6.4 Improper Riemann Integrals

Motivation: Standard Riemann integration requires a function f to be **bounded** on a **closed and bounded** interval $[a, b]$. If f is unbounded or the interval is infinite, we extend the definition.

I. Unbounded Functions on Finite Intervals

Definition 6.4.1 (Singularity at Endpoint)

Let f be defined on (a, b) such that $f \in \mathcal{R}[c, b]$ for every $c \in (a, b)$. The **improper Riemann integral** is defined as:

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f$$

- **Convergent:** The limit exists.
- **Divergent:** The limit does not exist.

Note: A similar definition applies if f is unbounded at b (limit as $c \to b^-$).

Interior Singularities

If f is unbounded at p where $a < p < b$, split the integral:

$$\int_a^b f = \int_a^p f + \int_p^b f$$

The integral converges only if **both** parts converge independently.

Examples on Finite Intervals

1. Example (Divergence): $f(x) = 1/x$ on $(0, 1]$

- f is unbounded at 0.
- $\int_c^1 \frac{1}{x} \, dx = \ln(1) - \ln(c) = -\ln c$.
- Limit: $\lim_{c \to 0^+} (-\ln c) = \infty$.
- **Conclusion:** The integral diverges.

2. Example (Convergence): $f(x) = \ln x$ on $(0, 1]$

- f is continuous on $(0, 1]$. Use Integration by Parts with $u = \ln x, dv = dx$:

$$\int_c^1 \ln x\,dx = [x \ln x]_c^1 - \int_c^1 dx = -c \ln c - (1 - c)$$

- Using L'Hospital's Rule ($c = 1/t$): $\lim_{c \to 0^+} c \ln c = 0$.
- Limit: $\lim_{c \to 0^+} (-c \ln c - 1 + c) = -1$.
- **Conclusion:** Converges to -1.

3. Example (Piecewise):
Let $f(x) = 0$ on $[-1, 0]$ and $f(x) = 1/x$ on $(0, 1]$.

- The integral over $[-1, 1]$ fails to exist because the sub-interval integral $\int_0^1 (1/x)$ diverges.

Comparison with Standard Riemann Integral

Properties of standard Riemann integrals do **not** always hold for improper integrals.

- **Property failure: $f \in \mathcal{R} \implies f^2 \in \mathcal{R}$ is FALSE.**
  - *Counter-example:* $f(x) = 1/\sqrt{x}$ on $(0, 1]$.
  - $\int_0^1 x^{-1/2}\,dx = \lim_{c \to 0^+} [2\sqrt{x}]_c^1 = 2$ (Converges).
  - However, $f^2(x) = 1/x$, which diverges on $(0, 1]$.
- **Property failure: $f \in \mathcal{R} \implies |f| \in \mathcal{R}$ is FALSE.**
  - Convergence of $\int f$ does not imply convergence of $\int |f|$.

II. Infinite Intervals

Definition 6.4.3
Let f be defined on $[a, \infty)$ and $f \in \mathcal{R}[a, c]$ for all $c > a$.

$$\int_a^\infty f = \lim_{c \to \infty} \int_a^c f$$

- For $(-\infty, b]$: Take limit as $c \to -\infty$.
- For $(-\infty, \infty)$: Split at any fixed $p \in \mathbb{R}$.

$$\int_{-\infty}^\infty f = \int_{-\infty}^p f + \int_p^\infty f$$

Condition: Both separate integrals must converge.

Important Warning:
One cannot compute $\int_{-\infty}^{\infty} f$ as $\lim_{c\to\infty} \int_{-c}^{c} f$.

- *Counter-example:* $f(x) = x$.
  - $\lim_{c\to\infty} \int_{-c}^{c} x\,dx = 0$.
  - But $\int_{0}^{\infty} x\,dx = \infty$. Thus, the improper integral on $(-\infty, \infty)$ **diverges**.

Examples on Infinite Intervals

1. Example: $f(x) = 1/x^2$ on $[1, \infty)$

- $\int_1^c x^{-2}\,dx = [-1/x]_1^c = 1 - \frac{1}{c}$.
- $\lim_{c\to\infty}(1 - 1/c) = 1$.
- **Conclusion:** Converges to 1.

2. Example: $f(x) = \frac{\sin x}{x}$ on $[\pi, \infty)$

- *Fact:* The integral of f converges (conditional convergence).
- *Proof focus:* The integral of the absolute value **diverges** ($\int |f| = \infty$).

Proof of Absolute Divergence for $\frac{\sin x}{x}$:
We estimate the sum of integrals over intervals $[k\pi, (k+1)\pi]$.

1. Consider $\int_{\pi}^{\infty} \frac{|\sin x|}{x}\,dx$.
2. On the sub-interval $[(k+\frac{1}{4})\pi, (k+\frac{3}{4})\pi]$, we know $|\sin x| \geq \frac{\sqrt{2}}{2}$.
3. Also, $\frac{1}{x} \geq \frac{1}{(k+1)\pi}$.
4. Bounding the integral for the k-th segment:

$$\text{Area} \geq \left(\frac{\sqrt{2}}{2}\right) \cdot \left(\frac{1}{(k+1)\pi}\right) \cdot \left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{4(k+1)}$$

5. Summing from $k = 1$ to n:

$$\int_{\pi}^{(n+1)\pi} |f| \geq \frac{\sqrt{2}}{4} \sum_{k=1}^{n} \frac{1}{k+1}$$

6. Since the Harmonic series $\sum \frac{1}{k}$ diverges, the integral diverges to ∞.

III. Absolute Integrability & Comparison Test

Definition: f is **absolutely integrable** on $[a, \infty)$ if $f \in \mathcal{R}[a, c]$ for all c and $\int_a^\infty |f|$ converges.

- *Note:* Absolute integrability implies convergence of the improper integral (similar to series).

Theorem 6.4.5: Comparison Test
Let $g \geq 0$ be a function where $\int_a^\infty g(x)\,dx < \infty$ (converges).
If f satisfies:

1. $f \in \mathcal{R}[a, c]$ for every $c > a$, and
2. $|f(x)| \leq g(x)$ for all $x \in [a, \infty)$,

Then:

- $\int_a^\infty f(x)\,dx$ **converges**.
- $|\int_a^\infty f(x)\,dx| \leq \int_a^\infty g(x)\,dx$.