Section 6.1 The Riemann Integral

1. Upper and Lower Sums

Definition (Partition):
A partition $\mathcal{P}$ of a closed interval $[a, b]$ is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ such that:

$$a = x_0 < x_1 < \cdots < x_n = b$$

Let $\Delta x_i = x_i - x_{i-1}$.

Definition (Bounds):
For a bounded function f on $[a, b]$, on each subinterval $[x_{i-1}, x_i]$, define:

- $m_i = \inf\{f(t) : x_{i-1} \leq t \leq x_i\}$
- $M_i = \sup\{f(t) : x_{i-1} \leq t \leq x_i\}$

Definition (Sums):

- **Lower Sum:** $\mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i$ (Inscribed rectangles)
- **Upper Sum:** $\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i$ (Circumscribed rectangles)

Basic Inequality:
For any partition $\mathcal{P}$:

$$\mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}, f)$$

2. Upper and Lower Integrals

Boundedness:
If $m \leq f(t) \leq M$ for all t, then for any partition $\mathcal{P}$:

$$m(b-a) \leq \mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}, f) \leq M(b-a)$$

Refinement Lemma:
A partition $\mathcal{P}^*$ is a **refinement** of $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}^*$. Adding points improves the approximation:

$$\mathcal{L}(\mathcal{P}, f) \leq \mathcal{L}(\mathcal{P}^*, f) \leq \mathcal{U}(\mathcal{P}^*, f) \leq \mathcal{U}(\mathcal{P}, f)$$

Key Proof Idea: Adding a point x^* in $[x_{k-1}, x_k]$ splits the interval. Since the infimum over a subset is larger (or equal) and the supremum is smaller (or equal), the lower sum increases and the upper sum decreases.

Comparison of Any Two Partitions:
For *any* partitions $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{Q}, f)$.
(Proof uses the common refinement $\mathcal{P} \cup \mathcal{Q}$).

Definition (Integrals):

- **Lower Integral:** $\underline{\int_a^b} f = \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f)$
- **Upper Integral:** $\overline{\int_a^b} f = \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f)$

Theorem 6.1.4:
Always, $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

3. Definition of the Riemann Integral

Definition 6.1.5:
A bounded function f is **Riemann integrable** on $[a, b]$ (denoted $f \in \mathcal{R}[a, b]$) if:

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

The common value is denoted $\int_a^b f(x) dx$.

4. Examples

(a) Dirichlet Function (Not Integrable):
$f(x) = 1$ if $x \in \mathbb{Q}$, 0 if $x \notin \mathbb{Q}$ on $[a, b]$.

- For any interval, density of rationals/irrationals implies $m_i = 0$ and $M_i = 1$.

- $\mathcal{L}(\mathcal{P}, f) = 0 \implies \underline{\int} f = 0$.
- $\mathcal{U}(\mathcal{P}, f) = b - a \implies \overline{\int} f = b - a$.
- Since $0 \neq b - a$, $f \notin \mathcal{R}[a, b]$.

(b) Step Function:
$f(x) = 0$ for $x < 1/2$, $f(x) = 1$ for $x \geq 1/2$ on $[0, 1]$.

- Proof involves isolating the discontinuity at $1/2$ within a small interval of the partition. The error term $\mathcal{U} - \mathcal{L}$ can be made arbitrarily small by shrinking the interval covering $1/2$.
- Result: $\int_0^1 f = 1/2$.

(c) $f(x) = x$ on $[a, b]$:

- f is increasing $\implies m_i = x_{i-1}, M_i = x_i$.
- $\mathcal{U} - \mathcal{L} = \sum(x_i - x_{i-1})\Delta x_i$.
- By choosing equal width $\Delta x, \mathcal{U} - \mathcal{L} = \Delta x(b - a)$. As $\Delta x \to 0$, difference goes to 0.
- $\int_a^b x dx = \frac{1}{2}(b^2 - a^2)$.

(d) $f(x) = x^2$ on $[0, 1]$:

- Uses equal partitions. $m_i = (\frac{i-1}{n})^2, M_i = (\frac{i}{n})^2$.
- Calculations use sum of squares formula $\sum i^2 = \frac{1}{6}m(m+1)(2m+1)$.
- Taking limits as $n \to \infty$, both sums converge to $1/3$.

5. Riemann's Criterion for Integrability

Theorem 6.1.7:
$f \in \mathcal{R}[a, b]$ **if and only if** for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that:

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon$$

- *Significance:* Allows checking integrability without knowing the value of the integral.
- *Proof Key Idea:*
  - $\Rightarrow$: Use definitions of supremum and infimum to find partitions close to the integrals, then combine them.
  - $\Leftarrow$: If $\mathcal{U} - \mathcal{L} < \epsilon$, then $0 \leq \overline{\int} - \underline{\int} < \epsilon$ for all ϵ, forcing equality.

6. Integrability of Specific Classes of Functions

Theorem 6.1.8:

1. **Continuous Functions:** If f is continuous on $[a, b]$, it is integrable.
   - *Proof Idea:* Use **Uniform Continuity**. Given ϵ, choose δ such that $|x - t| < \delta \implies |f(x) - f(t)| < \frac{\epsilon}{b-a}$. Then $M_i - m_i < \frac{\epsilon}{b-a}$. Summing gives $\mathcal{U} - \mathcal{L} < \epsilon$.
2. **Monotone Functions:** If f is monotone on $[a, b]$, it is integrable.
   - *Proof Idea:* Telescoping sum. For uniform partition with width h, $\sum(M_i - m_i)\Delta x_i = h\sum(f(x_i) - f(x_{i-1})) = h(f(b) - f(a))$. We can make h small enough to satisfy the ϵ condition.

7. The Composition Theorem

Theorem 6.1.9
Let $f \in \mathcal{R}[a, b]$ with range in $[c, d]$ and let $\varphi : [c, d] \to \mathbb{R}$ be **continuous**. Then the composition $\varphi \circ f$ is Riemann integrable on $[a, b]$ (i.e., $\varphi \circ f \in \mathcal{R}[a, b]$).

Proof Strategy

Since φ is continuous on a closed interval, it is **uniformly continuous** and **bounded**. We use these properties to control the upper and lower sums of the composition.

1. Setup & Uniform Continuity

- Let $K = \sup\{|\varphi(t)| : t \in [c, d]\}$.
- Let $\epsilon > 0$ be given. Define a strict tolerance $\epsilon' = \frac{\epsilon}{b-a+2K}$.
- By **uniform continuity** of φ, there exists $\delta \in (0, \epsilon')$ such that:

$$|s - t| < \delta \implies |\varphi(s) - \varphi(t)| < \epsilon'$$

2. Partitioning & The "Small Oscillation" Trick

Since f is integrable, choose a partition $P = \{x_0, \dots, x_n\}$ such that:

$$U(P, f) - L(P, f) < \delta^2$$

Note: We use δ^2 to force the total width of "bad" intervals to be small later.

3. Splitting the Indices (The Crucial Step)

Let M_k, m_k be bounds for f, and M_k^*, m_k^* be bounds for $\varphi \circ f$ on interval k.
We split the partition indices $k \in \{1, \ldots, n\}$ into two disjoint sets:

- **Set A (Small Oscillation):** k where $M_k - m_k < \delta$.
  - Here, $f(t)$ varies little, so $\varphi(f(t))$ varies by less than ϵ'.
  - $\implies M_k^* - m_k^* \leq \epsilon'$.
- **Set B (Large Oscillation):** k where $M_k - m_k \geq \delta$.
  - Here, we only know φ is bounded by K.
  - $\implies M_k^* - m_k^* \leq 2K$.

4. The Estimates

We split the difference between Upper and Lower sums:

$$U(P, \varphi \circ f) - L(P, \varphi \circ f) = \sum_{k \in A} (M_k^* - m_k^*) \Delta x_k + \sum_{k \in B} (M_k^* - m_k^*) \Delta x_k$$

- **Estimate for A (Good behavior):**

$$\sum_A (M_k^* - m_k^*) \Delta x_k \leq \epsilon' \sum_A \Delta x_k \leq \epsilon'(b-a)$$

- **Estimate for B (Bad behavior, but small width):**
  First, bound the total width of set B:

$$\sum_B \delta \Delta x_k \leq \sum_B (M_k - m_k) \Delta x_k \leq U(P, f) - L(P, f) < \delta^2$$

  Dividing by δ, we get $\sum_B \Delta x_k < \delta < \epsilon'$.
  Thus:

$$\sum_B (M_k^* - m_k^*) \Delta x_k \leq 2K \sum_B \Delta x_k < 2K\epsilon'$$

5. Conclusion

Combining the sums:

$$U - L < \epsilon'(b-a) + 2K\epsilon' = \epsilon'(b - a + 2K) = \epsilon$$

Since the difference is arbitrarily small, $\varphi \circ f \in \mathcal{R}[a, b]$. ■

Corollary 6.1.10
If $f \in \mathcal{R}[a, b]$, then:

1. $|f| \in \mathcal{R}[a, b]$ (using $\varphi(t) = |t|$)
2. $f^2 \in \mathcal{R}[a, b]$ (using $\varphi(t) = t^2$)

Important Warning:
The composition of two Riemann integrable functions is **not** necessarily integrable.

- *Counter-example (6.1.14b):* If f is the Riemann function and g is an indicator function, $g \circ f$ may fail integrability.

8. Lebesgue's Theorem

Definition (Measure Zero):
A set $E \subset \mathbb{R}$ has measure zero if for any $\epsilon > 0$, E can be covered by a countable union of open intervals $\{I_n\}$ such that $\sum length(I_n) < \epsilon$.

- *Examples:* Finite sets, countable sets (like $\mathbb{Q}$), and even the Cantor set (uncountable) have measure zero.

Theorem 6.1.13 (Lebesgue):
A bounded function f on $[a, b]$ is Riemann integrable **if and only if** the set of its discontinuities has **measure zero**.

Applications:

- Continuous functions: Discontinuity set is empty (measure 0) $\implies$ Integrable.
- Monotone functions: Discontinuities are countable (measure 0) $\implies$ Integrable.
- Function with finite discontinuities $\implies$ Integrable.

9. Advanced Examples (Lebesgue Application)

(a) Thomae's Function (Popcorn Function):

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ (lowest terms)} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

- Continuous at all irrationals (measure of rationals is 0).
- Discontinuous at all rationals.
- By Lebesgue's Theorem, it is **integrable** and $\int_0^1 f = 0$.

(b) Counter-example for Composition:
Let f be Thomae's function (integrable). Let $g(y) = 1$ if $y \in (0, 1]$, $g(0) = 0$.

- g is integrable (only discontinuous at 0).
- $g \circ f$ results in the Dirichlet function (1 at rationals, 0 at irrationals).
- $g \circ f$ is **not** integrable.
- Reason: f maps rationals to non-zero values (where $g = 1$) and irrationals to 0 (where $g = 0$).