuous on $K \implies f$ is uniformly continuous on K.
- **Heine's Theorem:** Continuous function on $[a, b]$ is uniformly continuous.

3. Lipschitz Functions

- **Def:** $|f(x) - f(y)| \le Md(x, y)$ for constant M.
- **Implication:** Lipschitz $\implies$ Uniformly Continuous (Take $\delta = \epsilon/M$).
  - *Note:* Converse is false (e.g., $\sqrt{x}$ on $[0, 1]$ is unif. cont. but not Lipschitz).

4. Examples & Counter-Examples

- $f(x) = x^2$**:**
  - On **Bounded** set E: **Uniform**. $|x^2 - y^2| = |x + y||x - y| \le 2C|x - y|$.
  - On **Unbounded** (e.g., $[0, \infty)$): **Not Uniform**. Values grow too fast.
- $f(x) = 1/x$**:**
  - On $(0, 1)$: **Not Uniform**. As $x \to 0$, slope $\to \infty$. δ cannot be constant.
  - On $[a, \infty)$ with $a > 0$: **Uniform**.
- $f(x) = \sin x$**: Uniform** on $\mathbb{R}$ (bounded derivative).

4.4 Monotone Functions & Discontinuities

1. One-Sided Limits

- **Right Limit ($f(p+)$):** Limit as $x \to p$ with $x > p$.
- **Left Limit ($f(p-)$):** Limit as $x \to p$ with $x < p$.
- **Continuity:** f is continuous at $p \iff f(p+) = f(p-) = f(p)$.

2. Classification of Discontinuities

- **Simple (1st Kind):** Both $f(p+)$ and $f(p-)$ exist.
  - **Removable:** $f(p+) = f(p-) \neq f(p)$.
  - **Jump:** $f(p+) \neq f(p-)$.
    - *Ex:* $f(x) = [x]$ (Greatest Integer). Jump size 1 at integers.
- **Second Kind:** At least one of $f(p+)$ or $f(p-)$ does not exist.
  - *Ex:* $f(x) = \sin(1/x)$ at $x = 0$.

3. Monotone Functions

- **Properties:** Let f be monotone increasing on I.
  - **Existence of Limits:** For any $p \in \text{Int}(I)$, $f(p-)$ and $f(p+)$ exist.
  - **Ordering:** $\sup_{x<p} f(x) = f(p-) \leq f(p) \leq f(p+) = \inf_{x>p} f(x)$.
  - **Interval Relation:** $p < q \implies f(p+) \leq f(q-)$.
- **Corollary:** The set of discontinuities of a monotone function is **countable** (at most).

4. Construction of Discontinuities

- **Unit Jump Function:** $I(x) = 0$ for $x < 0$, 1 for $x \geq 0$.
- **General Construction:** $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$ with $\sum c_n$ convergent.
  - f is monotone increasing.
  - f is discontinuous exactly at the countable set $\{x_n\}$.
  - Jump at x_n is $f(x_n) - f(x_n-) = c_n$.
  - f is continuous at $x \notin \{x_n\}$ and right-continuous everywhere.

5. Inverse Functions

- **Theorem:** If $f : I \to \mathbb{R}$ is strictly monotone and continuous:
  i. Image $J = f(I)$ is an interval.
  ii. Inverse $f^{-1} : J \to I$ exists.

iii. f^{-1} is **strictly monotone** and **continuous**.

- *Ex:* $f(x) = x^n$ on $[0, \infty) \implies \sqrt[n]{x}$ is continuous.