6.3 Fundamental Theorem of Calculus

This section establishes the connection between differentiation and integration (inverse operations) using Riemann or Darboux sums.

I. The First Fundamental Theorem (Evaluation)

Definition 6.3.1 (Antiderivative)
A function F on an interval I is an antiderivative of f if $F'(x) = f(x)$ for all $x \in I$.

- **Note:** Antiderivatives are not unique ($F(x) + C$).

Theorem 6.3.2 (Fundamental Theorem of Calculus - Part 1)
If $f \in \mathcal{R}[a, b]$ (Riemann integrable) and F is an antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- **Proof (Key Idea):**
  i. Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be any partition of $[a, b]$.
  ii. Apply the **Mean Value Theorem** to F on each subinterval $[x_{i-1}, x_i]$:

  $$F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i = f(t_i)\Delta x_i$$

  for some $t_i \in (x_{i-1}, x_i)$.
  iii. Summing over i yields a telescoping sum:

  $$\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = F(b) - F(a)$$

  iv. Since this Riemann sum equals $F(b) - F(a)$ for *any* partition, and f is integrable:

  $$\mathcal{L}(\mathcal{P}, f) \le F(b) - F(a) \le \mathcal{U}(\mathcal{P}, f) \implies \int_a^b f = F(b) - F(a)$$

Examples 6.3.3 (Notable Cases)

- **(b) Discontinuous Derivative:**
  Let $F(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $F(0) = 0$.
  Then $f(x) = F'(x)$ exists everywhere but is discontinuous at $x = 0$. Since f is bounded and

continuous almost everywhere, $f \in \mathcal{R}[0, 1]$.
Result: $\int_0^1 f = F(1) - F(0) = \sin 1$.

- **(c) Piecewise Function:**
  Let $f(x) = 1$ for $x \in [0, 1)$ and $f(x) = x - 1$ for $x \in [1, 2]$.
  Define $F(x) = \int_0^x f(t)dt$.
  Result: Even though f is discontinuous at $x = 1$, the accumulation function $F(x)$ is **continuous everywhere**.

II. The Second Fundamental Theorem (Differentiation)

Theorem 6.3.4 (Fundamental Theorem of Calculus - Part 2)
Let $f \in \mathcal{R}[a, b]$ and define F on $[a, b]$ by:

$$F(x) = \int_a^x f(t)\, dt$$

Then F is continuous on $[a, b]$. Furthermore, if f is **continuous** at $c \in [a, b]$, then F is differentiable at c and:

$$F'(c) = f(c)$$

- **Proof (Key Idea):**
  To show $F'(c) = f(c)$, consider the difference quotient:

$$\frac{F(c+h) - F(c)}{h} - f(c) = \frac{1}{h} \int_c^{c+h} (f(t) - f(c))\, dt$$

  Since f is continuous at c, for any $\epsilon > 0$, $|f(t) - f(c)| < \epsilon$ for sufficiently small h.

$$\left| \frac{1}{h} \int_c^{c+h} (f(t) - f(c))\, dt \right| < \frac{1}{h}(h \cdot \epsilon) = \epsilon$$

  Thus, limit as $h \to 0$ is 0, implying $F'(c) = f(c)$.
- **Remarks:**
  - Integrability of f implies continuity of F.
  - Continuity of f implies differentiability of F.

III. Application: The Natural Logarithm

Example 6.3.5 (Definition of $\ln x$)
For $x > 0$, define the natural logarithm as:

$$L(x) = \int_1^x \frac{1}{t}\,dt$$

Since $1/t$ is continuous, $L'(x) = 1/x$.

Properties:

1. **Log Rules:** $L(ab) = L(a) + L(b)$.
   - *Proof Idea:* Differentiate $L(ax)$ to get $1/x$, implying $L(ax) = L(x) + C$. Use $L(1) = 0$ to find $C = L(a)$.
2. **Power Rule:** $L(b^r) = rL(b)$ for $r \in \mathbb{R}$.
3. **Euler's Number:** $L(e) = 1$.
   - *Derivation:* $1 = L'(1) = \lim_{n\to\infty} nL(1 + 1/n) = L(\lim(1 + 1/n)^n) = L(e)$.

IV. Consequences of FTC

Theorem 6.3.6 (Mean Value Theorem for Integrals)
If f is continuous on $[a, b]$, there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

- **Proof:** Let $F(x) = \int_a^x f$. By standard MVT, $F(b) - F(a) = F'(c)(b-a)$. Since $F'(c) = f(c)$, the result follows.

Theorem 6.3.7 (Integration by Parts)
If f, g are differentiable and $f', g' \in \mathcal{R}[a, b]$:

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'$$

- **Proof:** Integrate the product rule $(fg)' = fg' + gf'$ and apply FTC.

Theorem 6.3.8 (Change of Variable)
Let φ be differentiable on $[a, b]$ with $\varphi' \in \mathcal{R}$. If f is continuous on the range of φ:

$$\int_a^b f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$$

- **Proof:** Let $F(x) = \int f(s)ds$. By Chain Rule, $\frac{d}{dt}F(\varphi(t)) = f(\varphi(t))\varphi'(t)$. Apply FTC.

Examples 6.3.9

- **(a) Substitution:** Evaluating $\int_0^2 t/(1+t^2)dt$ using $\varphi(t) = 1 + t^2$.
- **(b) Trig Substitution:** Evaluating $\int_0^a \sqrt{a^2 - x^2}dx$ using $x = a\sin t$. Converts to $\int \cos^2 t$, solved via half-angle identity.