Here are the organized notes on **Section 8.1: Pointwise Convergence and Interchange of Limits**, based on the provided text.

1. Definitions: Sequences and Series of Functions

Pointwise Convergence of Sequences

Let $E \subset X$ be a metric space (e.g., $E \subseteq \mathbb{R}$). A sequence of real-valued functions $\{f_n\}$ defined on E **converges pointwise** to a function f on E if the sequence of real numbers $\{f_n(x)\}$ converges for every $x \in E$.

- **Notation:** $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$.
- **Formal Definition ($\epsilon - n_0$):**
  The sequence converges pointwise to f if for every $x \in E$ and given $\epsilon > 0$, there exists a positive integer n_0 such that:

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } n \geq n_0$$

  - *Crucial Note:* The integer n_0 depends on both ϵ and x. Notation: $n_0 = n_0(x, \epsilon)$.

Pointwise Convergence of Series

For a sequence $\{f_n\}$, we define the sequence of partial sums $\{S_n\}$ by:

$$S_n(x) = \sum_{k=1}^{n} f_k(x)$$

The series $\sum_{k=1}^{\infty} f_k$ converges pointwise on E if the sequence $\{S_n(x)\}$ converges for each $x \in E$. The sum is denoted as:

$$S(x) = \sum_{k=1}^{\infty} f_k(x)$$

2. The Interchange of Limits Questions

Historically, mathematicians (including Cauchy) incorrectly believed that properties of f_n (like continuity) automatically transferred to the limit function f. The text investigates three major questions regarding the validity of interchanging limits:

1. **Continuity:** If each f_n is continuous at p, is $f = \lim f_n$ continuous at p?
   - *Mathematically:* Does $\lim_{t \to p}(\lim_{n\to\infty} f_n(t)) = \lim_{n\to\infty}(\lim_{t\to p} f_n(t))$?
2. **Differentiability:** If each f_n is differentiable, is f differentiable?
   - *Mathematically:* Does $f'(p) = \lim_{n\to\infty} f_n'(p)$?
3. **Integrability:** If each f_n is Riemann integrable, is f integrable? And do the integrals converge?
   - *Mathematically:* Does $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$?

General Answer: No. Pointwise convergence is insufficient to preserve these properties.

3. Counter-Examples (Proofs by Example)

The following examples demonstrate why the answer to the above questions is "No."

(a) Continuity is NOT preserved

- **Function:** $f_n(x) = x^n$ on domain $E = [0, 1]$.
- **Properties of f_n:** Each f_n is continuous on $[0, 1]$.
- **Pointwise Limit (f):**
  - If $0 \le x < 1$, $x^n \to 0$.
  - If $x = 1$, $1^n \to 1$.

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

- **Conclusion:** The limit function f is **discontinuous** at $x = 1$, despite all f_n being continuous.

(b) Continuity is NOT preserved (Series Example)

- **Function:** Series defined by partial sums of $f_k(x) = \frac{x^2}{(1+x^2)^k}$ on $\mathbb{R}$.
- **Sum Calculation:**
  - If $x = 0$: All terms are 0, so Sum $= 0$.
  - If $x \neq 0$: This is a geometric series with ratio $r = \frac{1}{1+x^2} < 1$.

$$\sum_{k=0}^{\infty} x^2 \left(\frac{1}{1+x^2}\right)^k = x^2 \left[\frac{1}{1 - \frac{1}{1+x^2}}\right] = x^2 \left[\frac{1+x^2}{x^2}\right] = 1 + x^2$$

- **Limit Function:**

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 + x^2, & x \neq 0 \end{cases}$$

- **Conclusion:** The sum function f is **discontinuous** at $x = 0$.

(c) Riemann Integrability is NOT preserved

- **Function:** Let $\{x_k\}$ be an enumeration of rationals in $[0, 1]$.

$$f_n(x) = \begin{cases} 1, & \text{if } x = x_k \text{ for } 1 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

- **Properties of f_n:** Each f_n is continuous except at finite points ($x_1 ... x_n$), so f_n is Riemann integrable with $\int_0^1 f_n = 0$.
- **Pointwise Limit (f):** As $n \to \infty$, f becomes the Dirichlet function:

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \text{ (rational)} \\ 0, & \text{if } x \notin \mathbb{Q} \text{ (irrational)} \end{cases}$$

- **Conclusion:** The limit function f is **not Riemann integrable** (it is nowhere continuous).

(d) Integral Value is NOT preserved ($\lim \int \neq \int \lim$)

- **Function:** $f_n(x) = nx(1 - x^2)^n$ on $[0, 1]$.
- **Properties of f_n:** Continuous and integrable.
- **Pointwise Limit (f):**
  - For $0 < x < 1$: limit is 0 (exponential decay dominates linear growth).
  - For $x = 0$ or $x = 1$: $f_n(x) = 0$.
  - Thus, $f(x) = 0$ for all x, and $\int_0^1 f(x)dx = 0$.
- **Sequence of Integrals:**
  Using substitution ($u = 1 - x^2$):

$$\int_0^1 f_n(x)dx = \int_0^1 nx(1 - x^2)^n dx = \frac{n}{2} \int_0^1 u^n du = \frac{n}{2(n + 1)}$$

$$\lim_{n \to \infty} \int_0^1 f_n(x)dx = \lim_{n \to \infty} \frac{n}{2n + 2} = \frac{1}{2}$$

- **Conclusion:**

$$\lim_{n\to\infty}\int_0^1 f_n \neq \int_0^1 (\lim_{n\to\infty} f_n) \quad \left(\frac{1}{2} \neq 0\right)$$

(e) Differentiability is NOT preserved

- **Function:** $f_n(x) = \frac{\sin(nx)}{n}$ on $\mathbb{R}$.
- **Pointwise Limit (f):** Since $|\sin(nx)| \le 1$, $|f_n(x)| \le 1/n$. Thus $f(x) = 0$.
  - The derivative of the limit is $f'(x) = 0$.
- **Sequence of Derivatives:**

$$f_n'(x) = \cos(nx)$$

At $x = 0$, $f_n'(0) = \cos(0) = 1$.

$$\lim_{n\to\infty} f_n'(0) = 1$$

- **Conclusion:**

$$\frac{d}{dx}(\lim_{n\to\infty} f_n(x)) \neq \lim_{n\to\infty} f_n'(x) \quad (0 \neq 1)$$