Here is a comprehensive, condensed mathematical cheat sheet based on the provided material.

Cheat Sheet: Real Analysis & Series

1. Series of Real Numbers (Fundamentals)

Definitions

- **Series:** $\sum_{k=1}^{\infty} a_k$. **Partial Sum:** $s_n = \sum_{k=1}^{n} a_k$.
- **Convergence:** $\sum a_k = s \iff \lim_{n\to\infty} s_n = s$.
- **Linearity:** $\sum(ca_k + b_k) = c\sum a_k + \sum b_k$ (if convergent).

Comparison Tests (for $a_k \geq 0$)

1. **Direct Comparison:** Given $0 \leq a_k \leq Mb_k$:
   - $\sum b_k < \infty \implies \sum a_k < \infty$.
   - $\sum a_k = \infty \implies \sum b_k = \infty$.
2. **Limit Comparison Test (LCT):** Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.
   - $0 < L < \infty$: $\sum a_k \iff \sum b_k$ (Converge/Diverge together).
   - $L = 0$: $\sum b_k < \infty \implies \sum a_k < \infty$.
   - *Example 1:* $\sum \frac{k}{3^k}$ vs $\sum (1/2)^k \implies$ **Converges**.
   - *Example 2:* $\sum \sqrt{\frac{k+1}{2k^3+1}} \approx \sum \frac{1}{k\sqrt{2}} \implies$ **Diverges**.

Integral Test

Let $f(x)$ be continuous, non-negative, decreasing on $[1, \infty)$ where $f(k) = a_k$.

$$\sum_{k=1}^{\infty} a_k < \infty \iff \int_1^{\infty} f(x)\,dx < \infty$$

- **p-Series:** $\sum \frac{1}{k^p}$ converges $\iff p > 1$.
- **Log Series:** $\sum \frac{1}{k \ln k}$ diverges ($\int \frac{dx}{x \ln x} = \ln(\ln x) \to \infty$).

Root and Ratio Tests

Let $R = \limsup \left|\frac{a_{k+1}}{a_k}\right|$, $r = \liminf \left|\frac{a_{k+1}}{a_k}\right|$, and $\alpha = \limsup \sqrt[k]{|a_k|}$.

Test	Condition	Conclusion	Note
Ratio	$R < 1$	**Converges**	Good for factorials.
	$r > 1$	**Diverges**	
	$r \leq 1 \leq R$	**Inconclusive**	p-series fails here.
Root	$\alpha < 1$	**Converges**	**Stronger test.**
	$\alpha > 1$	**Diverges**	
	$\alpha = 1$	**Inconclusive**	

- **Hierarchy:** Root test is strictly stronger.
    - *Ex:* a_n alternating $1/2^k, 1/3^k$. Ratio oscillates (inconclusive), Root gives $\alpha = 1/\sqrt{2} < 1$ (Converges).

2. Dirichlet Test & Applications

Abel's Partial Summation Formula

Discrete analogue of Integration by Parts. Let $A_n = \sum_{i=1}^{n} a_i$.

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p$$

Dirichlet Test

$\sum a_k b_k$ converges if:

1. **Bounded Sums:** $\left|\sum_{k=1}^{n} a_k\right| \leq M$ for all n.
2. **Monotonic:** $b_1 \geq b_2 \geq \cdots \geq 0$.
3. **Vanishing:** $\lim_{k\to\infty} b_k = 0$.

Alternating Series Test (AST)

Case of Dirichlet where $a_k = (-1)^{k+1}$.

- **Condition:** $b_k \downarrow 0$.

- **Result:** $\sum(-1)^{k+1}b_k$ converges.
- **Error Estimate:** $|S - S_n| \leq b_{n+1}$.
    - *Ex:* $\sum \frac{(-1)^{k+1}}{2k-1} \to \frac{\pi}{4}$. Very slow convergence (need $n \approx 50$ for 0.01 accuracy).

Trigonometric Series

1. **Sine:** $\sum b_k \sin(kt)$ converges $\forall t \in \mathbb{R}$ (if $b_k \downarrow 0$).
2. **Cosine:** $\sum b_k \cos(kt)$ converges $\forall t \in \mathbb{R} \setminus \{2p\pi\}$.
    - *Reason:* Partial sums of $\sin(kt)$ and $\cos(kt)$ are bounded by $\csc(t/2)$ unless $t = 2p\pi$.
    - *Ex:* $\sum \frac{1}{k}\cos(kt)$ diverges at $t = 0$ (Harmonic) but converges elsewhere.

3. Absolute vs. Conditional Convergence

Definitions

- **Absolute Convergence:** $\sum |a_k| < \infty$. Implies convergence ($\sum a_k < \infty$).
- **Conditional Convergence:** $\sum a_k < \infty$ BUT $\sum |a_k| = \infty$.
    - *Ex:* $\sum \frac{(-1)^{k+1}}{k}$ (Alternating Harmonic).

Rearrangements

- **Absolutely Convergent:** Any rearrangement sums to the same value.
- **Conditionally Convergent (Riemann's Theorem):** Can be rearranged to converge to **any** $\alpha \in \mathbb{R}$ or diverge to $\pm\infty$.
    - *Mechanism:* Greedy algorithm taking enough positive terms to exceed α, then negative terms to drop below, repeating indefinitely.

4. The Space l^2 (Square Summable Sequences)

Structure

- **Definition:** $l^2 = \{\{a_k\} : \sum a_k^2 < \infty\}$.
- **Norm:** $||a||_2 = \sqrt{\sum_{k=1}^{\infty} a_k^2}$.
- **Convergence Examples:**

    - $\{1/k\} \in l^2$ ($\sum 1/k^2 < \infty$).
    - $\{1/\sqrt{k}\} \notin l^2$ ($\sum 1/k = \infty$).
    - $\{1/k^q\} \in l^2 \iff q > 1/2$ (since $\sum 1/k^{2q}$ requires $2q > 1$).

Inequalities

1. **Cauchy-Schwarz:** Fundamental for geometry/angles.

$$\sum |a_k b_k| \leq ||a||_2 \cdot ||b||_2$$

    - *Implication:* Inner product $\langle a, b\rangle = \sum a_k b_k$ is well-defined.

2. **Minkowski (Triangle Inequality):**

$$||a + b||_2 \leq ||a||_2 + ||b||_2$$

    - *Proof:* Uses Cauchy-Schwarz on expanded square $(a_k + b_k)^2$.

Normed Linear Spaces

l^2 is a vector space with a norm satisfying:

1. **Non-negativity:** $||x|| \geq 0$, equals 0 iff $x = 0$.
2. **Homogeneity:** $||cx|| = |c| \cdot ||x||$.
3. **Triangle Inequality:** $||x + y|| \leq ||x|| + ||y||$.

- **Metric:** $d(x, y) = ||x - y||$.
- **Convergence:** $x_n \to x$ in $l^2 \iff ||x_n - x||_2 \to 0$.