Instructions

- Classroom Problems C9.1–C9.2 will be discussed and solved at the tutorial session on Wed 18 Mar, 14–16, Zoom. No credit is given for these problems.
- Homework Problems H9.1–H9.3 you should solve on your own, and submit your solutions via the MyCourses interface by the deadline of Tue 24 Mar, 23:59, These problems will be individually graded on a scale of 0–2 points per problem.
- In preparing your solutions to the Homework Problems:
  1. Justify your solutions, be precise, and provide sufficient detail so that it is easy to follow your reasoning.
  2. Submit your solutions as an easily readable, single pdf file, which is either typeset or written in full sentences and clean handwriting.
  3. **[Code of Conduct]** You can discuss the problems with your colleagues and the course's teaching staff, but you must write the presentations of your solutions *independently* and *individually*, without any notes from such discussions.

Classroom Problems

C9.1 Consider the following MAX SAT optimisation problem:

> Given a CNF formula ϕ, find an assignment to the variables of ϕ that maximises the number of satisfied clauses.

Here's a very naive algorithm for this problem:

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for each variable:
  set its value to either 0 or 1 by flipping a coin
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Suppose the formula ϕ has m clauses, of which the jth has k_j literals. Show that the *expected* number of clauses satisfied by this simple algorithm is

$$\sum_{j=1}^{m} \left(1 - \frac{1}{2^{k_j}}\right) \geq \frac{m}{2}.$$

In other words, this is a 2-approximation in expectation! And if the clauses all contain k literals, then this approximation factor improves to $1+1/(2^k - 1)$. (*Hint:* Indicator random variables, linearity of expectation.)

C9.2 In the MIN STEINER TREE problem, the input consists of: a complete graph $G = (V, E)$ with distances d_{uv} between all pairs of vertices; and a distinguished set of *terminal vertices* $V' \subseteq V$. The goal is to find a minimum-cost tree that includes the vertices V'. The tree may or may not include vertices in $V \setminus V'$.

Suppose the distances in the input satisfy the conditions of a *metric*:

(i) $d_{uv} \geq 0$ for all u, v,

(ii) $d_{uv} = 0$ if and only if $u = v$,

(iii) $d_{uv} = d_{vu}$ for all u, v,

(iv) $d_{uv} \leq d_{uw} + d_{wv}$ for all u, v, w (triangle inequality).

Show that an efficient ratio-2 approximation algorithm for Minimum Steiner Tree can be obtained by ignoring the nonterminal vertices and simply returning a minimum spanning tree on V'. (*Hint:* Recall the well-known "twice-around-the-tree" approximation method for metric TSP.)

Homework Problems

H9.1 Make the MAX SAT approximation algorithm in Problem C9.1 deterministic. (*Hint:* Express the expected number of satisfied clauses using two conditional expectations. At least one of the conditional expectations has to be at least the expected number of satisfied clauses.) (2 points)

H9.2 Show that for any integer n that is a power of 2, there is an instance of the MIN SET COVER problem with the following properties:

(i) There are n elements in the base set.

(ii) The optimal cover uses just two sets.

(iii) The greedy approximation algorithm presented at Lecture 16 may pick $\log_2 n$ sets.

Thus the logarithmic approximation ratio derived for the approximation algorithm at Lecture 16 is tight, up to a constant factor. (2 points)

H9.3 Let us consider the parameterised complexity of the 3SAT problem.

(a) Show that 3SAT is in class FPT, when parameterised by the number of variables $n = |X|$ in an input formula $\phi = \phi(X)$.

(b) Let $\phi = \phi(X)$ be a 3cnf formula over a variable set X. A set of variables $Z \subseteq X$ is a *backdoor* for ϕ, if every truth assignment to the variables in Z reduces ϕ, after simplification, to either **true** or **false**.

Show that 3SAT is in class XP when parameterised by a number k bounding the size of the smallest backdoor set $Z \subseteq X$ for a satisfiable input formula $\phi = \phi(X)$.[1] (Note that for a given value of k, we do not know *which* set of variables Z would constitute the backdoor, we just aim to determine the satisfiability of ϕ if such a set exists.) (2 points)

(c) **[Bonus problem]** A set of variables $Z \subseteq X$ in a 3cnf formula $\phi = \phi(X)$ is a *2cnf-backdoor* for ϕ, if every truth assignment to the variables in Z reduces ϕ, after simplification, to a 2cnf-formula. (Which is then of course polynomially solvable.)

Show that 3SAT is in class FPT when parameterised by a number k bounding the size of the smallest 2cnf-backdoor set $Z \subseteq X$ for an input formula $\phi = \phi(X)$. (*Hint:* Pick a clause and try to identify a 2cnf-backdoor variable in it. Use recursion.) (+1 point)

[1] I.e. there is an algorithm that (i) accepts a given input pair $\langle \phi, k \rangle$ if ϕ has a backdoor set of size k and is satisfiable, otherwise rejects; and (ii) runs in time $\mathcal{O}(|\phi|^{f(k)})$, where f is some computable function.