# 6.5 The Riemann-Stieltjes Integral

## 1. Motivation & Concept

The Riemann-Stieltjes integral unifies discrete and continuous summation into a single formula.

- **Physical Example (Moment of Inertia $I$):**
  - *Discrete:* For point masses $m_i$ at $r_i$: $I = \sum r_i^2 m_i$.
  - *Continuous:* For a wire with density $\rho(x)$: $I = \int x^2 \rho(x) dx$.
  - *Unified:* Using a mass distribution function $m(x)$, both can be written as:

$$I = \int_0^l x^2 dm(x)$$

## 2. Definition of the Integral

Let $\alpha$ be a **monotone increasing** function on $[a, b]$ and $f$ be a bounded real-valued function.

### Partition and Sums

For a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ of $[a, b]$:

- Define $\Delta\alpha_i$ instead of $\Delta x_i$:

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$$

- **Upper Sum:** $\mathcal{U}(\mathcal{P}, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$, where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$.
- **Lower Sum:** $\mathcal{L}(\mathcal{P}, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$, where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.

### Integrability

$f$ is Riemann-Stieltjes integrable with respect to $\alpha$, denoted $f \in \mathcal{R}(\alpha)$, if the upper and lower integrals meet:

$$\underline{\int_a^b} f d\alpha = \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f, \alpha) = \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f, \alpha) = \overline{\int_a^b} f d\alpha$$

The common value is denoted $\int_a^b f d\alpha$.

> **Note:** If $\alpha(x) = x$, this reduces to the standard Riemann integral.

# 3. Key Examples (Existence & Non-Existence)

## A. The Unit Jump Function (Discrete behavior)

Let $I_c(x)$ be the unit jump at $c$ ($a < c \leq b$):

$$I_c(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

- **Result:** If $f$ is continuous at $c$, then:

$$\int_a^b f dI_c = f(c)$$

- **Proof Idea:** For any partition where $c \in (x_{k-1}, x_k]$, only the $k$-th term has $\Delta\alpha_k \neq 0$ (specifically $\Delta\alpha_k = 1$). Thus sums collapse to $M_k$ and $m_k$. Since $f$ is continuous, as $\Delta x \to 0$, $M_k, m_k \to f(c)$.

## B. The Dirichlet Function

Let $f(x) = 1$ if $x \in \mathbb{Q}$, and $0$ if $x \notin \mathbb{Q}$.

- **Result:** Not integrable for any non-constant $\alpha$.
- **Reason:** In every interval, density of rationals/irrationals implies $M_i = 1$ and $m_i = 0$. Thus $\mathcal{U} = \alpha(b) - \alpha(a)$ and $\mathcal{L} = 0$.

# 4. Conditions for Integrability

## Theorem: Cauchy Criterion

$f \in \mathcal{R}(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that:

$$\mathcal{U}(\mathcal{P}, f, \alpha) - \mathcal{L}(\mathcal{P}, f, \alpha) < \epsilon$$

## Theorem: Sufficient Classes for Integrability

1. **Continuous $f$:** If $f$ is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha)$.
2. **Monotone $f$ / Continuous $\alpha$:** If $f$ is monotone and $\alpha$ is **continuous**, then $f \in \mathcal{R}(\alpha)$.
   - *Proof Idea:* Since $\alpha$ is continuous, use the Intermediate Value Theorem to choose a partition where $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$. The telescoping sum of $f$ then bounds the difference $U - L$.

# 5. Properties of the Integral

If $f, g \in \mathcal{R}(\alpha)$ and $c \in \mathbb{R}$:

1. **Linearity:** $\int(f+g)d\alpha = \int fd\alpha + \int gd\alpha$ and $\int cfd\alpha = c\int fd\alpha$.
2. **Additivity of $\alpha$:** $\int fd(\alpha_1 + \alpha_2) = \int fd\alpha_1 + \int fd\alpha_2$.
3. **Domain Splitting:** $\int_a^b = \int_a^c + \int_c^b$.
4. **Comparison:** If $f(x) \le g(x)$, then $\int fd\alpha \le \int gd\alpha$.
5. **Boundedness:**

$$\left|\int_a^b fd\alpha\right| \le \int_a^b |f|d\alpha \le M[\alpha(b) - \alpha(a)]$$

   (where $|f(x)| \le M$).

# 6. Fundamental Theorems

## Mean Value Theorem for Integrals

If $f$ is continuous and $\alpha$ is monotone increasing, there exists $c \in [a, b]$ such that:

$$\int_a^b fd\alpha = f(c)[\alpha(b) - \alpha(a)]$$

- *Proof Idea:* Use the Intermediate Value Theorem on the range of values bounded by $\inf f$ and $\sup f$.

## Integration by Parts

If $\alpha$ and $\beta$ are monotone increasing, then $\alpha \in \mathcal{R}(\beta) \iff \beta \in \mathcal{R}(\alpha)$.
**Formula:**

$$\int_a^b \alpha d\beta = \alpha(b)\beta(b) - \alpha(a)\beta(a) - \int_a^b \beta d\alpha$$

- *Proof Idea:* Relies on the summation identity for partitions:
  $\mathcal{U}(\mathcal{P}, \alpha, \beta) - \mathcal{L}(\mathcal{P}, \alpha, \beta) = \mathcal{U}(\mathcal{P}, \beta, \alpha) - \mathcal{L}(\mathcal{P}, \beta, \alpha)$.
  If the difference for one integral approaches 0, it must for the other.

# 7. Evaluating the Integral (Computational Theorems)

The Riemann-Stieltjes integral usually reduces to one of two forms for calculation:

## Case A: $\alpha$ is a Step Function (Summation)

If $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$ where $\sum c_n$ converges and $f$ is continuous:

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

- *Proof Idea:* Use the Unit Jump Example result and linearity. The infinite series case is handled by bounding the tail of the series.
- **Example:** $\int_0^2 e^x d[x]$ where $[x]$ is the floor function.
  - $[x]$ jumps at $x = 1$ and $x = 2$ within (0, 2].
  - Result: $e^1 + e^2$.

## Case B: $\alpha$ is Differentiable (Reduction to Riemann)

If $\alpha$ is differentiable and $\alpha' \in \mathcal{R}[a, b]$ (Riemann integrable), then:

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$$

- *Proof Idea:* By MVT, $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$. The Riemann-Stieltjes sum $\sum f(s_i)\Delta\alpha_i$ approximates the Riemann sum $\sum f(s_i)\alpha'(t_i)\Delta x_i$.

- **Example:** $\int_0^1 \sin(\pi x) d(x^2) = \int_0^1 \sin(\pi x) \cdot (2x) dx$.

# 8. Riemann-Stieltjes Sums and Limits

A Riemann-Stieltjes sum is defined as $\mathcal{S}(\mathcal{P}, f, \alpha) = \sum f(t_i)\Delta\alpha_i$ where $t_i$ are arbitrary tags in the intervals.

- **Warning:** Unlike the standard Riemann integral, the existence of the integral $\int f d\alpha$ does **not** guarantee that $\lim_{||\mathcal{P}|| \to 0} S(\mathcal{P}, f, \alpha)$ exists.
- **Counter-Example:**
  - $f(x) = 1$ for $x > 1$, $0$ else.
  - $\alpha(x) = 1$ for $x \geq 1$, $0$ else.
  - Integral exists (value 0 via continuity properties).
  - Sum limit does not exist: Depending on whether the tag $t_k$ is chosen at 1 or slightly right of 1, the sum oscillates between 0 and 1.
- **Convergence Condition:** However, if $f$ continuous and $\alpha$ monotone, the limit of sums *does* equal the integral.