Here is a comprehensive, mathematically rigorous cheat sheet based on Real Analysis Sections 8.1–8.8. It is designed for quick reference/exam preparation.

# Real Analysis Cheat Sheet: Sequences & Series of Functions

## 8.1 Pointwise Convergence & The Interchange Problem

### Definitions

- **Pointwise Convergence:** A sequence $\{f_n\}$ converges pointwise to $f$ on $E$ if for every $x \in E$:

$$\lim_{n\to\infty} f_n(x) = f(x)$$

  - *Formal:* $\forall x \in E, \forall \epsilon > 0, \exists n_0(x, \epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for $n \geq n_0$.
  - *Note:* $n_0$ depends on $x$.

### Failure of Pointwise Convergence (Counter-Examples)

Pointwise convergence is **insufficient** to preserve analytic properties.

1. **Continuity is NOT preserved**
   - *Example:* $f_n(x) = x^n$ on $[0, 1]$.
   - *Limit:* $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$.
   - *Result:* Each $f_n$ is continuous, but limit $f$ is discontinuous at $x = 1$.
2. **Integrals do NOT converge ($\lim \int \neq \int \lim$)**
   - *Example:* $f_n(x) = nx(1 - x^2)^n$ on $[0, 1]$. Pointwise limit $f(x) = 0$.
   - *Integration:* $\int_0^1 f_n(x)dx = \frac{n}{2(n+1)} \to \frac{1}{2}$.
   - *Result:* $\lim \int f_n = 1/2 \neq \int f = 0$.
3. **Differentiability is NOT preserved**
   - *Example:* $f_n(x) = \frac{\sin(nx)}{n}$. Pointwise limit $f(x) = 0$ ($f'(x) = 0$).
   - *Differentiation:* $f_n'(x) = \cos(nx)$. At $x = 0$, $f_n'(0) = 1$.
   - *Result:* $\lim f_n'(0) = 1 \neq f'(0) = 0$.

# 8.2 Uniform Convergence

## Definition & Interpretation

- **Uniform Convergence:** $\{f_n\} \to f$ uniformly on $E$ if:

$$\forall \epsilon > 0, \exists n_0(\epsilon) \text{ s.t. } \forall x \in E, \forall n \geq n_0, |f_n(x) - f(x)| < \epsilon$$

    - *Note:* $n_0$ depends **only** on $\epsilon$, not $x$.
    - *Geometric:* Graph of $f_n$ lies in a "tube" of width $2\epsilon$ around $f$.

## Convergence Tests

1. **Supremum Norm Test (The "Sup Test")**
   Define $M_n = \sup_{x \in E} |f_n(x) - f(x)|$.

$$f_n \to f \text{ uniformly} \iff \lim_{n \to \infty} M_n = 0$$

    - *Ex (Failure):* "Sliding Hump" $S_n(x) = nxe^{-nx^2}$. Max height $\sqrt{n/2e} \to \infty$. Not uniform.
2. **Cauchy Criterion**
   $\{f_n\}$ converges uniformly on $E$ iff $\forall \epsilon > 0, \exists n_0$ such that:

$$|f_n(x) - f_m(x)| < \epsilon \quad \forall n, m \geq n_0, \forall x \in E$$

3. **Weierstrass M-Test (For Series)**
   Let $f(x) = \sum f_k(x)$. If there exist constants $M_k$ such that:
    i. $|f_k(x)| \leq M_k$ for all $x \in E$.
    ii. $\sum M_k < \infty$.
    - **Then:** $\sum f_k$ converges **uniformly** and **absolutely**.

# 8.3 Continuity & $\mathcal{C}(K)$ Space

## Theorems

1. **Preservation of Continuity:**
   If $f_n \to f$ uniformly and each $f_n$ is continuous, then $f$ is continuous.

- *Corollary:* Intechange of limits: $\lim_{t \to p} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to p} f_n(t)$.

2. **Dini's Theorem:**
   Uniform convergence is guaranteed if:
   - $K$ is **compact**.
   - $f_n$ are continuous and converge **pointwise** to continuous $f$.
   - Sequence is **monotone** ($f_n(x) \geq f_{n+1}(x)$ for all $n, x$).

## The Space $\mathcal{C}(K)$

Let $K$ be compact. $\mathcal{C}(K)$ is the space of continuous functions equipped with the **Uniform Norm**:

$$||f||_u = \max_{x \in K} |f(x)|$$

- **Completeness:** $\mathcal{C}(K)$ is a **complete** metric space (Banach space). Every uniformly Cauchy sequence of continuous functions converges to a continuous function.

## Contraction Mappings

- **Banach Fixed Point Theorem:**
  Let $X$ be a complete normed space. If $T : X \to X$ is a contraction ($||T(x) - T(y)|| \leq c||x - y||$ with $0 < c < 1$), then $T$ has a **unique fixed point** ($T(x) = x$).

# 8.4 Integration

## Theorems

1. **Uniform Convergence & Integration:**
   If $f_n \in \mathcal{R}[a, b]$ and $f_n \to f$ **uniformly**, then $f \in \mathcal{R}[a, b]$ and:

$$\lim_{n \to \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$$

   - Also applies to series: $\int \sum f_k = \sum \int f_k$.

2. **Bounded Convergence Theorem (BCT):**
   Allows limit interchange *without* uniform convergence under specific conditions:
   - $f_n \to f$ pointwise.
   - $f_n$ and $f$ are Riemann integrable.
   - **Uniformly Bounded:** $\exists M$ s.t. $|f_n(x)| \leq M$ for all $x, n$.

- **Result:** $\lim \int f_n = \int f$.
- *Revisits $x^n$:* Converges pointwise to discontinuous function, but $|x^n| \leq 1$. Integrals converge $0 \to 0$. BCT holds.

# 8.5 Differentiation

**Uniform convergence of $f_n$ is NOT sufficient for $f_n' \to f'$.**

## Theorem 8.5.1 (Sufficient Condition)

Let $f_n$ be differentiable on $[a, b]$. If:

1. $\{f_n(x_0)\}$ converges for some $x_0$.
2. $\{f_n'\}$ converges **uniformly** on $[a, b]$.
   **Then:** $\{f_n\}$ converges uniformly to $f$, and $f'(x) = \lim f_n'(x)$.

## Example 8.5.3 (Weierstrass Function)

A function continuous everywhere but differentiable nowhere:

$$f(x) = \sum_{k=0}^{\infty} \frac{\cos(a^k \pi x)}{2^k}$$

- Continuity: Proven via Weierstrass M-test (Uniform conv).
- Nowhere Differentiable: Oscillations become infinitely steep ($f_n'$ diverges).

# 8.6 Weierstrass Approximation

## Weierstrass Approximation Theorem

If $f$ is continuous on $[a, b]$, for every $\epsilon > 0$, there exists a polynomial $P$ such that:

$$||f - P||_u < \epsilon$$

- *Meaning:* Polynomials are dense in $\mathcal{C}[a, b]$.

## Proof Method: Approximate Identities

Uses convolution with a kernel $\{Q_n\}$ (Dirac sequence):

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt$$

- Properties of $Q_n$: $\int Q_n = 1$, $Q_n \geq 0$, mass concentrates at 0 as $n \to \infty$.
- Landau Kernel: $Q_n(t) = c_n(1-t^2)^n$.

# 8.7 Power Series

Series form: $\sum_{k=0}^{\infty} a_k(x-c)^k$.

## Radius of Convergence ($R$)

Cauchy-Hadamard Formula:

$$\frac{1}{R} = \limsup_{k\to\infty} \sqrt[k]{|a_k|} \quad \text{or} \quad \frac{1}{R} = \lim \left| \frac{a_{k+1}}{a_k} \right|$$

- $|x-c| < R$: Absolute convergence.
- $|x-c| > R$: Divergence.
- $|x-c| \leq \rho < R$: **Uniform convergence**.

## Key Theorems

1. **Differentiation:** Power series are $C^\infty$ inside $R$. Derivatives are taken term-by-term; $R$ remains unchanged.
2. **Abel's Theorem:** If series converges at endpoint $x = c + R$, the limit function is continuous at that point.

$$\lim_{x\to R^-} \sum a_k x^k = \sum a_k R^k$$

3. **Uniqueness:** If $\sum a_k x^k = \sum b_k x^k$, then $a_k = b_k$. Specifically $a_k = \frac{f^{(k)}(c)}{k!}$.

## Taylor Series Remainder

$f(x) = T_n(x) + R_n(x)$. Conv $\iff R_n \to 0$.

- **Lagrange Form:** $R_n(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$.
- **Cauchy's Counter-example:** $f(x) = e^{-1/x^2}$ (at $x = 0$). $f$ is $C^\infty$, all derivatives at 0 are 0. Taylor series is 0, but $f(x) \neq 0$.

# 8.8 The Gamma Function

Generalization of factorial to real numbers.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (x > 0)$$

## Properties

1. **Functional Eq:** $\Gamma(x+1) = x\Gamma(x)$.
2. **Factorial:** $\Gamma(n+1) = n!$.
3. **Value:** $\Gamma(1/2) = \sqrt{\pi}$.

## Beta Function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$