Portfolio Construction with Robust Covariance

CQF Project

Overview

This project is part of CQF module 6 final Project on Portfolio Construction with Robust Covariance. Developed Black-Litterman model using python libraries for matrices, vectors and optimization. We tried best to choose the assets that diversified and less correlated. The assets are chosen from different ETFs. FM(Frontier Markets: iShares MSCI Frontier 100 Index Fund), GLD(SPDR Gold Trust ETF), IFEU (iShares FTSE EPRS/NAREIT Europe Index ETF),IJR(iShares Core S&P Small-Cap ETF),JXI(iShares Global Utilities ETF),PIN(Invesco India Portfolio ETF),TLT(iShares 20+ Year Treasury Bond ETF).Used 3 years of data from Jan 2016 to Dec 2018.Developed prior, posterior based on views and performed mean variance and Max sharp ratio risk budgeting.

Black-Litterman Model

The Black-Litterman Model is an asset allocation model developed in 1990 by Fischer Black and Robert Litterman at Goldman Sachs. The Markowitz portfolio optimization model was developed with two basic objectives of investing, maximizing expected returns and minimizing the risk. The Black-Litterman Model is an improvement to the Markowitz portfolio model because it can incorporate more information into the model rather than simple past stock returns. The model uses a Bayesian approach to combine the subjective views of an investor regarding the expected returns of one or more assets with the market equilibrium vector of expected returns (the prior distribution) to form a new, mixed estimate of expected returns. The resulting new vector of returns (the posterior distribution), leads to intuitive portfolios with sensible portfolio weights.

We started developing the model by using the daily returns and calculated the covariance matrix using the statistical technique called shrinkage. In general, the sample covariance matrix is estimated with a lot of error when the number of stocks under consideration is large, especially relative to the number of historical return observations available. It implies that most extreme coefficients in the matrix tends to take extreme values not because of truth but because of error. We utilized LedoitWolf Estimator which is available in python library sklearn. Also calculated correlation among assets to check how well the assets are diversified.

Constructed the prior by taking the equal weights to all the assets. By setting up views, risk aversion coefficient and tao constructed posterior, new covariance Matrix and market equilibrium weights. But we used the existing covariance matrix which we got from LedoitWolf Estimator.

Then it depends on the model user which risk budgeting strategy to use. In this project we developed code for 2 of the below optimization procedures. (Maximizing sharp ratio, Mean variance allocation)

The Black-Litterman inputs feeds the optimization procedure subject to a certain target:

1- Minimum Variance Allocation

$$\text{argmin}_{W}\{\lambda W'\Sigma W\}$$

2- Maximize Sharpe-Ratio

$$\text{argmax}_{W}\left\{\frac{W'\mu_{BL}}{\sqrt{\lambda W'\Sigma W}}\right\}$$

3- Target Return

$$\text{argmax}_{W}\{W'\mu_{BL} = Target\}$$

4- Minimize the VaR

$$\text{argmin}_{W}\{W'\mu_{BL} - \sqrt{\lambda W'\Sigma W} * Factor\}$$

Where the Factor is $\phi^{-1}(1-c)$ is the Student's ϕ^{-1} with $(1-c)$ confidence interval.

Black-Litterman Mathematics

Choosing the prior distribution is one of the crucial point in Black-Litterman model. As a starting point we calculate the equilibrium returns instead of historical returns which are implied from reverse optimization. The function which we use for reverse optimization is below.

$$\text{argmax}_{W}\{W'\pi - \lambda W'\Sigma W\}$$

Maximizing the function and taking the first derivative w.r.t weight(W) gives the below equation.

$$\pi = 2\lambda\Sigma\widetilde{w}$$

We assume that the returns of assets are normally distributed with π and covariance matrix $\tau\Sigma$.Where τ is a measure of investor´s confidence in prior estimates.

$$\mu \sim \mathrm{N}(\pi, \tau\Sigma)$$

The implied weights for this optimization can be calculated by below equation.

$$w^{*} = \frac{1}{2\lambda}\Sigma^{-1}\pi$$

The investor's views are implemented into model to adjust the equilibrium returns for investor's views on future returns. Views can be relative or absolute.

$$V|\mu \sim \mathrm{N}(\mathrm{P}\mu, \boldsymbol{\Omega})$$

$$\boldsymbol{\Omega} = \mathrm{diag}(\mathrm{P}\tau\Sigma\mathrm{P}')$$

By combining the prior and investor's views constructed posterior distribution by applying Bayesian theorem. The probability distribution of the expected excess return can be written as a product of two multivariate normal distribution.

$$\mu|\mathrm{v}; \boldsymbol{\Omega} \sim \mathrm{N}\left(\mu_{BL}, \boldsymbol{\Sigma}_{\boldsymbol{BL}}^{\mu}\right)$$

$$\mu_{BL} = \pi + \tau\Sigma\mathrm{P}'(\tau\mathrm{P}\Sigma\mathrm{P}' + \boldsymbol{\Omega})^{-1}(\mathrm{v} - \mathrm{P}\pi)$$

$$\Sigma_{BL} = (1 + \tau)\Sigma - \tau^{2}\Sigma\mathrm{P}'(\tau\mathrm{P}\Sigma\mathrm{P}' + \boldsymbol{\Omega})^{-1}\mathrm{P}\Sigma$$

Now by using the posterior distribution develop different risk budgeting strategies like Max sharp ratio, Mean variance optimization, etc.,

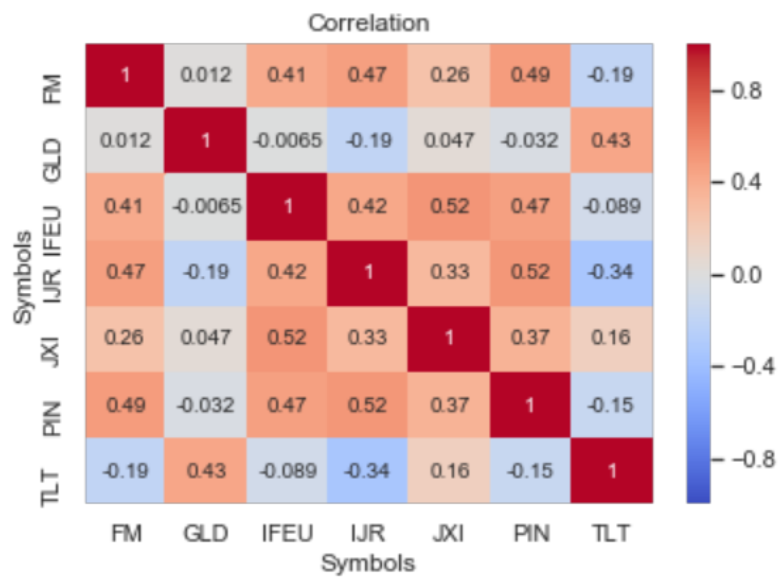
Results

Assets were chosen from 7 different type of ETFs. Used 3 years of daily data from Jan 2016 to Dec 2018. Below table describes ETFs and their types.

Ticker	**ETF Name**	**Type**
FM	iShares MSCI Frontier 100 Index Fund	Frontier Market Equity

GLD	SPDR Gold Trust ETF	Commodity
IFEU	iShares FTSE EPRS/NAREIT Europe Index ETF	Europe Real Estate ETF
IJR	iShares Core S&P Small-Cap ETF	U.S. Equities Small cap ETF
JXI	iShares Global Utilities ETF	Utilities ETF
PIN	Invesco India Portfolio ETF	Emerging Markets India
TLT	iShares 20+ Year Treasury Bond ETF	Bonds

To make sure that assets are less correlated and with optimal diversification performed correlation among the daily returns of the assets. From below correlation plot it shows that assets are very less correlated.



The sample covariance matrix is estimated with a lot of error, as alternative, practical approach to obtain the structured covariance estimator is shrinkage method by Ledoit-Wolf. We utilized LedoitWolf Estimator which is available in python library sklearn.

	FM	GLD	IFEU	IJR	JXI	PIN	TLT
FM	0.017445	0.000182	0.008996	0.009563	0.003825	0.011190	-0.002576
GLD	0.000182	0.015330	-0.000134	-0.003570	0.000645	-0.000689	0.005456
IFEU	0.008996	-0.000134	0.029914	0.011364	0.010140	0.014154	-0.001603
IJR	0.009563	-0.003570	0.011364	0.025651	0.005903	0.014444	-0.005641
JXI	0.003825	0.000645	0.010140	0.005903	0.014084	0.007488	0.001960
PIN	0.011190	-0.000689	0.014154	0.014444	0.007488	0.032523	-0.002833
TLT	-0.002576	0.005456	-0.001603	-0.005641	0.001960	-0.002833	0.011939

Structured covariance matrix using Ledoit-Wolf shrinkage method.

Symbols	FM	GLD	IFEU	IJR	JXI	PIN	TLT
Symbols							
FM	0.000129	-0.000008	-0.000398	-0.000423	-0.000169	-0.000496	0.000114
GLD	-0.000008	0.000222	0.000006	0.000158	-0.000029	0.000031	-0.000242
IFEU	-0.000398	0.000006	-0.000423	-0.000503	-0.000449	-0.000627	0.000071
IJR	-0.000423	0.000158	-0.000503	-0.000235	-0.000261	-0.000640	0.000250
JXI	-0.000169	-0.000029	-0.000449	-0.000261	0.000278	-0.000332	-0.000087
PIN	-0.000496	0.000031	-0.000627	-0.000640	-0.000332	-0.000539	0.000125
TLT	0.000114	-0.000242	0.000071	0.000250	-0.000087	0.000125	0.000373

Error/difference between structured covariance and historical covariance.

While developing black litterman model we choose risk aversion λ=1.12, (scalar) τ=0.4. For relative view the IFEU (European REIT ETF) will outperform GLD (SPDR Gold Trust ETF). The relative performance was taken based on one standard deviation between these two assets. The absolute view PIN (Invesco India Portfolio ETF) will perform 18.18%, the value is taken based on standard deviation of asset.

$$P=\begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$V=\begin{pmatrix} 5.12\% \\ 18.18\% \end{pmatrix}$$

Risk Budgeting:

While developing optimization we used SLSQP (Sequential Least Squares Programming) which is available in one of the python library scipy.optimize. It is an algorithm for nonlinearly constrained, gradient-based optimization, supports both equality and inequality constraints.

Mean Variance Optimization:

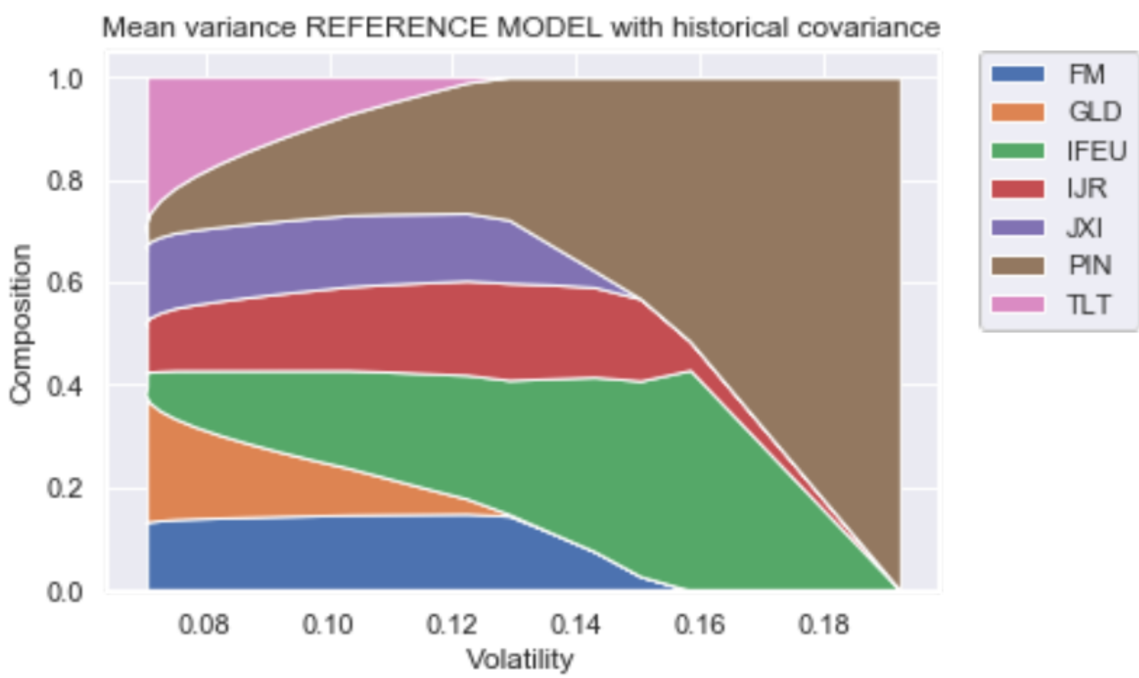


Figure 1

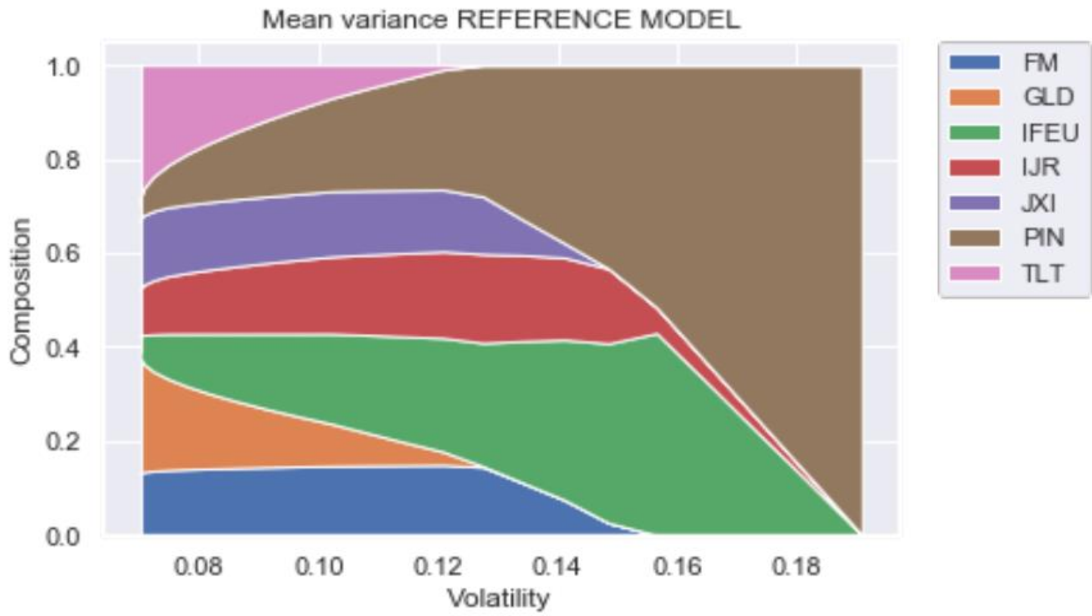


Figure 2

There is no much difference between Figure 1 and Figure 2 for reference model (without blacklitterman) as the error between historical covariance and structured covariance.

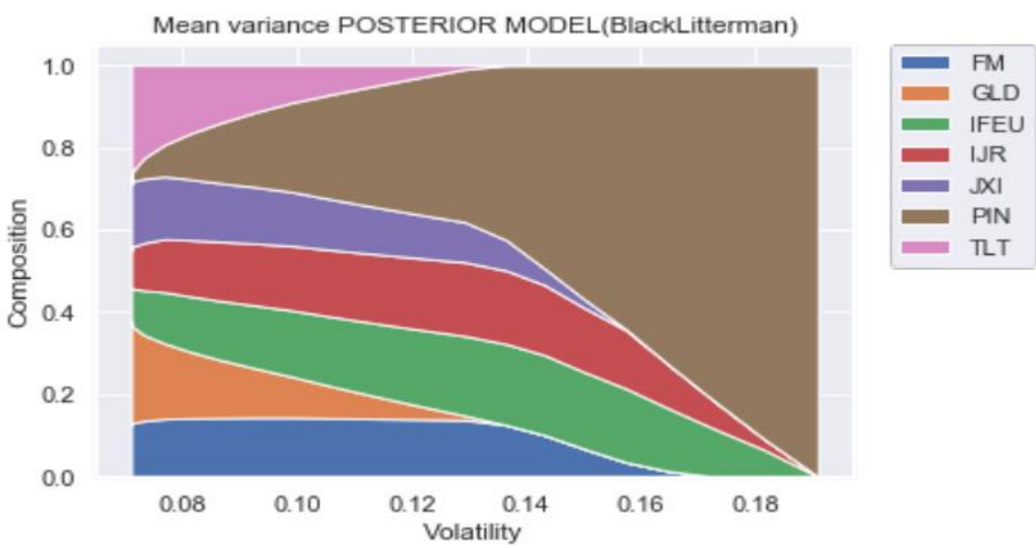


Figure 3

The Figure 2 is without Black-Litterman views and Figure 3 is with Black-Litterman views with mean variance optimization portfolio composition verse volatility. The weights were changing more for

IFEU and IJR as the volatility increase. The allocation for PIN(India) is high in both as volatility increases.

Below table shows the change in weights and returns for Posterior and prior for mean variance optimization.

	E[R]	PI	New Weights MV	Initial weights	Diff E[R]-PI	New Weights MV -Initial weights
FM	0.042465	0.0155596	1.660425e-01	0.142857	0.0269055	2.318540e-02
GLD	0.00427197	0.0055108	1.729828e-01	0.142857	-0.00123884	3.012569e-02
IFEU	0.0568097	0.0233059	3.461055e-03	0.142857	0.0335038	-1.393961e-01
IJR	0.053099	0.0184684	1.769040e-01	0.142857	0.0346306	3.404688e-02
JXI	0.0319984	0.0140941	1.122262e-01	0.142857	0.0179042	-3.063093e-02
PIN	0.102911	0.0244087	4.743385e-20	0.142857	0.0785026	-1.428571e-01
TLT	-0.00453219	0.00214466	3.683833e-01	0.142857	-0.00667685	2.255262e-01
Total			1.000000e+00	1.000000		5.551115e-17

Max Sharp Ratio Optimization:

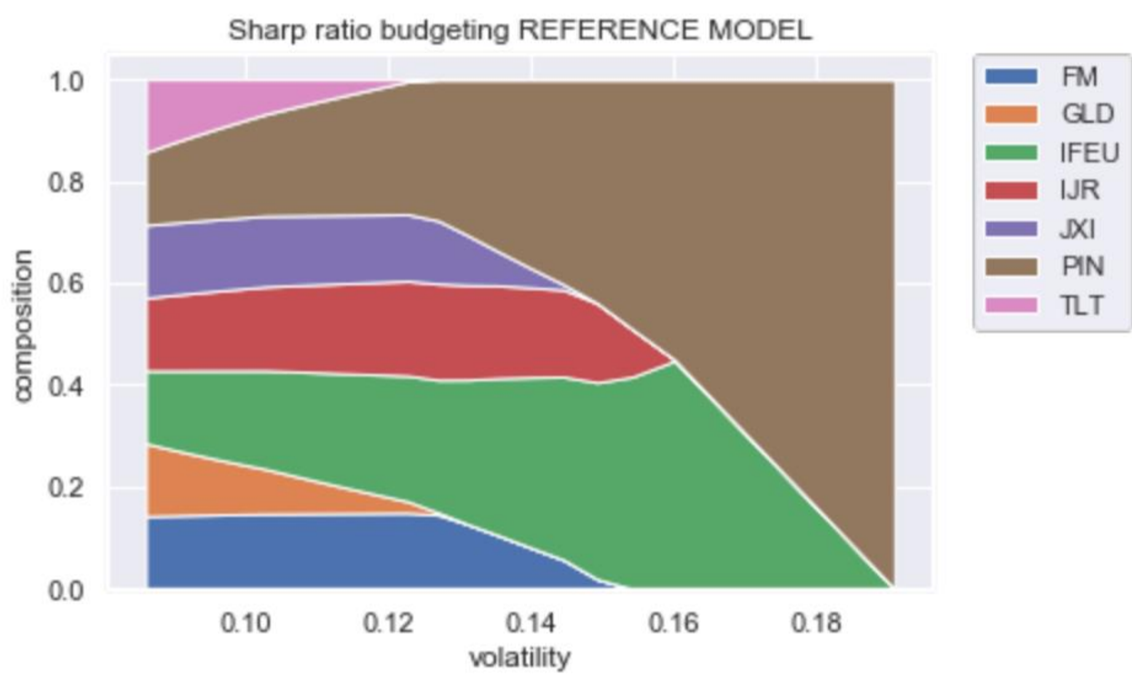


Figure 4

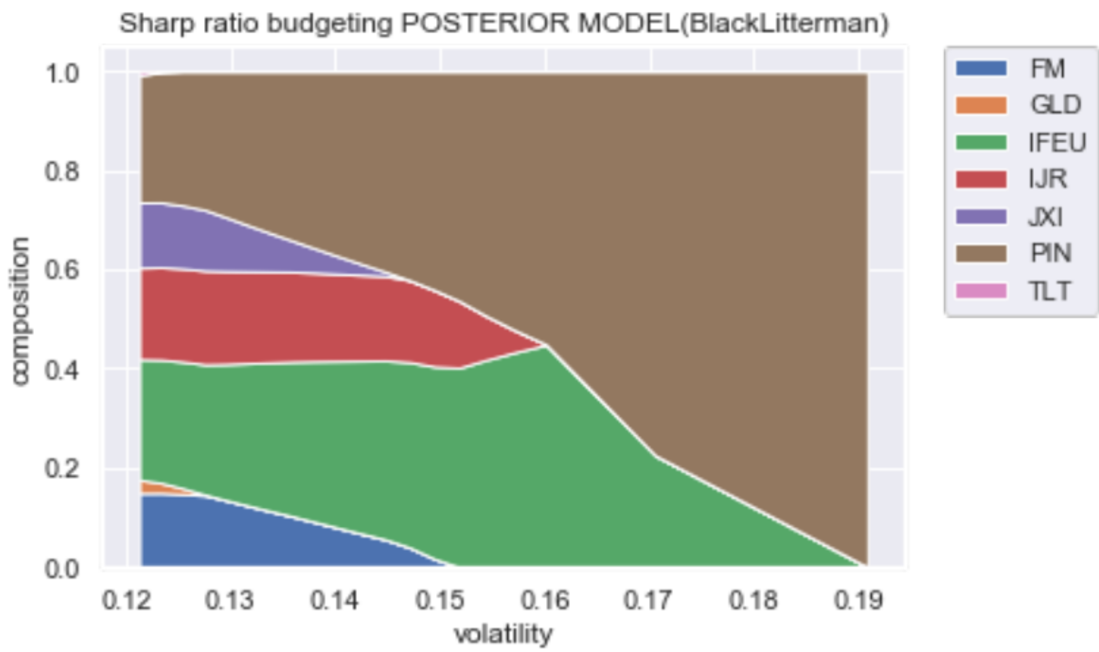


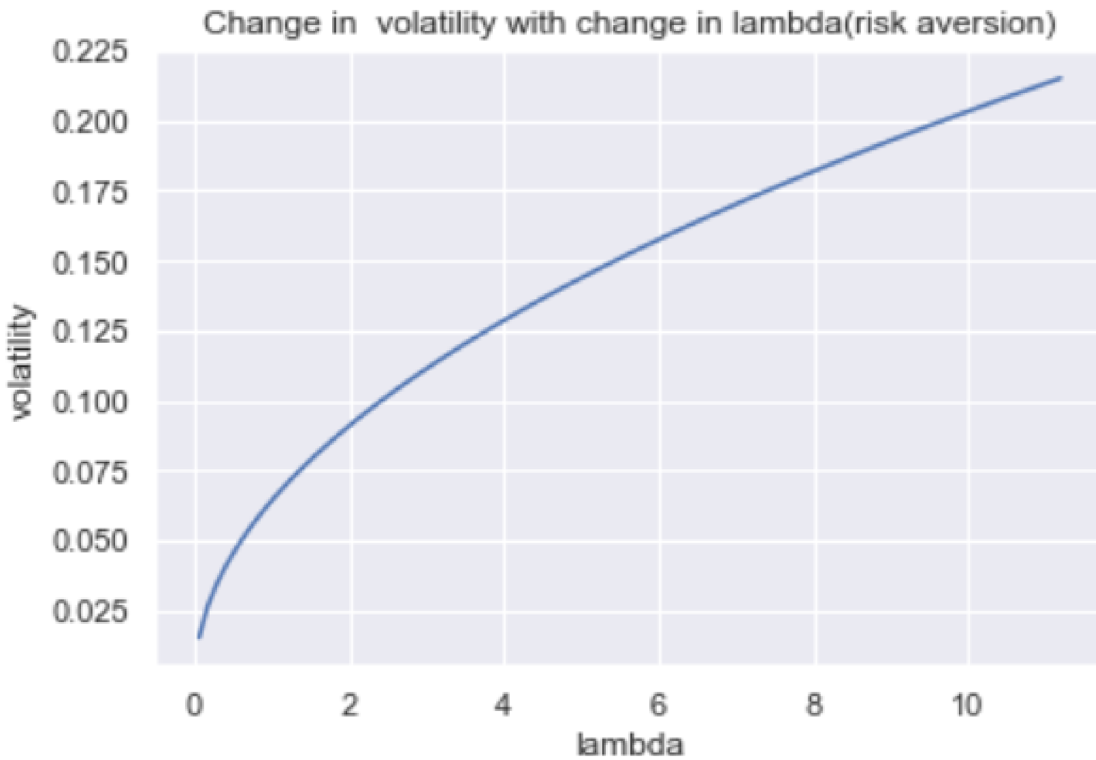
Figure 5

The Figure 4 is without Black-Litterman views and Figure 5 is with Black-Litterman views with Max sharp ratio optimization portfolio composition verse volatility. The weights for GLD and TLD are very less in Posterior model and the weights for IFEU and PIN are dominating.

Below table shows the change in weights and returns for Posterior and prior for Sharp Ratio optimization.

	E[R]	PI	New Weights SR	Initial weights	Diff E[R]-PI	New Weights SR -Initial weights
FM	0.042465	0.0155596	0.068576	0.142857	0.0269055	-7.428156e-02
GLD	0.00427197	0.0055108	0.074577	0.142857	-0.00123884	-6.827967e-02
IFEU	0.0568097	0.0233059	0.062571	0.142857	0.0335038	-8.028610e-02
IJR	0.053099	0.0184684	0.068572	0.142857	0.0346306	-7.428514e-02
JXI	0.0319984	0.0140941	0.068572	0.142857	0.0179042	-7.428545e-02
PIN	0.102911	0.0244087	0.588559	0.142857	0.0785026	4.457015e-01
TLT	-0.00453219	0.00214466	0.068574	0.142857	-0.00667685	-7.428360e-02
Total			1.000000	1.000000		-2.775558e-17

Change in volatility with change in Lambda: The volatility increases as the lambda value increase.



References:

1) Honey, I Shrunk the Sample Covariance Matrix http://www.ledoit.net/honey.pdf
2) T. M. Idzorek, “A step-by-step guide to the black-litterman model,” 2004
3) Beyond Black-Litterman in Practice: a Five-Step Recipe to Input Views on non-Normal Markets
4) http://www.blacklitterman.org/