Section 5.3 L'Hospital's Rule

1. Infinite Limits (Definitions)

Before introducing L'Hospital's rule, formal definitions for infinite limits are established to handle the indeterminate form ∞/∞.

- **Definition 5.3.1:** Let f be defined on a subset $E \subset \mathbb{R}$ and p be a limit point of E.
  - $\lim_{x \to p} f(x) = \infty$ if for every $M \in \mathbb{R}$, there exists $\delta > 0$ such that $f(x) > M$ for all $x \in E$ with $0 < |x - p| < \delta$.
  - $\lim_{x \to p} f(x) = -\infty$ is defined similarly (where $f(x) < M$).
- **Note:** These definitions extend to limits at infinity ($\lim_{x \to \infty}$) and one-sided limits ($\lim_{x \to p^+}$).

2. L'Hospital's Rule (Theorem 5.3.2)

This rule evaluates limits of indeterminate forms $0/0$ or ∞/∞.

Hypotheses:

1. f, g are real-valued differentiable functions on (a, b).
2. $g'(x) \neq 0$ for all $x \in (a, b)$.
3. $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, where $L \in \mathbb{R} \cup \{-\infty, \infty\}$.

Conditions (Indeterminate Forms):

- **(a) Case $0/0$:** $\lim_{x \to a^+} f(x) = 0$ and $\lim_{x \to a^+} g(x) = 0$.
- **(b) Case ∞/∞:** $\lim_{x \to a^+} g(x) = \pm\infty$ (Note: $f(x)$ does not strictly need to tend to ∞, but usually does).

Conclusion:

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

(Note: The rule applies equally to $x \to b^-$, $x \to p$, or $x \to \pm\infty$).

3. Proofs: Core Ideas

Case A: Indeterminate Form $0/0$ (Finite a)

- **Key Tool:** Generalized Mean Value Theorem (GMVT).
- **Method:**
  i. Since we are dealing with limits approaching a, we define $f(a) = g(a) = 0$ to make the functions continuous at a.
  ii. Consider a sequence $\{x_n\} \to a^+$. Apply GMVT on the interval $[a, x_n]$.
  iii. There exists c_n between a and x_n such that:

$$\frac{f(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(c_n)}$$

  iv. Since $f(a) = g(a) = 0$, this simplifies to $\frac{f(x_n)}{g(x_n)} = \frac{f'(c_n)}{g'(c_n)}$.
  v. As $n \to \infty$, $x_n \to a$ implies $c_n \to a$. Therefore, the limit of the ratio of functions equals the limit of the ratio of derivatives.

Case B: Limits at Infinity ($x \to -\infty$)

- **Key Tool:** Substitution.
- **Method:** Let $x = -1/t$. As $t \to 0^+$, $x \to -\infty$.
  - Define $\phi(t) = f(-1/t)$ and $\psi(t) = g(-1/t)$.
  - Using chain rule differentiation, the problem reduces to a limit at 0^+, which allows the use of the previous proof logic.

Case C: Indeterminate Form ∞/∞

- **Key Tool:** GMVT + Bounding Argument (No need for f, g to be continuous at a).
- **Method:**
  i. Assume $\lim \frac{f'(x)}{g'(x)} = L$.
  ii. Fix y and let x vary. Apply GMVT on interval (x, y) to get $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(\zeta)}{g'(\zeta)}$.
  iii. **Algebraically rearrange** the GMVT equation to isolate $\frac{f(x)}{g(x)}$:

$$\frac{f(x)}{g(x)} = \frac{f'(\zeta)}{g'(\zeta)} \left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)}$$

  iv. Since $g(x) \to \infty$ as $x \to a^+$, the term $\frac{g(y)}{g(x)} \to 0$ and $\frac{f(y)}{g(x)} \to 0$.
  v. This implies that for x sufficiently close to a, $\frac{f(x)}{g(x)}$ becomes arbitrarily close to $\frac{f'(\zeta)}{g'(\zeta)} \cdot (1 - 0) + 0$, which converges to L.

4. Examples

(a) Basic Application ($0/0$)

Problem: Compute $\lim_{x\to 0^+} \frac{\ln(1+x)}{x}$.

- **Form:** $0/0$ (since $\ln(1) = 0$).
- **Apply L'Hospital's:**

$$\lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 1$$

- **Note:** This can also be proven using inequalities derived from the Taylor expansion or Mean Value Theorem (i.e., $\frac{x}{1+x} \leq \ln(1+x) \leq x$), but L'Hospital's is more direct.

(b) Repeated Application

Problem: Compute $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

- **Form:** $0/0$.
- **First Application:**

$$\lim_{x\to 0} \frac{\sin x}{2x}$$

  This is *still* form $0/0$.
- **Second Application:**

$$\lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

- **Result:** The limit is $1/2$.

(c) Importance of Substitution (Avoiding Complexity)

Problem: Compute $\lim_{x\to 0^+} \frac{e^{-1/x}}{x}$.

- **Form:** $0/0$ (since $e^{-\infty} \to 0$).
- **Direct L'Hospital's Failure:** Differentiating directly gives $\frac{e^{-1/x} \cdot (1/x^2)}{1}$, which simplifies to $\frac{e^{-1/x}}{x^2}$. This is *more* complicated than the original.
- **Correct Approach:** Use substitution $t = 1/x$. As $x \to 0^+$, $t \to \infty$.

$$\lim_{t \to \infty} \frac{t}{e^t}$$

- **New Form:** ∞/∞.
- **Apply L'Hospital's:**

$$\lim_{t \to \infty} \frac{1}{e^t} = 0$$

- **Result:** The original limit is 0.