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Assignment 5

H.5.1

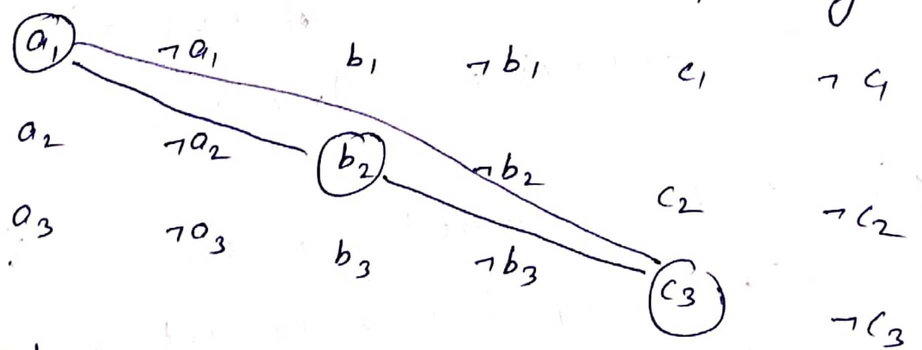
Aim : To show MAX CUT is NP-complete, for simple graphs.

Sol : We will show a reduction from NP-complete NAESAT to MAXCUT, such that $x \in NAESAT$ iff $R(x) \in MAXCUT$ with a given K.

steps for $G = R(x)$:

1. For any clause $(a \vee b \vee c)$ in x, create 3 copies of each variable and ~~its~~ their negations.

   Then make a triangle in following way:



That is, use 1st copy for 1st variable in clause, use 2nd copy for 2nd variable and so on.

- This will give us $3m$ edges in G $(R(x))$.

(2) After completing step 1 for all clauses in φ;
Create an edge from a variable to all its negations from all clauses, if it doesn't already exist.
This will give us $(3m)^2 n$ edges in G.
max

Thus we get our $R(\varphi) = G = (V, E)$ where $|V| = 6mn$ and $|E| = (3m)^2 n + 3m$

Claim: $\varphi \in NAESAT \iff R(\varphi) = G \in MAXCUT$ with size $(3m^2)n + 2m$.

Proof
($\Rightarrow$) Means $\exists$ a truth assignment for φ.
Using that, we can easily construct a cut of size $(3m^2)n + 2m$.
i.e. If a variable x_1 is true, then keep it and all its copy in S and all copies of its negation in $V-S$.
And each clause (triangles) gives a cut of size 2,
so size(cut) $= (3m^2)n + 2m$.

($\Leftarrow$) G has a maxcut with size $(3m)^2 n + 2m$.
(1) If a copy of a variable and a copy of its negation are in one set (S or $V-S$), then sending one with less neighbours on other side doesn't decrease the size of the cut.

This way, all positive copies of a variable will be on one side and negative copies on other side.

(2) This gives a size of $(3m)^2 n$.
Rest $2m$ size can come only by cutting every triangle.

3. Assign all vertices in S (truth or false) and ~~Therefore~~ ~~we~~ ~~get~~ all vertices in V−S the opposite.

4. Since every triangle is cut, so truth assignment satisfies x according to NAESAT.

∴ MAXCUT is NP-hard.

MAXCUT is in NP (for given K):

A NTM can guess a cut and check if its size $\geq K$.
Checking can be done in $O(|E|)$ time and space for guess is $O(|V| \log |V|)$.

∴ MAXCUT is in NP.

∴ MAXCUT is NP-complete.

H5.2

To show: Clique - Cover is NP-complete.

Sol :

① Clique-cover is in NP.

A NTM can guess a partition of size atmost k and check if each partition makes a CLIQUE.

Guessing space (certificate size) = $O(|V|)$

Checking time = $O(K \cdot |V|^2)$

② k-colouring ≤ CLIQUE-COVER

Given a graph $G = (V, E)$, we construct $R(G) = G' = (V', E')$ s.t. $G \in$ k-colouring $\iff R(G) \in$ Clique-co.

For a given $G = (V, E)$, create G' s.t.

① $V' = V$

② $E' = (V \times V) - E$

That is $G' = \overline{G}$.

Claim: $G \in$ k-colouring $\iff G' \in$ CLIQUE-COVER for a given k.

Proof:

(⇒) G is k-colouring means there are k-independent sets (disjoint) (each colour corresponds to 1 indepen. set).

A For any indepen. set S, if $x, y \in S$ then $(x, y) \notin E$.

B Therfore in G', if $(x, y) \in S$, $(x, y) \in E'$.

∴ Each independent set corresponds to one CLIQUE in G'.

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($\Leftarrow$) G' can be partitioned in K disjoint sets s.t. each set forms a CLIQUE.

Let S is a CLIQUE set in G'.

Then we can assign same colour to each vertex of S in G, because if $x, y \in S$ then $(x, y) \in E'$
$\Rightarrow (x, y) \notin E$.

$\therefore$ Each CLIQUE set corresponds to one color in G.

Since, K-colouring is NP-comp. and K-colouring $\leq$ CLIQUE COVER
$\therefore$ CLIQUE-COVER is NP-complete.

HS.3 MENTION ALL

To show : MENTION ALL is NP- complete.

Sol :

① It is in NP.

A NTM can guess an upside/down assignment for all pages and check if all names appear atleast once, on the top.

Certificate size = $O(m)$ (m = # pages)

Checking time = $O(mn)$ (n = # names)

② It is NP-hard.

We show that $SAT \leq$ MENTION ALL.

Let π is given in CNF form.

i.e. $\pi = C_1 \wedge C_2 \wedge C_3 \ldots \wedge C_m$

Construct $R(\pi)$ in following way:

① For each variable, assign one page.

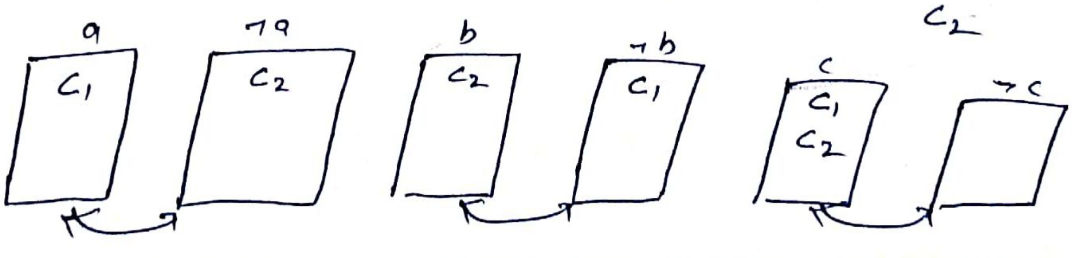
Up side $\equiv$ positive assignment

Down side $\equiv$ negative "

② For any clause C_i in π, if $C_i = (a \vee \neg b \vee \neg c)$ for example, write C_i on page corresponding to those variable.

For example, if $\phi = \underbrace{(a \vee \neg b \vee c)}_{C_1} \wedge \underbrace{(\neg a \vee b \vee c)}_{C_2}$

Pages :



claim: $x \in SAT \iff R(x) \in$ MENTION ALL

($\Leftarrow$) It is easy to see that either a or $\neg a$ will be facing up.
And clauses written on up-side are true for that assignment of that variable.
since, all clauses are appearing on top, means all clauses are true.
$\therefore x \in SAT$.

($\Rightarrow$) $x \in SAT$ means a truth assignment exists for all variables.
True means up-side of page.
False " down " " ".
since, every clause is true for that assignment, therefore every clause will be facing up on some page.
$\therefore R(x) \in$ MENTION ALL.

$\therefore$ MENTION ALL is NP-hard.

$\therefore$ MENTION-ALL is NP-complete.