Chapter 5.2: The Mean Value Thm.

1. Local Maxima and Minima

Definition 5.2.1:
A function $f : E \to \mathbb{R}$ has a:

- **Local maximum** at $p \in E$ if $\exists \delta > 0$ such that $f(x) \leq f(p)$ for all $x \in E \cap N_\delta(p)$.
- **Absolute maximum** if $f(x) \leq f(p)$ for all $x \in E$.
  (Analogous definitions apply for minimums).

Theorem 5.2.2 (Relationship to Derivative):
Let f be defined on interval I. If f has a local extremum at an interior point $p \in \text{Int}(I)$ and f is differentiable at p, then:

$$f'(p) = 0$$

- **Proof Idea:** Analyze the difference quotient. If p is a max, $\frac{f(t)-f(p)}{t-p} \leq 0$ for $t > p$ (implies $f'_+(p) \leq 0$) and ≥ 0 for $t < p$ (implies $f'_-(p) \geq 0$). Since $f'(p)$ exists, limits must be equal, thus 0.

Corollary 5.2.3:
For continuous f on $[a, b]$, relative extrema at $p \in (a, b)$ imply either $f'(p)$ does not exist or $f'(p) = 0$.

- *Note:* This does not apply to endpoints. At endpoints, we can only conclude inequality (e.g., if max at a, $f'(a) \leq 0$).

2. Rolle's Theorem

Theorem 5.2.5 (Rolle's Theorem):
Suppose f is:

1. Continuous on $[a, b]$
2. Differentiable on (a, b)
3. $f(a) = f(b)$

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

- **Geometric Interpretation:** There is at least one point where the tangent line is horizontal.
- **Proof Idea:**
  - Since $[a, b]$ is compact, f attains max and min (EVT).
  - If f is constant, $f'(x) = 0$ everywhere.
  - If not constant, extremum occurs at an interior point c. By Theorem 5.2.2, $f'(c) = 0$.
- **Example:** $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$. Derivative undefined at endpoints, but theorem holds ($c = 0$).

3. The Mean Value Theorem (MVT)

Theorem 5.2.6 (Lagrange):
Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that:

$$f(b) - f(a) = f'(c)(b - a)$$

- **Geometric Interpretation:** There is a point c where the tangent slope equals the secant slope connecting $(a, f(a))$ and $(b, f(b))$.
- **Proof Idea:**
  Construct an auxiliary function $g(x)$ representing the vertical distance between the curve and the secant line:

$$g(x) = f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

  Since $g(a) = g(b) = 0$, applying Rolle's Theorem to g yields $g'(c) = 0$, which rearranges to the MVT equation.

Example 5.2.7 (Inequalities):
Using MVT to prove $\frac{x}{1+x} \leq \ln(1 + x) \leq x$ for $x > -1$.

- Let $f(x) = \ln(1 + x)$. $f(0) = 0$.
- By MVT, $\ln(1 + x) = f(x) - f(0) = f'(c)x = \frac{x}{1+c}$ for some c between 0 and x.
- Analyze bounds of $\frac{1}{1+c}$ based on $0 < c < x$ or $x < c < 0$ to derive the inequality.

4. Cauchy Mean Value Theorem

Theorem 5.2.8:
If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If $g'(x) \neq 0$, this can be written as:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

- **Proof Idea:** Define $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. Since $h(a) = h(b)$, apply Rolle's Theorem to find $h'(c) = 0$.
- **Geometric Interpretation:** For a parametric curve defined by $(g(t), f(t))$, there is a point where the tangent slope equals the slope of the chord connecting the endpoints.

5. Applications: Monotonicity

1. Monotonicity on Intervals (Theorem 5.2.9)

Let f be differentiable on an interval I. The sign of the derivative determines the monotonicity of the function on that interval:

- $f'(x) \geq 0, \forall x \in I \implies f$ is **monotone increasing**.
- $f'(x) > 0, \forall x \in I \implies f$ is **strictly increasing**.
- $f'(x) \leq 0, \forall x \in I \implies f$ is **monotone decreasing**.
- $f'(x) = 0, \forall x \in I \implies f$ is **constant**.

Proof Idea: Apply the Mean Value Theorem (MVT) to arbitrary $x_1 < x_2$. The sign of $f(x_2) - f(x_1)$ is determined entirely by $f'(c)(x_2 - x_1)$.

2. Pointwise vs. Neighborhood Behavior (Crucial Distinction)

Observation:
The condition $f'(c) > 0$ at a **single point** c behaves differently than $f'(x) > 0$ on an **interval**.

A. What $f'(c) > 0$ implies:
If $f'(c) > 0$, there exists a $\delta > 0$ such that:

- $f(x) < f(c)$ for all $x \in (c - \delta, c)$
- $f(x) > f(c)$ for all $x \in (c, c + \delta)$
  (See Exercise 17).

B. What $f'(c) > 0$ does NOT imply:
It does **not** imply that f is increasing on the interval $(c - \delta, c + \delta)$.

- **Reason:** $f'(x)$ may assume both positive and negative values in every neighborhood of c.
- **Counter-Example (Exercise 20):**

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  Here, $f'(0) = 1 > 0$, yet f is **not monotone** on any interval containing 0 due to rapid oscillation.

C. Sufficient Condition for Monotonicity:
If we require **continuity** of the derivative:

- If $f'(c) > 0$ **AND** f' is continuous at c, then there exists a $\delta > 0$ such that $f'(x) > 0$ for all $x \in (c - \delta, c + \delta)$.
- $\therefore f$ is increasing on $(c - \delta, c + \delta)$.

3. Relative Extrema & The First Derivative Test

The First Derivative Test:
Used to classify critical points where $f'(c) = 0$ or $f'(c)$ does not exist.
Suppose f is continuous on (a, b).

1. If $f'(x) < 0$ on (a, c) AND $f'(x) > 0$ on (c, b):
   - f is decreasing to the left and increasing to the right.
   - $\implies f$ has a **relative minimum** at c.
2. (Similarly for relative maximum if signs switch from $+$ to $-$).

The False Converse:
One naturally assumes: *"If f has a relative minimum at c, then f must be decreasing to the left and increasing to the right."*

- **This is FALSE.**
- A function can have a relative minimum at c without being monotone on the immediate left or right sides.

Counter-Example (Example 5.2.10):

$$f(x) = \begin{cases} x^4(2 + \sin(1/x)) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- f has an absolute minimum at $x = 0$ (since $f(x) > 0$ for $x \neq 0$).
- However, $f'(x)$ oscillates between positive and negative values in every neighborhood of 0.
- Thus, f is **not** "decreasing then increasing" in the standard monotonic sense near 0.

6. Limits of Derivatives

Theorem 5.2.11:
If f is continuous on $[a, b)$ and differentiable on (a, b), and $\lim_{x\to a^+} f'(x) = L$, then the right-hand derivative exists and:

$$f'_+(a) = \lim_{x\to a^+} f'(x)$$

- **Proof Idea:** Use MVT on $[a, a+h]$. $f(a+h) - f(a) = f'(\zeta_h)h$. As $h \to 0$, $\zeta_h \to a$, so the difference quotient converges to L.
- **Implication:** Derivatives cannot have simple jump discontinuities; discontinuities must be of the second kind (oscillatory).

7. Intermediate Value Theorem for Derivatives

Theorem 5.2.13 (Darboux's Theorem):
If f is differentiable on I and $a, b \in I$ with $a < b$, then for any λ between $f'(a)$ and $f'(b)$, there exists $c \in (a, b)$ such that:

$$f'(c) = \lambda$$

- **Significance:** Derivatives possess the Intermediate Value Property even if they are **not continuous**.

- **Proof Idea:** Construct $g(x) = f(x) - \lambda x$. Depending on signs of $g'(a)$ and $g'(b)$, g attains a local extremum interior to the interval. At that extremum c, $g'(c) = f'(c) - \lambda = 0$.

8. Inverse Function Theorem

Theorem 5.2.14:
If f is differentiable on interval I and $f'(x) \neq 0$ for all $x \in I$:

1. f is one-to-one.
2. f^{-1} is continuous and differentiable on $J = f(I)$.
3. The derivative is given by:

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

- **Proof Idea:** $f' \neq 0$ implies f' maintains a single sign (by Darboux's Theorem), so f is strictly monotone (one-to-one). Differentiability follows from limits.