Here is a comprehensive and structured study guide for **Chapter 9: Fourier Series**. This guide organizes the definitions, theorems, and examples into a logical flow for review.

# Chapter 9: Fourier Series

## 9.1 Orthogonal Functions

**Core Concept:** We can approximate complex functions using a linear combination of simpler, "orthogonal" functions, much like decomposing a vector into components in $\mathbb{R}^n$.

### 1. Inner Product and Norm

To treat functions like vectors, we define the **Inner Product** for Riemann integrable functions on $[a, b]$:

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

- **Norm (Length):** $||f||_2 = \sqrt{\langle f, f \rangle} = \left[\int_a^b f^2(x)\,dx\right]^{1/2}$
- **Orthogonality:** Two functions are orthogonal if $\langle f, g \rangle = 0$.
- **Orthonormality:** A set $\{\phi_n\}$ is orthonormal if $\langle \phi_n, \phi_m \rangle = 0$ (for $n \neq m$) and $||\phi_n||^2 = 1$.

### 2. Standard Orthogonal Systems

- **Polynomials on $[-1, 1]$:** $\{1, x\}$ are orthogonal because $\int_{-1}^1 x\,dx = 0$.
- **Trigonometric System on $[-\pi, \pi]$:** The set $\{1, \cos nx, \sin nx\}$ is orthogonal.
  - $\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = 0$ for $n \neq m$.
  - **Norms:** $||1||^2 = 2\pi$, while $||\sin nx||^2 = \pi$ and $||\cos nx||^2 = \pi$.

### 3. Best Approximation (Least Squares)

We wish to approximate $f(x)$ with a sum $S_N(x) = \sum_{n=1}^N c_n\phi_n(x)$ by minimizing the **Mean Square Error**:

$$E_N = ||f - S_N||_2^2 = \int_a^b [f(x) - S_N(x)]^2\,dx$$

**Theorem:** The error is minimized if $c_n$ are the **Fourier Coefficients**:

$$c_n = \frac{\langle f, \phi_n \rangle}{||\phi_n||^2}$$

**Example 9.1.6:** Approximating $f(x) = x^3 + 1$ on $[-1, 1]$ using $\{1, x\}$:

1. Calculate $c_1$: $\langle x^3 + 1, 1\rangle / \langle 1, 1\rangle = 1$.
2. Calculate $c_2$: $\langle x^3 + 1, x\rangle / \langle x, x\rangle = 3/5$.
3. **Result:** $S_2(x) = 1 + \frac{3}{5}x$.

## 4. Bessel's Inequality

The error cannot be negative ($E_N \geq 0$), implying the sum of the coefficients converges:

$$\sum_{n=1}^{\infty} c_n^2 ||\phi_n||^2 \leq ||f||^2$$

- **Corollary:** Fourier coefficients must decay to zero ($c_n \to 0$) as $n \to \infty$.

# 9.2 Completeness and Parseval's Equality

## 1. Convergence in the Mean

A sequence $f_n$ converges to $f$ **in the mean** if the "area" of the squared difference goes to zero:

$$\lim_{n \to \infty} \int_a^b [f(x) - f_n(x)]^2 \, dx = 0$$

- **Contrast with Pointwise:** A sequence can converge in the mean but fail to converge pointwise (e.g., the "moving bump" counter-example).

## 2. Completeness

An orthogonal system is **Complete** if approximations can get arbitrarily close to *any* integrable function in the mean. This is equivalent to **Parseval's Equality**:

$$\sum_{n=1}^{\infty} c_n^2 ||\phi_n||^2 = ||f||^2$$

*(This is the infinite-dimensional Pythagorean Theorem: sum of squared components = squared length of vector.)*

# 9.3 Trigonometric Fourier Series

## 1. Definitions

For a function $f$ on $[-\pi, \pi]$, the Fourier Series is:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

**Formulas:**

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$

## 2. Key Calculation Examples

- **Step Function (Example 9.3.3a):**
  - $f(x) = 0$ on $[-\pi, 0)$, $1$ on $[0, \pi)$.
  - Result: $f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$ (contains only odd sine terms).
- **Linear Function (Example 9.3.3b):**
  - $f(x) = x$ on $[-\pi, \pi]$.
  - Since $f$ is odd, $a_n = 0$.
  - Result: $x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

## 3. Riemann-Lebesgue Lemma

For any integrable function, the oscillatory integrals decay to zero:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = 0$$

This confirms that high-frequency noise contributes less to the function's structure.

# 9.4 Convergence in the Mean (Theoretical Core)

## 1. The Dirichlet Kernel ($D_n$)

The partial sum $S_n$ is an integral convolution of $f$ with $D_n$:

$$D_n(t) = \frac{\sin((n + 1/2)t)}{2\sin(t/2)}$$

- **Issue:** $\int |D_n| \to \infty$. This makes $D_n$ a "bad" kernel; it does not guarantee pointwise convergence for continuous functions easily.

## 2. The Fejér Kernel ($F_n$)

To fix this, we average the partial sums (Cesàro Means, $\sigma_n$). This leads to the Fejér Kernel:

$$F_n(t) = \frac{1}{2(n+1)} \left[ \frac{\sin(\frac{n+1}{2}t)}{\sin(t/2)} \right]^2$$

- **Why it works:** $F_n \geq 0$ and behaves like a true probability distribution (Approximate Identity).

## 3. Fejér's Theorem

**Theorem:** If $f$ is continuous and periodic, the averages $\sigma_n$ converge to $f$ **uniformly**.

- **Consequence:** Since the averages converge, the original partial sums $S_n$ must converge to $f$ **in the mean**.

## 4. Application of Parseval's

Using the series for $f(x) = x$ and Parseval's equality ($\sum c_n^2 = \int f^2$), we can solve the famous Basel problem:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

# 9.5 Pointwise Convergence

## 1. Dirichlet's Convergence Theorem

When does the series equal $f(x)$ at a specific point?
If $f$ is periodic and at a point $x_0$:

1. Left/Right limits exist.
2. Left/Right derivatives exist (or $f$ is Lipschitz).

Then the series converges to the **average of the jump**:

$$\lim_{n\to\infty} S_n(x_0) = \frac{f(x_0+) + f(x_0-)}{2}$$

## 2. Examples

- **Square Wave:** At a jump discontinuity (like $x = 0$), the series converges to the midpoint of the jump.
- **Triangular Wave ($|x|$):** Since it is continuous everywhere, the series converges to $|x|$ everywhere.

## 3. Differentiation

We can differentiate a Fourier series term-by-term **only if**:

1. $f$ is continuous everywhere (including the boundary $f(-\pi) = f(\pi)$).
2. $f'$ is piecewise continuous.

- *Warning:* If $f$ has jumps (like the square wave), differentiating its series will produce a divergent series.

## Next Step

Would you like to try working through a specific calculation example, such as finding the Fourier series for $f(x) = x^2$ to see how it leads to the sum of $1/n^4$?