Here is a comprehensive cheat sheet for Chapter 6 (The Riemann Integral), covering Sections 6.1 through 6.5.

6.1 The Riemann Integral

1. Definitions

- **Partition $\mathcal{P}$:** A set $\{x_0, \dots, x_n\}$ where $a = x_0 < \cdots < x_n = b$.
- **Bounds:** On subinterval $[x_{i-1}, x_i]$:
  - $m_i = \inf f(x)$
  - $M_i = \sup f(x)$
- **Sums:**
  - Lower Sum: $\mathcal{L}(\mathcal{P}, f) = \sum m_i \Delta x_i$
  - Upper Sum: $\mathcal{U}(\mathcal{P}, f) = \sum M_i \Delta x_i$
- **Inequality:** $\mathcal{L}(\mathcal{P}, f) \le \underline{\int} f \le \overline{\int} f \le \mathcal{U}(\mathcal{P}, f)$

2. Riemann Integrability ($f \in \mathcal{R}[a, b]$)

- **Definition:** f is integrable if Upper Integral = Lower Integral.

$$\overline{\int_a^b} f = \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f) = \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f) = \underline{\int_a^b} f$$

- **Riemann's Criterion (Key for Proofs):**
  $f \in \mathcal{R}[a, b] \iff \forall \epsilon > 0, \exists \mathcal{P}$ such that:

$$\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) < \epsilon$$

3. Integrable Classes

- **Continuous Functions:** Integrable (proof uses Uniform Continuity).
- **Monotone Functions:** Integrable (proof uses telescoping sum).
- **Composition Theorem:** If $f \in \mathcal{R}$ and φ is continuous, then $\varphi \circ f \in \mathcal{R}$.
  - *Corollary:* If $f \in \mathcal{R}$, then $|f| \in \mathcal{R}$ and $f^2 \in \mathcal{R}$.

4. Lebesgue's Theorem

- **Measure Zero:** Set E covers by intervals with total length $< \epsilon$. (e.g., Finite sets, $\mathbb{Q}$, Cantor set).
- **Theorem:** $f \in \mathcal{R} \iff$ The set of discontinuities of f has measure zero.

5. Classical Examples

- **Dirichlet Function:** $f(x) = 1$ if $x \in \mathbb{Q}$, 0 else.
  - $m_i = 0, M_i = 1 \implies \mathcal{L} = 0, \mathcal{U} = b - a$. **Not Integrable.**
- **Thomae's Function:** $f(x) = 1/n$ if $x = m/n$, 0 irrational.
  - Discontinuous only at $\mathbb{Q}$ (measure 0). **Integrable**, $\int f = 0$.

6.2 Properties of the Integral

1. Algebraic Properties

If $f, g \in \mathcal{R}[a, b]$:

- **Linearity:** $\int (cf + g) = c \int f + \int g$.
- **Product:** $fg \in \mathcal{R}$.
- **Additivity:** $\int_a^b f = \int_a^c f + \int_c^b f$ for $a < c < b$.

2. Order Properties

- **Positivity:** If $f(x) \geq 0$, then $\int f \geq 0$.
- **Comparison:** If $f \leq g$, then $\int f \leq \int g$.
- **Absolute Value Inequality:**

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

3. Riemann Sums (Equivalence)

- **Tagged Sum:** $S(\mathcal{P}, f) = \sum f(t_i)\Delta x_i$ where $t_i \in [x_{i-1}, x_i]$.
- **Theorem:** Darboux integrability $\iff \lim_{||\mathcal{P}|| \to 0} S(\mathcal{P}, f)$ exists.
- **Example:** $\int_a^b x\, dx = \frac{b^2 - a^2}{2}$ via midpoint tags $t_i = \frac{x_i + x_{i-1}}{2}$.

6.3 Fundamental Theorem of Calculus (FTC)

1. FTC Part 1 (Evaluation)

If $f \in \mathcal{R}$ and F is an antiderivative ($F' = f$):

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

- *Proof:* Use MVT on subintervals: $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$. Sum telescopically.

2. FTC Part 2 (Differentiation)
If $f \in \mathcal{R}$ and $F(x) = \int_a^x f(t)\,dt$:

- F is continuous on $[a, b]$.
- If f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$.

3. Integration Techniques

- **Integration by Parts:** $\int_a^b fg' = [fg]_a^b - \int_a^b gf'$.
- **Substitution (Change of Variable):** $\int_a^b f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$.
- **MVT for Integrals:** If f continuous, $\exists c$ such that $\int_a^b f = f(c)(b-a)$.

4. The Natural Logarithm

- **Definition:** $L(x) = \int_1^x \frac{1}{t}\,dt$.
- **Properties:** $L(ab) = L(a) + L(b)$, $L'(x) = 1/x$, $L(e) = 1$.

6.4 Improper Riemann Integrals

1. Definitions

- **Unbounded Interval:** $\int_a^\infty f = \lim_{c\to\infty} \int_a^c f$.
- **Singularity (at a):** $\int_a^b f = \lim_{c\to a^+} \int_c^b f$.
- **Note:** Properties of standard Riemann integral (e.g., $f \in \mathcal{R} \implies f^2 \in \mathcal{R}$) **fail** here.

2. Comparison Test
If $|f(x)| \le g(x)$ and $\int g$ converges, then $\int f$ converges (Absolute Integrability).

3. Key Examples

- **$1/x$ on $(0, 1]$:** Diverges ($\ln c \to \infty$).
- **$\ln x$ on $(0, 1]$:** Converges to -1 (Integration by parts).
- **$1/x^2$ on $[1, \infty)$:** Converges to 1.
- **$\sin x/x$ on $[\pi, \infty)$:** Conditionally convergent (integral exists, but $\int \left|\frac{\sin x}{x}\right|$ diverges due to Harmonic series).
- **Counter-Example:** $f(x) = 1/\sqrt{x}$ converges on $(0, 1]$, but $f^2(x) = 1/x$ diverges.

6.5 The Riemann-Stieltjes Integral

1. Definition

Integrate f with respect to monotone increasing α: $\int_a^b f\,d\alpha$.

- **Sums:** $\mathcal{U}(\mathcal{P}, f, \alpha) = \sum M_i \Delta\alpha_i$ where $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.
- **Condition:** $\int f d\alpha$ exists if $\inf \mathcal{U} = \sup \mathcal{L}$.

2. Calculation Methods (Reductions)

- **If α is Differentiable:**

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx$$

- **If α is a Step Function:** (Jumps at s_n with height c_n)

$$\int_a^b f\,d\alpha = \sum c_n f(s_n)$$

(Requires f continuous at s_n).

3. Integration by Parts (Stieltjes)

$$\int_a^b \alpha\,d\beta = [\alpha\beta]_a^b - \int_a^b \beta\,d\alpha$$

4. Key Examples

- **Unit Jump:** $\alpha = I_c(x)$ (jump at c). $\int f\,dI_c = f(c)$.
- **Dirichlet:** If f is Dirichlet and $\alpha(x) = x$, **not** integrable.