Population Growth Models with Differential Equations

Let us explore some ways to predict population.. Shall we?

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How do we approach?

We try to make a deterministic model of the population that is based on some basic assumptions of how the population changes.

We will discuss these assumptions as well how these models work in detail….

...But are these models accurate and reliable?

Bad news!

These models provide a conceptual idea of the system rather than a perfect description.

Good news!

More deviation in the results of the model means better insight of the system!

The deviations can be examined to take into account the factors which were earlier ignored

Let’s start with a simple growth model…..

The Exponential Growth Model

What the model says?

The **per capita growth rate of the population remains a constant value** as long as the birth rates and death rates are held constant (independent of the population density)

The Assumptions:

Birth rates and death are held constant. *i.e.* per capita birth rates and death rates are independent of population density.

…….But birth and death rates can be influenced by other factors such as availability of **food, environmental conditions etc**. which affects the overall population density.

Let's get into the maths now…

$$\boxed{\frac{1}{N}\frac{dN(t)}{dt} = \beta - \delta}$$

$\frac{1}{N}\frac{dN(t)}{dt}$ stands for per capita growth of the population

β denotes the per capita birth rate

δ denotes the per capita death rate

This is a simple linear differential equation which can be solved by separation of variables

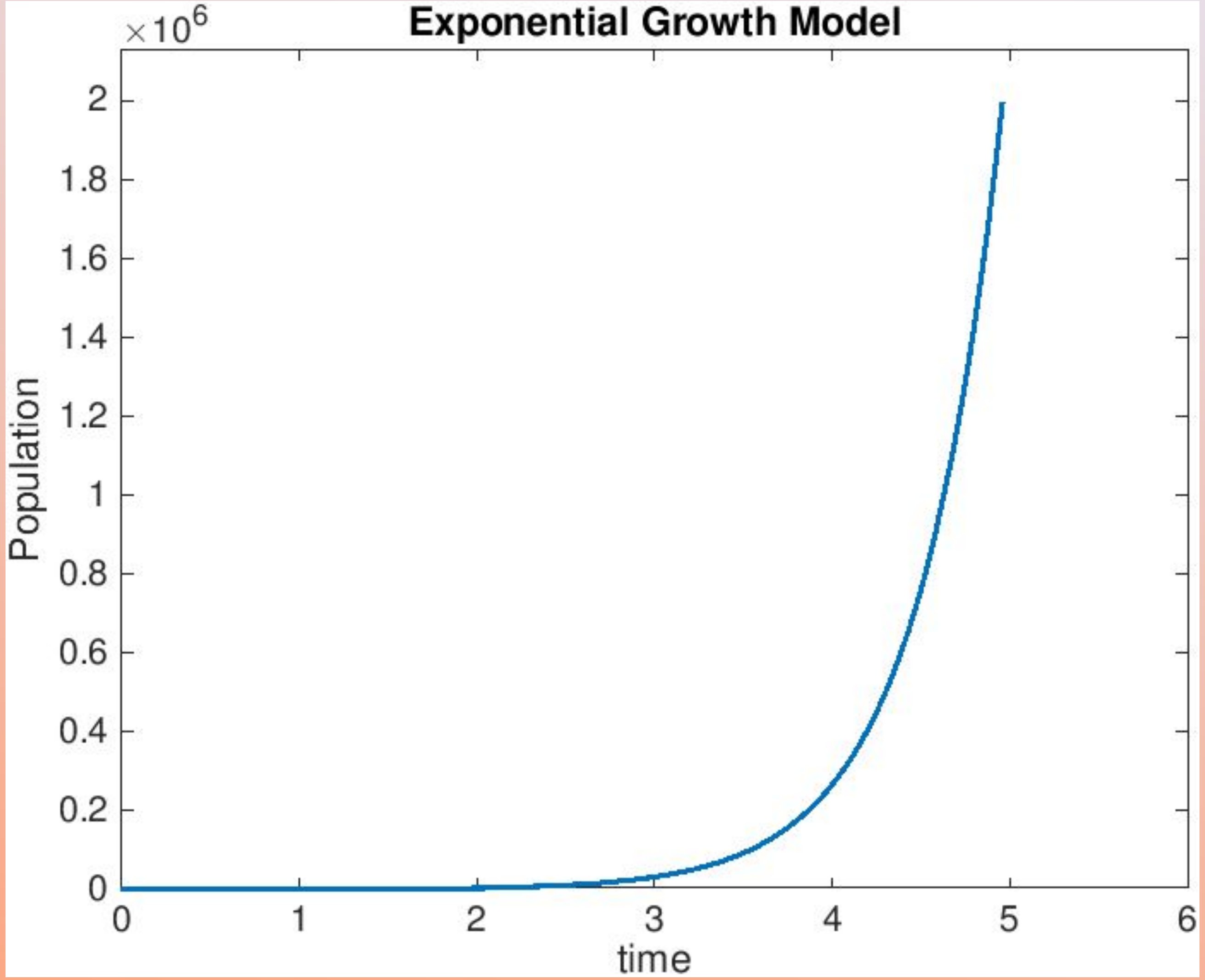
$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN, \qquad where\ \ r = \beta - \delta$$

$$\Rightarrow \qquad \int \frac{\mathrm{d}N}{N} = \int r\,\mathrm{d}t$$

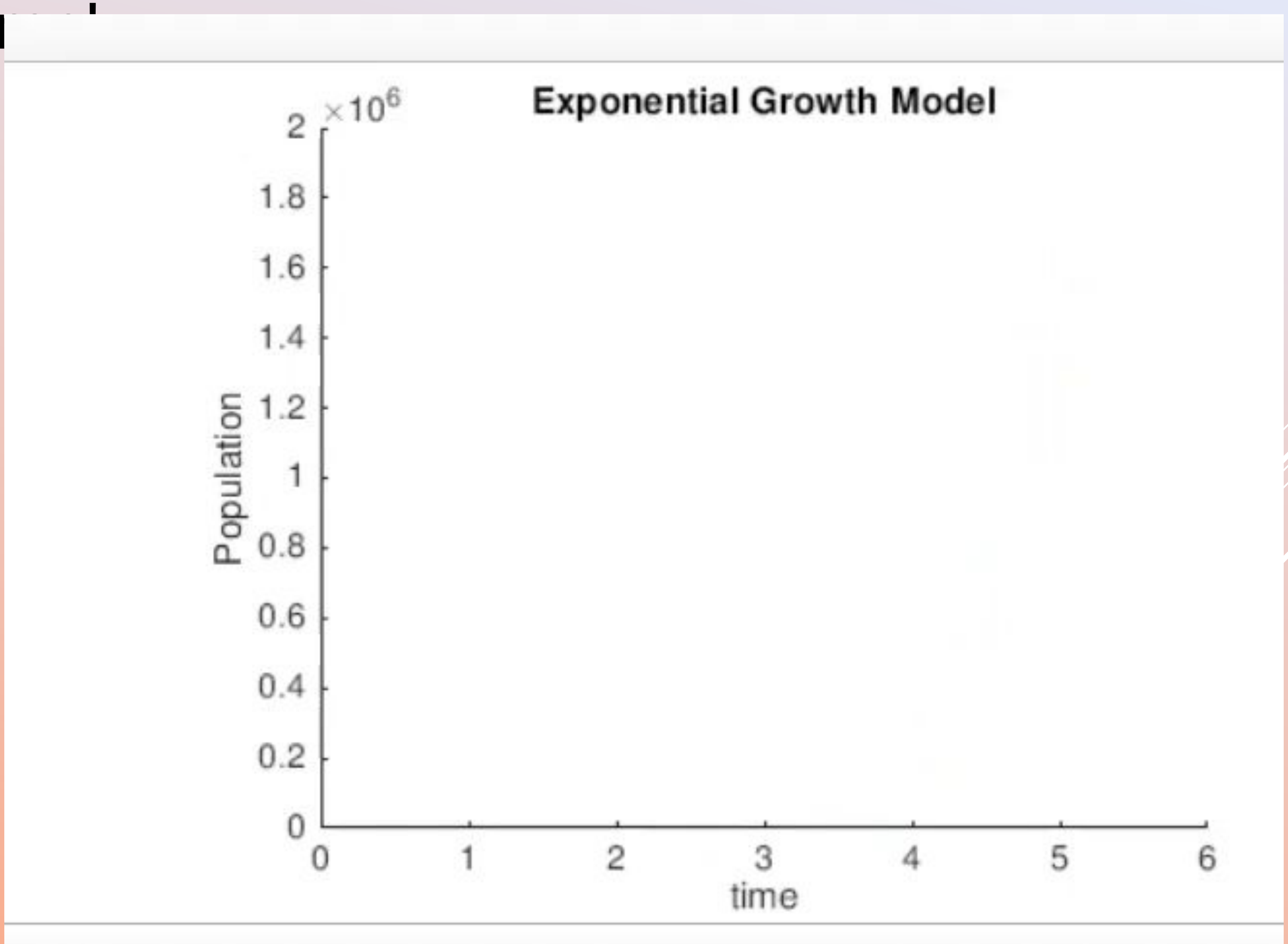
$$\Rightarrow \qquad \boxed{N(t) = N_0 \mathrm{e}^{rt}}$$

N_0 is a constant determined from the initial condition

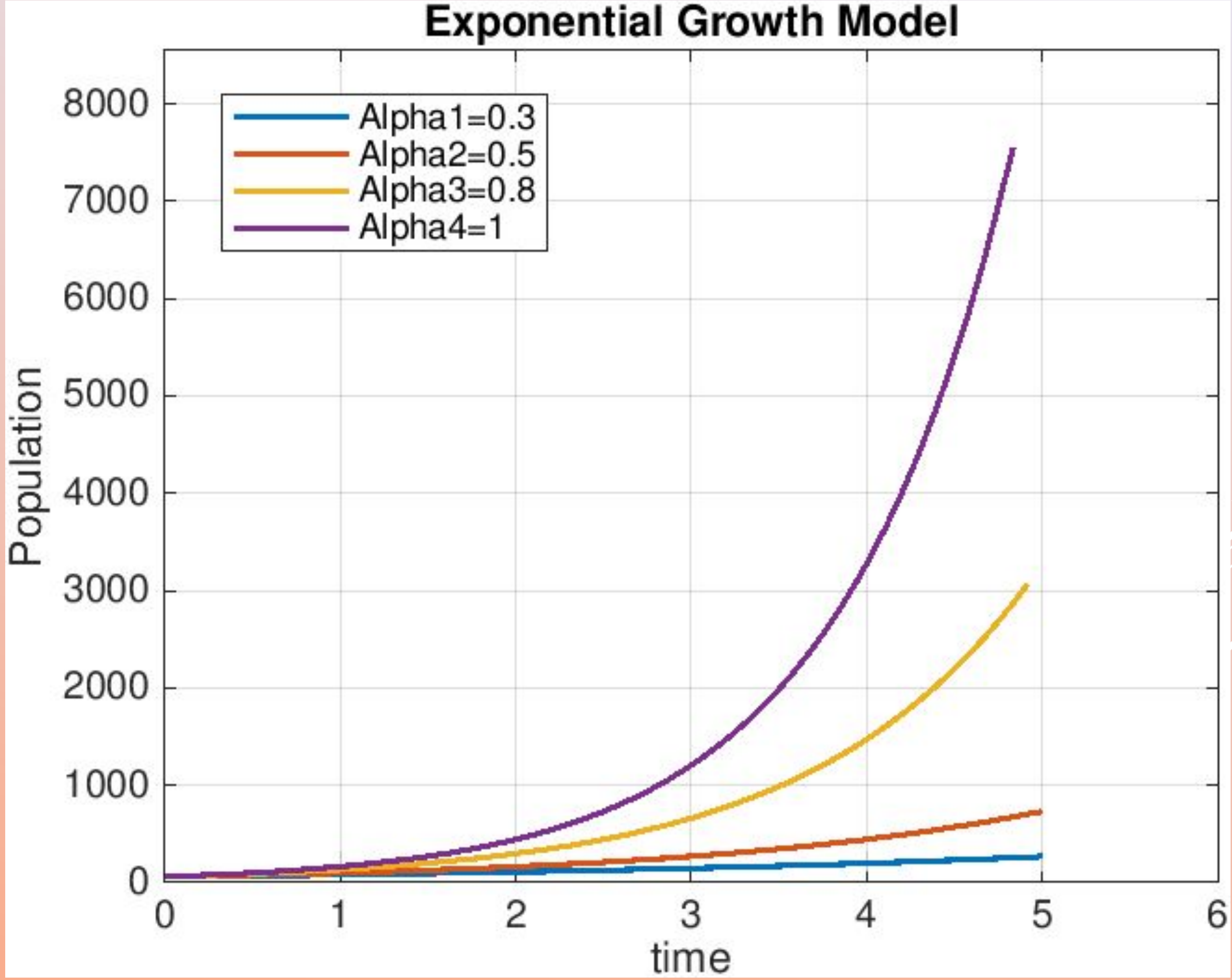
Let's put this equation on a graph!



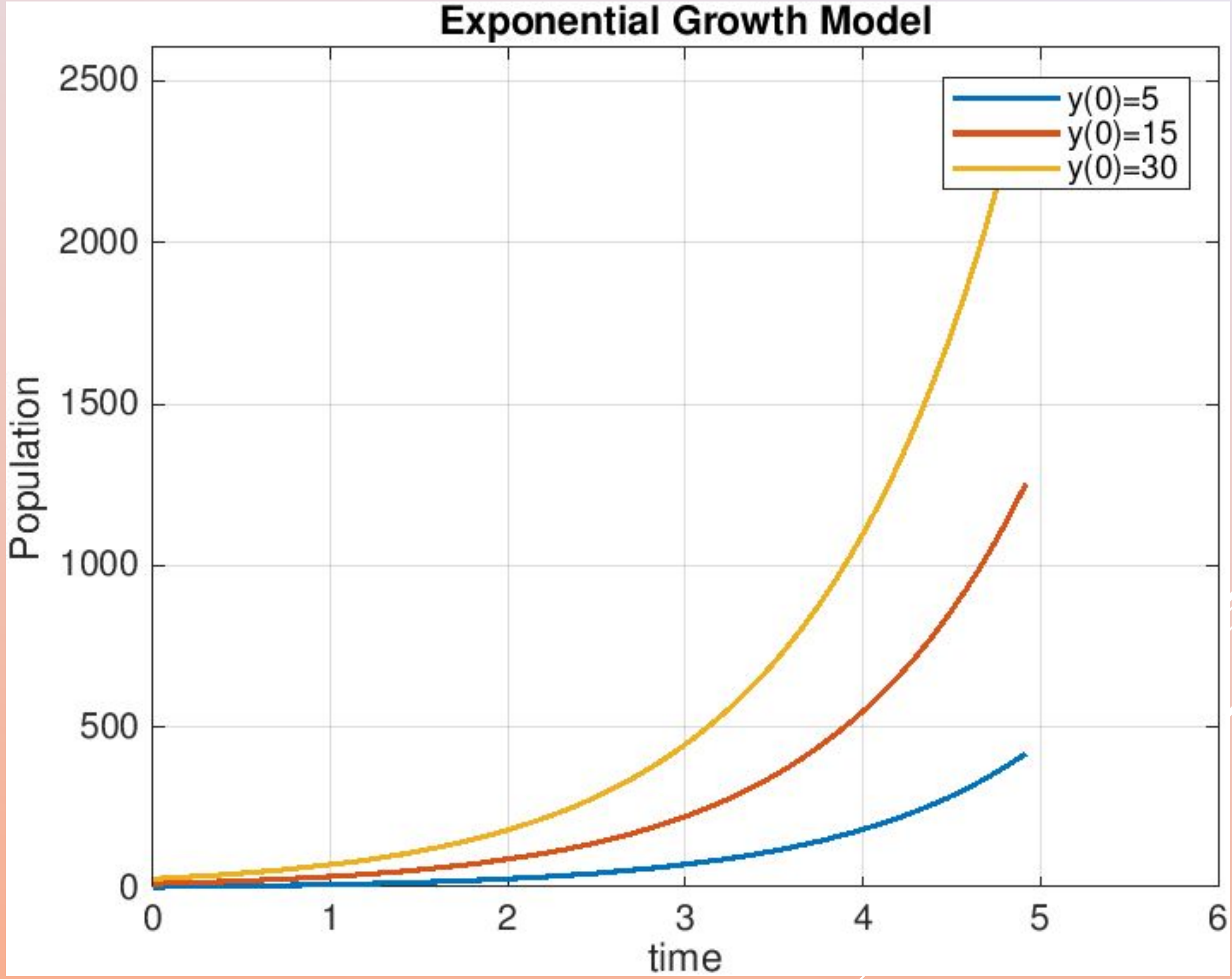
Better see it frame by fra



Lets compare it for different growth rates!



What if the Initial Population was different?



MATLAB at your service

```
clear
clc
clf
syms y(t)
alpha=2.1;
ode = diff(y,t) ==alpha*y;
cond = y(0) == 60;
ySol(t) = dsolve(ode,cond);
fplot(t,ySol(t),'linewidth',1.5)
title('Exponential Growth Model')
xlabel('time')
xlim([0 6]);
ylabel('Population')

fprintf(2,"The Solution of Differential equation is:  ")
disp(ySol(t))
```

*The Solution of Differential equation is: 60*exp((21*t)/10)*

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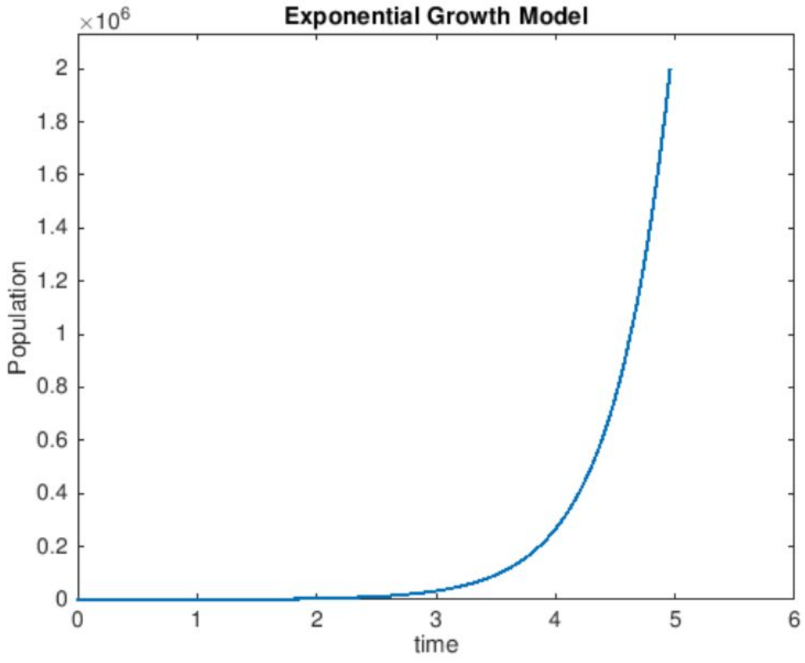

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Something strange….?

As you may have observed, the curve keeps on growing and growing!
Certainly the world isn’t filled with infinitely many of us!

But seriously, what is the issue?

Of course! The food runs out before we go beyond a certain point!
Our model did not take into account about any such limit

So we see Exponential model had some serious errors!
Let’s have a second attempt mastering some issues we faced….

The Logistic Growth Model

What modifications have we made?

We noticed that the exponential model did not take into account the scarcity of resources…..

So, a mathematician named Verhulst added an extra factor to the exponential model equation to account for scarce resources!

Here is a look to our improved equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN - \frac{r}{K}N^2$$

i.e.

$$\boxed{\frac{\mathbf{d}\boldsymbol{N}}{\mathbf{d}\boldsymbol{t}} = \boldsymbol{rN}\left(\mathbf{1} - \frac{\boldsymbol{N}}{\boldsymbol{K}}\right)}$$

The extra term $\left(\frac{r}{K}N^2\right)$ accounts for a limit on resources

K stands for the *"carrying capacity"* which is defined as the maximum amount of population that environment can sustain subject to number of resources available.

But how does this solve our problem?

Now, environment cannot sustain the population number greater than K and if the population goes beyond this,

There will be lack of availability of resources leading to decrease in rate of change of population density.

Let’s see how the solution comes out to be!

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right)$$

This differential equation for logistic growth model can be solved by the method partial fraction

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right)$$

$$\Rightarrow \quad \frac{K}{N(\mathrm{K} - N)}\mathrm{d}N = r\,\mathrm{d}t$$

$$\Rightarrow \quad \int\left(\frac{1}{N} + \frac{1}{\mathrm{K} - N}\right)\mathrm{d}N = \int r\,\mathrm{d}t$$

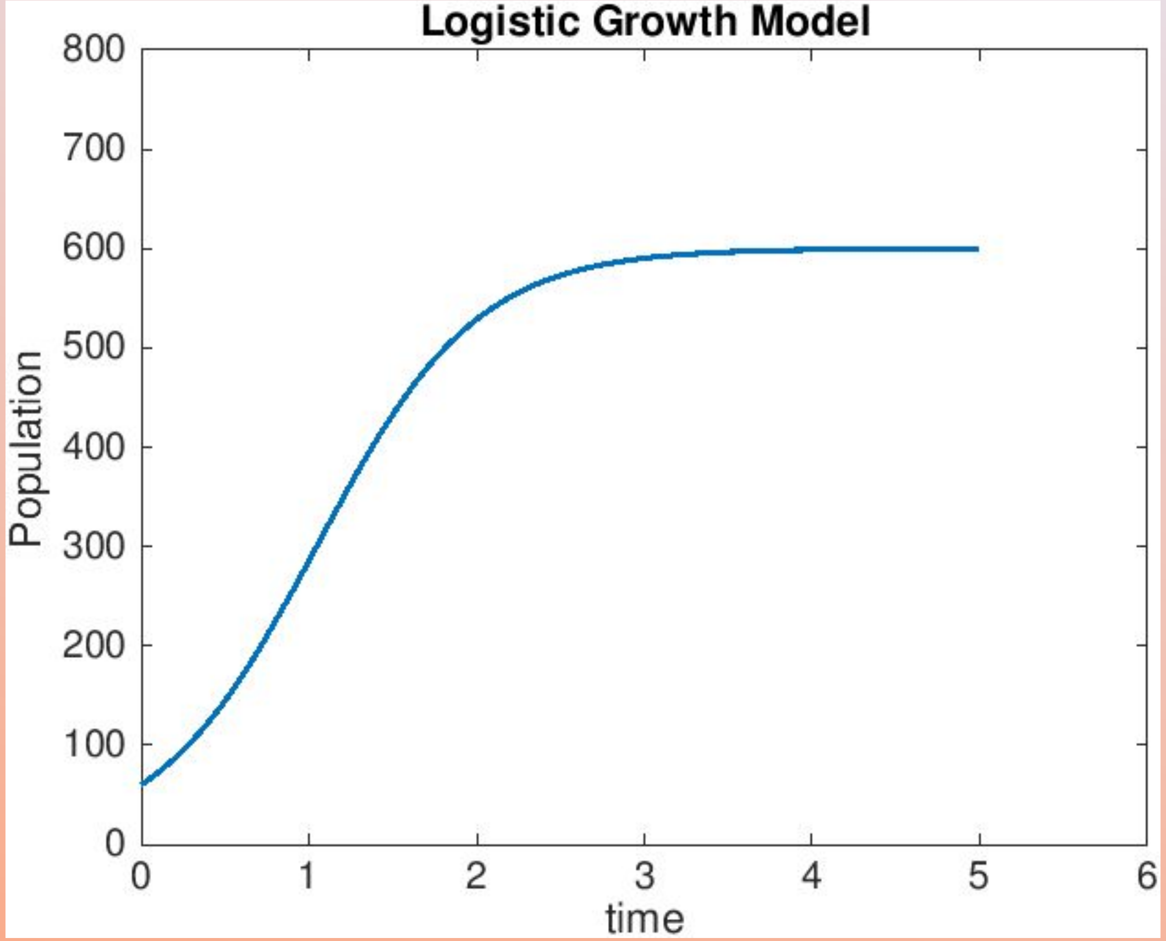
$$\Rightarrow \quad \ln(N)\Big|_{N_0}^{N} - \ln(\mathrm{K} - N)\Big|_{N_0}^{N} = rt$$

And…… finally

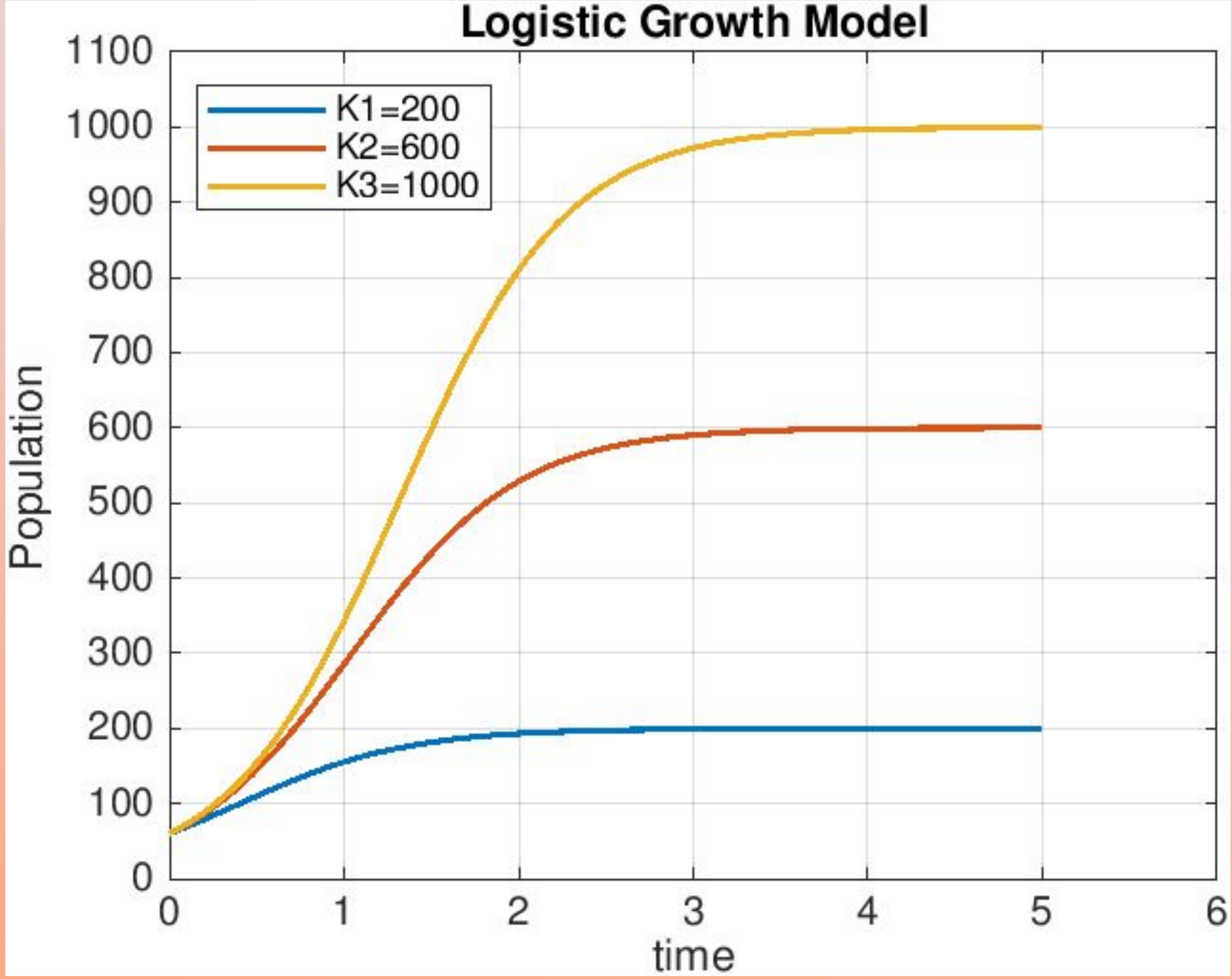
$$\boxed{\boldsymbol{N(t)} = \frac{\boldsymbol{N_o K}}{\boldsymbol{N_o} + (\mathbf{K} - \boldsymbol{N_0})\mathbf{e}^{-\boldsymbol{rt}}}}$$

Looks a bit complicated, but let’s understand it.

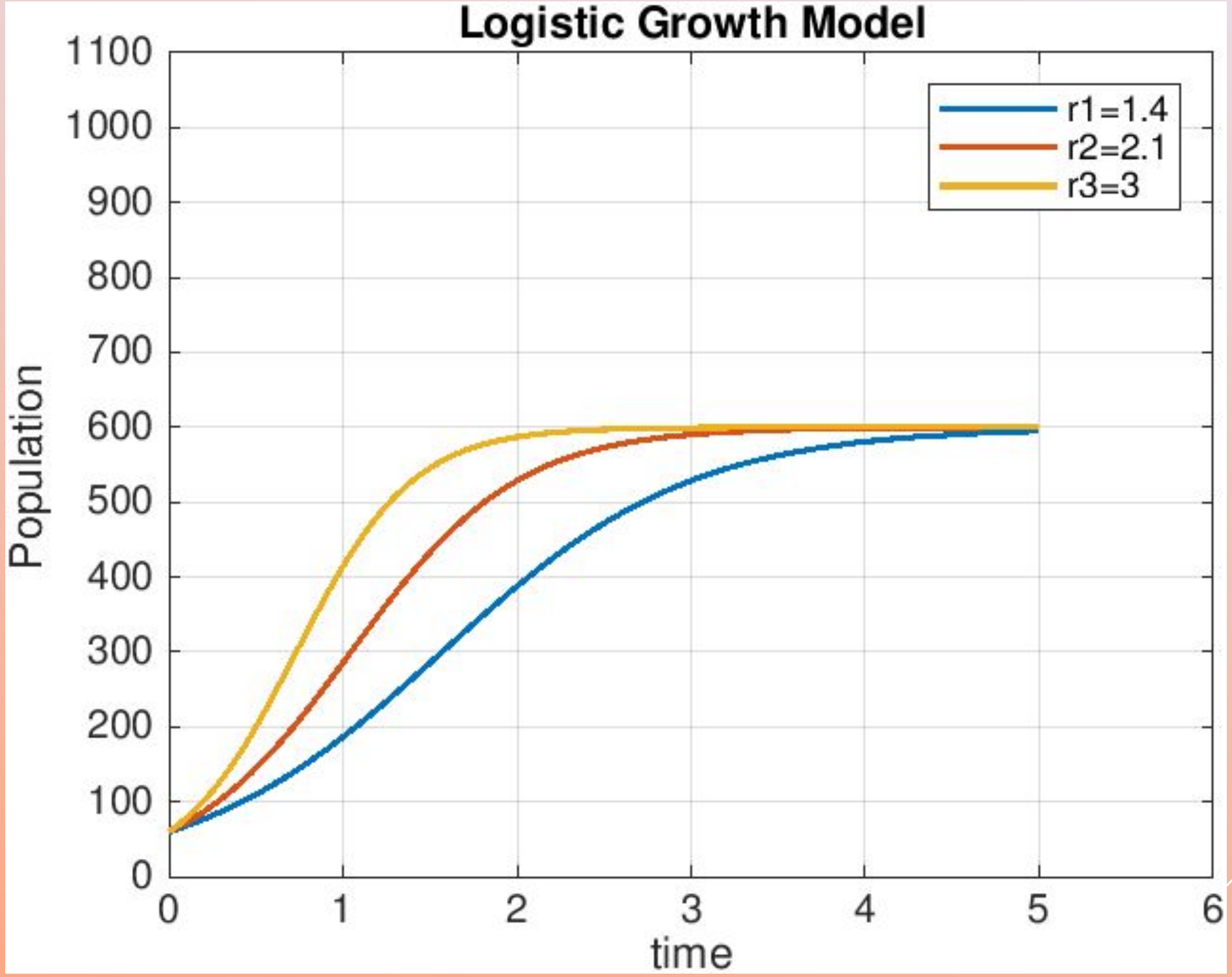
Time for some graphs!



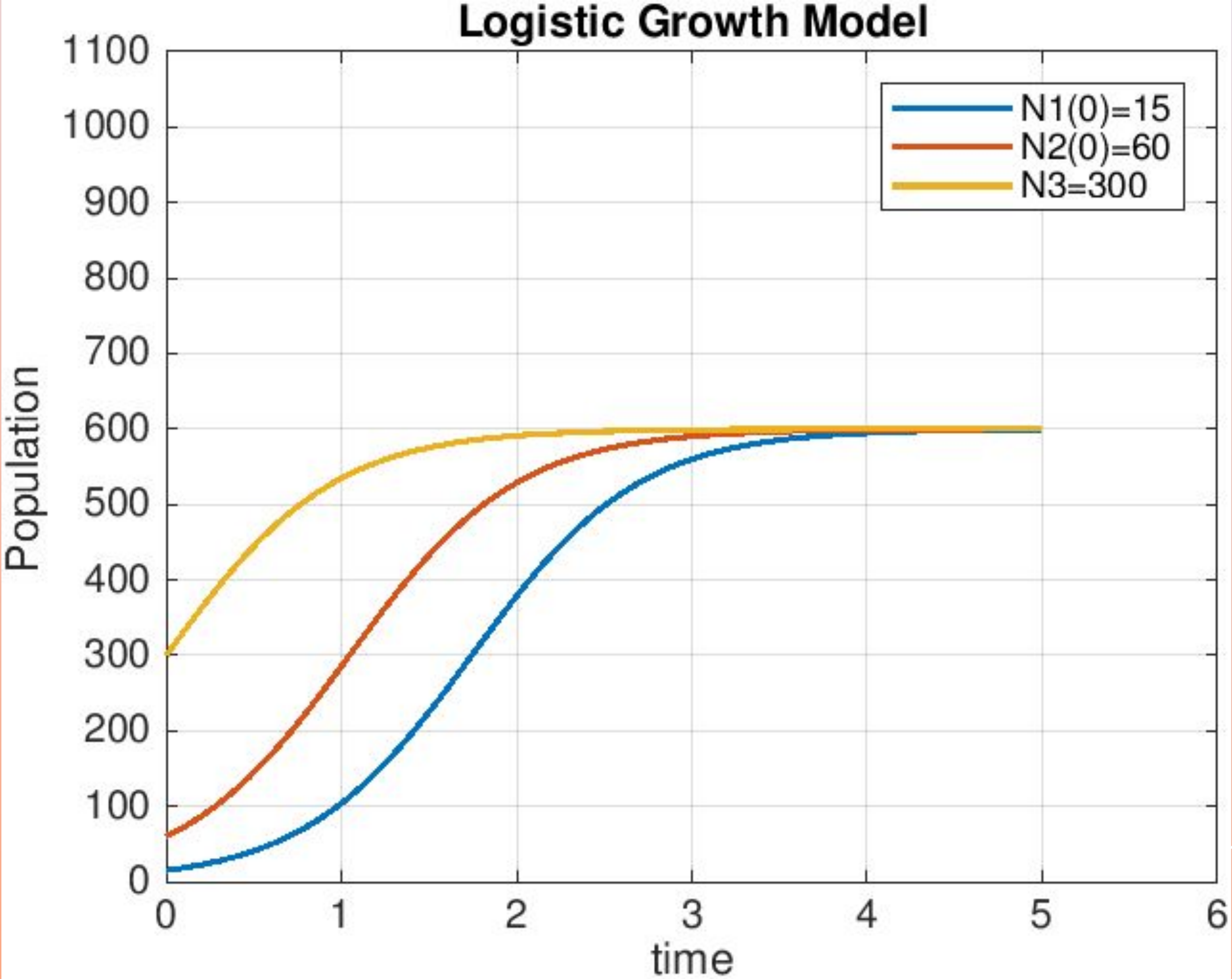
For different carrying capacity!



For different growth rates



What If the initial population was different?

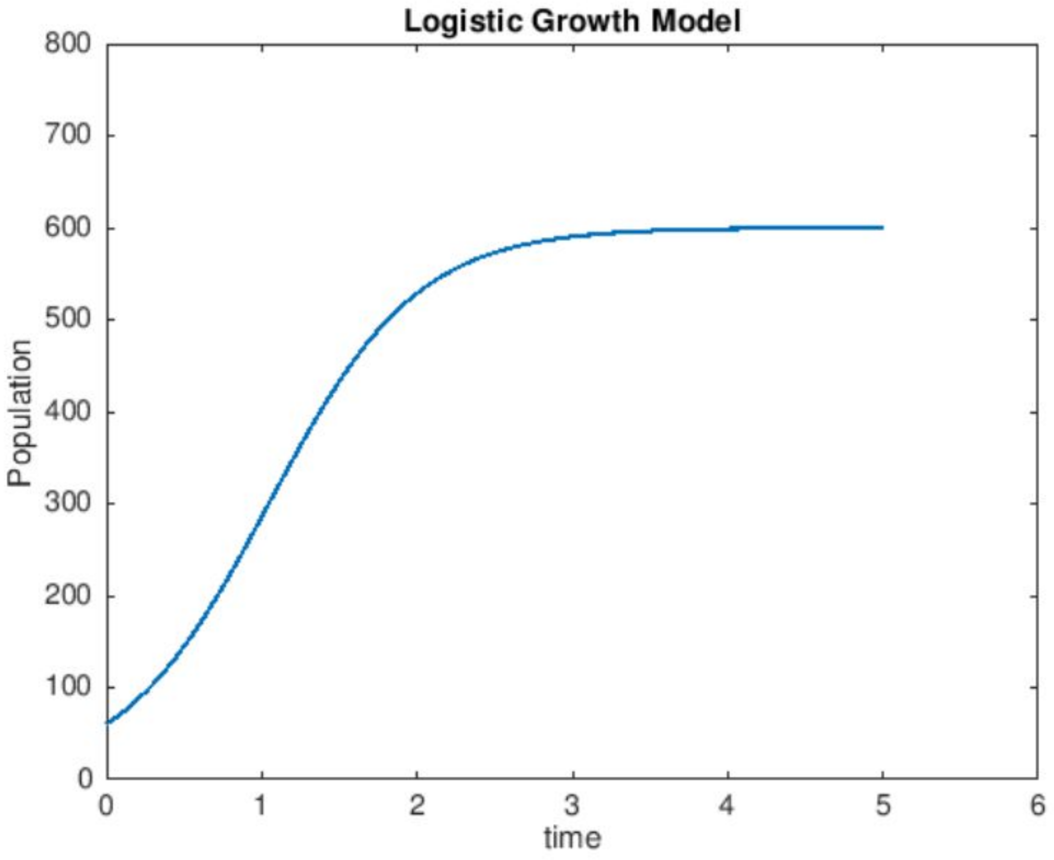


Courtesy: MATLAB

```
syms N(t)
r=2.1;
K=600;
b=r/K;
ode = diff(N,t) ==r*N-b*N*N;
cond = N(0) == 60;
ySol(t) = dsolve(ode,cond);
fplot(t,ySol(t),'linewidth',1.5)
title('Logistic Growth Model')
xlabel('time')
xlim([0 6]);
ylim([0 800]);
ylabel('Population')

fprintf("The Solution of Differential equation is:\n ")
disp(ySol(t))
```

The Solution of Differential equation is:
*600/(exp(log(9) - (21*t)/10) + 1)*



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Some imperfections still left?

It is very well possible that the population is prey to another species OR it hunts other species!

In that case, population of both species would depend on each other

Let’s go a step further and try to incorporate prey predator relationship…

The Lotka-Volterra Model

Characteristics of Lotka-Volterra model

The Lotka-Volterra equations describe population of two species in which one acts as prey and other as predator.

The change in population is governed by pair of first order nonlinear differential equations

The equations are…

$$\boxed{\frac{\mathrm{d}F}{\mathrm{d}t} = rF - aFC}$$

The prey equation

$$\boxed{\frac{\mathrm{d}C}{\mathrm{d}t} = \epsilon_a FC - \mu C}$$

The predator equation

F is the number of prey,

C is the number of its predator,

$\frac{\mathrm{d}F}{\mathrm{d}t}$ and $\frac{\mathrm{d}C}{\mathrm{d}t}$ represent the instantaneous growth rate of the respective populations,

r, a, ϵ_a and μ are positive real parameters

Let's understand what the equations means

$$\frac{\mathbf{d}\boldsymbol{F}}{\mathbf{d}\boldsymbol{t}} = \boldsymbol{rF} - \boldsymbol{aFC}$$

The prey equation

The prey equation assumes exponential growth of the prey population if there is no predation which is depicted by term rF

aFC represent the rate of predation whenever both the prey and predator meet

the rate of change of prey's population is its own growth rate minus the rate of predation.

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \epsilon_a F C - \mu C$$

The prey equation

$\epsilon_a F C$ represent the rate of growth of predator population

A different constant is used because the rate of predation need not be equal to the growth of predator population exactly

μC represents exponential death rate of the predator population if prey is not available

the rate of change of predator population is rate of predation minus death rate.

Some mathematical analysis..

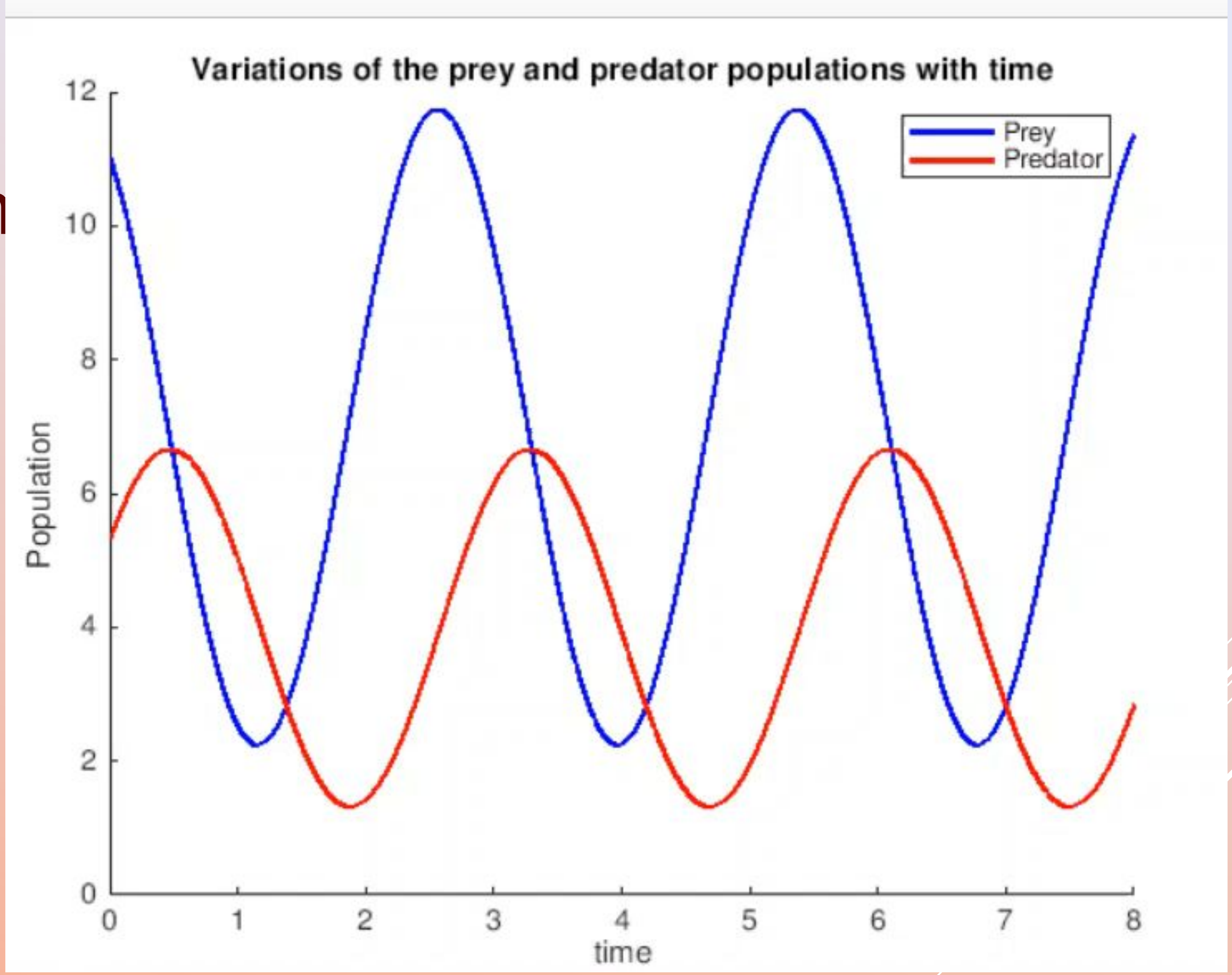
The nullclines for the prey equation, i.e. where $\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}\boldsymbol{t}} = \mathbf{0}$, indicate either $\boldsymbol{F} = \mathbf{0}$ or $\boldsymbol{C} = \frac{\boldsymbol{r}}{\boldsymbol{a}}$

the nullclines for predator equation, i.e. where $\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}\boldsymbol{t}} = \mathbf{0}$, indicate either $\boldsymbol{C} = \mathbf{0}$ or $\boldsymbol{F} = \frac{\boldsymbol{\mu}}{\boldsymbol{\epsilon_a}}$

This indicates that there are two steady states for the model. The first is the trivial state (0,0) and the second is the internal steady state $(\frac{\boldsymbol{\mu}}{\boldsymbol{\epsilon_a}}, \frac{\boldsymbol{r}}{\boldsymbol{a}})$

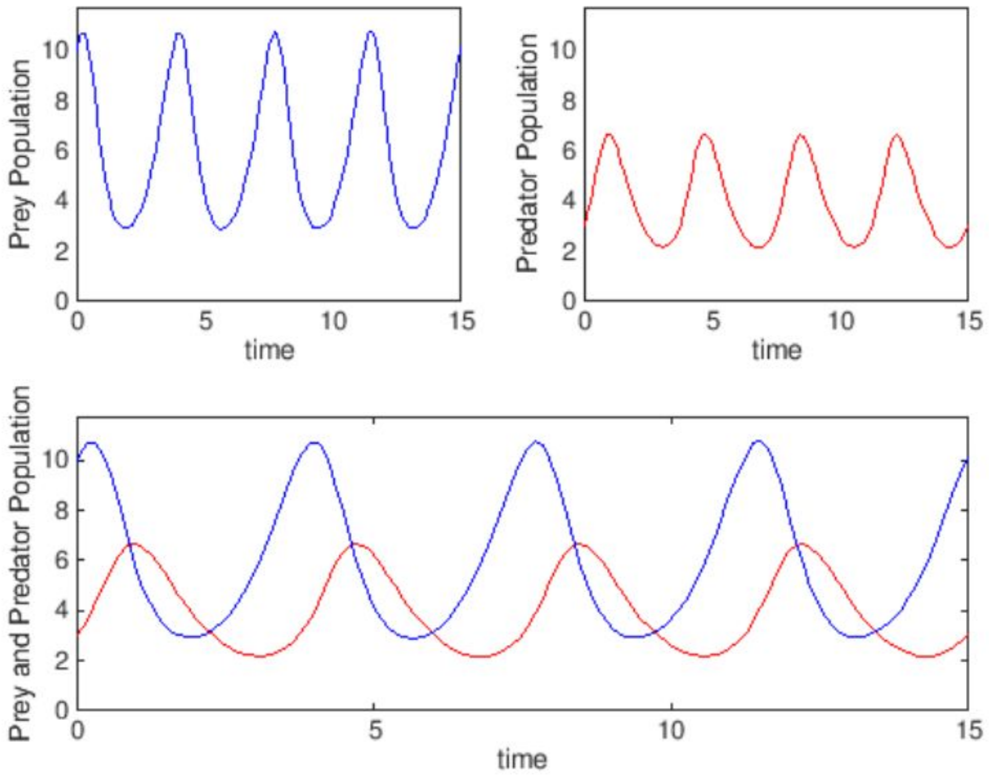
Let’s visualise
 this model through som

Courtesy: MATLAB



Can’t solve it genius?....
Let MATLAB do it for you….

```
function LotkaVolterra_JA
clear
clc
clf
a=2;
c=1.5;
alpha=0.5;
gamma=0.25;
initialPrey = 10;
initialPredator = 3;
deqs=@(t,x) [x(1)*(a -alpha*x(2)); x(2)*(-c +gamma*x(1));]
[t,sol] = ode45(deqs,[0 15],[initialPrey initialPredator])
subplot(2,2,1)
    plot(t(:,1),sol(:,1),'b')
    xlabel('time')
    ylabel('Prey Population')
    ymax=max(max(sol(:,1)),max(sol(:,2)))+1;
    axis([min(t(:,1)) max(t(:,1)) 0 ymax])
subplot(2,2,2)
    plot(t(:,1),sol(:,2),'r')
    xlabel('time')
    ylabel('Predator Population')
    axis([min(t(:,1)) max(t(:,1)) 0 ymax])
subplot(2,1,2)
    plot(t(:,1),sol(:,2),'r',t(:,1),sol(:,1),'b')
    xlabel('time')
    ylabel('Prey and Predator Population')
    axis([min(t(:,1)) max(t(:,1)) 0 ymax])
end
```



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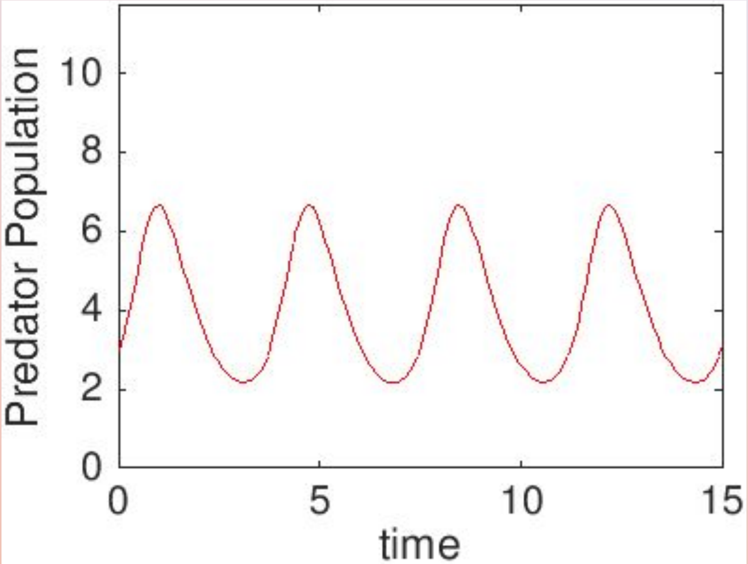
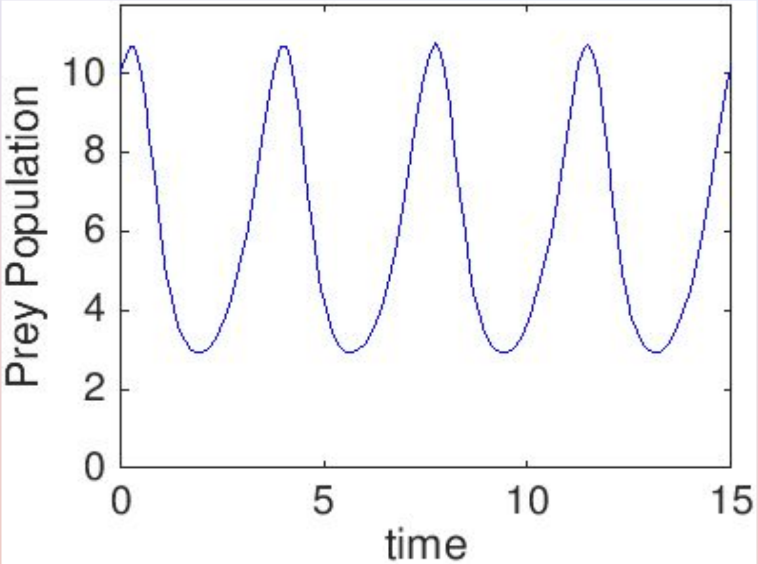
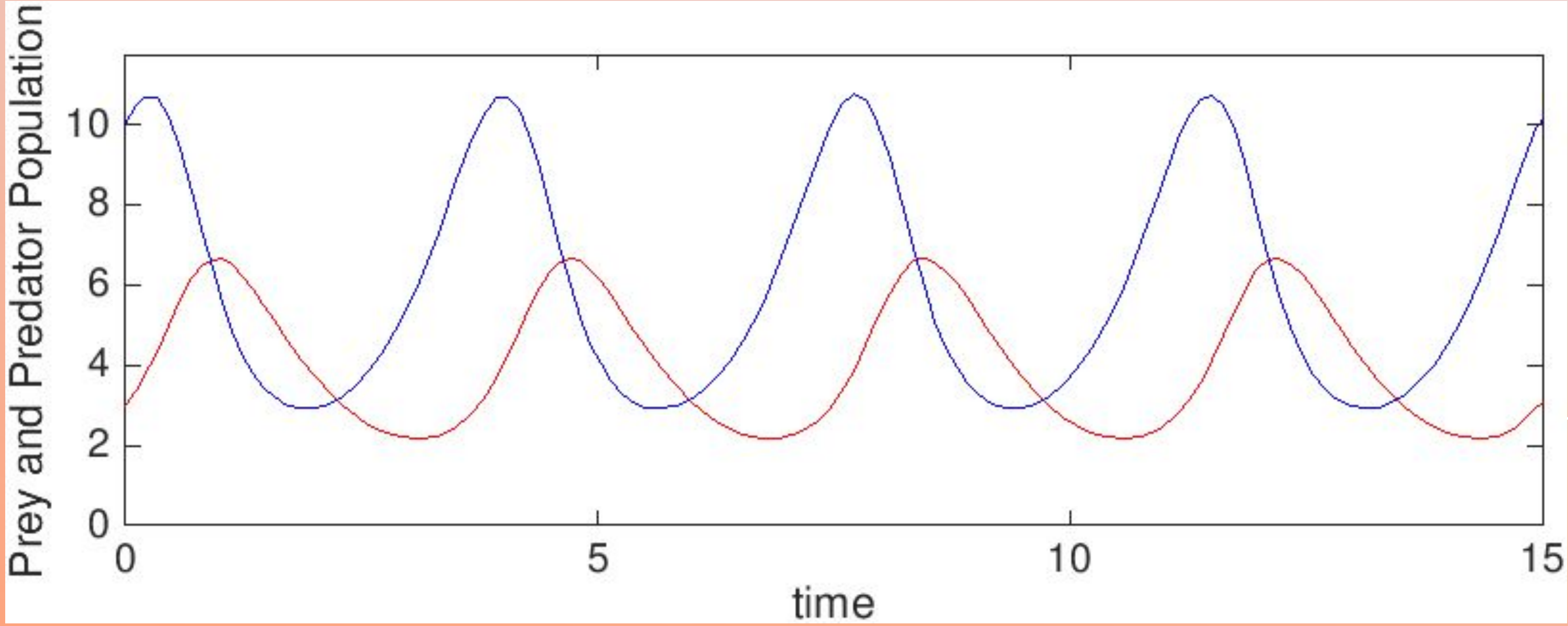

Predator Population
0
2
4
6
8
10
0
5
10
15
time


Prey Population
0
2
4
6
8
10
0
5
10
15
time


Prey and Predator Population
0
2
4
6
8
10
0
5
10
15
time

But what about the assumptions?... Here are some assumptions of this model

Ample food is available for the prey population

Rate of change of population is proportional to its size

Predator population is entirely dependent on the prey population for its survival

Environmental conditions are neutral to both the populations

Predators have infinite appetite

These were some of the models used to model population of species but there are few other’s as well like…

The Two Sexes Model which accounts for sexual reproduction in the species

The analysis of this model about how the population changes is directly/indirectly related to the number of times organisms of opposite sex configuration mate with each other

Now Let’s see some applications for these models and how they help us on many occasions…

Epidemics

Kermack-McKendrick model considers as prey being the susceptible population affected by the predators as infectives

This concept uses the fact that the infectives can only be removed from the system via death of susceptible population or quarantine.

The critical value R = $\frac{\epsilon_a F}{\mu}$ = 1 is called the epidemic threshold or tipping point which is defines for various diseases and epidemic control measure.

The term herd immunity is related to this epidemic threshold and states that the there is level for the susceptible population below which the infectives will not increase

Bacteria growth

Jacob-Monod model modifies the prey and predator model for the organisms like bacteria and their nutrient uptakes

$$\dot{x} = \frac{Vy}{k+y}x \qquad \dot{y} = -\frac{1}{Y}\frac{Vy}{k+y}x$$

where x is the size of population of bacteria,
y is concentration of feeding nutrients,
V is the uptake velocity,
K is saturation constant,
Y is yield of x per unit y

Here is a quick video of bacteria growth using the logistic model

Water resources

According to Population Reference Bureau, world population is expected to increase to 7.8 billion by 2025 and 9 billion by 2050

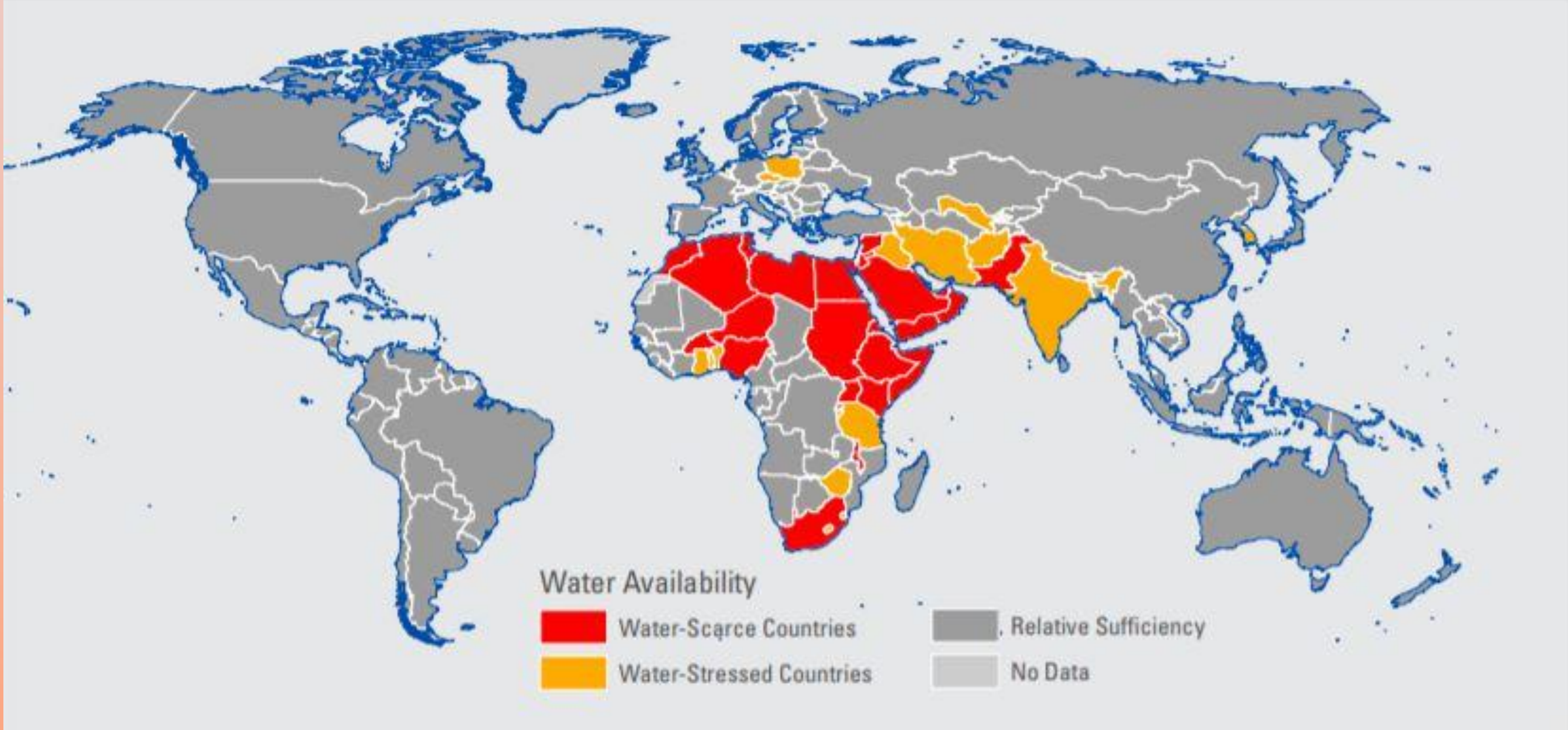
So around 508 billion gallons per day will be required by the US in 2050 according to the estimates made using the population models discussed above

To account for the aforementioned issues water resource models are constructed which analytically predict the quality and quantity of water in a country's water reserves.

Thus the population models can be used to predict the burden on water reserves of an area thereby helping in the examination of potential problems and their rectification.

Figure 1: Population Growth Impacts Water-Scarce and Water-Stressed Countries

Data on the amount of total renewable freshwater available in each country (2008-2012) is from the Food and Agriculture Organization of the United Nations. Total renewable freshwater includes the amount of both internal and external renewable water available to a country. This value is then divided by 2010 population figures provided by the United Nations Population Division to produce a per capita rate.



Source: Population Action International (PAI). *Mapping Population and Climate Change*. Washington, DC: PAI.

After understanding how population growth models function, We now see how valid they are….

Advantages of population models

By adjusting the amount of data input the complexity of these models can be modified suiting to one's requirements

The response of a system due to varying temporal and spatial configurations can be evaluated and predicted

The knowledge of previous models can be integrated into a single model to provide a hybrid solution for the functioning of the system

But there is other side of the coin also…

Disadvantages of population models

Although extensive data exists but any biological input always comes with some degree of uncertainty, thus these models are prone to errors

Many variables and inputs may have to be assumed so as to reach an acceptable solution

High precision and process time is required for the analysis of a complete model and still the results may not be accurate

Complicated interactions may be oversimplified during the process of modelling

Any Questions?