10907 Pattern Recognition

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Introductory Problem Set

This is the first weekly problem set. Its purpose is to outline the mathematical and programming background needed to navigate the course confidently. There are no weekly hand-in assignments; instead, a problem set will be posted each week and discussed in the exercise sessions. You are strongly encouraged to engage deeply with the material, as it is a prerequisite for developing the skills required to pass the exams. Remember that each exercise will not appear on the exam in exactly the same form, but it serves to outline the relevant area of knowledge. Each exercise has a difficulty level from $\star$ to $\star\star\star$.

1 Math

1 Linear Algebra

Exercise 1 (System of Linear Equations $\star$)

Consider the following system of linear equations,

$$\begin{aligned} 2x - 7y - 9z &= k_1 \\ 3x + 3y + 4z &= k_2 \\ 11x + 2y + 3z &= k_3 \end{aligned}$$

What are the values of k_1, k_2, k_3, for which the system above has solutions? If we set $k_1 = k_2 = k_3 = 0$, what are the possible values of x, y and z?

Exercise 2 (Elementary Row and Column Operations $\star$)

Let $\boldsymbol{M}$ be a block matrix given as

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{AB} \end{bmatrix},$$

where $\boldsymbol{A} \in \mathbb{R}^{s\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times t}$ and $\boldsymbol{I}_n$ is an $n \times n$ identity matrix. Use elementary row and column operations to transform this matrix into

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{I}_n \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}.$$

Provide an explanation of the row and column operations you used.

Exercise 3 (Rank of a Matrix $\star$)

Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be as in the previous exercise, $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{AB} \end{bmatrix}$ and $\boldsymbol{N} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{I}_n \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}$, with dimensions as before.

1. What conclusions can you draw regarding the rank of matrix $\boldsymbol{N}$ from the ranks of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$?
2. Using the results from part 1 of the exercise and the previous exercise, prove theinequality: $\operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B}) - \operatorname{rank}(\boldsymbol{AB}) \leq n$.
3. If $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{AB}) = s$, what is the rank of matrix $\boldsymbol{B}$?

Exercise 4 (Eigenvalues and Positive Definiteness)

Let $\boldsymbol{A} \in \mathbb{R}^{s\times s}$ be an $s \times s$ real matrix such that $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^s$. Show that $\boldsymbol{A}$ does not have any negative eigenvalues.

2 Probability

Exercise 5 (System of Linear Equations with Random Variables $\star\star$)

For the system of linear equations given below:

$$\begin{aligned} 2X - 7Y + 9Z &= K_1 \\ 3X + 3Y + 4Z &= K_2 \\ 5X + 2Y + 5Z &= K_3 \end{aligned}$$

The random variables X, Y, Z are independent and normally distributed such that $X \sim \mathcal{N}(0, 1), Y \sim \mathcal{N}(1, 4)$ and $Z \sim \mathcal{N}(0, 9)$.

- What are the distributions of K_1, K_2 and K_3?
- What is the joint probability density function (pdf) of K_1, K_2 and K_3, $p(k_1, k_2, k_3)$?

Recall that a normally distributed random variable X with mean μ and variance σ^2 (written $X \sim \mathcal{N}(\mu, \sigma^2)$) has the probability density function (pdf)

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

so that for some a, b such that $a \leq b$ we have

$$\mathbb{P}[a \leq X \leq b] = \int_a^b p_X(x)dx.$$

Exercise 6 (Sum of Bernoulli Distribution $\star\star\star$)

Alice and Bob decide to play a game. They toss a biased coin n times. If it comes up heads at least $n/2$ times, Bob wins; otherwise Alice wins. The total number of heads is

$$S = \sum_{i=1}^{n} X_i$$

where X_i is an independent Bernoulli random variable with mean p, that is,

$$\mathbb{P}(X_i = 1) = p, \quad \mathbb{P}(X_i = 0) = 1 - p.$$

Assume that $p = 0.52$.

1. What is the law of the random variable S? Give its name, its probability mass function (p.m.f.) and its mean.

2. Bob only wants to play if he has a 90% chance of winning.
   For a given n, what is the expected number of heads?
   Compute the probability that Bob wins when $n = 2$.
   Can you figure out for which values of n Bob will participate in the game?

   Hint: One way to do this is by using Hoeffding's inequality: let Z_i be independent and identically distributed random variables such that $0 \leq Z_i \leq 1$, then

   $$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i - \mathbb{E}(Z_i) \geq t\right) \leq \exp(-nt^2).$$

3. Alice and Bob continue the game for a long, long time. What can you say about the value or distribution of S as $n \to \infty$? What about the value or distribution of $\frac{S}{n}$? And what about the value or distribution of $\frac{S - np}{\sqrt{n}}$?

Exercise 7 (Expectations ⋆⋆)

The expectation of a non-negative random variable X with probability density function $f_X(x)$ is

$$\mathbb{E}(X) = \int_0^\infty x f_X(x) dx.$$

Show that an alternative way to compute it is by the following useful identity:

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > t) dt.$$

2 Coding

To get your hands dirty with Python, you are asked to write some basic array operations realted to linear algebra and probability. Specifically, you need to complete the blank functions in `sample.py`.

Exercise 8 (Basic Computation ⋆)

1. Write a function `check_span` that takes as the inputs a vector $\boldsymbol{y}$ and a matrix $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots \boldsymbol{a}_n]$, where $\boldsymbol{a}_i$'s are the column vectors of the matrix. The function outputs a Boolean variable. The output is `True` if $\boldsymbol{y}$ is in the span of column vectors in $\boldsymbol{A}$. Otherwise, the output is `False`.

2. Write a function `estimate_mean_and_covariance` to compute the mean and the unbiased covariance matrix of a set of vectors $\{\boldsymbol{x}_1, \ldots \boldsymbol{x}_n\}$. The set of vectors are passed to the function as a matrix $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots \boldsymbol{x}_n]$. The unbiased covariance matrix is defined as:

   $$\boldsymbol{Q} = \frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^{\mathrm{T}}$$

   where $\bar{\boldsymbol{x}} = \frac{1}{n}\sum_i^n \boldsymbol{x}$.