9.2. HEINE-BOREL THM. ~~期~~期末考

5/3. 2023.

9.9 Lemma [BOREL COVERING LEMMA]

Let E be a closed, bdd subset of $\mathbb{R}^n$. If r is any function from $E \to (0,\infty)$, then $\exists$ finitely many point $y_1, y_2, \cdots y_N \in E$ s.t.

$E \subseteq \bigcup_{j=1}^{N} B_{r(y_j)}(y_j)$. (以 r 為半徑, y_j 為心的球).

[☆ 可能會考証明, 要額外做 Goodnote Apple watch 筆記.]

9.10 Def.

Let E be a subset of $\mathbb{R}^n$

1) An "open covering of E" is a collection of sets $\{V_\alpha\}_{\alpha \in A}$ s.t. each V_α is open and

$$E \subseteq \bigcup_{\alpha \in A} V_\alpha.$$

☆ open covering: 每個元素都開, 且聯集後蓋住E.
開覆蓋.

ii) The set E is said to be "緊緻的 compact" iff. every open covering of E has a finite subcovering ; that is, iff. given any open covering $\{V_\alpha\}_{\alpha\in A}$ of E, there is a finite subset $A_0=\{\alpha_1,\dots,\alpha_N\}$ of A s.t.

$E \subseteq \bigcup_{j=1}^{N} V_{\alpha_j}$ ：每個開覆蓋，都有個有限子覆蓋.

或者說：Given 任意開覆蓋 $\{V_\alpha\}_{\alpha\in A}$ 的 E，存在 A 的有限子集 $A_0=\{\alpha_1,\dots,\alpha_N\}$ s.t.

$E \subseteq \bigcup_{j=1}^{N} V_{\alpha_j}$. 不懂？問人囉.

☆ 9.11 Thm. [HEINE-BOREL THM].

Let E be a subset of $\mathbb{R}^n$, Then E is "compact" iff. E is closed and bounded.

證明 ~~不考~~. 可能考一部分.

pf: ($\Leftarrow$) Borel covering Lemma

($\Rightarrow$) 1. E is compact $\Rightarrow$ E is bounded.

2. E is compact $\Rightarrow$ E is closed.

Suppose E is not closed $\Leftrightarrow$ $E \neq \phi$ and (By thm 9.8)

$\exists \{x_k\} \subseteq E$, $x_k \to x \notin E$

For each $y \in E$, set $r(y) := \|y-x\|/2$ "定義"

$\because x \notin E$

$\therefore r(y) > 0$, $\forall y \in E$

$\Rightarrow B_{r(y)}(y)$ is open and $y \in B_{r(y)}(y)$

$\Rightarrow \therefore \{B_{r(y)}(y)\}_{y\in E}$ is an open covering of E

(Since $E = \bigcup_{y\in E}\{y\} \subseteq \bigcup_{y\in E} B_{r(y)}(y)$).

$\because E$ is a compact

$\therefore \exists\ y_j$ and $r_j := r(y_j)$, for $j=1,\dots,N$.

set $E \subseteq \bigcup_{j=1}^{N} B_{r_j}(y_j)$.

set $r = \min\{r_1, r_2, \dots, r_N\}$

$\because x_k \to x$

$\therefore x_k \in B_r(x)$ for large k

But $x_k \in B_r(x) \cap E$

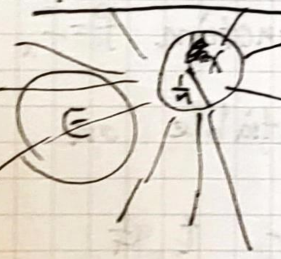
$\Rightarrow \cancel{x_k \in B_{r_j}(x) \cap E} \Rightarrow x_k \in B_{r_j}(y_j)$ for some $j \in \mathbb{N}$.

$\therefore r_j \geq \|x_k - y_j\| \geq \|x - y_j\| - \|x - x_k\|$

$= 2r_j - \|x - x_k\| > 2r_j - r$

$\geq 2r_j - r_j = r_j \Rightarrow r_j > r_j$ (※) #.

ANOTHER WAY:



$V_n : \{y \in \mathbb{R}^n : \|y - x\| > \frac{1}{n}\}$

n 愈大, V_n 愈大

目標: E^c is open. Let $x \in E^c$ and set

$V_n = \{y \in \mathbb{R}^n : \|y - x\| > \frac{1}{n}\}$, $n \in \mathbb{N}$.

$\forall z \in E \Rightarrow \|z - x\| > 0$ ($\because x \notin E$)

$\Rightarrow \exists\ n_0 \in \mathbb{N}$ s.t. $\|z - x\| > \frac{1}{n_0}$

$\Rightarrow z \in V_{n_0}$

$\therefore \{V_n\}_{n\in\mathbb{N}}$ is an open covering of E.

$\because E$ is compact.

$\therefore \exists\, n_1, n_2, \dots, n_N \in \mathbb{N}$, s.t.

$E \subseteq \bigcup_{j=1}^{N} V_{n_j}$

Let $m := \max_{1\le j\le N} \{n_j\}$,

Then $\bigcup_{j=1}^{N} V_{n_j} = V_m$

$\Rightarrow E \subseteq V_m$

$\Rightarrow V_m^c \subseteq E^c$

$\Rightarrow B_{\frac{1}{m}}(x) \subseteq V_m^c \subseteq E^c$

Hence, E^c is open. ※

在歐氏空間，只要 closed and bdd 則 ~~po~~ compact.

(metric space is not)

☆可能會考 $\boxed{}$

compact.

①

<u>EX 9.12</u>. Suppose that E is a closed bonneded subset of $\mathbb{R}$. If for every $x \in E$, $\exists$ a nonnegative function $f = f_x$ and a number $r = r(x) > 0$ s.t. f is differentiable on $\mathbb{R}$, $f_x(t) > 0$ for $t \in (x-r, x+r)$ and $f_x(t) = 0$ for $t \notin (x-2r, x+2r)$. prove that $\exists$ differentiable function f and an open set V which contains E s.t. f is nonzero and bounded on E and $f(x) = 0$ for $x \notin V$.

pf: For each $x \in E$, choose $r_x > 0$ and $f_x \geq 0$ s.t. f_x is differentiable on $\mathbb{R}$, $f_x(t) > 0$ for $t \in I(x-r, x+r)$, and $f_x(t) = 0$ for $t \notin (x-2r_x, x+r_x)$

Set $I_{r_x} := (x-r_x, x+r_x)$

$J_{r_x} := (x-2r_x, x+2r_x)$

Then $\{I_{r_x}\}_{x\in E}$ is an open covering of E.

$\because E$ is a compact

$\therefore \exists\, x_1, \dots, x_N \in E$ s.t. $E \subseteq \bigcup_{j=1}^{N} I_{r_{x_j}}$

Let $f = \sum_{j=1}^{N} f_{x_j}$ and $V = \bigcup_{j=1}^{N} J_{r_{x_j}}$

Then f is diff. on $\mathbb{R}$ and $V \supseteq E$ is open.

先用其的條件抄一編,

If $x \in E$, then $x \in I_{r_{x_j}}(x_j)$ for some $1 \leq j \leq N$,

$\Rightarrow f_{x_j}(x) > 0$

$\therefore f(x) = \sum_{j=1}^{N} f_{x_j}(x)$ ~~on $\mathbb{R}$~~ $\geq f_{x_{j_0}}(x) > 0$

$(\Rightarrow f(x) > 0,\ \forall x \in E)$

$\because f_{x_j}$ is diff. on $\mathbb{R}$

$\therefore f_{x_j}$ is continuous on $H = \bigcup_{j=1}^{N} [x_j - V_{x_j}, x_j + V_{x_j}]$.

THE EXTREME VALUE THM.

$\Rightarrow \exists M_j > 0$ s.t. $|f_{x_j}(x)| \leq M_j$ on H for $j = 1, 2, \dots, N$.

Thus, $|f(x)| \leq \sum_{j=1}^{N} |f(x_j)| \leq \Sigma$

Finally, if $x \notin V$, then $x \notin J_{r_{x_j}}(x_j)\ \forall\ 1 \leq j \leq N$

Thus, $f(x) = \sum_{j=1}^{N} f_{x_j}(x) = 0$ #.

可能会考.

- open
- bound~~ary~~ed
- closed
- compact
- thm 9.10
  - prove
  - applicate

EXERCISE.

EXAMPLE

其实还行吧

就還行唄

補充：~~也~~可能会考

極限負起了很重要的角色.

Prove $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact by definition.

（唯一可以用 def 証的 compact set）.

pf.

Let $\{V_\alpha\}_{\alpha \in A}$ is a open covering of E.

$(E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\})$.

必考

$\because 0 \in E \subseteq \bigcup_{\alpha \in A} V_\alpha$

$\therefore \exists \alpha_0 \in A$ s.t. $x \in V_{\alpha_0}$

$\because V_{\alpha_0}$ is open

$\therefore \exists \varepsilon > 0$ s.t. $B_\varepsilon(0) \subseteq V_\alpha$

$\because \lim_{n \to \infty} \frac{1}{n} = 0$

$\therefore \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |\frac{1}{n} - 0| < \varepsilon$

$\Rightarrow \frac{1}{n} \in B_\varepsilon(0) \subseteq V_{\alpha_0}$

$$\begin{array}{ccccccc} 1, & \frac{1}{2}, & \frac{1}{3}, & \cdots\cdots & \frac{1}{N-1} & \frac{1}{N} & \frac{1}{N+1} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ V_{\alpha_1} & V_{\alpha_2} & V_{\alpha_3} & \cdots\cdots & V_{\alpha_{N-1}} & V_{\alpha_N} & V_{\alpha_{N+1}} \end{array} \quad V_{\alpha_0}$$

$\because \{\frac{1}{i} : 1 \leq i \leq N-1\} \subseteq E \subseteq \bigcup_{\alpha \in A} V_\alpha$

$\therefore \exists \alpha_i \in A$ s.t.

$\frac{1}{i} \in V_{\alpha_i}, \quad 1 \leq i \leq N-1$

Hence $E \subseteq \bigcup_{i=0}^{N-1} V_{\alpha_i} \Rightarrow E$ is compact #.

§ 9.3. Limit of function.

Def 9.14.

Let $n, m \in \mathbb{N}$ and $a \in \mathbb{R}^n$, let V be an open set which contains a, and suppose that $f: V \setminus \{a\} \to \mathbb{R}^m$. Then $f(x)$ is said to converge to L, as x approach to a, iff. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < \|x - a\| < \delta \Rightarrow \|f(x) - L\| < \varepsilon$

$f(x) \to L$ as $x \to a$ or $L = \lim_{x \to a} f(x)$.

Thm 9.15. [自己看].

i)

ii)

iii)

iv) Squeeze thm.

v)

Thm 9.16.

Let $a \in \mathbb{R}^n$, let V be an open set which contains a, ~~s~~and suppose that $f = (f_1, \dots, f_m): V \setminus \{a\} \to \mathbb{R}^m$. then $\lim_{x \to a} f(x) = L := (L_1, L_2, \dots, L_m)$ exists in $\mathbb{R}^m$ iff. $\lim_{x \to a} f_j(x) = L_j$ exists in $\mathbb{R}$ for $j = 1, \dots, m$.

EX 9.18.

Prove that $f(x,y)=\frac{3x^2y}{x^2+y^2}$ converges as $(x,y)\to(0,0)$

$$0 \le |f(x,y)| \le \frac{|3x^2y|}{|x^2+y^2|} \le \frac{|3x^2y|}{|x^2|}$$

pf. Fix $(x,y)\neq(0,0)$, we have $|f(x,y)| = \frac{|3x^2y|}{|x^2+y^2|} \le \frac{|3x^2y|}{|x^2|}$

$= 3|y|$.

Let $\varepsilon>0$, $\exists\ \delta=\frac{\varepsilon}{3}$ s.t. $0<\|(x,y)-(0,0)\|<\delta \Rightarrow$

$|f(x,y)-0| \le 3|y| \le 3\sqrt{x^2+y^2} < 3\delta = \varepsilon$.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)=0$.

通常跑到原点高机率收斂到0.

EX 9.19. Prove that the function $f(x,y)=\frac{2xy}{x^2+y^2}$

has no limit as $(x,y)\to(0,0)$.

當找到2個不同的路徑 limit，則極限不存在.
(都收斂到A)

pf. $\because \lim\limits_{\substack{y=x\\x\to0}} f(x,y) = \lim\limits_{x\to0} f(x,x) = \lim\limits_{x\to0}\frac{2x^2}{2x^2}=1$ and

$\lim\limits_{\substack{x=0\\y\to0}} f(x,y) = \lim\limits_{y\to0} f(0,y) = \lim\limits_{y\to0} \quad =0$

找兩個路徑不一樣，則 limit 不存在.
且兩個極限不一樣，

通常都用夾幾定理證 limit 存在.

方法：Let $m \in \mathbb{R}$,

$$\lim_{\substack{y=mx \\ x\to 0}} f(x,y) = \lim_{x\to 0} \frac{2x(mx)}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{2mx^2}{x^2(1+m^2)} = \frac{2m}{1+m^2}$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ 不存在！

EX9.20.

Determine whether $f(x,y) = \dfrac{xy^2}{x^2+\cancel{y^2}\, y^4}$ has a limit as $(x,y)\to(0,0)$

$\boxed{\text{搞懂}}$

pf. $\because \lim_{\substack{x=0 \\ y\to 0}} f(x,y) = \lim_{y\to 0} f(0,y) = 0$ and

$\lim_{\substack{x=y^2 \\ y\to 0}} f(x,y) = \lim_{y\to 0} f(y^2,y) = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$ $\qquad 0 \neq \frac{1}{2}$

$\therefore f$ has no limit ~~as~~ as $(x,y)\to(0,0)$.

$$\cancel{\lim_{y=mx} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{xm^2x^2}{1+m^4x^4} = \lim_{x\to 0} \frac{xm^2}{1+m^4x^2} = 0}$$

"沒分數，不能判別"

考古題的習題"

Definition.

Let V be an open set in $\mathbb{R}^2$.

let $(a,b) \in V$, and suppose that $f: V \setminus \{(a,b)\} \to \mathbb{R}^m$

The "iterated" limits of f at (a,b)

are defined to be

$$\lim_{x\to a}\lim_{y\to b} f(x,y) \underset{①}{=} \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$$

and

$$\lim_{y\to b}\lim_{x\to a} f(x,y) \underset{②}{=} \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right).$$

EX 9.21

Evaluate the iterated limit of $f(x,y) = \frac{x^2}{x^2+y^2}$

at $(0,0)$.

sol/

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2}{x^2+y^2}\right) = \lim_{x\to 0} 1 = 1.$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2}{x^2+y^2}\right) = \lim_{y\to 0}\frac{0}{0+y^2} = 0$$

極限在 → 則所有路徑求出的極限相同.

Rmk 9.22. Suppose that I and J are open intervals,

that $a \in I$, $b \in J$ and $f: (I \times J) \setminus \{(a,b)\} \to \mathbb{R}$,

If $g(x) := \lim_{y\to b} f(x,y)$.

exists for each $x \in I \setminus \{a\}$

if $\lim_{x\to a} f(x,y)$ exists for each $y \in J \setminus \{b\}$

and if $f(x,y) \to L$ as $(x,y) \to (a,b)$

then $L = \lim_{x\to a} \lim_{y\to b} f(x,y) = \lim_{y\to b} \lim_{x\to a} f(x,y)$

若沒辦法用夾擠証明lim ~~不存~~存在，

則找2個路徑使得lim值不同，則可証lim不存在

· 只考收斂到0.

5/19 小考 9.3 (習題，課本).

5/26 小考 9.2 & 9.4

6/2 大考.

§9.4 Continuous function.

Def 9.23

Let $\phi \neq E \subseteq \mathbb{R}^n$ and $f: E \to \mathbb{R}^n$

i) f is said to be continuous at $a \in E$

iff. $\forall \varepsilon > 0$, $\exists \delta > 0$ (可能與 ε, f, E, a 有關)

s.t $\|x-a\| < \delta \Rightarrow \|f(x) - f(a)\| < \varepsilon$ $\quad \lim_{x \to a} f(x) = f(a)$.

ii) f is said to be continuous on E

(notation $f: E \to \mathbb{R}^m$ is continuous).

iff. f is continuous at every $x \in E$.

Note: (不証明).

f is continuous at $a \in E$ iff.

$f(x_k) \to f(a)$ for all $x_k \in E$ which converge to a.

Definition 9.24

let $E \neq \phi$, $E \subseteq \mathbb{R}^n$ and $f: E \to \mathbb{R}^m$

Then f is said to be uniformly continuous on E

iff. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$\|x-a\| < \delta$ and $x, a \in E \Rightarrow \|f(x) - f(a)\| < \varepsilon$

2 点間的距離小於 δ，則其函數值小於 ε.

(notation : $f: E \to \mathbb{R}^m$ is uniformly continuous).

Thm 9.25. 参考 證明.

Let E be a nonempty compact subset of $\mathbb{R}^n$

If f is continuous on E,

then f is uniformly continuous on E.

pf. Suppose f is continuous on E.

Given $\varepsilon > 0$, $a \in E$, $\exists\, \delta(a) > 0$ s.t.

$\|x-a\| < \delta(a) \Rightarrow \|f(x) - f(a)\| < \varepsilon/2$.

$\Rightarrow$ ~~δ(a)~~ $x \in B_{\delta(a)}(a) \cap E \Rightarrow \|f(x) - f(a)\| < \varepsilon/2$

$\because \delta(a) > 0$, $\forall a \in E$

$\therefore$ the collection $\{B_{\frac{\delta(a)}{2}}(a)\}_{a\in E}$ is

an open covering of E.

$\because$ E is a compact

$\therefore \exists\, a_1, a_2, \dots, a_n \in E$ s.t.

$E \subseteq \bigcup_{j=1}^{N} B_{\delta_j}(a_j)$, 其中 $\delta_j \equiv \dfrac{\delta(a_j)}{2}$ ⋯ (4).

Set $\delta = \min\{\delta_1, \delta_2, \dots, \delta_N\} > 0$

Suppose that $x, a \in E$ with $\|x-a\| < \delta$

By (4), $x \in B_{\delta_j}(a_j)$ for some $1 \le j \le N$,

$\therefore \|a - a_j\| \le \|a - x\| + \|x - a_j\|$

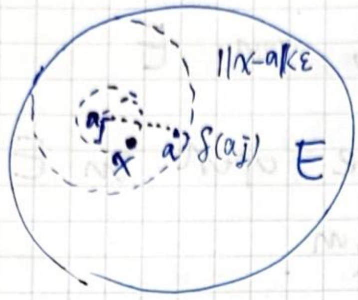
$< \delta + \delta_j \le 2\delta_j = \delta(a_j)$

$\therefore a \in B_{\delta(a_j)}(a_j)$

$$\therefore \|f(x)-f(a)\| \le \|f(x)-f(a_j)\| + \|f(a_j)-f(a)\| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon .$$

Hence, f is uniformly continuous on E #.

• Tips：要讓 x 和 a 在同一個球裡面.



對於 x 在小球內, 且對於 $\|x-a\|<\varepsilon$ 的 a 點, 則 a 必定在 2 倍半徑的 $B_{\delta(a_j)}(a_j)$ 內.

算了, 做小抄 or 背起來吧. 5/9/2023.

5/11. 2023

Thm 9.26

Suppose that $\phi \neq E \subseteq \mathbb{R}^n$ and that $f: E \to \mathbb{R}^m$

Then f is continuous on E iff. $f^{-1}(V)$ is relative open in E for every V open in $\mathbb{R}^m$.

$f^{-1}(V) = \{x \in E \mid f(x) \in V\}$

- f is continuous, $f: E \to \mathbb{R}^m$ 以下皆等價：

① for all $x \in E$, f is continuous on x

$\Rightarrow \forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\|y - x\| < \delta \Rightarrow \|f(y) - f(x)\| < \varepsilon$

$\langle y \in B_\delta(x) \cap E \Rightarrow f(y) \in B_\varepsilon(f(x)).\rangle$

② $\forall \{x_k\} \subseteq E$, $x_k \to x$

$\Rightarrow f(x_k) \to f(x)$.

③ for every open set V in $\mathbb{R}^m$

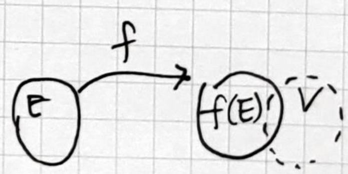
f^{-1} is relative open in E.

④ for every closed set F in $\mathbb{R}^m$

$f^{-1}(F)$ is relatively closed in E.

pf. ($\Rightarrow$) Suppose f is continuous on E.

and V is open in $\mathbb{R}^m$.



$\because \phi$ is open, $\therefore$ we may suppose $a \in f^{-1}(V) \neq \phi$

(By Remark 8.27, to show $f^{-1}(V)$ is relatively open in E, we just need to show there is a $\delta > 0$ s.t. $B_\delta(a) \cap E \subseteq f^{-1}(V)$.) → 考試不用寫

∵ $f(a)\in V$ and V is open

∴ $\exists\, \varepsilon>0$ s.t. $B_\varepsilon(f(a))\subseteq V$

∵ f is continuous at a

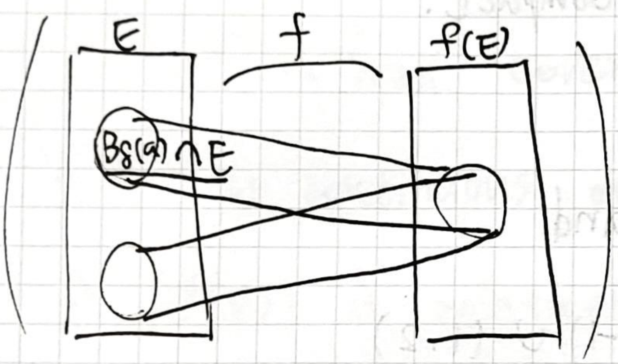
∴ $\exists\, \delta>0$ s.t. $\|x-a\|<\delta$ and $x\in E \Rightarrow \|f(x)-f(a)\|<\varepsilon$

i.e. $x\in B_\delta(a)\cap E \Rightarrow f(x)\in B_\varepsilon(f(a))$

$\Leftrightarrow f(B_\delta(a)\cap E)\subseteq B_\varepsilon(f(a))\subseteq V$

$\Rightarrow \cancel{f^{-1}(f(B_\delta(a)\cap E))}$

$\Rightarrow B_\delta(a)\cap E\subseteq f^{-1}(f(B_\delta(a)\cap E))\subseteq f^{-1}(V)$



∴ $f^{-1}(V)$ is relatively open in E.

$\Leftarrow$) if $a\in E$, $\varepsilon>0$.

then $B_\varepsilon(f(a))$ is open in $\mathbb{R}^m$

By hypothesis, $f^{-1}(B_\varepsilon(f(a))$ is relatively open in E. By Remark 8.27, $\exists\,\delta>0$ s.t. $B_\delta(a)\cap E\subseteq f^{-1}(B_\delta(f(a))) \Rightarrow f(B_\delta(a)\cap E)\subseteq f\circ f^{-1}(B_\delta(f(a)))\subseteq B$

∴ if $\|x-a\|<\delta \Rightarrow \cancel{f(a)}\, \|f(x)-f(a)\|<\varepsilon$ #.

★ 不考 close, 只考 open.

EX9.27.

i) open set 換成 bounded set，

Thm 9.26 不成立！

If $f(x)=\frac{1}{1+x^2}$ and $E=[0,1]$

then $f^{-1}(E)=(-\infty,\infty)$ E is bounded but $f^{-1}(E)$ is not bounded.

找一個例子：f is continuous,

C is compact $\not\Rightarrow$ $f^{-1}(C)$ is compact.

ii) If $f(x)=x^2$ and $E=(1,4)$

then f is continuous on $\mathbb{R}$ and E is connected, but $f^{-1}(E)=(-2,-1)\cup(1,2)$ is not connected.

Note. 連續函數不一定把開(閉)集送到開(閉)集.

EX9.28.

i) If $f(x)=x^2$ and $V=(-1,1)$

then f is continuous on V and V is open

but $f(V)=[0,1)$ is neither open or closed set

If $f(x)=\frac{1}{x}$ and $E=[1,\infty)$, then f is continuous on ... and E is closed, but $f(E)=(0,1]$, X closed Y ...

Thm 9.29. ☆ $\boxed{\text{会考}}$

If H is compact in $\mathbb{R}^n$ and

$f: H \to \mathbb{R}^m$ is continuous on H

the $f(H)$ is compact.

pf. Suppose that $\{V_\alpha\}_{\alpha\in A}$ is an open covering of $f(H)$, then $f(H) \subseteq \bigcup_{\alpha\in A} V_\alpha \Rightarrow H \subseteq f^{-1}(f(H)) \subseteq$

$f^{-1}(\bigcup_{\alpha\in A} V_\alpha) = \bigcup_{\alpha\in A} f^{-1}(V_\alpha)$.

$\therefore \{f^{-1}(V_\alpha)\}_{\alpha\in A}$ covers H.

$\because$ f is continuous on H

$\therefore$ $f^{-1}(V_\alpha)$ is relatively open in H

$\therefore$ $\exists$ open sets O_α s.t.

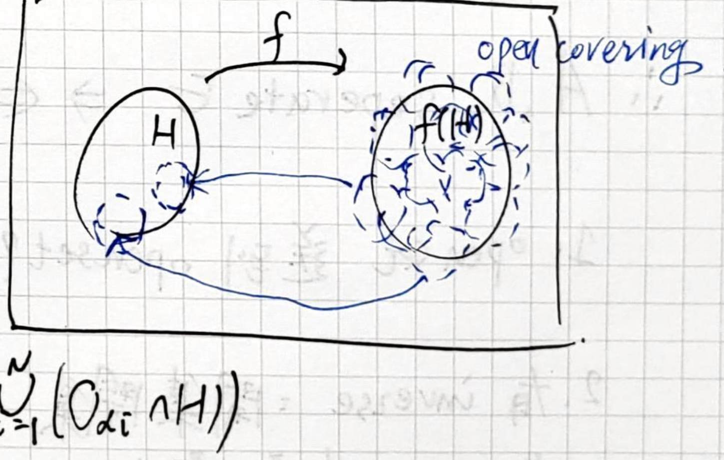
$f^{-1}(V_\alpha) = O_\alpha \cap H$.

$\therefore$ $\{O_\alpha\}_{\alpha\in A}$ is a open covering of H.

$\because$ H is compact

$\therefore$ $\exists \alpha_1, \alpha_2, \dots, \alpha_N \in A$ s.t.

$H \subseteq \bigcup_{i=1}^{N} O_{\alpha_i}$



$\Rightarrow f(H) \subseteq f((\bigcup_{i=1}^{N} O_{\alpha_i}) \cap H) = f(\bigcup_{i=1}^{N} (O_{\alpha_i} \cap H))$

$= \bigcup_{i=1}^{N} f(O_{\alpha_i} \cap H) = \bigcup_{i=1}^{N} f(f^{-1}(V_{\alpha_i})) \subseteq \bigcup_{i=1}^{N} V_{\alpha_i}$

$\therefore$ $f(H)$ is compact #.

Thm 9.30.

If E is connected in $\mathbb{R}^n$ and $f: E \to \mathbb{R}^n$ is continuous, then $f(E)$ is connected. (証明不考).

pf. Suppose $f(E)$ is not connected.

By def 8.28. $\exists$ relative open sets U, V in $f(E)$ separate $f(E)$ i.e. $U \neq \phi$, $V \neq \phi$, $f(E) = U \cup V$ and $U \cap V = \phi$. Set $A := f^{-1}(U)$ and $B := f^{-1}(V)$

$\because f$ is continuous on E (By ex 9.4.5 b)

$\therefore A$ and B are relatively open in E.

$\because f(E) = U \cup V$

$\therefore \underbrace{f^{-1}(f(E))}_{= E} = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$

$\therefore U \cap V = \phi \Rightarrow f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\phi) = \phi$

$\therefore A, B$ seperate $E \Rightarrow \Leftarrow$ #.

1. open set 送到 open set? 連續函數的特性.

2. 有 inverse：開集閉集

3. 有 f.：連通. 緊緻.

4.

下禮拜会講完，(四) 小考 9.3

Remark 9.31 考.

The graph $y=f(x)$ of a continuous real function f on $[a,b]$ is compact and connected

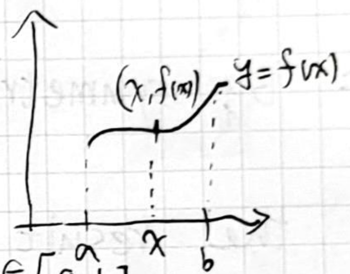
$f:[a,b]\to\mathbb{R}$

graph of $f = \{(x,y) : y=f(x),\ x\in[a,b]\}$

$F:[a,b]\to\mathbb{R}^2$, $F(x) = (x, f(x))$.

pf. $\because$ f is continuous on $[a,b]$

$\therefore$ $F(x) = (x, f(x))$ is continuous from $[a,b]$ into $\mathbb{R}^2$,



and the graph of $y=f(x)$ for $x\in[a,b]$ is $F([a,b])$

$\because$ $[a,b]$ is compact and connected

($\because$ Thm 9.29 & Thm 9.30)

$\Rightarrow$ $F([a,b])$ is compact and connected.

Thm 9.32 [Extreme Value Thm] 極值定理 考

~~Choose~~ Suppose that $\phi \neq H \subseteq \mathbb{R}^n$ and that $f: H \to \mathbb{R}$

If H is compact, and f is continuous on H

then $M := \sup\{f(x) : x \in H\}$ and $m := \inf\{f(x) : x \in H\}$

are finite real number.

Moreover, there exist $x_M, x_m \in H$ s.t.

$f(x_M) = M$ and $f(x_m) = m$.

pf: By symmetricy, it suffies to prove

the result for M.

$\because H$ is compact, f is continuous on H.

$\therefore f(H)$ is compact

$\because f(H) \subseteq \mathbb{R}$, the Heine - Borel Thm,

$\therefore f(H)$ is bounded and closed.

$\because f(H)$ is bounded,

$\therefore M < \infty$

By the Approximateion Property,

$\exists x_k \in H$ s.t. $f(x_k) \to M$

$\because f(H)$ is closed

$\therefore M \in f(H)$

i.e. $\exists x_M \in H$ s.t. $f(x_M) = M$.

☆ 9-4 會考多一點，9-2 考 compact，9-3 大都微積分考一題
(4~5題)　(1~2題)　(1題)

Remark 9.34　考

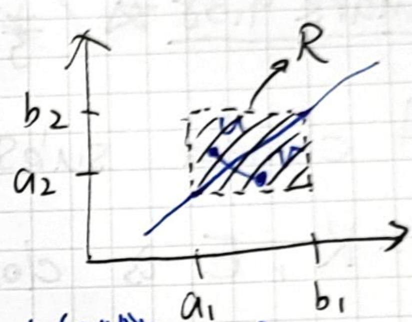
If $a_j \le b_j$ for $j = 1, 2, \dots, n$

then $R := \{(x_1, x_2, \dots, x_n) : a_j \le x_j \le b_j\}$ is connected.

pf: Suppose R is not connected then there are

~~nonem~~ nonempty sets U, V relative open in R,

s.t. $R = U \cup V$, $U \cap V = \phi$.



Let $a \in U$, $b \in V$,

Consider the line segment

$f(t) = ta + (1-t)b$. ~~$f(t) = t(a \cdots (1-t)b)$~~

$E := \{ta + (1-t)b : t \in [0,1]\}$　　$E = f([0,1])$

$\because$ E is a continuous image of $[0,1]$

$\therefore$ E is connected by thm 8.30 and 9.30.

On the other hand

$\because E \subseteq R$ by the definition of R

$\therefore$ It's easy to check that

$U_0 = U \cap E$ and $V_0 = V \cap E$ are nonempty sets

which are relative open in E and satisfy

$E = U_0 \cup V_0$, $V_0 \cap U_0 = \phi$

$\therefore$ E is not connected $(\Rightarrow\Leftarrow)$.　　9.4 finished.

§9.5 Compact Sets

Remark 9.36

The empty set and all finite subsets of $\mathbb{R}^n$ are compact

Remark 9.37 証考

A compact set is always closed

☆ **Remark 9.38** ☆ 考

A closed subset of a compact is compact.
$E \subseteq K$, E is closed, K is compact.
$\Rightarrow E$ is compact

$\Updownarrow$ E^c is open

mini pf. $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ K is closed and bounded
$E \subseteq K \Rightarrow E$ is bounded + E is closed $\Rightarrow E$ is compact

Def

pf: Let E ~~is~~ be a closed subset of K
and K is compact.
Suppose $\{V_\alpha\}_{\alpha \in A}$ is an open covering of E
$\because E$ is closed, $\therefore E^c$ is open
$\Rightarrow \{V_\alpha\}_{\alpha \in A} \cup E^c$ is an open
$\because K$ is ~~compact~~ compact
$\therefore \exists$ finite set $A_0 \subseteq A$ s.t. $K \subseteq E^c \cup (\bigcup_{\alpha \in A_0} V_\alpha)$
$(\exists \alpha_1, \dots, \alpha_N \in A$ s.t. $K \subseteq E^c \cup \bigcup_{i=1}^{N} V_{\alpha_i})$

$\Rightarrow \cancel{K \cap E}$ $E \subseteq K \subseteq E^c \cup \bigcup_{i=1}^{N} V_{\alpha_i}$

$\because E \cap E^c = \phi$

$\therefore E \subseteq \bigcup_{i=1}^{N} V_{\alpha_i}$

$\therefore E$ is compact.

期末考到這結束.

9.3 習題講解 (Good note).