Here are the structured study notes based on the provided text, focusing on **Square Summable Sequences** and **Normed Linear Spaces**.

1. The Space l^2 (Square Summable Sequences)

Definition

The set l^2 consists of all sequences of real numbers $\{a_k\}_{k=1}^{\infty}$ such that the sum of their squares converges.

$$l^2 = \left\{ \{a_k\} : \sum_{k=1}^{\infty} a_k^2 < \infty \right\}$$

The Norm

For a sequence $\{a_k\} \in l^2$, the norm (magnitude) is defined as:

$$||\{a_k\}||_2 = \sqrt{\sum_{k=1}^{\infty} a_k^2}$$

Examples of Convergence in l^2

- **Case 1:** $\{1/k\}$.
  The norm squared is $\sum (1/k)^2 = \sum 1/k^2$. Since this is a p-series with $p = 2$, it converges.
  - $\therefore \{1/k\} \in l^2$.
- **Case 2:** $\{1/\sqrt{k}\}$.
  The norm squared is $\sum (1/\sqrt{k})^2 = \sum 1/k$. This is the harmonic series (diverges).
  - $\therefore \{1/\sqrt{k}\} \notin l^2$.
- **General Case:** $\{1/k^q\}$ for fixed q.
  The series is $\sum 1/k^{2q}$. By p-series test, this converges if $2q > 1$.
  - $\therefore \{1/k^q\} \in l^2 \iff q > 1/2$.

2. The Cauchy-Schwarz Inequality

This is the fundamental inequality allowing us to define angles and geometry in infinite-dimensional spaces.

Finite Version (Theorem 7.4.3)
For real numbers $a_1, ..., a_n$ and $b_1, ..., b_n$:

$$\sum_{k=1}^{n} |a_k b_k| \leq \sqrt{\sum_{k=1}^{n} a_k^2} \sqrt{\sum_{k=1}^{n} b_k^2}$$

Proof (Key Idea):
We construct a non-negative quadratic function.

1. Let $\lambda \in \mathbb{R}$. Consider the square of a linear combination:

$$0 \leq \sum_{k=1}^{n} (|a_k| - \lambda |b_k|)^2$$

2. Expand the square:

$$0 \leq \sum a_k^2 - 2\lambda \sum |a_k b_k| + \lambda^2 \sum b_k^2$$

   Let $A = \sum a_k^2$, $B = \sum b_k^2$, and $C = \sum |a_k b_k|$.

$$0 \leq A - 2\lambda C + \lambda^2 B$$

3. **Optimization:** To minimize the expression, choose $\lambda = C/B$ (assuming $B \neq 0$; if $B = 0$, the inequality is trivial).

$$0 \leq A - 2\left(\frac{C}{B}\right) C + \left(\frac{C}{B}\right)^2 B \implies 0 \leq A - \frac{C^2}{B}$$

$$C^2 \leq AB \implies C \leq \sqrt{A}\sqrt{B}$$

   This yields the desired inequality.

Infinite Version (Corollary 7.4.4)
If $\{a_k\}, \{b_k\} \in l^2$, then the series $\sum a_k b_k$ converges absolutely and:

$$\sum_{k=1}^{\infty} |a_k b_k| \leq ||\{a_k\}||_2 \cdot ||\{b_k\}||_2$$

Proof: Apply the finite theorem to partial sums S_n and let $n \to \infty$.

Inner Product

Based on this, we define the inner product of two sequences in l^2 as:

$$\langle a, b \rangle = \sum_{k=1}^{\infty} a_k b_k$$

Consequently: $|\langle a, b \rangle| \leq ||a||_2 ||b||_2$.

3. Minkowski's Inequality (Triangle Inequality)

This theorem proves that the "length" of a sum is less than the sum of the "lengths."

Theorem 7.4.5

If $\{a_k\}, \{b_k\} \in l^2$, then their sum $\{a_k + b_k\} \in l^2$ and:

$$||\{a_k + b_k\}||_2 \leq ||\{a_k\}||_2 + ||\{b_k\}||_2$$

Proof (Key Idea):

1. Start with the square of the sum term: $(a_k + b_k)^2$.
2. Expand and apply absolute values:

$$(a_k + b_k)^2 = a_k^2 + 2a_k b_k + b_k^2 \leq a_k^2 + 2|a_k b_k| + b_k^2$$

3. Sum over all k:

$$||\{a + b\}||_2^2 \leq ||a||_2^2 + 2 \sum |a_k b_k| + ||b||_2^2$$

4. **Apply Cauchy-Schwarz** to the middle term:

$$||\{a + b\}||_2^2 \leq ||a||_2^2 + 2||a||_2 ||b||_2 + ||b||_2^2$$

5. Recognize the perfect square on the right side:

$$||\{a + b\}||_2^2 \leq (||a||_2 + ||b||_2)^2$$

6. Taking the square root proves the theorem.

4. Normed Linear Spaces

The space l^2 is an example of a broader algebraic structure called a Normed Linear Space.

Vector Space Definition

A set X is a vector space over $\mathbb{R}$ if it satisfies standard axioms (commutativity, associativity, existence of zero element and additive inverses, distributivity of scalar multiplication).

- l^2 is a vector space where addition is component-wise: $\{a_k\} + \{b_k\} = \{a_k + b_k\}$.
- The zero element is the sequence of all zeros.

Norm Definition (Def 7.4.8)

Let X be a vector space. A function $||\cdot|| : X \to \mathbb{R}$ is a **norm** if it satisfies four properties for all $x, y \in X$ and $c \in \mathbb{R}$:

1. **Non-negativity:** $||x|| \geq 0$.
2. **Zero property:** $||x|| = 0 \iff x = 0$.
3. **Homogeneity:** $||cx|| = |c| \cdot ||x||$.
4. **Triangle Inequality:** $||x + y|| \leq ||x|| + ||y||$.

Verification for l^2 (Theorem 7.4.6)

- Properties 1, 2 are obvious from the definition of the sum of squares.
- Property 3 is easily proven by factoring out constants from the series.
- Property 4 is exactly Minkowski's Inequality.
- Therefore, $(l^2, ||\cdot||_2)$ is a normed linear space.

Distance and Convergence

- **Distance (Metric):** Defined as $d(x, y) = ||x - y||$. This satisfies metric space properties (positivity, symmetry, triangle inequality).
- **Norm Convergence:** A sequence of vectors $\{x_n\}$ converges to x if:

$$\lim_{n \to \infty} ||x_n - x|| = 0$$

  This is analogous to standard convergence in $\mathbb{R}$, but care must be taken (e.g., the Bolzano-Weierstrass theorem fails in infinite-dimensional spaces like l^2).