Here are detailed study notes on **Uniform Convergence**, **The Cauchy Criterion**, and the **Weierstrass M-Test**, based on the provided text.

# 1. Pointwise vs. Uniform Convergence

To preserve properties like continuity or to interchange limit operations (integrals, derivatives), pointwise convergence is often insufficient. Stronger conditions are required.

## Pointwise Convergence (Recap)

A sequence $\{f_n\}$ converges pointwise to $f$ on a set $E$ if for every $x \in E$ and $\epsilon > 0$, there exists an integer $n_0$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq n_0$.

- **Crucial Limitation:** The integer $n_0$ depends on both $\epsilon$ **and** $x$.
- Notation: $n_0 = n_0(x, \epsilon)$.

## Uniform Convergence (Definition 8.2.1)

A sequence $\{f_n\}$ converges **uniformly** to $f$ on $E$ if for every $\epsilon > 0$, there exists an integer $n_0$ such that:

$$|f_n(x) - f(x)| < \epsilon$$

for **all** $x \in E$ and all $n \geq n_0$.

- **Key Distinction:** The integer $n_0$ depends **only** on $\epsilon$, not on $x$. The dependence on $x$ is removed.
- **Geometric Interpretation:** For $n \geq n_0$, the entire graph of $f_n(x)$ lies within a "tube" of width $2\epsilon$ centered around $f(x)$.

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon$$

# 2. Analysis of Specific Examples

## Example A: $f_n(x) = x^n$ on $[0, 1]$

- **Pointwise Limit:**

$$f(x) = \lim_{n \to \infty} x^n = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

The limit function $f$ is discontinuous, even though each $f_n$ is continuous.

- **Uniform Convergence Test:**
  If convergence were uniform, there would exist $n_0$ such that $|x^{n_0} - 0| < \epsilon$ for all $x \in [0, 1)$.
  However, if we pick $\epsilon < 1$, the inequality $x^{n_0} < \epsilon$ fails as $x$ approaches 1.
  - **Conclusion:** Convergence is **not uniform** on $[0, 1]$.
  - **Note:** Convergence *is* uniform on $[0, a]$ for any fixed $0 < a < 1$, because $|x^n| \le a^n$, and $a^n \to 0$ independent of $x$.

## Example B: The "Sliding Hump" ($S_n(x) = nxe^{-nx^2}$)

Consider the partial sums $S_n(x) = nxe^{-nx^2}$ on $[0, 1]$.

- **Pointwise Limit:** For any fixed $x$, $\lim_{n \to \infty} S_n(x) = 0$. The limit function is $S(x) = 0$.
- **Uniform Convergence Test:**
  We examine the maximum value of the function to see if it stays within $\epsilon$ distance of 0.
  Using the derivative, $S_n$ has a maximum at $x_n = \sqrt{1/2n}$.

$$M_n = \max_{x \in [0,1]} S_n(x) = S_n\left(\sqrt{\frac{1}{2n}}\right) = \sqrt{\frac{n}{2e}}$$

* **Conclusion:** Since $M_n \to \infty$ as $n \to \infty$, the graph does not flatten out into an $\epsilon$-strip. The convergence is **not uniform**.
  - **Terminology:** These are called "sliding-hump" functions because the peak moves toward 0 (slides) but grows infinitely high (hump) as $n$ increases.

# 3. Criteria for Uniform Convergence

## The Supremum Test (Theorem 8.2.5)

This is the most practical method for determining uniform convergence.
Let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$.
The sequence $\{f_n\}$ converges uniformly to $f$ on $E$ if and only if:

$$\lim_{n\to\infty} M_n = 0$$

- **Application to Example A ($x^n$):** On $[0,1]$, $\sup|x^n - f(x)| = 1$ (near $x=1$). Since limit is 1, not 0, it is not uniform.
- **Application to Example B ($nxe^{-nx^2}$):** On $[0,\infty)$, $M_n = \sqrt{n/2e} \to \infty$. Not uniform. However, on $[a,\infty)$ for fixed $a>0$, the maximum eventually occurs at the endpoint $a$, and decreases to 0. Thus, it is uniform on $[a,\infty)$.

## The Cauchy Criterion (Theorem 8.2.3)

A sequence $\{f_n\}$ converges uniformly on $E$ if and only if for every $\epsilon > 0$, there exists $n_0$ such that:

$$|f_n(x) - f_m(x)| < \epsilon$$

for all $x \in E$ and all $n, m \geq n_0$.

**Proof Idea:**

1. **Forward ($\Rightarrow$):** Follows from the triangle inequality, similar to standard Cauchy sequences.
2. **Reverse ($\Leftarrow$):**
   - Fix $x$. The sequence $\{f_n(x)\}$ is a Cauchy sequence of real numbers, so it converges to some value $y$. Define $f(x) = y$.
   - To prove uniformity, take the limit as $m \to \infty$ in the Cauchy inequality $|f_n(x) - f_m(x)| < \epsilon$. This yields $|f_n(x) - f(x)| \leq \epsilon$, valid for all $x$, proving uniform convergence.

**Corollary for Series (8.2.4):**
A series $\sum f_k$ converges uniformly on $E$ if and only if for large enough $n, m$:

$$|\sum_{k=n+1}^{m} f_k(x)| < \epsilon \quad \forall x \in E$$

# 4. The Weierstrass M-Test

A powerful tool for proving the uniform convergence of series.

## Theorem 8.2.7

Suppose $\{f_k\}$ is defined on $E$ and there exists a sequence of constants $\{M_k\}$ such that:

1. $|f_k(x)| \leq M_k$ for all $x \in E$ and $k \in \mathbb{N}$.
2. $\sum_{k=1}^{\infty} M_k < \infty$ (The series of constants converges).

Then, $\sum_{k=1}^{\infty} f_k(x)$ converges **uniformly** and **absolutely** on $E$.

**Proof Idea:**
Let $S_n(x)$ be the partial sums. For $n > m$:

$$|S_n(x) - S_m(x)| = \left| \sum_{k=m+1}^{n} f_k(x) \right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} M_k$$

Since $\sum M_k$ converges, the tail of the series approaches 0. By the Cauchy Criterion, $\sum f_k$ converges uniformly.

### Examples of M-Test

1. **Trigonometric Series:** $\sum \frac{\cos kx}{k^p}$ for $p > 1$.
   - Bound: $\left|\frac{\cos kx}{k^p}\right| \leq \frac{1}{k^p}$.
   - Since $\sum \frac{1}{k^p}$ converges (p-series), the trig series converges uniformly on $\mathbb{R}$.
2. **Geometric Series:** $\sum (x/2)^k$ on $[-a, a]$ where $0 < a < 2$.
   - Bound: $|(x/2)^k| \leq (a/2)^k$.
   - Since $a/2 < 1$, $\sum (a/2)^k$ converges. The series converges uniformly on $[-a, a]$.
   - *Note:* It does **not** converge uniformly on the open interval $(-2, 2)$ because the terms are not bounded by a single convergent geometric series near the endpoints.

## 5. Uniform vs. Absolute Convergence

Uniform convergence and absolute convergence are distinct concepts. One does not necessarily imply the other.

### Case A: Uniform $\not\Rightarrow$ Absolute

**Example:** $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$ on $[0, 1]$.

- **Uniform:** Let $a_k(x) = x^k/k$. The sequence $a_k(x)$ is decreasing and goes to 0 uniformly. By the Alternating Series Estimation Theorem, the error $|S(x) - S_n(x)| \leq a_{n+1}(x) \leq \frac{1}{n+1}$. Since the error bound $\frac{1}{n+1}$ is independent of $x$, convergence is uniform.

- **Not Absolute:** At $x = 1$, the series of absolute values is $\sum \frac{1}{k}$ (Harmonic Series), which diverges.

## Case B: Absolute $\not\Rightarrow$ Uniform

**Example:** $\sum x^2(1 + x^2)^{-k}$ (from Example 8.1.2b).

- **Absolute:** The series consists of non-negative terms and converges pointwise to a discontinuous function ($0$ at $x = 0$, $1 + x^2$ elsewhere).
- **Not Uniform:** A series of continuous functions converging to a discontinuous function **cannot** be uniformly convergent (a key theorem to be discussed in later sections). Thus, absolute convergence does not guarantee uniformity.