6.5 The Riemann-Stieltjes Integral

1. Motivation

The Riemann-Stieltjes integral unifies discrete summations and continuous integrals into a single mathematical framework.

- **Example (Moment of Inertia I):**
  - **Discrete:** System of n masses m_i at distance r_i: $I = \sum r_i^2 m_i$.
  - **Continuous:** Wire of length l with density $\rho(x)$: $I = \int_0^l x^2 \rho(x) dx$.
  - **Unified:** Using mass distribution $m(x)$, both become $I = \int x^2 dm(x)$.

2. Definition

Let α be a **monotone increasing** function on $[a, b]$ and f be bounded.
For a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$, define $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Note that $\Delta\alpha_i \geq 0$.

- **Upper Sum:** $\mathcal{U}(\mathcal{P}, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$, where $M_i = \sup_{[x_{i-1}, x_i]} f$.
- **Lower Sum:** $\mathcal{L}(\mathcal{P}, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$, where $m_i = \inf_{[x_{i-1}, x_i]} f$.

Integrability:
f is Riemann-Stieltjes integrable with respect to α ($f \in \mathcal{R}(\alpha)$) if the lower and upper integrals meet:

$$\sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f, \alpha) = \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f, \alpha) = \int_a^b f d\alpha$$

If $\alpha(x) = x$, this reduces to the standard Riemann integral.

3. Conditions for Existence

- **Cauchy Criterion (Theorem 6.5.5):** $f \in \mathcal{R}(\alpha) \iff \forall \epsilon > 0, \exists \mathcal{P}$ such that $\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) < \epsilon$.
- **Theorem 6.5.6:** Integrability is guaranteed if:
  i. f is **continuous** (proof relies on uniform continuity).
  ii. f is **monotone** and α is **continuous**.

4. Key Properties

If $f, g \in \mathcal{R}(\alpha)$ and $c \in \mathbb{R}$:

1. **Linearity:** $\int (f+g)d\alpha = \int fd\alpha + \int gd\alpha$ and $\int cfd\alpha = c\int fd\alpha$.
2. **Additivity of α:** $\int fd(\alpha_1 + \alpha_2) = \int fd\alpha_1 + \int fd\alpha_2$.
3. **Interval Additivity:** $\int_a^b = \int_a^c + \int_c^b$.
4. **Order:** $f \leq g \implies \int fd\alpha \leq \int gd\alpha$.
5. **Boundedness:** $|\int fd\alpha| \leq \int |f|d\alpha \leq M[\alpha(b) - \alpha(a)]$.

5. Calculation Methods & Major Theorems

A. The Unit Jump Function (Discrete Case)

Let $I_c(x)$ be the unit jump at c (0 if $x < c$, 1 if $x \geq c$).

- If f is continuous at c ($a < c \leq b$), then:

$$\int_a^b fdI_c = f(c)$$

- **Proof Idea:** For any partition where $c \in (x_{k-1}, x_k]$, only $\Delta\alpha_k = 1$ (others are 0). The sums squeeze $f(c)$ due to continuity.
- **General Series (Theorem 6.5.11):** If $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$ (a step function with jumps at s_n):

$$\int_a^b fd\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

B. The Differentiable α (Continuous Case)

- **Theorem 6.5.12:** If α is differentiable and $\alpha' \in \mathcal{R}[a, b]$:

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

- **Proof Idea:** Uses Mean Value Theorem. $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$. The Riemann-Stieltjes sum transforms into a Riemann sum for $f\alpha'$.

C. Integration by Parts (Theorem 6.5.10)

$$\int_a^b \alpha d\beta = \alpha(b)\beta(b) - \alpha(a)\beta(a) - \int_a^b \beta d\alpha$$

- **Proof Idea:** Uses the identity for partition sums: $\sum \alpha_i \Delta\beta_i = \alpha\beta - \sum \beta_i \Delta\alpha_i$ (Abel's transformation logic).

D. Mean Value Theorem (Theorem 6.5.9)

If f is continuous, $\exists c \in [a, b]$ such that:

$$\int_a^b f d\alpha = f(c)[\alpha(b) - \alpha(a)]$$

6. Illustrative Examples

Example 6.5.4 (Integrability check)

1. **Jump Function:** If f is continuous at c, $\int f dI_c = f(c)$.
2. **Dirichlet Function:** $f(x) = 1$ if $x \in \mathbb{Q}$, 0 if irrational. This is **not** integrable w.r.t any non-constant α because upper sums use $M_i = 1$ and lower sums use $m_i = 0$.

Example 6.5.13 (Computational Examples)

1. **Step Function:** $\int_0^2 e^x d[x]$.
   - $[x]$ has jumps of size 1 at $x = 1$ and $x = 2$.
   - Result: $e^1 + e^2$.
2. **Smooth α:** $\int_0^1 \sin(\pi x) d(x^2)$.
   - Convert using $\alpha'(x) = 2x$.
   - Integral becomes $2\int_0^1 x \sin(\pi x) dx = \frac{1}{\pi}$.
3. **Mixed:** $\int_0^3 [x] d(e^{2x})$.
   - Use Theorem 6.5.12 (convert $d(e^{2x})$ to $2e^{2x}dx$).
   - Split integral based on values of $[x]$: 0 on $[0, 1)$, 1 on $[1, 2)$, 2 on $[2, 3]$.
   - Result: $2e^6 - e^4 - e^2$.

7. Riemann-Stieltjes Sums & Limits

A Riemann-Stieltjes sum is $S(\mathcal{P}, f, \alpha) = \sum f(t_i)\Delta\alpha_i$ where $t_i \in [x_{i-1}, x_i]$.

- **Warning:** Unlike standard calculus, knowing the limit of sums exists does **not** imply $f \in \mathcal{R}(\alpha)$ in all cases, nor does integrability always imply the limit of sums equals the integral for *any* choice of

tags t_i if f and α share discontinuities.
- **Theorem:** If f is continuous, then $\lim_{||\mathcal{P}||\to 0} S(\mathcal{P}, f, \alpha) = \int f d\alpha$.

Example 6.5.15 (Failure of Sums)

- $f(x) = 0$ on $[0, 1]$, 1 on $(1, 2]$ (jump at 1).
- $\alpha(x) = 0$ on $(0, 1)$, 1 on $[1, 2]$ (jump at 1).
- f is left-continuous $\implies$ Integrable. $\int f d\alpha = 0$.
- However, choosing tags t_k differently at the discontinuity point $x = 1$ yields different sum limits (0 or 1). Thus, the limit of sums **does not exist** even though the integral is defined.