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Assignment 1

(1) $K_1(x, y) = (\langle x, y\rangle + c)^m$

where $c \geq 0$, m is a +ve integer, $x, y \in \mathbb{R}^d$

Expanding RHS :

$$K_1(x,y) = {}^nC_0 (\langle x,y\rangle)^m c^0 + {}^nC_1 (\langle x,y\rangle)^{m-1} c^1 + \cdots + {}^nC_n \langle x,y\rangle^0 c^m \quad \text{--- (1)}$$

Now,

we know that $\langle x, y\rangle$ is a linear kernel.

We can show that for any $n \geq 0$, $(\langle x, y\rangle)^n$ is also a valid kernel.

$x = [x_1, x_2, \ldots, x_d]$

$y = [y_1, y_2, \ldots, y_d]$

$$\langle x, y\rangle^n = (x_1 y_1 + x_2 y_2 + \cdots + x_d y_d)^n$$

$$= \langle \phi_1(x), \phi_2(y)\rangle$$

where $\phi_1(x) = \underbrace{\begin{bmatrix} x_1 . x_1 . x_1 \cdots x_1 \\ x_1 . x_1 . x_1 \cdots x_2 \\ \vdots \\ x_d . x_d . \cdots x_d \end{bmatrix}}_{n\text{-times}}$

$\therefore$ $\phi_1(x)$ is a (d^n)- dimn. vector.

$\therefore$ $\langle x, y\rangle^n$ is a valid- kernel.

$\therefore$ All terms in binomial expansion of $K_1(x,y)$ in eq. (1) are valid kernels. (Because scalar times a kernel is a kernel).

$\therefore$ $K_1(x,y)$ is a valid kernel (Because sum of kernels is kernel).

(2) $h(x) = \begin{cases} +1 & \text{if } \|\phi(x) - c_-\|^2 > \|\phi(x) - c_+\|^2 \\ -1 & \text{otherwise} \end{cases}$

$$h(x) = sgn\left(\|\phi(x) - c_-\|^2 - \|\phi(x) - c_+\|^2\right)$$

$$= sgn\left(\cancel{\|\phi(x)\|^2} + \|c_-\|^2 - 2\langle\phi(x), c_-\rangle - \cancel{\|\phi(x)\|^2} - \|c_+\|^2 + 2\langle\phi(x), c_+\rangle\right)$$

$$= sgn\left(\langle\phi(x), c_+\rangle - \langle\phi(x), c_-\rangle + \frac{\|c_-\|^2 - \|c_+\|^2}{2}\right)$$

Substituting value of centroid:

$$= sgn\left(\left\langle\phi(x), \frac{\sum_{x_i \in I_+} \phi(x_i)}{n_+}\right\rangle - \left\langle\phi(x), \frac{\sum_{x_i \in I_-} \phi(x_i)}{n_-}\right\rangle + \frac{\|c_-\|^2 - \|c_+\|^2}{2}\right)$$

Using distributive property of dot product:

$$= sgn\left(\sum_{x_i \in I_+} \left\langle\phi(x), \frac{\phi(x_i)}{n_+}\right\rangle - \sum_{x_i \in I_-} \left\langle\phi(x), \frac{\phi(x_i)}{n_-}\right\rangle + \frac{\|c_-\|^2 - \|c_+\|^2}{2}\right)$$

Comparing with given form of $h(x) = sgn\left(\sum_{i=1}^{n} \alpha_i \langle\phi(x), \phi(x_i)\rangle + b\right)$

$$\alpha_i = \begin{cases} \dfrac{1}{n_+} & \text{if } y_i = +1 \\ \dfrac{-1}{n_-} & \text{if } y_i = -1 \end{cases}$$

$$b = \frac{\|c_-\|^2 - \|c_+\|^2}{2}$$

$$= \frac{1}{2}\left\| \frac{\sum_{x_i \in I_-} \phi(x_i)}{n_-} \right\|^2 - \frac{1}{2}\left\| \frac{\sum_{x_i \in I_+} \phi(x_i)}{n_+} \right\|^2$$

$$= \frac{1}{2n_-^2} \left\| \sum_{x_i \in I_-} \phi(x_i) \right\|^2 - \frac{1}{2n_+^2} \left\| \sum_{x_i \in I_+} \phi(x_i) \right\|^2$$

$$= \frac{1}{2n_-^2} \sum_{i,j \in I^-} \langle \phi(x_i), \phi(x_j) \rangle - \frac{1}{2n_+^2} \sum_{i,j \in I^+} \langle \phi(x_i), \phi(x_j) \rangle$$

$$= \frac{1}{2n_-^2} \sum_{i,j \in I^-} K(x_i, x_j) - \frac{1}{2n_+^2} \sum_{i,j \in I^+} K(x_i, x_j)$$

We got the required values.
showed that $h(x)$ can be written as
$h(x) = sgn\left(\sum_{i=1}^{n} \alpha_i K(x, x_i) + b \right)$

(3) $K_3(x,y) = \cos(x+y)$

Property of kernel function:

All kernel functions are positive definate functions

i.e. $\forall n \geq 1$ samples in feature space, $\forall (a_1, \cdots, a_n) \in \mathbb{R}^n$
$(x_1, \cdots, x_n) \in \mathbb{R}^n$ (because $x \in \mathbb{R}$ given)

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j K(x_i, x_j) \geq 0$$

We want to see $\underline{\text{if}}$ $K_3(x,y)$ satisfies this property.

Let's choose $n=1$ point, in $\mathbb{R}$, say point 't'.

Then

$$\sum_{i=1}^{1} \sum_{j=1}^{1} a_i a_j K(x_i, x_j)$$

$$= a_1^2 K(t, t)$$

$$= a_1^2 \cos(t+t)$$

$$= a_1^2 \cos(2t)$$

This should be ≥ 0, acc. to above property.

But for, say, $t = \frac{3\pi}{4}$.

$$a_1^2 \left(\cos\left(\frac{3\pi}{4} \times 2\right)\right) = a_1^2 \cos\left(\frac{3\pi}{2}\right) = -a_1^2$$

which is always $-ve$.

$\therefore$ $K_3(x,y)$ doesn't satisfy above property.

$\therefore$ $K_3(x,y)$ is $\underline{\text{not}}$ a valid kernel function.

④ for $x, y \in (-1, 1)$

$$k_4(x,y) = \frac{1}{1-xy}$$

Expanding RHS using Taylor's expansion :

$$k_4(x,y) = 1 + xy + (xy)^2 + (xy)^3 + \cdots \qquad \left(\begin{array}{l} \because x \in (-1,1) \\ \quad y \in (-1,1) \\ \therefore xy \in (-1,1) \end{array} \right)$$

$$= \left\langle \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix}, \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \\ \vdots \end{bmatrix} \right\rangle$$

$$= \langle \phi(x), \phi(y) \rangle$$

i.e. $k_4(x,y)$ can be written as $\langle \phi(x), \phi(y) \rangle$ where $\phi(x)$ is an infinite dimentional vector.

$\therefore$ $k_4(x,y)$ is a valid kernel.