Section 4.1: Limit of a Function

1. Definition of a Limit

Context: Let (X, d) be a metric space, $E \subset X$, and $f : E \to \mathbb{R}$. Let p be a **limit point** of E.

Definition 4.1.1 ($\epsilon - \delta$ Definition):
$\lim_{x \to p} f(x) = L$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$|f(x) - L| < \epsilon$$

for all $x \in E$ satisfying $0 < d(x, p) < \delta$.

- **Metric Space Extension:** If $f : E \to (Y, \rho)$, replace $|f(x) - L|$ with $\rho(f(x), L)$.
- **Neighborhood Notation:** $f(x) \in N_\epsilon(L)$ for all $x \in E \cap (N_\delta(p) \setminus \{p\})$.

Key Remarks:

1. **Dependence:** δ depends on ϵ, the function f, and often the point p.
2. **Isolated Points:** If p is not a limit point (isolated), the limit is meaningless.
3. **Value at p:** It is **not** required that $p \in E$. Even if $p \in E$, $\lim_{x \to p} f(x)$ need not equal $f(p)$.

2. $\epsilon - \delta$ Proof Strategies (Examples)

A. Rational Functions (Algebraic Manipulation)

Example 4.1.2(c): $h(x) = \frac{\sqrt{x+1}-1}{x}$ for $x \in (-1, 0) \cup (0, \infty)$.

- **Claim:** $\lim_{x \to 0} h(x) = 1/2$.
- **Analysis:** Rationalize the numerator:

$$|h(x) - \frac{1}{2}| = \left| \frac{1}{\sqrt{x+1}+1} - \frac{1}{2} \right| = \frac{|x|}{2(\sqrt{x+1}+1)^2}$$

  Since $(\sqrt{x+1}+1)^2 > 1$, we have $|h(x) - 1/2| < |x|/2$.
- **Choice:** Set $\delta = \min\{1, 2\epsilon\}$. Then $|h(x) - 1/2| < \epsilon$.

B. The Dirichlet Function (Nowhere Continuous)

Example 4.1.2(d): $f(x) = 1$ if $x \in \mathbb{Q}$, 0 if $x \notin \mathbb{Q}$.

- **Claim:** $\lim_{x \to p} f(x)$ does not exist for any $p \in \mathbb{R}$.
- **Proof:** Fix L. Let $\epsilon = \max\{|L-1|, |L|\}$.
  - Density of $\mathbb{Q}$ implies $\exists x \in \mathbb{Q}$ near $p \implies |1-L| = \epsilon$.
  - Density of irrationals implies $\exists x \notin \mathbb{Q}$ near $p \implies |0-L| = \epsilon$.
  - Regardless of δ, $|f(x) - L| \geq \epsilon$ is always possible. Thus, limit fails.

C. Modified Dirichlet Function (Limit exists at one point)

Example 4.1.2(e): $f(x) = 0$ if $x \in \mathbb{Q}$, x if $x \notin \mathbb{Q}$.

- **At $x = 0$:** Since $|f(x)| \leq |x|$, choosing $\delta = \epsilon$ proves $\lim_{x \to 0} f(x) = 0$.
- **At $p \neq 0$:** Limit does not exist (similar argument to Dirichlet function).

D. Dependence of δ on p (Uniformity issue)

Example 4.1.2(f): $f(x) = 1/x$ on $(0, 1)$. Show $\lim_{x \to p}(1/x) = 1/p$.

- **Inequality:** $|\frac{1}{x} - \frac{1}{p}| = \frac{|x-p|}{xp}$.
  Restricting $x > p/2$ implies $\frac{1}{xp} < \frac{2}{p^2}$.
- **Choice:** $\delta = \min\{p/2, p^2\epsilon/2\}$.
- **Key Insight:** As $p \to 0$, $p^2\epsilon/2 \to 0$. δ *must* shrink as p approaches 0; it cannot be independent of p for this domain.

E. Multivariable Limit

Example 4.1.2(g): $f(x, y) = \frac{xy}{x^2+y^2}$ on $\mathbb{R}^2 \setminus (0, 0)$. Show limit at $(1, 2)$ is $2/5$.

- **Technique:** Algebraic factorization and Triangle Inequality.
  i. **Common Denominator:**

$$|f(x,y) - \frac{2}{5}| = \left|\frac{5xy - 2x^2 - 2y^2}{5(x^2+y^2)}\right|$$

  ii. **Numerator Decomposition:**
  The numerator is rewritten to isolate terms approaching zero, $(x-1)$ and $(y-2)$:

$$5xy - 2x^2 - 2y^2 = (x-2y)(y-2) + (4y-2x)(x-1)$$

  iii. **Triangle Inequality ($|a + b| \leq |a| + |b|$):**

$$\leq \frac{|x-2y||y-2|}{5(x^2+y^2)} + \frac{|4y-2x||x-1|}{5(x^2+y^2)} \leq \frac{(|x|+2|y|)|y-2|+(4|y|+2|x|)|x-1|}{5(x^2+y^2)}$$

- **Bounding:** Restrict (x, y) to the neighborhood $N_{1/2}(1, 2)$.
  - This implies $1/2 < |x| < 3/2$ and $3/2 < |y| < 5/2$.
  - Using these values, we bound the coefficients (e.g., $5(x^2+y^2) > 25/2$) to find a constant $K = 26/25$.
  - $$|f(x,y) - \frac{2}{5}| < \frac{26}{25}(|y-2| + |x-1|)$$
- **Result:** Given ϵ, choose $\delta < \min\{1/2, \frac{25}{52}\epsilon\}$.

3. Sequential Criterion for Limits

Theorem 4.1.3: $\lim_{x\to p} f(x) = L$ **if and only if** for every sequence $\{p_n\}$ in E ($p_n \neq p$) with $p_n \to p$, the sequence $\{f(p_n)\} \to L$.

Proof Technique: Constructing a Sequence ($\delta = 1/n$)
The "If" direction ($\Leftarrow$) is often proven by contradiction (Contrapositive).

- **Assumption:** Suppose $\lim_{x\to p} f(x) \neq L$.
- **Negation of Limit:** There exists an $\epsilon_0 > 0$ such that for any $\delta > 0$, there is an x with $0 < |x - p| < \delta$ but $|f(x) - L| \geq \epsilon_0$.
- **Construction:** For each $n \in \mathbb{N}$, choose $\delta = 1/n$.
  - We can find a point p_n such that $0 < |p_n - p| < 1/n$ and $|f(p_n) - L| \geq \epsilon_0$.
- **Conclusion:** The sequence $\{p_n\}$ converges to p (since $|p_n - p| < 1/n \to 0$), but $\{f(p_n)\}$ does not converge to L. This contradicts the hypothesis.

Corollary 4.1.4 (Uniqueness): If a limit exists, it is unique.

Application: Disproving Existence of Limits
To show a limit **does not exist**:

1. Find a sequence $p_n \to p$ where $\{f(p_n)\}$ diverges.
2. Find two sequences $p_n \to p, r_n \to p$ where $\lim f(p_n) \neq \lim f(r_n)$.

Example 4.1.5(a): $f(x) = \sin(1/x)$ at $x \to 0$.

- Choose $p_n = \frac{2}{(2n+1)\pi}$.
- $f(p_n) = \sin((2n+1)\pi/2) = (-1)^n$.

- The sequence $(-1)^n$ does not converge. Thus, $\lim_{x \to 0} \sin(1/x)$ does not exist.

4. Limit Theorems

Let $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$.

1. **Algebra:**
   - $\lim(f+g) = A+B$
   - $\lim(fg) = AB$
   - $\lim(f/g) = A/B$ (provided $B \neq 0$).

   Proof Detail for Quotient (Bounding away from 0):
   To prove $\lim \frac{1}{g(x)} = \frac{1}{B}$, we must ensure $g(x)$ does not vanish near p.
   - **Specific Choice of ϵ:** Set $\epsilon = \frac{|B|}{2}$.
   - **Logic:** Since $\lim g(x) = B$, $\exists \delta_1 > 0$ such that whenever $0 < |x-p| < \delta_1$, we have $|g(x) - B| < \frac{|B|}{2}$.
   - **Triangle Inequality:**

   $$|g(x)| = |B - (B - g(x))| \geq |B| - |g(x) - B| > |B| - \frac{|B|}{2} = \frac{|B|}{2}$$

   - **Result:** $|g(x)| > \frac{|B|}{2} > 0$. The denominator is strictly bounded away from zero by a specific positive constant in this neighborhood.
2. **Boundedness Theorem (Thm 4.1.8):**
   If g is bounded on E ($|g(x)| \leq M$) and $\lim_{x \to p} f(x) = 0$, then $\lim_{x \to p} f(x)g(x) = 0$.
   - *Example 4.1.10(c):* $\lim_{x \to 0} x \sin(1/x) = 0$. Since $|\sin(1/x)| \leq 1$ is bounded and $x \to 0$, the product goes to 0.
3. **Squeeze Theorem (Thm 4.1.9):**
   If $g(x) \leq f(x) \leq h(x)$ and $\lim g(x) = \lim h(x) = L$, then $\lim f(x) = L$.

5. Essential Trigonometric Limit

Claim: $\lim_{t \to 0} \frac{\sin t}{t} = 1$.

- **Geometric Proof:** Using unit circle areas (Triangle OPQ, Sector OPR, Triangle ORS).

$$\text{Area}(\triangle OPQ) < \text{Area}(\text{Sector}) < \text{Area}(\triangle ORS)$$

$$\frac{1}{2}\sin t\cos t < \frac{1}{2}t < \frac{1}{2}\tan t$$

- **Inequality:** $\cos t < \frac{\sin t}{t} < \frac{1}{\cos t}$.
- **Conclusion:** Since $\lim_{t\to 0}\cos t = 1$, by Squeeze Theorem, limit is 1.

6. Limits at Infinity

Definition 4.1.11:
Let domain of f be unbounded above. $\lim_{x\to\infty} f(x) = L$ if $\forall\epsilon > 0, \exists M \in \mathbb{R}$ such that:

$$|f(x) - L| < \epsilon$$

for all $x \in \mathrm{Dom}(f) \cap (M, \infty)$.
(Analogous definition exists for $x \to -\infty$ using $x < M$).

Examples:

1. **Damped Sine:** $f(x) = \frac{\sin x}{x}$ on $(0, \infty)$.
   Since $|f(x)| \leq 1/x$, choosing $M = 1/\epsilon$ proves limit is 0.
2. **Oscillation at Infinity:** $f(x) = x\sin(\pi x)$.
   Choosing sequence $p_n = n + 1/2$ gives $f(p_n) = (-1)^n(n + 1/2)$. This is unbounded; limit does not exist.