Section 4.1: Limit of a Function

1. Definition of a Limit

Context: Let (X, d) be a metric space, $E \subset X$, and $f : E \to \mathbb{R}$. Let p be a **limit point** of E.

Definition 4.1.1 ($\epsilon - \delta$ Definition):
$\lim_{x \to p} f(x) = L$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$|f(x) - L| < \epsilon$$

for all $x \in E$ satisfying $0 < d(x, p) < \delta$.

- **Metric Space Extension:** If $f : E \to (Y, \rho)$, replace $|f(x) - L|$ with $\rho(f(x), L)$.
- **Neighborhood Notation:** $f(x) \in N_\epsilon(L)$ for all $x \in E \cap (N_\delta(p) \setminus \{p\})$.

Key Remarks:

1. **Dependence:** δ depends on ϵ, the function f, and often the point p.
2. **Isolated Points:** If p is not a limit point (isolated), the limit is meaningless.
3. **Value at p:** It is **not** required that $p \in E$. Even if $p \in E$, $\lim_{x \to p} f(x)$ need not equal $f(p)$.

2. $\epsilon - \delta$ Proof Strategies (Examples)

A. Rational Functions (Algebraic Manipulation)

Example 4.1.2(c): $h(x) = \frac{\sqrt{x+1}-1}{x}$ for $x \in (-1, 0) \cup (0, \infty)$.

- **Claim:** $\lim_{x \to 0} h(x) = 1/2$.
- **Analysis:** Rationalize the numerator:

$$|h(x) - \frac{1}{2}| = \left| \frac{1}{\sqrt{x+1}+1} - \frac{1}{2} \right| = \frac{|x|}{2(\sqrt{x+1}+1)^2}$$

  Since $(\sqrt{x+1}+1)^2 > 1$, we have $|h(x) - 1/2| < |x|/2$.
- **Choice:** Set $\delta = \min\{1, 2\epsilon\}$. Then $|h(x) - 1/2| < \epsilon$.

B. The Dirichlet Function (Nowhere Continuous)

Example 4.1.2(d): $f(x) = 1$ if $x \in \mathbb{Q}$, 0 if $x \notin \mathbb{Q}$.

- **Claim:** $\lim_{x \to p} f(x)$ does not exist for any $p \in \mathbb{R}$.
- **Proof:** Fix L. Let $\epsilon = \max\{|L-1|, |L|\}$.
  - Density of $\mathbb{Q}$ implies $\exists x \in \mathbb{Q}$ near $p \implies |1 - L| = \epsilon$.
  - Density of irrationals implies $\exists x \notin \mathbb{Q}$ near $p \implies |0 - L| = \epsilon$.
  - Regardless of δ, $|f(x) - L| \geq \epsilon$ is always possible. Thus, limit fails.

C. Modified Dirichlet Function (Limit exists at one point)

Example 4.1.2(e): $f(x) = 0$ if $x \in \mathbb{Q}$, x if $x \notin \mathbb{Q}$.

- **At $x = 0$:** Since $|f(x)| \leq |x|$, choosing $\delta = \epsilon$ proves $\lim_{x \to 0} f(x) = 0$.
- **At $p \neq 0$:** Limit does not exist (similar argument to Dirichlet function).

D. Dependence of δ on p (Uniformity issue)

Example 4.1.2(f): $f(x) = 1/x$ on $(0, 1)$. Show $\lim_{x \to p}(1/x) = 1/p$.

- **Inequality:** $|\frac{1}{x} - \frac{1}{p}| = \frac{|x-p|}{xp}$.
  Restricting $x > p/2$ implies $\frac{1}{xp} < \frac{2}{p^2}$.
- **Choice:** $\delta = \min\{p/2, p^2\epsilon/2\}$.
- **Key Insight:** As $p \to 0$, $p^2\epsilon/2 \to 0$. δ *must* shrink as p approaches 0; it cannot be independent of p for this domain.

E. Multivariable Limit

Example 4.1.2(g): $f(x, y) = \frac{xy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus (0, 0)$. Show limit at $(1, 2)$ is $2/5$.

- **Technique:** Algebraic factorization and Triangle Inequality.
  i. **Common Denominator:**

$$|f(x,y) - \frac{2}{5}| = \left| \frac{5xy - 2x^2 - 2y^2}{5(x^2+y^2)} \right|$$

  ii. **Numerator Decomposition:**
  The numerator is rewritten to isolate terms approaching zero, $(x-1)$ and $(y-2)$:

$$5xy - 2x^2 - 2y^2 = (x - 2y)(y - 2) + (4y - 2x)(x - 1)$$

  iii. **Triangle Inequality ($|a + b| \leq |a| + |b|$):**

$$\leq \frac{|x-2y||y-2|}{5(x^2+y^2)} + \frac{|4y-2x||x-1|}{5(x^2+y^2)} \leq \frac{(|x|+2|y|)|y-2|+(4|y|+2|x|)|x-1|}{5(x^2+y^2)}$$

- **Bounding:** Restrict (x, y) to the neighborhood $N_{1/2}(1,2)$.
  - This implies $1/2 < |x| < 3/2$ and $3/2 < |y| < 5/2$.
  - Using these values, we bound the coefficients (e.g., $5(x^2+y^2) > 25/2$) to find a constant $K = 26/25$.
  - $$|f(x,y) - \frac{2}{5}| < \frac{26}{25}(|y-2| + |x-1|)$$
- **Result:** Given ϵ, choose $\delta < \min\{1/2, \frac{25}{52}\epsilon\}$.

3. Sequential Criterion for Limits

Theorem 4.1.3: $\lim_{x \to p} f(x) = L$ **if and only if** for every sequence $\{p_n\}$ in E ($p_n \neq p$) with $p_n \to p$, the sequence $\{f(p_n)\} \to L$.

Proof Technique: Constructing a Sequence ($\delta = 1/n$)
The "If" direction ($\Leftarrow$) is often proven by contradiction (Contrapositive).

- **Assumption:** Suppose $\lim_{x \to p} f(x) \neq L$.
- **Negation of Limit:** There exists an $\epsilon_0 > 0$ such that for any $\delta > 0$, there is an x with $0 < |x - p| < \delta$ but $|f(x) - L| \geq \epsilon_0$.
- **Construction:** For each $n \in \mathbb{N}$, choose $\delta = 1/n$.
  - We can find a point p_n such that $0 < |p_n - p| < 1/n$ and $|f(p_n) - L| \geq \epsilon_0$.
- **Conclusion:** The sequence $\{p_n\}$ converges to p (since $|p_n - p| < 1/n \to 0$), but $\{f(p_n)\}$ does not converge to L. This contradicts the hypothesis.

Corollary 4.1.4 (Uniqueness): If a limit exists, it is unique.

Application: Disproving Existence of Limits
To show a limit **does not exist**:

1. Find a sequence $p_n \to p$ where $\{f(p_n)\}$ diverges.
2. Find two sequences $p_n \to p, r_n \to p$ where $\lim f(p_n) \neq \lim f(r_n)$.

Example 4.1.5(a): $f(x) = \sin(1/x)$ at $x \to 0$.

- Choose $p_n = \frac{2}{(2n+1)\pi}$.
- $f(p_n) = \sin((2n+1)\pi/2) = (-1)^n$.

- The sequence $(-1)^n$ does not converge. Thus, $\lim_{x\to 0} \sin(1/x)$ does not exist.

4. Limit Theorems

Let $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$.

1. **Algebra:**
   - $\lim(f+g) = A+B$
   - $\lim(fg) = AB$
   - $\lim(f/g) = A/B$ (provided $B \neq 0$).

   Proof Detail for Quotient (Bounding away from 0):
   To prove $\lim \frac{1}{g(x)} = \frac{1}{B}$, we must ensure $g(x)$ does not vanish near p.
   - **Specific Choice of ϵ:** Set $\epsilon = \frac{|B|}{2}$.
   - **Logic:** Since $\lim g(x) = B$, $\exists \delta_1 > 0$ such that whenever $0 < |x-p| < \delta_1$, we have $|g(x) - B| < \frac{|B|}{2}$.
   - **Triangle Inequality:**

   $$|g(x)| = |B - (B - g(x))| \geq |B| - |g(x) - B| > |B| - \frac{|B|}{2} = \frac{|B|}{2}$$

   - **Result:** $|g(x)| > \frac{|B|}{2} > 0$. The denominator is strictly bounded away from zero by a specific positive constant in this neighborhood.
2. **Boundedness Theorem (Thm 4.1.8):**
   If g is bounded on E ($|g(x)| \leq M$) and $\lim_{x\to p} f(x) = 0$, then $\lim_{x\to p} f(x)g(x) = 0$.
   - *Example 4.1.10(c):* $\lim_{x\to 0} x\sin(1/x) = 0$. Since $|\sin(1/x)| \leq 1$ is bounded and $x \to 0$, the product goes to 0.
3. **Squeeze Theorem (Thm 4.1.9):**
   If $g(x) \leq f(x) \leq h(x)$ and $\lim g(x) = \lim h(x) = L$, then $\lim f(x) = L$.

5. Essential Trigonometric Limit

Claim: $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

- **Geometric Proof:** Using unit circle areas (Triangle OPQ, Sector OPR, Triangle ORS).

$$\text{Area}(\triangle OPQ) < \text{Area}(\text{Sector}) < \text{Area}(\triangle ORS)$$

$$\frac{1}{2}\sin t\cos t < \frac{1}{2}t < \frac{1}{2}\tan t$$

- **Inequality:** $\cos t < \frac{\sin t}{t} < \frac{1}{\cos t}$.
- **Conclusion:** Since $\lim_{t\to 0}\cos t = 1$, by Squeeze Theorem, limit is 1.

6. Limits at Infinity

Definition 4.1.11:
Let domain of f be unbounded above. $\lim_{x\to\infty} f(x) = L$ if $\forall\epsilon > 0, \exists M \in \mathbb{R}$ such that:

$$|f(x) - L| < \epsilon$$

for all $x \in \mathrm{Dom}(f) \cap (M, \infty)$.
(Analogous definition exists for $x \to -\infty$ using $x < M$).

Examples:

1. **Damped Sine:** $f(x) = \frac{\sin x}{x}$ on $(0, \infty)$.
   Since $|f(x)| \leq 1/x$, choosing $M = 1/\epsilon$ proves limit is 0.
2. **Oscillation at Infinity:** $f(x) = x\sin(\pi x)$.
   Choosing sequence $p_n = n + 1/2$ gives $f(p_n) = (-1)^n(n + 1/2)$. This is unbounded; limit does not exist.

Section 4.2: Continuous Functions

1. Definition of Continuity

- **Metric Space Definition:** Let (X, d) be a metric space and $E \subset X$. A function $f : E \to \mathbb{R}$ is continuous at $p \in E$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$|f(x) - f(p)| < \epsilon \quad \text{for all } x \in E \text{ with } d(x, p) < \delta$$

- **Topological Phrasing:** f is continuous at p if and only if f maps the δ-neighborhood of p into the ϵ-neighborhood of $f(p)$:

$$x \in N_\delta(p) \cap E \implies f(x) \in N_\epsilon(f(p))$$

- **Sequential Criterion:** f is continuous at p if and only if for every sequence $\{p_n\}$ in E with $p_n \to p$, we have $\lim_{n\to\infty} f(p_n) = f(p)$.
- **Isolated Points:** If p is an isolated point of E, every function f is continuous at p because there exists a δ such that $N_\delta(p) \cap E = \{p\}$.

2. Specific Examples of Continuity (4.2.2)

- **(a) Removable Discontinuity:**

$$g(x) = \frac{x^2 - 4}{x - 2}, \quad x \neq 2; \quad g(2) = 2$$

  - $\lim_{x\to 2} g(x) = 4 \neq g(2)$. Thus, discontinuous at $x = 2$.
  - *Correction:* Redefining $g(2) = 4$ makes it continuous.
- **(b) Rational Indicator:**

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ x, & x \notin \mathbb{Q} \end{cases}$$

  - Continuous at $p = 0$ since $\lim_{x\to 0} f(x) = 0 = f(0)$.
  - Discontinuous at every $p \neq 0$.
- **(c) Dirichlet Function:**

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

  - Discontinuous at every $p \in \mathbb{R}$ because rationals and irrationals are dense in $\mathbb{R}$.
- **(d) Reciprocal:** $f(x) = 1/x$ is continuous on $(0, 1)$.
- **(e) Oscillating Function:**

$$f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

  - Continuous at $x = 0$ because $|x \sin(1/x)| \leq |x|$, so the limit is 0.
- **(f) Sine Function:** $f(x) = \sin x$ is continuous on $\mathbb{R}$.
  - *Proof:* Using $|\sin y - \sin x| = 2|\cos \frac{y+x}{2} \sin \frac{y-x}{2}| \leq |y - x|$, we can choose $\delta = \epsilon$.
- **(g) Thomae's Function (Popcorn Function):**

$$f(x) = \begin{cases} 1/n, & x = m/n \in \mathbb{Q} \cap (0, 1) \text{ (lowest terms)} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

  - **Discontinuous** at every rational p (since $f(p) \neq 0$).
  - **Continuous** at every irrational p (limit is 0).
  - *Proof Sketch:* For $\epsilon > 0$, only finitely many rationals have a denominator n such that $1/n \geq \epsilon$. We can choose δ to exclude these specific rationals from the neighborhood of irrational p.

3. Algebra and Composition

- **Algebraic Operations:** If f, g are continuous at p, then $f + g$, $f - g$, fg, and f/g (provided $g(p) \neq 0$) are continuous at p.
- **Composition (Theorem 4.2.4):**
  If f is continuous at p and g is continuous at $f(p)$, then $h = g \circ f$ is continuous at p.
  - *Examples:* Polynomials, Rational functions (on domain), and $\sin(p(x))$ are continuous.

4. Topological Characterization (Theorem 4.2.6)

- **Theorem:** A function $f : E \to \mathbb{R}$ is continuous on E **if and only if** $f^{-1}(V)$ is open in E for every open subset V of $\mathbb{R}$.
- **Proof:**
  1. Continuity $\implies$ Open Pre-images

- Let $p \in f^{-1}(V)$, which means $f(p) \in V$.

$$\underbrace{f(N_\delta(p) \cap E) \subseteq N_\epsilon(f(p))}_{\text{Since } f \text{ is continuous}} \subseteq \underbrace{V}_{\text{Since } V \text{ is open}}$$

- By definition of inverse image ($A \subseteq B \iff f^{-1}(A) \subseteq f^{-1}(B)$):

$$N_\delta(p) \cap E \subseteq f^{-1}(N_\epsilon(f(p))) \subseteq f^{-1}(V)$$

Since $N_\delta(p) \cap E \subseteq f^{-1}(V)$, the set $f^{-1}(V)$ **is open**.

2. **Open Pre-images $\implies$ Continuity**
   - Let $\epsilon > 0$. Set $V = N_\epsilon(f(p))$ (which is open).

$$\underbrace{N_\delta(p) \cap E \subseteq f^{-1}(V)}_{\text{Since } f^{-1}(V) \text{ is open}}$$

   - Applying f to both sides gives the continuity definition directly:

$$f(N_\delta(p) \cap E) \subseteq V = N_\epsilon(f(p))$$

- **Example:** For $f(x) = \sqrt{x}$ on $[0, \infty)$, let $V = (a, b)$.
  - If $a \le 0 < b$, then $f^{-1}(V) = [0, b^2)$.
  - While $[0, b^2)$ is not open in $\mathbb{R}$, it **is** open in $[0, \infty)$ because it can be written as $(-b^2, b^2) \cap [0, \infty)$.
- **Warning:** The forward image of an open set is **not** necessarily open.
  - *Example*: $f(x) = x^2$ for $x \le 2$ and $6 - x$ for $x > 2$. $f((-1, 1)) = [0, 1)$, which is not open.

5. Continuity and Compactness

- **Theorem 4.2.8 (Preservation of Compactness):**
  If K is a compact subset of a metric space X and $f : K \to \mathbb{R}$ is continuous on K, then the image $f(K)$ is compact.
  - **Proof:**
    a. Let $\{V_\alpha\}$ be an arbitrary open cover of $f(K)$.
    b. Since f is continuous, each $f^{-1}(V_\alpha)$ is open in K, forming an open cover of K.
    c. Since K is compact, there exists a finite subcover corresponding to indices $\alpha_1, \ldots, \alpha_n$.
    d. The collection $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$ covers $f(K)$. Thus, $f(K)$ is compact.

- **Corollary 4.2.9 (Extreme Value Theorem):**
  If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous, then f attains its maximum and minimum on K.

$$\exists p, q \in K \text{ s.t. } f(q) \le f(x) \le f(p) \quad \forall x \in K$$

  - **Concise Proof:**
    By Theorem 4.2.8, the image $f(K)$ is compact in $\mathbb{R}$.
    - **Bounded:** Since compact sets in $\mathbb{R}$ are bounded, $M = \sup f(K)$ and $m = \inf f(K)$ exist.
    - **Closed:** Since compact sets in $\mathbb{R}$ are closed, $f(K)$ contains its limit points, so $M, m \in f(K)$.
    - **Conclusion:** Therefore, there exist $p, q \in K$ such that $f(p) = M$ and $f(q) = m$.
- **Counter-examples (4.2.10):**
  The theorem fails if K is not compact (i.e., not closed or not bounded).
  - **Unbounded Domain:** $f(x) = \frac{x^2}{1+x^2}$ on $[0, \infty)$. The supremum is 1, but $f(x) < 1$ for all x, so the maximum is never attained.
  - **Not Closed Domain:** $g(x) = x$ on $(0, 1)$. The supremum is 1 and infimum is 0, but neither is contained in the range $(0, 1)$.

6. Intermediate Value Theorem (IVT)

- **Theorem 4.2.11 (Intermediate Value Theorem):**
  Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < \gamma < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = \gamma$.
  - **Key Intuition:** Continuity implies **local persistence of inequalities**. If $f(c) \neq \gamma$, the function forces values to stay strictly above or below γ in a small neighborhood, contradicting the precise boundary nature of the supremum.
  - **Rigorous Proof Sketch:**
    a. **Construct Set:** Let $A = \{x \in [a, b] : f(x) \le \gamma\}$. Since $a \in A$, $A \neq \emptyset$. Since A is bounded by b, let $c = \sup A$.
    b. **Contradiction (Case 1: $f(c) < \gamma$):**
       Let $\epsilon = \frac{1}{2}(\gamma - f(c)) > 0$. By continuity, $\exists \delta > 0$ such that $f(x) < f(c) + \epsilon < \gamma$ for $x \in N_\delta(c)$.
       This implies there exist points $x > c$ where $f(x) < \gamma$, meaning $x \in A$. This contradicts that c is the upper bound of A.

c. **Contradiction (Case 2: $f(c) > \gamma$):**
Similar logic shows that for some neighborhood, $f(x) > \gamma$. This implies c is not a limit point of A (or $c \notin A$), contradicting the properties of the supremum.
d. **Conclusion:** Therefore, $f(c) = \gamma$.

- **Corollary 4.2.12 (Topological Characterization):**
If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is continuous on I, then $f(I)$ is an interval.
  - **Proof:**
To show $f(I)$ is an interval, let $s, t \in f(I)$ with $s < t$ and pick any γ such that $s < \gamma < t$. There exist $a, b \in I$ with $f(a) = s$, $f(b) = t$. Since I is an interval, the closed segment between a and b lies in I. By the **Intermediate Value Theorem**, there exists c between a and b such that $f(c) = \gamma$. Thus $\gamma \in f(I)$, implying $f(I)$ is an interval.
- **Corollary 4.2.13 (Existence of Roots):**
For every $\gamma > 0$ and $n \in \mathbb{N}$, there exists a unique $y > 0$ such that $y^n = \gamma$.
- **Corollary 4.2.14 (Fixed Point Theorem):**
If $f : [0, 1] \to [0, 1]$ is continuous, there exists $y \in [0, 1]$ such that $f(y) = y$.
  - **Proof Technique (Auxiliary Function):**
    a. Define $g(x) = f(x) - x$.
    b. **Evaluate Endpoints:**
      - $g(0) = f(0) - 0 \geq 0$ (since $f(0) \in [0, 1]$).
      - $g(1) = f(1) - 1 \leq 0$ (since $f(1) \in [0, 1]$).
    c. **Apply IVT:** Since g is continuous and 0 lies between $g(1)$ and $g(0)$, there exists y such that $g(y) = 0$.
    d. **Result:** $f(y) - y = 0 \implies f(y) = y$.

Caveats & Counter-examples (4.2.15)

- **Converse is False:** A function can satisfy the intermediate value property (Darboux property) but be discontinuous.
  - *Example:* $f(x) = \sin(1/x)$ for $x > 0$ and $f(0) = 0$. This function takes every value between -1 and 1 in any neighborhood of 0, satisfying the property, but is discontinuous at 0.
- **Requires Completeness of $\mathbb{R}$:** The IVT relies on the Least Upper Bound Property.
  - *Example:* Let $f(x) = x^2$ on the rational interval $E = [0, 2] \cap \mathbb{Q}$.
  - We have $f(0) < 2 < f(2)$, but there is **no rational number** $c \in E$ such that $c^2 = 2$. Thus, IVT fails in $\mathbb{Q}$.

Section 4.3: Uniform Continuity

1. Concept: Pointwise vs. Uniform Continuity

- **Standard Continuity (Pointwise):**
  - A function $f : E \to \mathbb{R}$ is continuous on E if for each $p \in E$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ for all $x \in E \cap N_\delta(p)$.
  - **Dependency:** Here, δ depends on **both** ϵ and the specific point p. (i.e., $\delta = \delta(\epsilon, p)$).
  - *Problem:* As p changes, the required δ might get infinitely small (e.g., $f(x) = 1/x$ near 0).
- **Uniform Continuity (Definition 4.3.1):**
  - Let $E \subset (X, d)$ and $f : E \to \mathbb{R}$.
  - f is **uniformly continuous** on E if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in E, \text{ if } d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

  - **Dependency:** Here, δ depends **only** on ϵ, f, and the set E. It is **independent** of the point x.

2. Examples of Uniform Continuity

Case A: $f(x) = x^2$ on a Bounded Subset E

- **Claim:** f is uniformly continuous on E if E is bounded.
- **Proof:**
  - Since E is bounded, $\exists C > 0$ such that $|x| \leq C$ for all $x \in E$.
  - Consider $|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y|$.
  - Using the bound: $|x + y| \leq |x| + |y| \leq 2C$.
  - Thus, $|f(x) - f(y)| \leq 2C|x - y|$.
  - Choose $\delta = \epsilon/2C$. If $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Case B: $f(x) = \sin x$ on $\mathbb{R}$

- **Claim:** f is uniformly continuous on $\mathbb{R}$.
- **Proof:**
  - Standard trigonometric inequality: $|\sin x - \sin y| \leq |x - y|$.
  - Choose $\delta = \epsilon$.

- If $|x-y|<\delta$, then $|f(x)-f(y)| \leq |x-y| < \epsilon$.

Case C: Non-Example ($f(x) = 1/x$ on $(0,1)$)

- **Claim:** f is **not** uniformly continuous on $(0,1)$.
- **Proof (Contradiction):**
  - Assume f is uniformly continuous. Let $\epsilon = 1$.
  - There must exist $\delta > 0$ (assume $\delta < 1$) such that $|x-y| < \delta \implies |1/x - 1/y| < 1$.
  - Choose $x \in (0, 1/2)$ such that x is very small, and let $y = x + \delta/2$.
  - Then $|x-y| = \delta/2 < \delta$.
  - However, analysis shows that as $x \to 0$, the difference $|1/x - 1/(x+\delta/2)|$ becomes arbitrarily large, eventually exceeding 1.
  - Specifically, if x is small enough ($x < \delta/2$), the inequality fails, leading to a contradiction.

3. Lipschitz Functions

- **Definition:** A function $f: E \to \mathbb{R}$ satisfies a **Lipschitz condition** on E if there exists a constant $M > 0$ such that:

$$|f(x) - f(y)| \leq M d(x,y) \quad \forall x, y \in E$$

  - These are called Lipschitz functions.
  - Functions with bounded derivatives are Lipschitz.
- **Theorem 4.3.3:**
  - If f satisfies a Lipschitz condition on E, then f is **uniformly continuous** on E.
  - *Proof Logic:* Simply choose $\delta = \epsilon/M$.
- **Important Note (Converse is False):**
  - Not every uniformly continuous function is Lipschitz.
  - *Example:* $f(x) = \sqrt{x}$ on $[0, \infty)$ is uniformly continuous, but **not** Lipschitz (the derivative is unbounded near 0).

4. The Uniform Continuity Theorem

Determining uniform continuity for non-Lipschitz functions is difficult. Compactness provides a sufficient condition.

- **Theorem 4.3.4:**

- If K is a **compact** metric space and $f: K \to \mathbb{R}$ is **continuous** on K, then f is **uniformly continuous** on K.

- **Proof Sketch:**
  i. Fix $\epsilon > 0$. Since f is continuous, for every $p \in K$, there is a δ_p valid locally.
  ii. The neighborhoods $N_{\delta_p/2}(p)$ form an open cover of K.
  iii. Since K is compact, there is a **finite subcover** corresponding to points $p_1, \ldots, p_n$.
  iv. Define $\delta = \frac{1}{2}\min\{\delta_{p_1}, \ldots, \delta_{p_n}\}$. This δ works globally for the whole set.
- **Corollary 4.3.5 (Heine's Theorem):**
  - A continuous real-valued function on a **closed and bounded interval** $[a, b]$ is uniformly continuous.

5. Necessity of Compactness (Example 4.3.6)

To guarantee uniform continuity via Theorem 4.3.4, the domain must be **both** closed and bounded (Compact).

- **Closed but Not Bounded:**
  - Set: $[0, \infty)$.
  - Function: $f(x) = x^2$.
  - Result: Continuous, but **not** uniformly continuous (values grow too fast as $x \to \infty$).
- **Bounded but Not Closed:**
  - Set: $(0, 1)$.
  - Function: $f(x) = 1/x$.
  - Result: Continuous, but **not** uniformly continuous (values grow too fast as $x \to 0$).

Section 4.4: Monotone Functions and Discontinuities

I. Right and Left Limits

1. Definitions

Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$.

- **Right Limit ($f(p+)$):** Suppose p is a limit point of $E \cap (p, \infty)$.

$$\lim_{x \to p^+} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } p < x < p + \delta.$$

- **Left Limit ($f(p-)$):** Suppose p is a limit point of $E \cap (-\infty, p)$.

$$\lim_{x \to p^-} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } p - \delta < x < p.$$

2. Relationship to General Limits

For $p \in \text{Int}(I)$, $\lim_{x \to p} f(x)$ exists if and only if:

1. Both $f(p+)$ and $f(p-)$ exist.
2. $f(p+) = f(p-)$.

3. One-Sided Continuity

- **Right Continuous at p:** $\lim_{x \to p^+} f(x) = f(p)$.
- **Left Continuous at p:** $\lim_{x \to p^-} f(x) = f(p)$.
- **Theorem 4.4.3:** f is continuous at p iff $f(p+) = f(p-) = f(p)$.

II. Classification of Discontinuities

If f is discontinuous at p, it falls into one of two main categories:

1. Simple Discontinuities (First Kind)

Both $f(p+)$ and $f(p-)$ exist.

- **Removable Discontinuity:** $f(p+)$ and $f(p-)$ exist and are equal, but differ from $f(p)$ (or $f(p)$ is undefined).

- *Example:* $g(x) = \frac{x^2-4}{x-2}$. Limit is 4, but undefined at $x = 2$. Can be "removed" by defining $g(2) = 4$.

- **Jump Discontinuity:** $f(p+) \neq f(p-)$.
  - *Example:* The greatest integer function $f(x) = [x]$. At integer n:
    - $f(n-) = n - 1$
    - $f(n+) = n$
    - Jump size is 1.

2. Discontinuities of the Second Kind

Either $f(p+)$ or $f(p-)$ (or both) does not exist.

- *Example:* $f(x) = \sin(\frac{1}{x})$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.
  - $f(0-) = 0$.
  - $f(0+)$ does not exist (oscillates between -1 and 1).

3. Specific Examples Analyzed

- **Example 4.4.5(a):** Piecewise function defined as x for $x \leq 1$ and $3 - x^2$ for $x > 1$.
  - $f(1-) = 1$, $f(1+) = 2$.
  - Left continuous, but jump discontinuity at $x = 1$.
- **Example 4.4.5(d):** $g(x) = \sin(2\pi x[x])$.
  - Continuous on $(n, n+1)$ as $[x]$ is constant n.
  - At integers n, limits from both sides go to 0, so it is continuous at integers.
  - However, it is not *uniformly* continuous on $\mathbb{R}$.

III. Monotone Functions

1. Definitions

Let f be defined on an interval I.

- **Monotone Increasing:** $x < y \implies f(x) \leq f(y)$.
- **Monotone Decreasing:** $x < y \implies f(x) \geq f(y)$.
- **Strictly Increasing:** $x < y \implies f(x) < f(y)$.

2. Theorem 4.4.7 (Existence of Limits)

If f is monotone increasing on an open interval I, then for every $p \in I$, both $f(p+)$ and $f(p-)$ exist. The following inequality holds:

$$\sup_{x<p} f(x) = f(p-) \le f(p) \le f(p+) = \inf_{x>p} f(x)$$

Furthermore, if $p < q$, then $f(p+) \le f(q-)$.

Proof of Theorem 4.4.7

- **Part 1: Existence of Left Limit $f(p-)$**
  Let $S = \{f(x) : x < p\}$. Since f is monotonic increasing, $f(p)$ serves as an upper bound for S.
    - By the **Completeness Property**, the supremum $A = \sup S$ exists, and clearly $A \le f(p)$.
    - **Convergence:** For any $\epsilon > 0$, since A is the least upper bound, $A - \epsilon$ is not an upper bound. Thus, there exists $x_0 < p$ such that $A - \epsilon < f(x_0) \le A$.
    - By monotonicity, for all $x \in (x_0, p)$, we have $f(x_0) \le f(x) \le A$.
    - Consequently, $|f(x) - A| < \epsilon$, which implies $f(p-) = A \le f(p)$.
- **Part 2: Existence of Right Limit $f(p+)$**
  By an analogous argument using the **infimum** of the set $\{f(x) : x > p\}$, we establish that the right limit exists and satisfies:

$$f(p) \le f(p+) = \inf_{x>p} f(x)$$

- **Part 3: Relation for distinct points ($p < q$)**
  Let $p < q$. Choose any x such that $p < x < q$.
  From the definitions of the one-sided limits established above:

$$f(p+) = \inf_{z>p} f(z) \le f(x) \le \sup_{z<q} f(z) = f(q-)$$

3. Corollary 4.4.8 (Countability of Discontinuities)

The set of discontinuities of a monotone function is **at most countable**.

- *Proof Logic:* Each discontinuity is a jump $(f(p-), f(p+))$. Since intervals are disjoint for distinct points, we can map each jump to a rational number within that interval. Since $\mathbb{Q}$ is countable, the set of jumps is countable.

IV. Construction of Functions with Prescribed Discontinuities

We can construct a monotone function that is discontinuous exactly at a specific countable set of points, with controlled jump sizes.

1. The Unit Jump Function $I(x)$

$$I(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

- The function $I(x)$ is **right continuous** at 0 with a unit jump: $I(0+) = 1$ and $I(0-) = 0$.
- **Shifted Function:** $I_k(x) = I(x - a_k)$ represents a unit step occurring strictly at $x = a_k$.

2. Theorem 4.4.10 (General Construction)
Let $\{x_n\}_{n=1}^{\infty}$ be a countable subset of (a, b). Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} c_n$ converges.

Define the function f on $[a, b]$ by:

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

Detailed Analysis of f:

- **Well-Defined and Monotone:**
  Since $0 \leq c_n I(x - x_n) \leq c_n$, the partial sums $s_n(x)$ are bounded above by $\sum c_n$. Thus, $f(x)$ converges for all x. Furthermore, because each term $I(x - x_n)$ is non-decreasing, f is **monotone increasing** on $[a, b]$.
  - Boundary values: $f(a) = 0$ (since $x_n > a$) and $f(b) = \sum_{n=1}^{\infty} c_n$.
- **Continuity on $[a, b] \setminus \{x_n\}$:**
  The function is continuous at any point p where $p \neq x_n$ for any n. The proof relies on splitting the set of points $E = \{x_n\}$:
  - **Isolated Points:** If p is not a limit point of E, there exists a neighborhood $(p - \delta, p + \delta)$ containing no x_n. In this interval, f is constant, and therefore continuous.
  - **Limit Points:** If p is a limit point of E, continuity is proved using the Cauchy criterion. For any $\epsilon > 0$, we can choose an integer N such that the "tail" of the series is small ($\sum_{k=N+1}^{\infty} c_k < \epsilon$). By choosing a neighborhood δ small enough to exclude the first N points, the variation of f near p is bounded by ϵ, proving continuity.
- **Right Continuity Everywhere:**
  For any point x_n (or any p), f is right continuous:

$$f(x_n+) = f(x_n)$$

This is because as $x \to x_n$ from the right, the terms $I(x - x_k)$ do not change state (they remain

1 for $x_k \leq x_n$ and 0 for $x_k > x$ locally). The variation is again controlled by the tail of the convergent series $\sum c_n$.

- **Discontinuity at $\{x_n\}$:**
  At each prescribed point x_n, the function exhibits a jump discontinuity exactly equal to the weight c_n:

$$f(x_n) - f(x_n-) = c_n$$

  - **Left Limit:** As $y \to x_n$ from the left ($y < x_n$), the term $c_n I(y - x_n)$ is 0.
  - **Value at x_n:** Exactly at x_n, the term becomes $c_n I(0) = c_n$.
  - This confirms that f has a countable number of discontinuities strictly located at the set $\{x_n\}$.

3. Examples of Construction

- **Step Function:** If $\{x_n\}$ is a finite set, f is a standard step function with finitely many jumps.
- **Rational Discontinuities:** Let $\{x_n\}$ be an enumeration of the rational numbers $\mathbb{Q} \cap (0, 1)$ and $c_n = 2^{-n}$. The resulting function is strictly increasing, discontinuous at every rational number in $(0, 1)$, and continuous at every irrational number.
- **Distribution Functions:** If the weights are normalized such that $\sum c_n = 1$, the function behaves like a cumulative distribution function (CDF) used in probability theory.

V. Inverse Functions

1. Logic of Invertibility

- **Strict Monotonicity:** Let f be a strictly increasing real-valued function on an interval I. If $x, y \in I$ with $x < y$, then $f(x) < f(y)$. The same logic applies if f is strictly decreasing.
- **Injectivity (One-to-One):** Strictly monotone functions imply that $f(x) \neq f(y)$ for any distinct $x, y \in I$. Therefore, f is one-to-one.
- **Existence:** Because f is one-to-one, it possesses an inverse function f^{-1} defined on the range $f(I)$.

2. Theorem 4.4.12: Continuity of the Inverse Function

- **Statement:** Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then the inverse function f^{-1} is strictly monotone and continuous on the interval $J = f(I)$.
- **Proof Summary:**
  i. **Image is an Interval:** Since f is continuous on the interval I, the image $J = f(I)$ must also be an interval (by Corollary 4.2.12).

ii. **Monotonicity of Inverse:** If f is strictly increasing, f^{-1} is also strictly increasing. For $y_1, y_2 \in J$ with $y_1 < y_2$, their pre-images must satisfy $x_1 < x_2$.

iii. **Continuity of Inverse:** To prove f^{-1} is continuous at $y_0 \in J$:
   - Consider an ϵ-neighborhood around $x_0 = f^{-1}(y_0)$.
   - Due to the strict monotonicity and continuity of f, we can map the interval $(x_0 - \epsilon, x_0]$ to $(y_0 - \delta, y_0]$ where $\delta = f(x_0) - f(x_0 - \epsilon)$.
   - This ensures that for y within δ of y_0, $|f^{-1}(y_0) - f^{-1}(y)| < \epsilon$, proving continuity.

3. Example 4.4.13 (n-th Roots)

- $f(x) = x^n$ is strictly increasing and continuous on $I = [0, \infty)$.
- Therefore, the inverse $g(x) = \sqrt[n]{x}$ is strictly increasing and continuous on $J = [0, \infty)$.

4. Remark (Conditions for Monotonicity)

- **The Converse:** If f is one-to-one and continuous on an interval I, then f is necessarily strictly monotone (increasing or decreasing). This results from the Intermediate Value Theorem.
- **Essential Conditions:** This converse holds **only** if:
  i. f is continuous.
  ii. The domain of f is an interval.

  If either condition is unmet, a one-to-one function might not be strictly monotone.

Chapter 5.1: The Derivative

1. Definition of the Derivative

Historical Context: Formulated rigorously by Cauchy (1821) using limits, moving away from vague notions of tangent lines and velocity.

Definition 5.1.1 (The Derivative)
Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. For a fixed $p \in I$, the derivative $f'(p)$ is:

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

Alternatively, letting $x = p + h$:

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

- **Geometric interpretation:** Slope of the tangent line at $(p, f(p))$.
- **Physical interpretation:** Instantaneous velocity.

2. One-Sided Derivatives

If p is an endpoint or we need to analyze corner points, we use one-sided limits (Definition 5.1.2).

- **Right Derivative ($f'_+(p)$):** $\lim_{h \to 0^+} \frac{f(p+h) - f(p)}{h}$
- **Left Derivative ($f'_-(p)$):** $\lim_{h \to 0^-} \frac{f(p+h) - f(p)}{h}$

Key Property:
For an interior point $p \in I$, $f'(p)$ exists **if and only if** both $f'_+(p)$ and $f'_-(p)$ exist and are **equal**.

3. Worked Examples (Specific Functions)

The text analyzes the differentiability of several specific functions to illustrate the definition.

A. Power Function ($f(x) = x^2$)

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} (2x + h) = 2x$$

(Note: Generalizes to $f(x) = x^n \implies f'(x) = nx^{n-1}$).

B. Square Root ($f(x) = \sqrt{x}, x > 0$)

Uses rationalization:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \xrightarrow{h \to 0} \frac{1}{2\sqrt{x}}$$

C. Sine Function ($f(x) = \sin x$)

Uses the identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right)$$

Since $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$, $f'(x) = \cos x$.

D. Absolute Value ($f(x) = |x|$) at $x = 0$

- $f'_+(0) = \lim_{h \to 0^+} \frac{h}{h} = 1$
- $f'_-(0) = \lim_{h \to 0^-} \frac{-h}{h} = -1$
- **Result:** $f'_+(0) \neq f'_-(0)$, so f is **not differentiable** at 0.

E. Cusp ($g(x) = x^{3/2}$) at $x = 0$

$$g'(0) = \lim_{h \to 0^+} \frac{h^{3/2}}{h} = \lim_{h \to 0^+} \sqrt{h} = 0$$

Differentiable at 0.

F. Oscillating Discontinuity ($f(x) = x \sin(1/x)$ for $x \neq 0$, $f(0) = 0$)

$$f'(0) = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin \left(\frac{1}{h} \right)$$

Result: Limit does not exist. Not differentiable at 0.

G. Differentiable with Discontinuous Derivative ($g(x) = x^2 \sin(1/x)$ for $x \neq 0$, $g(0) = 0$)

- **At $x = 0$:**

$$g'(0) = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

  (By Squeeze Theorem).
- **For $x \neq 0$:** $g'(x) = 2x \sin(1/x) - \cos(1/x)$.
- **Observation:** $\lim_{x \to 0} g'(x)$ does not exist (due to $\cos(1/x)$).
- **Conclusion:** g is differentiable everywhere, but g' is **not continuous** at 0.

4. Relationship between Differentiability and Continuity

Theorem 5.1.4:
If f is differentiable at p, then f is continuous at p.

Proof Sketch:

$$\lim_{t \to p}(f(t) - f(p)) = \lim_{t \to p}\left[\frac{f(t) - f(p)}{t - p} \cdot (t - p)\right] = f'(p) \cdot 0 = 0$$

Thus $\lim_{t \to p} f(t) = f(p)$.

- **Converse:** False. Continuity does **not** imply differentiability (e.g., $f(x) = |x|$).
- **Weierstrass Function:** An example of a function continuous everywhere but differentiable nowhere.

5. Arithmetic of Derivatives

Theorem 5.1.5 (Algebraic Rules)
Let f, g be differentiable at x.

A. Sum Rule

$$(f + g)'(x) = f'(x) + g'(x)$$

B. Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Key Proof Idea: The "Add and Subtract" Trick
Direct substitution creates a mixed term $f(x+h)g(x+h)$ that cannot be factored. To fix this, we insert a "middle term" into the numerator.

Proof Sketch:

1. **Setup:** Start with the difference quotient:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

2. **The Trick:** Add and subtract $f(x+h)g(x)$:

$$\frac{f(x+h)g(x+h) - \mathbf{f(x+h)g(x)} + \mathbf{f(x+h)g(x)} - f(x)g(x)}{h}$$

3. **Group & Limit:** Separate into two parts:

$$f(x+h)\underbrace{\left[\frac{g(x+h)-g(x)}{h}\right]}_{\to g'(x)} + g(x)\underbrace{\left[\frac{f(x+h)-f(x)}{h}\right]}_{\to f'(x)}$$

Note: $f(x+h) \to f(x)$ because differentiability implies continuity.

C. Quotient Rule (Reciprocal Case)

To prove the quotient rule, we first focus on the derivative of the reciprocal.

The Reciprocal Rule:

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{[g(x)]^2}$$

Proof Sketch:

1. **Common Denominator:**

$$\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \frac{g(x) - g(x+h)}{h \cdot g(x+h)g(x)}$$

2. **Extract Derivative Definition:**
   Recognize that $g(x) - g(x+h) = -[g(x+h) - g(x)]$:

$$-\underbrace{\left[\frac{g(x+h) - g(x)}{h}\right]}_{\to g'(x)} \cdot \frac{1}{g(x+h)g(x)}$$

3. **Take Limit ($h \to 0$):**
   Since g is continuous, $g(x+h) \to g(x)$, giving the denominator $[g(x)]^2$.

(Note: The full Quotient Rule is simply the Product Rule applied to $f(x) \cdot [1/g(x)]$.)

6. The Chain Rule

Theorem 5.1.6 (Composition)

If f is differentiable at x and g is differentiable at $y = f(x)$, then $h = g \circ f$ is differentiable at x:

$$h'(x) = g'(f(x)) \cdot f'(x)$$

Proof: Linear Approximation (to avoid division by zero).

Standard limits fail if $f(t) - f(x) = 0$. Instead, use "error terms" (u, v) that go to 0:

- **f:** $f(t) - f(x) = (t - x)[f'(x) + u(t)]$
- **g:** $g(s) - g(y) = (s - y)[g'(y) + v(s)]$
- Set $s = f(t)$ and $y = f(x)$. The difference quotient becomes:

$$\frac{g(f(t)) - g(f(x))}{t - x} = [f'(x) + u(t)] \cdot [g'(y) + v(f(t))]$$

- As $t \to x$, the error terms vanish ($u, v \to 0$), leaving $f'(x)g'(y)$.

Examples:

- **Composite Trig:** $h(x) = \sin(1/x^2)$.

$$h'(x) = \cos\left(\frac{1}{x^2}\right) \cdot \frac{d}{dx}(x^{-2}) = \cos\left(\frac{1}{x^2}\right) \cdot (-2x^{-3})$$

- **Power Rule Extension:** $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$.

Chapter 5.2: The Mean Value Thm.

1. Local Maxima and Minima

Definition 5.2.1:
A function $f : E \to \mathbb{R}$ has a:

- **Local maximum** at $p \in E$ if $\exists \delta > 0$ such that $f(x) \leq f(p)$ for all $x \in E \cap N_\delta(p)$.
- **Absolute maximum** if $f(x) \leq f(p)$ for all $x \in E$.
  (Analogous definitions apply for minimums).

Theorem 5.2.2 (Relationship to Derivative):
Let f be defined on interval I. If f has a local extremum at an interior point $p \in \text{Int}(I)$ and f is differentiable at p, then:

$$f'(p) = 0$$

- **Proof Idea:** Analyze the difference quotient. If p is a max, $\frac{f(t)-f(p)}{t-p} \leq 0$ for $t > p$ (implies $f'_+(p) \leq 0$) and ≥ 0 for $t < p$ (implies $f'_-(p) \geq 0$). Since $f'(p)$ exists, limits must be equal, thus 0.

Corollary 5.2.3:
For continuous f on $[a, b]$, relative extrema at $p \in (a, b)$ imply either $f'(p)$ does not exist or $f'(p) = 0$.

- *Note:* This does not apply to endpoints. At endpoints, we can only conclude inequality (e.g., if max at a, $f'(a) \leq 0$).

2. Rolle's Theorem

Theorem 5.2.5 (Rolle's Theorem):
Suppose f is:

1. Continuous on $[a, b]$
2. Differentiable on (a, b)
3. $f(a) = f(b)$

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

- **Geometric Interpretation:** There is at least one point where the tangent line is horizontal.
- **Proof Idea:**
  - Since $[a, b]$ is compact, f attains max and min (EVT).
  - If f is constant, $f'(x) = 0$ everywhere.
  - If not constant, extremum occurs at an interior point c. By Theorem 5.2.2, $f'(c) = 0$.
- **Example:** $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$. Derivative undefined at endpoints, but theorem holds ($c = 0$).

3. The Mean Value Theorem (MVT)

Theorem 5.2.6 (Lagrange):
Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that:

$$f(b) - f(a) = f'(c)(b - a)$$

- **Geometric Interpretation:** There is a point c where the tangent slope equals the secant slope connecting $(a, f(a))$ and $(b, f(b))$.
- **Proof Idea:**
  Construct an auxiliary function $g(x)$ representing the vertical distance between the curve and the secant line:

  $$g(x) = f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

  Since $g(a) = g(b) = 0$, applying Rolle's Theorem to g yields $g'(c) = 0$, which rearranges to the MVT equation.

Example 5.2.7 (Inequalities):
Using MVT to prove $\frac{x}{1+x} \leq \ln(1 + x) \leq x$ for $x > -1$.

- Let $f(x) = \ln(1 + x)$. $f(0) = 0$.
- By MVT, $\ln(1 + x) = f(x) - f(0) = f'(c)x = \frac{x}{1+c}$ for some c between 0 and x.
- Analyze bounds of $\frac{1}{1+c}$ based on $0 < c < x$ or $x < c < 0$ to derive the inequality.

4. Cauchy Mean Value Theorem

Theorem 5.2.8:
If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If $g'(x) \neq 0$, this can be written as:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

- **Proof Idea:** Define $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. Since $h(a) = h(b)$, apply Rolle's Theorem to find $h'(c) = 0$.
- **Geometric Interpretation:** For a parametric curve defined by $(g(t), f(t))$, there is a point where the tangent slope equals the slope of the chord connecting the endpoints.

5. Applications: Monotonicity

1. Monotonicity on Intervals (Theorem 5.2.9)

Let f be differentiable on an interval I. The sign of the derivative determines the monotonicity of the function on that interval:

- $f'(x) \geq 0, \ \forall x \in I \implies f$ is **monotone increasing**.
- $f'(x) > 0, \ \forall x \in I \implies f$ is **strictly increasing**.
- $f'(x) \leq 0, \ \forall x \in I \implies f$ is **monotone decreasing**.
- $f'(x) = 0, \ \forall x \in I \implies f$ is **constant**.

Proof Idea: Apply the Mean Value Theorem (MVT) to arbitrary $x_1 < x_2$. The sign of $f(x_2) - f(x_1)$ is determined entirely by $f'(c)(x_2 - x_1)$.

2. Pointwise vs. Neighborhood Behavior (Crucial Distinction)

Observation:
The condition $f'(c) > 0$ at a **single point** c behaves differently than $f'(x) > 0$ on an **interval**.

A. What $f'(c) > 0$ implies:
If $f'(c) > 0$, there exists a $\delta > 0$ such that:

- $f(x) < f(c)$ for all $x \in (c - \delta, c)$
- $f(x) > f(c)$ for all $x \in (c, c + \delta)$
  (See Exercise 17).

B. What $f'(c) > 0$ does NOT imply:
It does **not** imply that f is increasing on the interval $(c - \delta, c + \delta)$.

- **Reason:** $f'(x)$ may assume both positive and negative values in every neighborhood of c.
- **Counter-Example (Exercise 20):**

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  Here, $f'(0) = 1 > 0$, yet f is **not monotone** on any interval containing 0 due to rapid oscillation.

C. Sufficient Condition for Monotonicity:
If we require **continuity** of the derivative:

- If $f'(c) > 0$ **AND** f' is continuous at c, then there exists a $\delta > 0$ such that $f'(x) > 0$ for all $x \in (c - \delta, c + \delta)$.
- $\therefore f$ is increasing on $(c - \delta, c + \delta)$.

3. Relative Extrema & The First Derivative Test

The First Derivative Test:
Used to classify critical points where $f'(c) = 0$ or $f'(c)$ does not exist.
Suppose f is continuous on (a, b).

1. If $f'(x) < 0$ on (a, c) AND $f'(x) > 0$ on (c, b):
   - f is decreasing to the left and increasing to the right.
   - $\implies f$ has a **relative minimum** at c.
2. (Similarly for relative maximum if signs switch from $+$ to $-$).

The False Converse:
One naturally assumes: *"If f has a relative minimum at c, then f must be decreasing to the left and increasing to the right."*

- **This is FALSE.**
- A function can have a relative minimum at c without being monotone on the immediate left or right sides.

Counter-Example (Example 5.2.10):

$$f(x) = \begin{cases} x^4(2+\sin(1/x)) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- f has an absolute minimum at $x=0$ (since $f(x) > 0$ for $x \neq 0$).
- However, $f'(x)$ oscillates between positive and negative values in every neighborhood of 0.
- Thus, f is **not** "decreasing then increasing" in the standard monotonic sense near 0.

6. Limits of Derivatives

Theorem 5.2.11:
If f is continuous on $[a,b)$ and differentiable on (a,b), and $\lim_{x\to a^+} f'(x) = L$, then the right-hand derivative exists and:

$$f'_+(a) = \lim_{x\to a^+} f'(x)$$

- **Proof Idea:** Use MVT on $[a, a+h]$. $f(a+h) - f(a) = f'(\zeta_h)h$. As $h \to 0$, $\zeta_h \to a$, so the difference quotient converges to L.
- **Implication:** Derivatives cannot have simple jump discontinuities; discontinuities must be of the second kind (oscillatory).

7. Intermediate Value Theorem for Derivatives

Theorem 5.2.13 (Darboux's Theorem):
If f is differentiable on I and $a, b \in I$ with $a < b$, then for any λ between $f'(a)$ and $f'(b)$, there exists $c \in (a,b)$ such that:

$$f'(c) = \lambda$$

- **Significance:** Derivatives possess the Intermediate Value Property even if they are **not continuous**.

- **Proof Idea:** Construct $g(x) = f(x) - \lambda x$. Depending on signs of $g'(a)$ and $g'(b)$, g attains a local extremum interior to the interval. At that extremum c, $g'(c) = f'(c) - \lambda = 0$.

8. Inverse Function Theorem

Theorem 5.2.14:
If f is differentiable on interval I and $f'(x) \neq 0$ for all $x \in I$:

1. f is one-to-one.
2. f^{-1} is continuous and differentiable on $J = f(I)$.
3. The derivative is given by:

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

- **Proof Idea:** $f' \neq 0$ implies f' maintains a single sign (by Darboux's Theorem), so f is strictly monotone (one-to-one). Differentiability follows from limits.

Section 5.3 L'Hospital's Rule

1. Infinite Limits (Definitions)

Before introducing L'Hospital's rule, formal definitions for infinite limits are established to handle the indeterminate form ∞/∞.

- **Definition 5.3.1:** Let f be defined on a subset $E \subset \mathbb{R}$ and p be a limit point of E.
    - $\lim_{x\to p} f(x) = \infty$ if for every $M \in \mathbb{R}$, there exists $\delta > 0$ such that $f(x) > M$ for all $x \in E$ with $0 < |x - p| < \delta$.
    - $\lim_{x\to p} f(x) = -\infty$ is defined similarly (where $f(x) < M$).
- **Note:** These definitions extend to limits at infinity ($\lim_{x\to\infty}$) and one-sided limits ($\lim_{x\to p^+}$).

2. L'Hospital's Rule (Theorem 5.3.2)

This rule evaluates limits of indeterminate forms $0/0$ or ∞/∞.

Hypotheses:

1. f, g are real-valued differentiable functions on (a, b).
2. $g'(x) \neq 0$ for all $x \in (a, b)$.
3. $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where $L \in \mathbb{R} \cup \{-\infty, \infty\}$.

Conditions (Indeterminate Forms):

- **(a) Case $0/0$:** $\lim_{x\to a^+} f(x) = 0$ and $\lim_{x\to a^+} g(x) = 0$.
- **(b) Case ∞/∞:** $\lim_{x\to a^+} g(x) = \pm\infty$ (Note: $f(x)$ does not strictly need to tend to ∞, but usually does).

Conclusion:

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

(Note: The rule applies equally to $x \to b^-$, $x \to p$, or $x \to \pm\infty$).

3. Proofs: Core Ideas

Case A: Indeterminate Form $0/0$ (Finite a)

- **Key Tool:** Generalized Mean Value Theorem (GMVT).
- **Method:**
  i. Since we are dealing with limits approaching a, we define $f(a) = g(a) = 0$ to make the functions continuous at a.
  ii. Consider a sequence $\{x_n\} \to a^+$. Apply GMVT on the interval $[a, x_n]$.
  iii. There exists c_n between a and x_n such that:

$$\frac{f(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(c_n)}$$

  iv. Since $f(a) = g(a) = 0$, this simplifies to $\frac{f(x_n)}{g(x_n)} = \frac{f'(c_n)}{g'(c_n)}$.
  v. As $n \to \infty$, $x_n \to a$ implies $c_n \to a$. Therefore, the limit of the ratio of functions equals the limit of the ratio of derivatives.

Case B: Limits at Infinity ($x \to -\infty$)

- **Key Tool:** Substitution.
- **Method:** Let $x = -1/t$. As $t \to 0^+$, $x \to -\infty$.
  - Define $\phi(t) = f(-1/t)$ and $\psi(t) = g(-1/t)$.
  - Using chain rule differentiation, the problem reduces to a limit at 0^+, which allows the use of the previous proof logic.

Case C: Indeterminate Form ∞/∞

- **Key Tool:** GMVT + Bounding Argument (No need for f, g to be continuous at a).
- **Method:**
  i. Assume $\lim \frac{f'(x)}{g'(x)} = L$.
  ii. Fix y and let x vary. Apply GMVT on interval (x, y) to get $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(\zeta)}{g'(\zeta)}$.
  iii. **Algebraically rearrange** the GMVT equation to isolate $\frac{f(x)}{g(x)}$:

$$\frac{f(x)}{g(x)} = \frac{f'(\zeta)}{g'(\zeta)} \left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)}$$

  iv. Since $g(x) \to \infty$ as $x \to a^+$, the term $\frac{g(y)}{g(x)} \to 0$ and $\frac{f(y)}{g(x)} \to 0$.
  v. This implies that for x sufficiently close to a, $\frac{f(x)}{g(x)}$ becomes arbitrarily close to $\frac{f'(\zeta)}{g'(\zeta)} \cdot (1 - 0) + 0$, which converges to L.

4. Examples

(a) Basic Application ($0/0$)

Problem: Compute $\lim_{x \to 0^+} \frac{\ln(1+x)}{x}$.

- **Form:** $0/0$ (since $\ln(1) = 0$).
- **Apply L'Hospital's:**

$$\lim_{x \to 0^+} \frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = \lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1} = 1$$

- **Note:** This can also be proven using inequalities derived from the Taylor expansion or Mean Value Theorem (i.e., $\frac{x}{1+x} \leq \ln(1+x) \leq x$), but L'Hospital's is more direct.

(b) Repeated Application

Problem: Compute $\lim_{x \to 0} \frac{1-\cos x}{x^2}$.

- **Form:** $0/0$.
- **First Application:**

$$\lim_{x \to 0} \frac{\sin x}{2x}$$

  This is *still* form $0/0$.
- **Second Application:**

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

- **Result:** The limit is $1/2$.

(c) Importance of Substitution (Avoiding Complexity)

Problem: Compute $\lim_{x \to 0^+} \frac{e^{-1/x}}{x}$.

- **Form:** $0/0$ (since $e^{-\infty} \to 0$).
- **Direct L'Hospital's Failure:** Differentiating directly gives $\frac{e^{-1/x} \cdot (1/x^2)}{1}$, which simplifies to $\frac{e^{-1/x}}{x^2}$. This is *more* complicated than the original.
- **Correct Approach:** Use substitution $t = 1/x$. As $x \to 0^+$, $t \to \infty$.

$$\lim_{t \to \infty} \frac{t}{e^t}$$

- **New Form:** ∞/∞.
- **Apply L'Hospital's:**

$$\lim_{t \to \infty} \frac{1}{e^t} = 0$$

- **Result:** The original limit is 0.