Chapter 5.1: The Derivative

1. Definition of the Derivative

Historical Context: Formulated rigorously by Cauchy (1821) using limits, moving away from vague notions of tangent lines and velocity.

Definition 5.1.1 (The Derivative)
Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. For a fixed $p \in I$, the derivative $f'(p)$ is:

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

Alternatively, letting $x = p + h$:

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

- **Geometric interpretation:** Slope of the tangent line at $(p, f(p))$.
- **Physical interpretation:** Instantaneous velocity.

2. One-Sided Derivatives

If p is an endpoint or we need to analyze corner points, we use one-sided limits (Definition 5.1.2).

- **Right Derivative ($f'_+(p)$):** $\lim\limits_{h \to 0^+} \dfrac{f(p+h) - f(p)}{h}$
- **Left Derivative ($f'_-(p)$):** $\lim\limits_{h \to 0^-} \dfrac{f(p+h) - f(p)}{h}$

Key Property:
For an interior point $p \in I$, $f'(p)$ exists **if and only if** both $f'_+(p)$ and $f'_-(p)$ exist and are **equal**.

3. Worked Examples (Specific Functions)

The text analyzes the differentiability of several specific functions to illustrate the definition.

A. Power Function ($f(x) = x^2$)

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0}(2x + h) = 2x$$

(Note: Generalizes to $f(x) = x^n \implies f'(x) = nx^{n-1}$).

B. Square Root ($f(x) = \sqrt{x}, x > 0$)

Uses rationalization:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \xrightarrow{h \to 0} \frac{1}{2\sqrt{x}}$$

C. Sine Function ($f(x) = \sin x$)

Uses the identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$\frac{\sin(x+h) - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right)$$

Since $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$, $f'(x) = \cos x$.

D. Absolute Value ($f(x) = |x|$) at $x = 0$

- $f'_+(0) = \lim_{h \to 0^+} \frac{h}{h} = 1$
- $f'_-(0) = \lim_{h \to 0^-} \frac{-h}{h} = -1$
- **Result:** $f'_+(0) \neq f'_-(0)$, so f is **not differentiable** at 0.

E. Cusp ($g(x) = x^{3/2}$) at $x = 0$

$$g'(0) = \lim_{h \to 0^+} \frac{h^{3/2}}{h} = \lim_{h \to 0^+} \sqrt{h} = 0$$

Differentiable at 0.

F. Oscillating Discontinuity ($f(x) = x \sin(1/x)$ for $x \neq 0$, $f(0) = 0$)

$$f'(0) = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

Result: Limit does not exist. Not differentiable at 0.

G. Differentiable with Discontinuous Derivative ($g(x) = x^2 \sin(1/x)$ for $x \neq 0$, $g(0) = 0$)

- **At $x = 0$:**

$$g'(0) = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

  (By Squeeze Theorem).
- **For $x \neq 0$:** $g'(x) = 2x \sin(1/x) - \cos(1/x)$.
- **Observation:** $\lim_{x \to 0} g'(x)$ does not exist (due to $\cos(1/x)$).
- **Conclusion:** g is differentiable everywhere, but g' is **not continuous** at 0.

4. Relationship between Differentiability and Continuity

Theorem 5.1.4:
If f is differentiable at p, then f is continuous at p.

Proof Sketch:

$$\lim_{t \to p}(f(t) - f(p)) = \lim_{t \to p}\left[\frac{f(t) - f(p)}{t - p} \cdot (t - p)\right] = f'(p) \cdot 0 = 0$$

Thus $\lim_{t \to p} f(t) = f(p)$.

- **Converse:** False. Continuity does **not** imply differentiability (e.g., $f(x) = |x|$).
- **Weierstrass Function:** An example of a function continuous everywhere but differentiable nowhere.

5. Arithmetic of Derivatives

Theorem 5.1.5 (Algebraic Rules)
Let f, g be differentiable at x.

A. Sum Rule

$$(f + g)'(x) = f'(x) + g'(x)$$

B. Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Key Proof Idea: The "Add and Subtract" Trick
Direct substitution creates a mixed term $f(x+h)g(x+h)$ that cannot be factored. To fix this, we insert a "middle term" into the numerator.

Proof Sketch:

1. **Setup:** Start with the difference quotient:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

2. **The Trick:** Add and subtract $f(x+h)g(x)$:

$$\frac{f(x+h)g(x+h) \mathbf{- f(x+h)g(x) + f(x+h)g(x)} - f(x)g(x)}{h}$$

3. **Group & Limit:** Separate into two parts:

$$f(x+h)\underbrace{\left[\frac{g(x+h)-g(x)}{h}\right]}_{\to g'(x)} + g(x)\underbrace{\left[\frac{f(x+h)-f(x)}{h}\right]}_{\to f'(x)}$$

Note: $f(x+h) \to f(x)$ because differentiability implies continuity.

C. Quotient Rule (Reciprocal Case)

To prove the quotient rule, we first focus on the derivative of the reciprocal.

The Reciprocal Rule:

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{[g(x)]^2}$$

Proof Sketch:

1. **Common Denominator:**

$$\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \frac{g(x) - g(x+h)}{h \cdot g(x+h)g(x)}$$

2. **Extract Derivative Definition:**
   Recognize that $g(x) - g(x+h) = -[g(x+h) - g(x)]$:

$$-\underbrace{\left[\frac{g(x+h) - g(x)}{h}\right]}_{\to g'(x)} \cdot \frac{1}{g(x+h)g(x)}$$

3. **Take Limit ($h \to 0$):**
   Since g is continuous, $g(x+h) \to g(x)$, giving the denominator $[g(x)]^2$.

(Note: The full Quotient Rule is simply the Product Rule applied to $f(x) \cdot [1/g(x)]$.)

6. The Chain Rule

Theorem 5.1.6 (Composition)

If f is differentiable at x and g is differentiable at $y = f(x)$, then $h = g \circ f$ is differentiable at x:

$$h'(x) = g'(f(x)) \cdot f'(x)$$

Proof: Linear Approximation (to avoid division by zero).

Standard limits fail if $f(t) - f(x) = 0$. Instead, use "error terms" (u, v) that go to 0:

- **f:** $f(t) - f(x) = (t - x)[f'(x) + u(t)]$
- **g:** $g(s) - g(y) = (s - y)[g'(y) + v(s)]$
- Set $s = f(t)$ and $y = f(x)$. The difference quotient becomes:

$$\frac{g(f(t)) - g(f(x))}{t - x} = [f'(x) + u(t)] \cdot [g'(y) + v(f(t))]$$

- As $t \to x$, the error terms vanish ($u, v \to 0$), leaving $f'(x)g'(y)$.

Examples:

- **Composite Trig:** $h(x) = \sin(1/x^2)$.

$$h'(x) = \cos\left(\frac{1}{x^2}\right) \cdot \frac{d}{dx}(x^{-2}) = \cos\left(\frac{1}{x^2}\right) \cdot (-2x^{-3})$$

- **Power Rule Extension:** $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$.