7.2 The Dirichlet Test and Applications

1. Abel Partial Summation Formula

This formula is the discrete analogue of **Integration by Parts**. It is the fundamental tool required to prove the Dirichlet Test.

Theorem 7.2.1
Let $\{a_k\}$ and $\{b_k\}$ be sequences of real numbers.
Define partial sums $A_0 = 0$ and $A_n = \sum_{k=1}^{n} a_k$ for $n \geq 1$.
If $1 \leq p \leq q$, then:

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p$$

Proof (Key Idea):
We substitute a_k using the difference of partial sums: $a_k = A_k - A_{k-1}$.

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} (A_k - A_{k-1}) b_k = \sum_{k=p}^{q} A_k b_k - \sum_{k=p}^{q} A_{k-1} b_k$$

By shifting the index of the second summation (letting $j = k - 1$), we align the terms to factor out A_k. The middle terms collapse into the form $A_k(b_k - b_{k+1})$, leaving the boundary terms $A_q b_q$ and $-A_{p-1} b_p$.

2. The Dirichlet Test

This test provides convergence criteria for series of the form $\sum a_k b_k$, usually where one part oscillates and the other decays.

Theorem 7.2.2
The series $\sum_{k=1}^{\infty} a_k b_k$ converges if the sequences satisfy three conditions:

1. **Bounded Partial Sums:** The sequence $A_n = \sum_{k=1}^{n} a_k$ is bounded (i.e., $|A_n| \leq M$ for some $M > 0$).
2. **Monotonicity:** $\{b_k\}$ is decreasing ($b_1 \geq b_2 \geq \cdots \geq 0$).
3. **Limit Zero:** $\lim_{k\to\infty} b_k = 0$.

Proof (Key Idea):
We use the **Cauchy Criterion**. We need to show that for large enough p, q, the tail sum $|\sum_{k=p}^{q} a_k b_k|$ is arbitrarily small.
Using Abel's Formula and the triangle inequality:

$$\left|\sum_{k=p}^{q} a_k b_k\right| \leq \sum_{k=p}^{q-1} |A_k|(b_k - b_{k+1}) + |A_q|b_q + |A_{p-1}|b_p$$

Since $|A_k| \leq M$ and terms $(b_k - b_{k+1})$ are non-negative (because b_k is decreasing):

$$\leq M \left[\sum_{k=p}^{q-1}(b_k - b_{k+1}) + b_q + b_p\right]$$

The summation inside is a telescoping sum that simplifies to $b_p - b_q$. Thus, the expression simplifies to $2Mb_p$. Since $b_p \to 0$, this value becomes smaller than any ϵ, proving convergence.

3. Application I: Alternating Series

The most common application of the Dirichlet Test is for alternating series.

Theorem 7.2.3 (Alternating Series Test)
If $\{b_k\}$ satisfies $b_1 \geq b_2 \geq \cdots \geq 0$ and $\lim_{k\to\infty} b_k = 0$, then:

$$\sum_{k=1}^{\infty} (-1)^{k+1} b_k \quad \text{converges.}$$

Proof:
Set $a_k = (-1)^{k+1}$. The partial sums A_n alternate between 1 and 0. Thus, $|A_n| \leq 1$ for all n. Since partial sums are bounded and $\{b_k\}$ decreases to zero, the Dirichlet Test applies directly.

Error Estimation

For alternating series, we can easily bound the error between the limit s and the partial sum s_n.

Theorem 7.2.4
Let $s = \sum_{k=1}^{\infty} (-1)^{k+1} b_k$ and s_n be the n-th partial sum. Then:

$$|s - s_n| \leq b_{n+1}$$

Proof (Key Idea):

- Consider the even partial sums $\{s_{2n}\}$. Because $s_{2n} = (b_1 - b_2) + \cdots + (b_{2n-1} - b_{2n})$ and terms are decreasing, $\{s_{2n}\}$ is **increasing**.
- Similarly, $\{s_{2n+1}\}$ is **decreasing**.
- Since the series converges to s, the limit is "trapped" between consecutive partial sums: $s_{2n} \leq s \leq s_{2n+1}$.
- The distance between consecutive sums is $|s_{k+1} - s_k| = b_{k+1}$. Therefore, the distance from s_k to s cannot exceed b_{k+1}.

Example 7.2.5
Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$.

- Since $\left\{\frac{1}{2k-1}\right\}$ decreases to 0, the series converges.
- (Note: It converges to $\pi/4$).
- **Error:** By Theorem 7.2.4, $\left|\frac{\pi}{4} - s_n\right| \leq \frac{1}{2n+1}$.
- **Result:** Convergence is very slow. To get 2 decimal places of accuracy (< 0.01), we need $2n + 1 > 100$, implying $n \approx 50$.

4. Application II: Trigonometric Series

We examine series of the form $\sum b_k \sin(kt)$ and $\sum b_k \cos(kt)$.

Theorem 7.2.6
Let $\{b_k\}$ be a sequence where $b_1 \geq b_2 \geq \cdots \geq 0$ and $\lim_{k\to\infty} b_k = 0$.

1. **Sine Series:** $\sum_{k=1}^{\infty} b_k \sin(kt)$ converges for all $t \in \mathbb{R}$.
2. **Cosine Series:** $\sum_{k=1}^{\infty} b_k \cos(kt)$ converges for all $t \in \mathbb{R}$, **except** possibly where $t = 2p\pi$ for integers p.

Proof (Key Idea):
We must prove that the partial sums of the trigonometric parts ($a_k = \sin kt$ or $\cos kt$) are bounded.
Key Identity for partial sums of sine (for $t \neq 2p\pi$):

$$A_n = \sum_{k=1}^{n} \sin kt = \frac{\cos \frac{1}{2}t - \cos(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t}$$

Bounding A_n:

$$|A_n| \leq \frac{|\cos \frac{1}{2}t| + |\cos(n + \frac{1}{2})t|}{2|\sin \frac{1}{2}t|} \leq \frac{2}{2|\sin \frac{1}{2}t|} = \frac{1}{|\sin \frac{1}{2}t|}$$

This is a finite constant for any fixed $t \neq 2p\pi$.

- Since A_n is bounded and $b_k \downarrow 0$, the **Dirichlet Test** proves convergence.
- **Case $t = 2p\pi$:**
  - For sine: $\sin(k \cdot 2p\pi) = 0$. The sum is $\sum 0$, which converges.
  - For cosine: $\cos(k \cdot 2p\pi) = 1$. The sum becomes $\sum b_k$. Since we only know $b_k \to 0$, this might diverge (e.g., if $b_k = 1/k$).

Example 7.2.7

1. **Series:** $\sum_{k=1}^{\infty} \frac{1}{k} \cos kt$
   - Converges for all $t \neq 2p\pi$ (by Theorem 7.2.6).
   - If $t = 2p\pi$, series becomes $\sum \frac{1}{k}$ (Harmonic Series), which **diverges**.
2. **Series:** $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos kt$
   - Converges for all $t \in \mathbb{R}$.
   - Even at $t = 2p\pi$, it becomes $\sum \frac{1}{k^2}$, which converges (p-series, $p = 2$).