Here is a concise, self-contained summary of **Section 8.4: Uniform Convergence and Integration** from the provided text.

1. Motivation: The Failure of Pointwise Convergence

Pointwise convergence is **not sufficient** to preserve Riemann integrability or to interchange the limit and the integral.

- **Problem 1 (Integrability):** A sequence of Riemann integrable functions $\{f_n\}$ can converge pointwise to a limit function f that is **not** Riemann integrable.
- **Problem 2 (Value of Integral):** Even if the limit function f is integrable, the integral of the limit may not equal the limit of the integrals.
  - *Example:* There exists a sequence of continuous functions on $[0, 1]$ where $\lim_{n\to\infty} f_n(x) = 0$ for all x, yet:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx$$

  - Specifically, in a previous example (8.1.2d), the integrals evaluated to $\frac{1}{2}\frac{n}{n+1} \to \frac{1}{2}$, while the integral of the zero limit function is 0.

2. Main Theorem: Uniform Convergence

Uniform convergence provides the sufficient condition to guarantee both the integrability of the limit and the interchange of operations.

Theorem 8.4.1

Suppose $f_n \in \mathcal{R}[a, b]$ (Riemann integrable) for all $n \in \mathbb{N}$, and $\{f_n\}$ converges **uniformly** to f on $[a, b]$. Then:

1. $f \in \mathcal{R}[a, b]$ (The limit is integrable).
2. The limit can be moved inside the integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

Proof Strategy (Key Ideas)

The proof relies on "squeezing" the function f between the sequence functions using the uniform error bound.

1. **Define the error:** Let $\epsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. Because convergence is uniform, $\lim_{n\to\infty} \epsilon_n = 0$.
2. **Establish bounds:** From the definition of absolute value, for all $x \in [a,b]$:

$$f_n(x) - \epsilon_n \le f(x) \le f_n(x) + \epsilon_n$$

3. **Integrability Argument:**
   Using the properties of Upper ($\overline{\int}$) and Lower ($\underline{\int}$) integrals:

$$\int_a^b (f_n - \epsilon_n) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \int_a^b (f_n + \epsilon_n)$$

   The difference between the upper and lower integrals of f is bounded by the difference of the outer terms:

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le 2\epsilon_n(b-a)$$

   Since $\epsilon_n \to 0$, the upper and lower integrals must be equal. Thus, $f \in \mathcal{R}[a,b]$.
4. **Convergence of Values:**
   Using the same inequality bounds:

$$\left| \int_a^b f(x)dx - \int_a^b f_n(x)dx \right| \le \int_a^b |f(x) - f_n(x)|dx \le \epsilon_n(b-a)$$

   As $n \to \infty$, the right side vanishes, proving the equality of the integrals.

3. Application to Series

This theorem extends naturally to infinite series of functions.

Corollary 8.4.2

If $f_k \in \mathcal{R}[a,b]$ and the series $f(x) = \sum_{k=1}^{\infty} f_k(x)$ converges **uniformly** on $[a,b]$, then f is integrable and we can integrate term-by-term:

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx$$

Proof Idea: Apply Theorem 8.4.1 to the sequence of partial sums, $S_n(x) = \sum_{k=1}^{n} f_k(x)$.

4. Nuance: Sufficient but Not Necessary

Uniform convergence guarantees the interchange of limits and integrals, but it is not strictly required. The interchange can sometimes work even if convergence is only pointwise.

Counter-Example: $f_n(x) = x^n$ on $[0,1]$

- **Convergence:** This sequence converges pointwise to:

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

  The convergence is **not uniform** (the function becomes arbitrarily steep near $x = 1$).
- **Integrability:** Despite the discontinuity at $x = 1$, the limit f is Riemann integrable and $\int_0^1 f(x)dx = 0$.
- **Limit of Integrals:**

$$\lim_{n\to\infty} \int_0^1 x^n\,dx = \lim_{n\to\infty} \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

- **Conclusion:** $\lim \int = \int \lim$ holds here, even without uniform convergence.

5. The Bounded Convergence Theorem (BCT)

To address cases where convergence is not uniform (like x^n), a stronger theorem from Lebesgue theory is introduced (Theorem 8.4.3).

Hypotheses:

1. f_n and f are Riemann integrable on $[a, b]$.
2. $f_n \to f$ pointwise.
3. **Boundedness:** There exists a constant $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in [a, b]$ and all n.

Conclusion:

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Analysis of Examples using BCT:

- **$f_n(x) = x^n$:** Satisfies BCT ($|x^n| \leq 1$ on $[0, 1]$). This explains why the integral limit works despite lack of uniformity.
- **"Moving Hump" Example (8.1.2d):** This sequence (which had $\int f_n \to 1/2$ but $\int f = 0$) fails the hypotheses of BCT. While not detailed in this section, such functions usually fail the uniform boundedness condition (M) or the specific integrability requirements of the limit in more complex scenarios.