Here is a detailed, mathematically rigorous study note based on the provided text, covering **Sections 8.5 through 8.8**.

8.5 Uniform Convergence and Differentiation

This section addresses the conditions under which limits and differentiation can be interchanged.

The Problem of Interchange

Uniform convergence of a sequence $\{f_n\}$ to f is **not sufficient** to guarantee that $\{f_n'\}$ converges to f'.

- **Example:** Consider $f_n(x) = \frac{x^n}{n}$ on $[0,1]$. $f_n \to 0$ uniformly, but $f_n'(1) = 1$ while $f'(1) = 0$.

Theorem 8.5.1: Sufficient Conditions

Let $\{f_n\}$ be a sequence of differentiable functions on $[a,b]$. If:

1. $\{f_n'\}$ converges **uniformly** on $[a,b]$, and
2. $\{f_n(x_0)\}$ converges for at least one point $x_0 \in [a,b]$,

Then $\{f_n\}$ converges uniformly to a function f on $[a,b]$, and $f'(x) = \lim_{n\to\infty} f_n'(x)$.

Proof Idea:
The proof relies on the **Mean Value Theorem (MVT)** applied to the difference $f_n - f_m$.

1. Let $\epsilon > 0$. Using the Cauchy criterion for uniform convergence of f_n', for large n, m, $|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)}$.
2. By MVT, $|(f_n(x) - f_m(x)) - (f_n(y) - f_m(y))| \le |x-y| \sup |f_n' - f_m'|$.
3. This inequality allows showing that $\{f_n\}$ is uniformly Cauchy. Let f be the limit.
4. To prove $f'(p) = \lim f_n'(p)$, define difference quotients $g_n(t)$. Uniform convergence of derivatives allows swapping the limit order: $\lim_{t\to p} \lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} \lim_{t\to p} g_n(t)$.

Example 8.5.3: A Continuous Nowhere Differentiable Function

Weierstrass constructed a function that is continuous everywhere but differentiable nowhere.

$$f(x) = \sum_{k=0}^{\infty} \frac{\cos(a^k \pi x)}{2^k}$$

where a is an odd integer and $a > 3\pi + 2$.

- **Continuity:** Since $|\frac{\cos(\dots)}{2^k}| \leq \frac{1}{2^k}$, the series converges uniformly by the Weierstrass M-test.
- **Non-differentiability:** The derivative fails to exist because the oscillations become infinitely steep at every scale. The proof involves constructing a sequence $h_n \to 0$ such that the difference quotient $\frac{f(x+h_n)-f(x)}{h_n} \to \infty$.

8.6 The Weierstrass Approximation Theorem

Theorem 8.6.1: If f is a continuous real-valued function on $[a, b]$, then for every $\epsilon > 0$, there exists a polynomial P such that $|f(x) - P(x)| < \epsilon$ for all $x \in [a, b]$.

- *Interpretation:* Polynomials are dense in the space of continuous functions $C[a, b]$ under the uniform norm.

To prove this, we utilize **Approximate Identities**.

Periodic Functions and Approximate Identities

Definition: A sequence $\{Q_n\}$ of nonnegative Riemann integrable functions on $[-1, 1]$ is an **Approximate Identity** (or Dirac sequence) if:

1. $\int_{-1}^{1} Q_n(t)dt = 1$
2. $\lim_{n\to\infty} \int_{\{\delta \leq |t|\}} Q_n(t)dt = 0$ for every $\delta > 0$.

Theorem 8.6.5: Let f be a continuous, periodic function (period 2). The convolution

$$S_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt$$

converges uniformly to $f(x)$ on $\mathbb{R}$.

Proof Idea:

1. Write $S_n(x) - f(x) = \int_{-1}^{1} [f(x+t) - f(x)] Q_n(t) dt$ (using property 1).
2. Split the integral into $[-\delta, \delta]$ and the "tails" $\{\delta \leq |t|\}$.
3. Near 0, continuity ensures $|f(x+t) - f(x)|$ is small.
4. In the tails, the approximate identity property ensures the integral vanishes as $n \to \infty$.

Proof of Weierstrass Approximation Theorem

1. **Normalization:** Transform f defined on $[a, b]$ to $[0, 1]$. Adjust f so $f(0) = f(1) = 0$ and extend it to be periodic on $\mathbb{R}$.
2. **Specific Kernel:** Choose the polynomial kernel $Q_n(t) = c_n(1 - t^2)^n$, where c_n normalizes the integral to 1.
3. **Estimate:** It is shown that $c_n < \sqrt{n}$, ensuring the mass concentrates at 0 properly.
4. **Polynomial Result:** The convolution $P_n(x) = \int_{-1}^{1} f(x+t) Q_n(t) dt$ becomes a polynomial in x because $Q_n(t)$ is a polynomial.

8.7 Power Series Expansions

Definition: A series $\sum_{k=0}^{\infty} a_k (x - c)^k$ is a power series centered at c.

Radius of Convergence

The radius of convergence R is defined by the Cauchy-Hadamard formula:

$$\frac{1}{R} = \limsup_{k \to \infty} \sqrt[k]{|a_k|}$$

Alternatively, using the ratio test (if the limit exists): $\frac{1}{R} = \lim \left|\frac{a_{k+1}}{a_k}\right|$.

Theorem 8.7.3:

- If $|x - c| < R$, the series converges absolutely.
- If $|x - c| > R$, the series diverges.
- If $0 < \rho < R$, the series converges **uniformly** on $|x - c| \leq \rho$.

Abel's Theorem (Theorem 8.7.5)

If a power series (radius R) converges at an endpoint (e.g., $x = c + R$), then the function defined by the series is **continuous** at that endpoint.

$$\lim_{x \to R^-} \sum a_k x^k = \sum a_k R^k$$

Differentiation and Uniqueness

Theorem 8.7.7: A power series can be differentiated term-by-term within its radius of convergence.

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - c)^{k-1}$$

- The derived series has the **same** radius of convergence R.
- **Corollary:** Functions defined by power series are infinitely differentiable (C^∞) inside the interval of convergence.
- **Uniqueness:** If $\sum a_k x^k = \sum b_k x^k$ inside a neighborhood, then $a_k = b_k$ for all k. Specifically, $a_k = \frac{f^{(k)}(c)}{k!}$.

Taylor Series and Remainder Estimates

Given $f \in C^\infty$, the **Taylor Series** is $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$.
Convergence to $f(x)$ depends on the remainder $R_n(x) = f(x) - T_n(x)$.

Counter-example (Cauchy):
$f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$.
f is C^∞ and $f^{(n)}(0) = 0$ for all n. The Taylor series is identically 0, which does not converge to $f(x)$ (except at $x = 0$).

Remainder Formulas:
To prove convergence ($R_n \to 0$), we use specific forms of the remainder:

1. **Lagrange Form:** Exists ζ between c and x such that:

$$R_n(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x - c)^{n+1}$$

   (Derived via repeated Mean Value Theorem)
2. **Integral Form:**

$$R_n(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)(x-t)^n dt$$

(Derived via Induction and Integration by Parts)

3. **Cauchy Form:** Exists ζ between c and x such that:

$$R_n(x) = \frac{f^{(n+1)}(\zeta)}{n!}(x-c)(x-\zeta)^n$$

Applications:

- **Binomial Series:** $(1+x)^n = \sum \binom{n}{k} x^k$.
- **Sine:** $\sin x = \sum \frac{(-1)^k}{(2k+1)!} x^{2k+1}$. Convergence shown because $|f^{(n+1)}(\zeta)| \leq 1$ and $\frac{x^n}{n!} \to 0$.
- **Natural Log:** $\ln(1+x) = \sum \frac{(-1)^{n+1}}{n} x^n$. Convergence on $(-1, 1]$.
  - For $x \in [0, 1]$, Lagrange form suffices.
  - For $x \in (-1, 0)$, Cauchy form is required to bound the remainder.

8.8 The Gamma Function

Definition: For $x > 0$,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Properties (Theorem 8.8.2)

1. **Functional Equation:** $\Gamma(x+1) = x\Gamma(x)$.
   - *Proof:* Integration by parts with $u = t^x, dv = e^{-t} dt$.
2. **Factorial Generalization:** $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.
3. **Value at 1/2:** $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
   - *Proof:* Substitution $t = s^2$ transforms the integral into a Gaussian integral $\int e^{-s^2} ds$, evaluated via polar coordinates in double integrals.

The Binomial Series (General)

For any real $\alpha > 0$:

$$\frac{1}{(1-x)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!} x^n \quad \text{for } |x| < 1$$

This generalizes the binomial theorem to non-integer exponents.

The Beta Function

Defined as $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ for $x, y > 0$.
Theorem 8.8.5: Relationship to Gamma:

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$