Section 4.4: Monotone Functions and Discontinuities

I. Right and Left Limits

1. Definitions

Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$.

- **Right Limit ($f(p+)$):** Suppose p is a limit point of $E \cap (p, \infty)$.

$$\lim_{x \to p^+} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } p < x < p + \delta.$$

- **Left Limit ($f(p-)$):** Suppose p is a limit point of $E \cap (-\infty, p)$.

$$\lim_{x \to p^-} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } p - \delta < x < p.$$

2. Relationship to General Limits

For $p \in \text{Int}(I)$, $\lim_{x \to p} f(x)$ exists if and only if:

1. Both $f(p+)$ and $f(p-)$ exist.
2. $f(p+) = f(p-)$.

3. One-Sided Continuity

- **Right Continuous at p:** $\lim_{x \to p^+} f(x) = f(p)$.
- **Left Continuous at p:** $\lim_{x \to p^-} f(x) = f(p)$.
- **Theorem 4.4.3:** f is continuous at p iff $f(p+) = f(p-) = f(p)$.

II. Classification of Discontinuities

If f is discontinuous at p, it falls into one of two main categories:

1. Simple Discontinuities (First Kind)

Both $f(p+)$ and $f(p-)$ exist.

- **Removable Discontinuity:** $f(p+)$ and $f(p-)$ exist and are equal, but differ from $f(p)$ (or $f(p)$ is undefined).

- *Example:* $g(x) = \frac{x^2-4}{x-2}$. Limit is 4, but undefined at $x = 2$. Can be "removed" by defining $g(2) = 4$.

- **Jump Discontinuity:** $f(p+) \neq f(p-)$.
  - *Example:* The greatest integer function $f(x) = [x]$. At integer n:
    - $f(n-) = n - 1$
    - $f(n+) = n$
    - Jump size is 1.

2. Discontinuities of the Second Kind

Either $f(p+)$ or $f(p-)$ (or both) does not exist.

- *Example:* $f(x) = \sin(\frac{1}{x})$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.
  - $f(0-) = 0$.
  - $f(0+)$ does not exist (oscillates between -1 and 1).

3. Specific Examples Analyzed

- **Example 4.4.5(a):** Piecewise function defined as x for $x \leq 1$ and $3 - x^2$ for $x > 1$.
  - $f(1-) = 1$, $f(1+) = 2$.
  - Left continuous, but jump discontinuity at $x = 1$.
- **Example 4.4.5(d):** $g(x) = \sin(2\pi x[x])$.
  - Continuous on $(n, n+1)$ as $[x]$ is constant n.
  - At integers n, limits from both sides go to 0, so it is continuous at integers.
  - However, it is not *uniformly* continuous on $\mathbb{R}$.

III. Monotone Functions

1. Definitions

Let f be defined on an interval I.

- **Monotone Increasing:** $x < y \implies f(x) \leq f(y)$.
- **Monotone Decreasing:** $x < y \implies f(x) \geq f(y)$.
- **Strictly Increasing:** $x < y \implies f(x) < f(y)$.

2. Theorem 4.4.7 (Existence of Limits)

If f is monotone increasing on an open interval I, then for every $p \in I$, both $f(p+)$ and $f(p-)$ exist. The following inequality holds:

$$\sup_{x<p} f(x) = f(p-) \le f(p) \le f(p+) = \inf_{x>p} f(x)$$

Furthermore, if $p < q$, then $f(p+) \le f(q-)$.

Proof of Theorem 4.4.7

- **Part 1: Existence of Left Limit $f(p-)$**
  Let $S = \{f(x) : x < p\}$. Since f is monotonic increasing, $f(p)$ serves as an upper bound for S.
  - By the **Completeness Property**, the supremum $A = \sup S$ exists, and clearly $A \le f(p)$.
  - **Convergence:** For any $\epsilon > 0$, since A is the least upper bound, $A - \epsilon$ is not an upper bound. Thus, there exists $x_0 < p$ such that $A - \epsilon < f(x_0) \le A$.
  - By monotonicity, for all $x \in (x_0, p)$, we have $f(x_0) \le f(x) \le A$.
  - Consequently, $|f(x) - A| < \epsilon$, which implies $f(p-) = A \le f(p)$.
- **Part 2: Existence of Right Limit $f(p+)$**
  By an analogous argument using the **infimum** of the set $\{f(x) : x > p\}$, we establish that the right limit exists and satisfies:

$$f(p) \le f(p+) = \inf_{x>p} f(x)$$

- **Part 3: Relation for distinct points ($p < q$)**
  Let $p < q$. Choose any x such that $p < x < q$.
  From the definitions of the one-sided limits established above:

$$f(p+) = \inf_{z>p} f(z) \le f(x) \le \sup_{z<q} f(z) = f(q-)$$

3. Corollary 4.4.8 (Countability of Discontinuities)

The set of discontinuities of a monotone function is **at most countable**.

- *Proof Logic:* Each discontinuity is a jump $(f(p-), f(p+))$. Since intervals are disjoint for distinct points, we can map each jump to a rational number within that interval. Since $\mathbb{Q}$ is countable, the set of jumps is countable.

IV. Construction of Functions with Prescribed Discontinuities

We can construct a monotone function that is discontinuous exactly at a specific countable set of points, with controlled jump sizes.

1. The Unit Jump Function $I(x)$

$$I(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

- The function $I(x)$ is **right continuous** at 0 with a unit jump: $I(0+) = 1$ and $I(0-) = 0$.
- **Shifted Function:** $I_k(x) = I(x - a_k)$ represents a unit step occurring strictly at $x = a_k$.

2. Theorem 4.4.10 (General Construction)
Let $\{x_n\}_{n=1}^{\infty}$ be a countable subset of (a, b). Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} c_n$ converges.

Define the function f on $[a, b]$ by:

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

Detailed Analysis of f:

- **Well-Defined and Monotone:**
  Since $0 \leq c_n I(x - x_n) \leq c_n$, the partial sums $s_n(x)$ are bounded above by $\sum c_n$. Thus, $f(x)$ converges for all x. Furthermore, because each term $I(x - x_n)$ is non-decreasing, f is **monotone increasing** on $[a, b]$.
  - Boundary values: $f(a) = 0$ (since $x_n > a$) and $f(b) = \sum_{n=1}^{\infty} c_n$.
- **Continuity on $[a, b] \setminus \{x_n\}$:**
  The function is continuous at any point p where $p \neq x_n$ for any n. The proof relies on splitting the set of points $E = \{x_n\}$:
  - **Isolated Points:** If p is not a limit point of E, there exists a neighborhood $(p - \delta, p + \delta)$ containing no x_n. In this interval, f is constant, and therefore continuous.
  - **Limit Points:** If p is a limit point of E, continuity is proved using the Cauchy criterion. For any $\epsilon > 0$, we can choose an integer N such that the "tail" of the series is small ($\sum_{k=N+1}^{\infty} c_k < \epsilon$). By choosing a neighborhood δ small enough to exclude the first N points, the variation of f near p is bounded by ϵ, proving continuity.
- **Right Continuity Everywhere:**
  For any point x_n (or any p), f is right continuous:

  $$f(x_n+) = f(x_n)$$

  This is because as $x \to x_n$ from the right, the terms $I(x - x_k)$ do not change state (they remain

1 for $x_k \le x_n$ and 0 for $x_k > x$ locally). The variation is again controlled by the tail of the convergent series $\sum c_n$.

- **Discontinuity at $\{x_n\}$:**
  At each prescribed point x_n, the function exhibits a jump discontinuity exactly equal to the weight c_n:

$$f(x_n) - f(x_n-) = c_n$$

  - **Left Limit:** As $y \to x_n$ from the left ($y < x_n$), the term $c_n I(y - x_n)$ is 0.
  - **Value at x_n:** Exactly at x_n, the term becomes $c_n I(0) = c_n$.
  - This confirms that f has a countable number of discontinuities strictly located at the set $\{x_n\}$.

3. Examples of Construction

- **Step Function:** If $\{x_n\}$ is a finite set, f is a standard step function with finitely many jumps.
- **Rational Discontinuities:** Let $\{x_n\}$ be an enumeration of the rational numbers $\mathbb{Q} \cap (0, 1)$ and $c_n = 2^{-n}$. The resulting function is strictly increasing, discontinuous at every rational number in $(0, 1)$, and continuous at every irrational number.
- **Distribution Functions:** If the weights are normalized such that $\sum c_n = 1$, the function behaves like a cumulative distribution function (CDF) used in probability theory.

V. Inverse Functions

1. Logic of Invertibility

- **Strict Monotonicity:** Let f be a strictly increasing real-valued function on an interval I. If $x, y \in I$ with $x < y$, then $f(x) < f(y)$. The same logic applies if f is strictly decreasing.
- **Injectivity (One-to-One):** Strictly monotone functions imply that $f(x) \neq f(y)$ for any distinct $x, y \in I$. Therefore, f is one-to-one.
- **Existence:** Because f is one-to-one, it possesses an inverse function f^{-1} defined on the range $f(I)$.

2. Theorem 4.4.12: Continuity of the Inverse Function

- **Statement:** Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then the inverse function f^{-1} is strictly monotone and continuous on the interval $J = f(I)$.
- **Proof Summary:**
  i. **Image is an Interval:** Since f is continuous on the interval I, the image $J = f(I)$ must also be an interval (by Corollary 4.2.12).

ii. **Monotonicity of Inverse:** If f is strictly increasing, f^{-1} is also strictly increasing. For $y_1, y_2 \in J$ with $y_1 < y_2$, their pre-images must satisfy $x_1 < x_2$.

iii. **Continuity of Inverse:** To prove f^{-1} is continuous at $y_0 \in J$:

- Consider an ϵ-neighborhood around $x_0 = f^{-1}(y_0)$.
- Due to the strict monotonicity and continuity of f, we can map the interval $(x_0 - \epsilon, x_0]$ to $(y_0 - \delta, y_0]$ where $\delta = f(x_0) - f(x_0 - \epsilon)$.
- This ensures that for y within δ of y_0, $|f^{-1}(y_0) - f^{-1}(y)| < \epsilon$, proving continuity.

3. Example 4.4.13 (n-th Roots)

- $f(x) = x^n$ is strictly increasing and continuous on $I = [0, \infty)$.
- Therefore, the inverse $g(x) = \sqrt[n]{x}$ is strictly increasing and continuous on $J = [0, \infty)$.

4. Remark (Conditions for Monotonicity)

- **The Converse:** If f is one-to-one and continuous on an interval I, then f is necessarily strictly monotone (increasing or decreasing). This results from the Intermediate Value Theorem.
- **Essential Conditions:** This converse holds **only** if:

  i. f is continuous.

  ii. The domain of f is an interval.

  If either condition is unmet, a one-to-one function might not be strictly monotone.