6.2 Properties of the Riemann Integral

This section derives fundamental algebraic and order properties of the Riemann integral and establishes the equivalence between the Darboux definition (upper/lower sums) and the original Riemann definition (limit of sums).

1. Linearity and Algebra of Integrable Functions

Theorem 6.2.1: Let $f, g \in \mathcal{R}[a, b]$.

1. **Sum:** $f + g \in \mathcal{R}[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
2. **Scalar Multiplication:** $cf \in \mathcal{R}[a, b]$ for any $c \in \mathbb{R}$, and $\int_a^b cf = c \int_a^b f$.
3. **Product:** $fg \in \mathcal{R}[a, b]$.

Proof of (1) - Sum:

- **Key Idea:** Relate the supremum of the sum to the sum of supremums.
- Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition.
- Inequality: $\sup_{[x_{i-1}, x_i]}(f + g) \leq \sup_{[x_{i-1}, x_i]} f + \sup_{[x_{i-1}, x_i]} g$.
- This implies $\mathcal{U}(\mathcal{P}, f + g) \leq \mathcal{U}(\mathcal{P}, f) + \mathcal{U}(\mathcal{P}, g)$.
- Using refinements $\mathcal{P}_f$ and $\mathcal{P}_g$ where upper sums are within $\epsilon/2$ of the integral, let $\mathcal{Q} = \mathcal{P}_f \cup \mathcal{P}_g$.
- $\overline{\int}(f + g) \leq \mathcal{U}(\mathcal{Q}, f + g) < \int f + \int g + \epsilon$.
- A similar argument with lower sums shows $\underline{\int}(f + g) \geq \int f + \int g$.
- Since $\underline{\int} \leq \overline{\int}$, equality holds.

Proof of (3) - Product:

- **Key Idea:** Use the algebraic identity $fg = \frac{1}{4}[(f + g)^2 - (f - g)^2]$.
- Since $f, g \in \mathcal{R}$, $f \pm g \in \mathcal{R}$.
- Square functions of integrable functions are integrable (Corollary 6.1.10).
- By linearity (proven in parts 1 and 2), the linear combination of these squares is integrable.

2. Absolute Value Property

Theorem 6.2.2: If $f \in \mathcal{R}[a, b]$, then $|f| \in \mathcal{R}[a, b]$ and:

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Proof:

- **Integrability:** $|f| \in \mathcal{R}$ follows from composition properties (Corollary 6.1.10).
- **Inequality:** Choose $c = \pm 1$ such that $\left| \int f \right| = c \int f$.
- Note that $cf(x) \leq |f(x)|$ for all x.
- Therefore, $\int cf \leq \int |f|$.
- Substituting back: $\left| \int f \right| = c \int f = \int cf \leq \int |f|$.

3. Additivity Over Intervals

Theorem 6.2.3: Let f be bounded on $[a, b]$ and let $a < c < b$. Then $f \in \mathcal{R}[a, b]$ if and only if $f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$.
If integrable, the identity holds:

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof:

- **Step 1: Upper Integrals:**
  - Let $\mathcal{P}$ be a partition of $[a, b]$ containing c. It splits into partitions $\mathcal{P}_1$ of $[a, c]$ and $\mathcal{P}_2$ of $[c, b]$.
  - $\mathcal{U}(\mathcal{P}, f) = \mathcal{U}(\mathcal{P}_1, f) + \mathcal{U}(\mathcal{P}_2, f)$.
  - Taking infimums yields: $\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f$.
  - (Note: If a partition doesn't contain c, taking the refinement $\mathcal{P} \cup \{c\}$ only decreases the sum, preserving the inequality for the infimum).
- **Step 2: Lower Integrals:**
  - Similarly, $\underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f$.
- **Conclusion:**
  - If f is integrable on the sub-intervals, the upper and lower sums match on the right-hand side, forcing them to match on the left (whole interval).
  - Conversely, if $f \in \mathcal{R}[a, b]$, equality of total upper/lower sums forces equality on the sub-intervals (since upper $\geq$ lower always).

4. Riemann's Definition of the Integral

While Darboux used sup/inf sums, Riemann used arbitrary tags.

Definitions:

- **Riemann Sum:** Given a partition $\mathcal{P} = \{x_0, \dots, x_n\}$ and tags $t_i \in [x_{i-1}, x_i]$:

$$\mathcal{S}(\mathcal{P}, f) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

- **Norm (Mesh):** $||\mathcal{P}|| = \max\{\Delta x_i : i = 1, \dots, n\}$.
- **Limit Definition:** $\lim_{||\mathcal{P}||\to 0} \mathcal{S}(\mathcal{P}, f) = I$ means:
  Given $\epsilon > 0$, there exists $\delta > 0$ such that for *any* partition $\mathcal{P}$ with $||\mathcal{P}|| < \delta$ and *any* choice of tags t_i:

$$|\mathcal{S}(\mathcal{P}, f) - I| < \epsilon$$

5. Equivalence of Definitions

Theorem 6.2.6: $f \in \mathcal{R}[a, b]$ (Darboux sense) if and only if the Riemann limit exists. If so, the limit equals the integral.

Proof Highlights:

Part A: Limit Exists $\implies$ Darboux Integrable

- Assume limit is I. Given ϵ, choose δ per definition.
- Pick a partition $\mathcal{P}$ with $||\mathcal{P}|| < \delta$.
- Because $M_i = \sup f$, we can choose tags t_i such that $f(t_i)$ is arbitrarily close to M_i.
- This implies $\mathcal{U}(\mathcal{P}, f) < \mathcal{S}(\mathcal{P}, f) + \epsilon < I + 2\epsilon$.
- Similarly for lower sums. Thus, $\mathcal{U} - \mathcal{L}$ can be made arbitrarily small.

Part B: Darboux Integrable $\implies$ Limit Exists

- Given $f \in \mathcal{R}[a, b]$, there exists a partition $\mathcal{Q} = \{x_0, \dots, x_N\}$ such that $\mathcal{U}(\mathcal{Q}) - \mathcal{L}(\mathcal{Q}) < \epsilon$.
- Let $|f(x)| \le M$. Choose $\delta = \epsilon/(NM)$.
- Let $\mathcal{P} = \{y_0, \dots, y_n\}$ be *any* partition with $||\mathcal{P}|| < \delta$.
- **Classification of Intervals:** Split intervals of $\mathcal{P}$ into two types:
  i. **Good intervals:** Subsets of intervals in $\mathcal{Q}$ (contain no points of $\mathcal{Q}$ in their interior).
  ii. **Bad intervals:** Contain points of $\mathcal{Q}$.

- **Bounding the Sum:**
  - There are at most $N-1$ "bad" intervals. Their total length is small (controlled by δ). The contribution to the sum is bounded by $2M(N-1)\delta < 2\epsilon$.
  - On "good" intervals, the difference between $f(t_k)$ and the sup/inf of $\mathcal{Q}$ is controlled by the original ϵ.
- Result: $|\mathcal{S}(\mathcal{P}, f) - \int f| < 3\epsilon$.

6. Application Example

Example 6.2.7: Calculating $\int_a^b x\,dx$

- Since $f(x) = x$ is continuous, it is integrable.
- We can compute the limit using a specific choice of tags (since the limit is unique).
- **Choice:** Midpoint rule. $t_i = \frac{x_{i-1}+x_i}{2}$.
- **Riemann Sum:**

$$\mathcal{S}(\mathcal{P}, f) = \sum_{i=1}^{n} \left(\frac{x_{i-1}+x_i}{2}\right)(x_i - x_{i-1})$$

- Using difference of squares $(x+y)(x-y) = x^2 - y^2$:

$$\mathcal{S}(\mathcal{P}, f) = \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2)$$

- **Telescoping Sum:**

$$= \frac{1}{2}[(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + \cdots + (x_n^2 - x_{n-1}^2)]$$

$$= \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}(b^2 - a^2)$$