Here is a comprehensive study guide for **Chapter 9: Fourier Series**, organized by logical flow and key concepts. It covers definitions, calculation examples, convergence types, and the logic behind major theorems.

9.1 Orthogonal Functions

1. Inner Product and Orthogonality

To discuss "approximation," we treat functions as vectors.

- **Inner Product:** For $f, g \in \mathcal{R}[a, b]$ (Riemann integrable functions):

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$$

- **Norm:** The "length" of a function is:

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \left[\int_a^b f^2(x)\, dx \right]^{1/2}$$

- **Orthogonality:** Two functions are orthogonal if $\langle f, g \rangle = 0$.
- **Orthonormal System:** A sequence $\{\phi_n\}$ is orthonormal if:

$$\langle \phi_n, \phi_m \rangle = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

2. Examples of Orthogonal Systems

- **Example A:** $\{1, x\}$ on $[-1, 1]$.

$$\int_{-1}^{1} 1 \cdot x\, dx = \left[\frac{x^2}{2} \right]_{-1}^{1} = 0$$

- **Example B (Trigonometric System):** $\{\sin nx\}_{n=1}^{\infty}$ on $[-\pi, \pi]$.
  For $n \neq m$, using product-to-sum identities:

$$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = 0$$

For $n = m$:

$$\int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi$$

- **Example C:** $\{1, \cos \frac{n\pi x}{L}, \sin \frac{n\pi x}{L}\}$ on $[-L, L]$.

3. Approximation in the Mean (Least Squares)

Given an orthogonal system $\{\phi_n\}$, we want to approximate f using a partial sum $S_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$.
We want to minimize the **mean square error**:

$$E_N = \|f - S_N\|_2^2 = \int_a^b [f(x) - S_N(x)]^2 \, dx$$

Theorem: The error is minimized if and only if c_n are the **Fourier Coefficients**:

$$c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} = \frac{\int_a^b f(x)\phi_n(x) \, dx}{\int_a^b \phi_n^2(x) \, dx}$$

Proof Idea:
Expand the integral $\int (f - \sum c_n \phi_n)^2$. By completing the square with respect to c_n, one finds that the minimum occurs exactly when c_n takes the form above.

4. Bessel's Inequality

Since the error $E_N \geq 0$:

$$\sum_{n=1}^{\infty} c_n^2 \|\phi_n\|^2 \leq \|f\|_2^2$$

Corollary: The coefficients c_n must decay such that $\lim_{n \to \infty} c_n = 0$.

9.2 Completeness and Parseval's Equality

1. Convergence in the Mean

A sequence $\{f_n\}$ converges to f **in the mean** if:

$$\lim_{n\to\infty}\int_a^b [f(x)-f_n(x)]^2\,dx = 0 \quad\iff\quad \lim_{n\to\infty}\|f-f_n\|_2 = 0$$

Note: Uniform convergence $\implies$ Mean convergence. However, Mean convergence does NOT imply Pointwise convergence.

2. Completeness

An orthogonal system $\{\phi_n\}$ is **complete** if the Fourier series partial sums S_N converge to f in the mean for every $f \in \mathcal{R}[a,b]$.

3. Parseval's Equality

The system is complete if and only if **Parseval's Equality** holds for all f:

$$\sum_{n=1}^{\infty} c_n^2\|\phi_n\|^2 = \int_a^b f^2(x)\,dx$$

Concept: This is the infinite-dimensional version of the Pythagorean theorem ($\|v\|^2 = \sum |components|^2$).

9.3 Trigonometric Fourier Series

1. The Definitions

For the system $\{1, \cos nx, \sin nx\}$ on $[-\pi, \pi]$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Formulas:

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$

2. Calculation Examples

Example A: Step Function

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

- **a_0:** $\frac{1}{\pi} \int_0^{\pi} 1\, dx = 1$.
- **a_n:** $\frac{1}{\pi} \int_0^{\pi} \cos nx\, dx = \frac{1}{n\pi} [\sin nx]_0^{\pi} = 0$.
- **b_n:** $\frac{1}{\pi} \int_0^{\pi} \sin nx\, dx = \frac{-1}{n\pi} [\cos nx]_0^{\pi} = \frac{1}{n\pi}(1 - (-1)^n)$.
  - If n is even, $b_n = 0$.
  - If n is odd $(2k+1)$, $b_n = \frac{2}{(2k+1)\pi}$.
- **Series:** $f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$.

Example B: $f(x) = x$ on $[-\pi, \pi]$

- **Symmetry:** $f(x)$ is **Odd**.
  - $a_n = 0$ (integral of odd $\times$ even is odd, symmetric integral is 0).
- **b_n:** Use integration by parts.

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx\, dx = \frac{2}{\pi} \left[-\frac{x \cos nx}{n} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx\, dx \right]$$

$$b_n = \frac{2}{\pi} \left(-\frac{\pi(-1)^n}{n} \right) = \frac{2(-1)^{n+1}}{n}$$

- **Series:** $x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

3. Riemann-Lebesgue Lemma

For any $f \in \mathcal{R}[-\pi, \pi]$:

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

Implication: Fourier coefficients $a_n, b_n \to 0$ as $n \to \infty$.

4. Sine and Cosine Series (Half-Range Expansions)

If f is defined only on $[0, \pi]$:

- **Even Extension (f_e):** Extends f to $[-\pi, \pi]$ as an even function. Generates a **Cosine Series**.
- **Odd Extension (f_o):** Extends f to $[-\pi, \pi]$ as an odd function. Generates a **Sine Series**.

9.4 Convergence in the Mean

1. The Dirichlet Kernel (D_n)

The partial sum $S_n(x)$ can be written as an integral convolution:

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\,dt$$

where $D_n(t) = \frac{1}{2} + \sum_{k=1}^{n}\cos kt = \frac{\sin((n+1/2)t)}{2\sin(t/2)}$.
Properties: $\frac{1}{\pi}\int D_n = 1$, but $\int |D_n| \to \infty$. This makes D_n "bad" for proving uniform convergence directly.

2. The Fejér Kernel (F_n)

To fix the behavior of D_n, we take the arithmetic mean (Cesàro mean) of the partial sums:

$$\sigma_n(x) = \frac{S_0(x) + \cdots + S_n(x)}{n+1}$$

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)F_n(x-t)\,dt$$

where $F_n(t) = \frac{1}{2(n+1)} \left[\frac{\sin((n+1)t/2)}{\sin(t/2)} \right]^2$.

Properties of F_n (Approximate Identity):

1. $F_n(t) \geq 0$.
2. $\frac{1}{\pi} \int_{-\pi}^{\pi} F_n(t)\, dt = 1$.
3. For any $\delta > 0$, $F_n(t) \to 0$ uniformly outside $[-\delta, \delta]$.

3. Fejér's Theorem

If f is continuous on $[-\pi, \pi]$ and periodic ($f(-\pi) = f(\pi)$), then:

$$\sigma_n(x) \to f(x) \quad \text{uniformly on } [-\pi, \pi].$$

Proof Idea: Use the properties of the Approximate Identity. Split the integral into a small region near 0 (where $f(x-t) \approx f(x)$) and the tail (where $F_n \to 0$).

4. Main Result: Mean Convergence

Theorem: For any $f \in \mathcal{R}[-\pi, \pi]$, the Fourier series converges to f **in the mean**.

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2\, dx = 0$$

Corollary (Parseval's Equality for Trig Series):

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx$$

Calculation Example using Parseval:
Using $f(x) = x$ and its coefficients $b_n = \frac{2(-1)^{n+1}}{n}$:

$$\sum_{n=1}^{\infty} \left(\frac{2}{n} \right)^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^2}{3}$$

$$4 \sum \frac{1}{n^2} = \frac{2\pi^2}{3} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

9.5 Pointwise Convergence

1. The Problem

Convergence in the mean does not guarantee that $S_n(x) \to f(x)$ at every specific point x. We need stronger conditions.

2. Dirichlet's Theorem (Sufficient Conditions)

If f is periodic (2π) and piecewise continuous on $[-\pi, \pi]$, and x_0 is a point where:

1. One-sided limits $f(x_0^+)$ and $f(x_0^-)$ exist.
2. One-sided derivatives (or Lipschitz conditions) exist.

$$|f(x_0 \pm t) - f(x_0^\pm)| \leq Mt$$

Then the Fourier series converges to the average of the jump:

$$\lim_{n\to\infty} S_n(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}$$

Note: If f is continuous at x_0, it converges to $f(x_0)$.

3. Application Examples

- **Example A (Discontinuous):**
  $f(x) = 0$ on $[-\pi, -\pi/2)$, 3 on $[-\pi/2, \pi/2]$, 0 on $(\pi/2, \pi)$.
  At $x = \pi/2$, the series converges to $\frac{3+0}{2} = 1.5$.
  At $x = 0$, the series converges to 3.
- **Example B (Continuous):**
  $f(x) = |x|$ on $[-\pi, \pi]$. Continuous everywhere and derivative is piecewise continuous (± 1). Series converges to $|x|$ everywhere. Here are concise, comprehensive study notes on **Chapter 9: Fourier Series**, based on the provided text. These notes focus on mathematical definitions, key theorems, proof logic, and all illustrative examples included in the source.

Chapter 9: Fourier Series

9.1 Orthogonal Functions

Core Concept: Approximating functions using a linear combination of orthogonal functions, analogous to vector decomposition in $\mathbb{R}^n$.

1. Inner Product and Orthogonality

For Riemann integrable functions on $[a, b]$, denoted $\mathcal{R}[a, b]$, the **inner product** is defined as:

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

- **Norm:** $||f||_2 = \sqrt{\langle f, f \rangle} = \left[\int_a^b f^2(x)\,dx\right]^{1/2}$.
- **Orthogonality:** Two functions ϕ, ψ are orthogonal if $\langle \phi, \psi \rangle = 0$.
- **Orthonormality:** A sequence $\{\phi_n\}$ is orthonormal if $\langle \phi_n, \phi_m \rangle = 0$ for $n \neq m$ and $||\phi_n||^2 = 1$.

2. Key Examples of Orthogonal Systems

- **Example 9.1.2(a):** $\{1, x\}$ on $[-1, 1]$.

$$\int_{-1}^{1} 1 \cdot x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

- **Example 9.1.2(b):** $\{\sin nx\}_{n=1}^{\infty}$ on $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = 0 \quad (\text{for } n \neq m)$$

  Norm squared: $\int_{-\pi}^{\pi} \sin^2 nx\,dx = \pi$.
- **Example 9.1.2(c):** $\{1, \sin\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\}$ on $[-L, L]$.
  This is the general trigonometric system.

3. Approximation in the Mean

We seek constants c_n to minimize the mean square error between f and the partial sum $S_N(x) = \sum_{n=1}^{N} c_n\phi_n(x)$:

$$E_N = ||f - S_N||_2^2 = \int_a^b [f(x) - S_N(x)]^2 \, dx$$

Theorem 9.1.4 (Best Approximation):
The error is minimized if and only if c_n are the **Fourier Coefficients**:

$$c_n = \frac{\langle f, \phi_n \rangle}{||\phi_n||^2} = \frac{\int_a^b f(x)\phi_n(x) \, dx}{\int_a^b \phi_n^2(x) \, dx}$$

With these coefficients, the error identity is:

$$||f - S_N||^2 = ||f||^2 - \sum_{n=1}^{N} c_n^2 ||\phi_n||^2$$

Example 9.1.6:
Approximating $f(x) = x^3 + 1$ on $[-1, 1]$ using orthogonal system $\{1, x\}$.

- $c_1 = \frac{\int (x^3+1)(1)}{\int 1^2} = 1$
- $c_2 = \frac{\int (x^3+1)(x)}{\int x^2} = \frac{3}{5}$
- Result: $S_2(x) = 1 + \frac{3}{5}x$.

4. Bessel's Inequality

Since the error $||f - S_N||^2 \geq 0$, it follows that:

$$\sum_{n=1}^{\infty} c_n^2 ||\phi_n||^2 \leq ||f||^2$$

Corollary: For orthogonal systems, Fourier coefficients $c_n \to 0$ as $n \to \infty$ (normalized by norm).

9.2 Completeness and Parseval's Equality

1. Convergence in the Mean

A sequence f_n converges to f **in the mean** if:

$$\lim_{n\to\infty} \int_a^b [f(x) - f_n(x)]^2 \, dx = 0 \quad \iff \quad \lim_{n\to\infty} ||f - f_n||_2 = 0$$

- **Theorem:** Uniform convergence $\implies$ Convergence in the mean.
- **Counter-Example 9.2.3 (Mean $\not\Rightarrow$ Pointwise):**
  A sequence of "moving bump" functions f_n on $[0, 1]$.
  - Constructed by dividing $[0, 1]$ into intervals of size $1/2^k$. f_n is 1 on a shrinking sub-interval and 0 elsewhere.
  - $\int f_n^2 \to 0$ (Converges in mean to 0).
  - For any x, $f_n(x)$ oscillates between 0 and 1 infinitely often (Diverges pointwise).

2. Completeness

An orthogonal system $\{\phi_n\}$ is **complete** if Parseval's Equality holds for *every* $f \in \mathcal{R}[a, b]$.
Parseval's Equality:

$$\sum_{n=1}^{\infty} c_n^2 ||\phi_n||^2 = ||f||^2$$

- Completeness is equivalent to the partial sums S_N converging to f in the mean.
- **Uniqueness Theorem:** If $\{\phi_n\}$ is complete and f is continuous, and all Fourier coefficients are zero ($\int f\phi_n = 0$), then $f(x) \equiv 0$.

9.3 Trigonometric and Fourier Series

1. The Trigonometric System

On $[-\pi, \pi]$, the system is $\{1, \cos nx, \sin nx\}$.

- $||1||^2 = 2\pi$
- $||\cos nx||^2 = \pi$, $||\sin nx||^2 = \pi$

2. Definitions

For $f \in \mathcal{R}[-\pi, \pi]$:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Fourier Series:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

3. Examples of Calculation

- **Example 9.3.3(a): Step Function**
  $f(x) = 0$ for $x \in [-\pi, 0)$, $f(x) = 1$ for $x \in [0, \pi)$.
  - $a_0 = 1$.
  - $a_n = 0$ (calculation yields $\sin n\pi = 0$).
  - $b_n = \frac{1}{n\pi}[1 - (-1)^n]$. (Non-zero only for odd n).
  - Series: $\frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$.
- **Example 9.3.3(b):** $f(x) = x$
  - f is odd $\implies a_n = 0$.
  - $b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$. Integration by parts yields $b_n = \frac{2(-1)^{n+1}}{n}$.
  - Series: $2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

4. Riemann-Lebesgue Lemma

For $f \in \mathcal{R}[-\pi, \pi]$:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\cos nx\,dx = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

- **Example 9.3.5:** If f is not integrable (e.g., $f(x) = 1/x$), the limit may not be zero ($\lim \to \pi/2$). This confirms integrability is required.

5. Sine and Cosine Series

- If f is defined on $[0, \pi]$:
  - **Odd Extension:** Yields **Sine Series** ($a_n = 0$).
  - **Even Extension:** Yields **Cosine Series** ($b_n = 0$).

9.4 Convergence in the Mean

Goal: Prove that for $f \in \mathcal{R}[-\pi, \pi]$, the Fourier series converges to f in the mean.

1. The Dirichlet Kernel (D_n)

The partial sum $S_n(x)$ can be written as an integral convolution:

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t)\, dt$$

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin((n+1/2)t)}{2\sin(t/2)}$$

- **Problem:** D_n is *not* a "good" kernel (approximate identity) because $\int |D_n| \to \infty$. This makes pointwise convergence difficult to prove directly.

2. The Fejér Kernel (F_n)

To solve the convergence problem, we look at **Cesàro means** (averages of partial sums):

$$\sigma_n(x) = \frac{S_0(x) + \cdots + S_n(x)}{n+1}$$

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) F_n(x-t)\, dt$$

$$F_n(t) = \frac{1}{2(n+1)} \left[\frac{\sin(\frac{n+1}{2}t)}{\sin(t/2)} \right]^2$$

Properties of F_n (Approximate Identity):

1. $F_n(t) \geq 0$.
2. $\frac{1}{\pi} \int F_n = 1$.
3. $F_n \to 0$ uniformly outside any neighborhood of $t = 0$.

3. Fejér's Theorem

Theorem 9.4.5: If f is continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$, then $\sigma_n(x)$ converges uniformly to $f(x)$.

Proof Logic:
Using the properties of the Fejér kernel (Approximate Identity), the convolution σ_n approximates any continuous periodic function uniformly.

Corollary 9.4.6: Since $\sigma_n \to f$ uniformly, S_n converges to f **in the mean** for continuous functions. (The mean square error of S_n is always $\leq$ the mean square error of σ_n).

4. General Convergence and Parseval's

Theorem 9.4.7: For *any* $f \in \mathcal{R}[-\pi, \pi]$, S_n converges to f in the mean.
Proof Idea: Any Riemann integrable function can be approximated in the mean by a continuous function (Step function approximation lemma). Combined with Fejér's theorem, the result follows.

Parseval's Equality (Corollary 9.4.9):

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

Example 9.4.10:
Using $f(x) = x$ (from 9.3.3b), $b_n = \frac{2(-1)^{n+1}}{n}$.
Apply Parseval's: $\sum \frac{4}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}$.
Result: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

9.5 Pointwise Convergence

Question: When does $S_n(x) \to f(x)$ at a specific point x?

1. Dirichlet's Theorem

Theorem 9.5.3: Let f be periodic (2π). If at a point x_0:

1. Left and right limits $f(x_0-), f(x_0+)$ exist.
2. Left and right derivatives (or Lipschitz conditions) exist.
   - $|f(x_0 \pm t) - f(x_0 \pm)| \leq Mt$.

Then:

$$\lim_{n\to\infty} S_n(x_0) = \frac{f(x_0+) + f(x_0-)}{2}$$

Proof Logic:
Using the Dirichlet integral representation:

$$S_n(x) - A = \frac{2}{\pi}\int_0^{\pi} \left[\frac{f(x+t)+f(x-t)}{2} - A\right] D_n(t)\, dt$$

Set $A = \text{avg limit}$. The term in the bracket divided by $2\sin(t/2)$ behaves like a Riemann integrable function $g(t)$ due to the derivative condition (boundedness near 0).
By Riemann-Lebesgue Lemma, $\int g(t)\sin((n+1/2)t)dt \to 0$.

2. Piecewise Continuity

- **Piecewise Continuous:** Finite number of simple discontinuities (finite jumps).
- **Piecewise Smooth:** f and f' are piecewise continuous.
- **Corollary 9.5.6:** If f is periodic and piecewise smooth, Fourier series converges to the average of limits everywhere.

3. Examples

- **Example 9.5.7(a):** Square wave (0, 3, 0).
  - At discontinuity (e.g., $x = \pi/2$), series converges to $3/2$.
  - Evaluating at $x = 0$ yields Leibniz series: $\sum \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$.
- **Example 9.5.7(b):** Triangular wave ($f(x) = |x|$ on $[-\pi, \pi]$).
  - Continuous everywhere. Series converges to $|x|$.
  - Evaluating at $x = 0$ yields $\sum \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$.

4. Differentiation of Fourier Series

Theorem 9.5.8:
Differentiation term-by-term is valid if:

1. f is continuous on $[-\pi, \pi]$ AND $f(-\pi) = f(\pi)$ (Periodic continuity).
2. f' is piecewise continuous.

Then at points where $f''(x)$ exists:

$$f'(x) = \sum_{n=1}^{\infty}(-na_n \sin nx + nb_n \cos nx)$$

Note: If f is discontinuous, the differentiated series generally diverges.

Example 9.5.9:

- $f(x) = x^2$ is continuous and periodic.
- Differentiating the series for x^2 leads to the series for $2x$ (which matches the calculated series for x).
- Conversely, integrating the series for $2x$ term-by-term yields the series for x^2.Evaluating at $x = 0$ often yields sums of series like $\sum \frac{1}{(2k-1)^2}$.

4. Differentiation of Fourier Series

Simply differentiating a Fourier series term-by-term is not always valid (the resulting series might diverge).
Theorem: If f is **continuous** everywhere (including $f(-\pi) = f(\pi)$) and f' is **piecewise continuous**, then the Fourier series of f' is obtained by differentiating the series of f term-by-term.

$$f'(x) \sim \sum(-na_n \sin nx + nb_n \cos nx)$$