Image Restoration

Degradation Model, Discrete Formulation, Algebraic Approach to Restoration - Unconstrained & Constrained; Constrained Least Square Restoration, Geometric Transformation - Spatial Transformation, Gray Level Interpolation.

What is image restoration?

Image restoration is task of recovering or reconstructing an image from its degraded version assuming some priori **knowledge of the degradation phenomenon**. The restoration technique m**odels** the degradation process and applies the inverse process to obtain the original from the degraded (observed) image. It differs from image **enhancement**–which does not fully account for the nature of the degradation. Image enhancement is largely a subjective process while image restoration is an objective process.

Give the difference between Enhancement and Restoration.

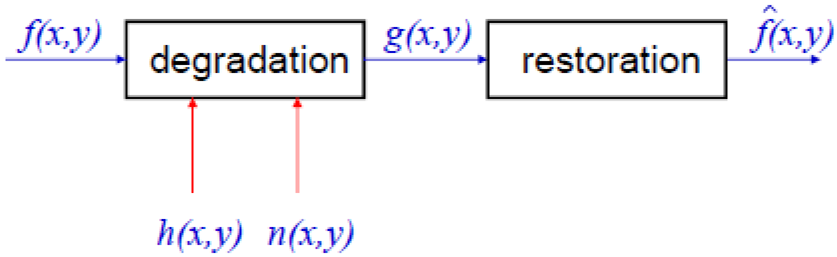
Enhancement technique is based primarily on the pleasing aspects it might present to the viewer. For example: Contrast Stretching. Where as Removal of image blur by applying a deblurrings function is considered a restoration technique.

A model of the Image Degradation/Restoration Process

The degradation process is modeled as a degradation function that, together with an additive noise term, operates on an input image f(x,y) to produce a degraded image g(x,y).Given g(x,y),some knowledge about the degradation function $\mathcal{H}$, and some knowledge about the additive noise term η(x,y),the objective of restoration is to obtain an estimate $\hat{f}(x,y)$ of the original image. We want the estimate to be as close as possible to the original input image.The approaches is based on various types of image restoration filters.

A model of the image degradation/restoration process.

- $f(x,y)$ – image before degradation, 'true image'
- $g(x,y)$ – image after degradation, 'observed image'
- $h(x,y)$ – degradation filter
- $\hat{f}(x,y)$ – estimate of $f(x,y)$ computed from $g(x,y)$
- $n(x,y)$ – additive noise



If H is a linear, postion invariant process ,then the degraded image is given in the spatial domain by

$$g(x,y) = h(x,y) * f(x,y) + n(x,y)$$

Where h(x,y) is the spatial representation of the degradation function,the symbol "*" indicates spatial convolution. *h* is the impulse response of the system, i.e. degraded image if *f*(*x*,*y*) was a unit impulse image – also called as **convolution kernel**.We know that the spatial convolution in the spatial domain is equal to multiplication in the frequency domain,So we may write the model in an equivalent frequency domain representation:

$$G(u,v) = H(u,v)\, F(u,v) + N(u,v)$$

Common Assumptions on $\mathcal{H}$:

(1) Linearity,

$$\mathcal{H}(k_1 f_1(x,y) + k_2 f_2(x,y)) = k_1 \mathcal{H}(f_1(x,y)) + k_2 \mathcal{H}(f_2(x,y)),$$

(2) Space Invariance

$$\mathcal{H}(f(x - x_1, y - y_1)) = g(x - x_1, y - y_1)$$

Degradations

- original

- optical blur

- motion blur

- spatial quantization (discrete pixels)

- additive intensity noise

Overview – Deconvolution

The objective is to restore a degraded image to its original form.

An observed image can often be modelled as:

$$g(x,y) = \int\int h(x-x', y-y') f(x',y')\, dx'\, dy' + n(x,y)$$

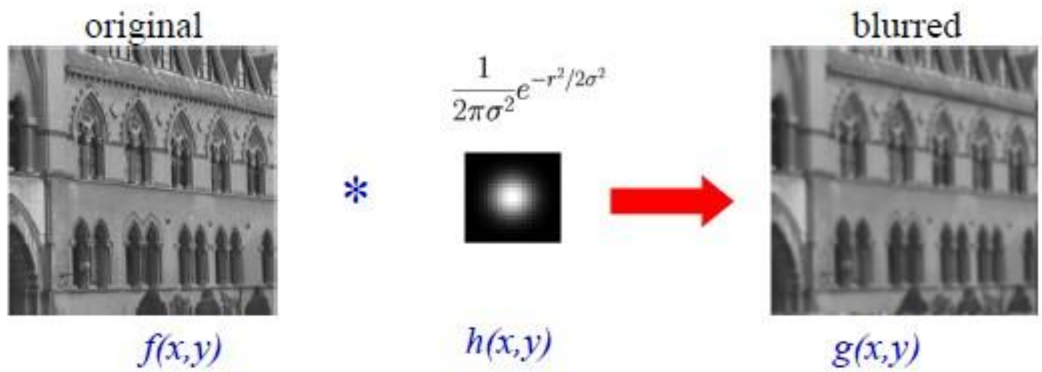
where the integral is a convolution, h is the point spread function of the imaging system, and n is additive noise.

The objective of image restoration in this case is to estimate the original image f from the observed degraded image g.

Degradation model

Model degradation as a convolution with a linear, shift invariant, filter $h(x,y)$

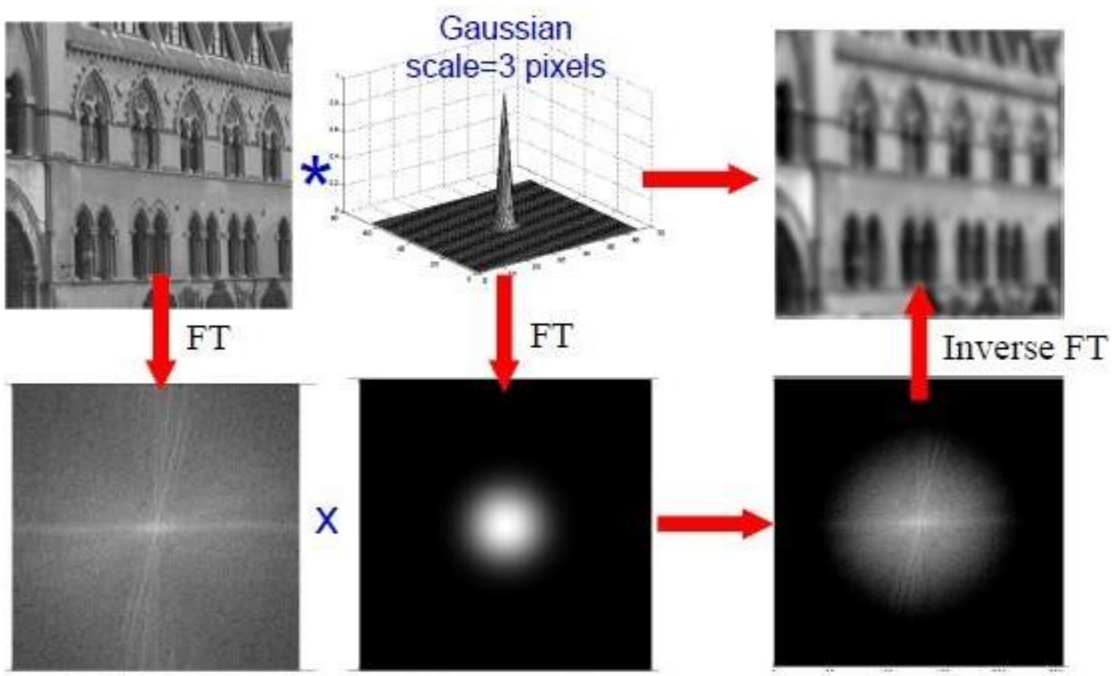
- Example: for out of focus blurring, model $h(x,y)$ as a Gaussian



i.e. : $g(x,y) = h(x,y) * f(x,y)$

$h(x,y)$ is the impulse response or point spread function of the imaging system

The challenge: loss of information and noise



Blurring acts as a low pass filter and attenuates higher spatial frequencies

What is meant by unconstrained restoration?

In the absence of any knowledge about the noise 'n', a meaningful criterion function is to seek an f^ such that H f^ approximates of in a least square sense by assuming the noise term is as small as possible.

It is also known as least square error approach.

n = g-Hf

To estimate the original image f^, noise n has to be minimized and

f^ = g/H

Where,

H = system operator.
f^ = estimated input image.
g = degraded image.

What is meant by constrained restoration?

It is also known as maximum square error approach n = g-Hf. To estimate the original image f^, noise n has to be maximized and f^ = g/H.

What is inverse filtering?

The simplest approach to restoration is direct inverse filtering, an estimate F^(u,v) of thetransform of the original image simply by dividing the transform of the degraded image G^(u,v) by the degradation function.

F^ (u,v) = G^(u,v)/H(u,v)

Inverse filtering is the process of recovering the input of the system from its output.

What are the methods to estimating the degradation function?

The three methods of degradation function are,

- Observation
- Experimentation
- Mathematical modeling

.How the blur is removed caused by uniform linear motion?

An image f(x,y) undergoes planar motion in the x and y-direction and $x_0(t)$ and $y_0(t)$ are the time varying components of motion. The total exposure at any point of the recording medium (digital memory) is obtained by integrating the instantaneous exposure over the time interval during which the imaging system shutter is open.

Algebraic Approach to Restoration

.What are the methods of algebraic approach?

- ❖ Unconstraint restoration approach
- ❖ Constraint restoration approach

.What are the types of noise models?

- ➢ Guassian noise
- ➢ Rayleigh noise
- ➢ Erlang noise
- ➢ Exponential noise
- ➢ Uniform noise
- ➢ Impulse noise

What is blur impulse response and noise levels?

Blur impulse response: This parameter is measured by isolating an image of a suspected object within a picture.

Noise levels: The noise of an observed image can be estimated by measuring the image covariance over a region of constant background luminance.

Explain degradation model for (i) continuous function (ii) discrete formulation

Restoration attempts to reconstruct or recover an image that has been degraded by using a priori knowledge of the degradation phenomenon.
Restoration techniques are oriented toward modeling the degradation and applying the inverse process in order to recover the original image.

Degradation models:

- Many types of degradation can be approximated by linear, space invariant processes
    - Can take advantages of the mature techniques developed for linear systems
- Non-linear and space variant models are more accurate
    - Difficult to solve
    - Unsolvable

Estimating degradation function

- Estimation by image observation
    - Degradation system H is completely characterized by its impulse response
    - Select a small section from the degraded image $g_S(x, y)$
    - Reconstruct an unblurred image of the same size $\hat{f}_S(x, y)$

- The degradation function can be estimated by

$$H_s(u,v) = \frac{G_s(u,v)}{\hat{F}_s(u,v)}$$

By ignoring the noise term, G(u,v) = F(u,v)H(u,v). If F(u,v) is the Fourier transform of point source (impulse), then G(u,v) is approximates H(u,v).

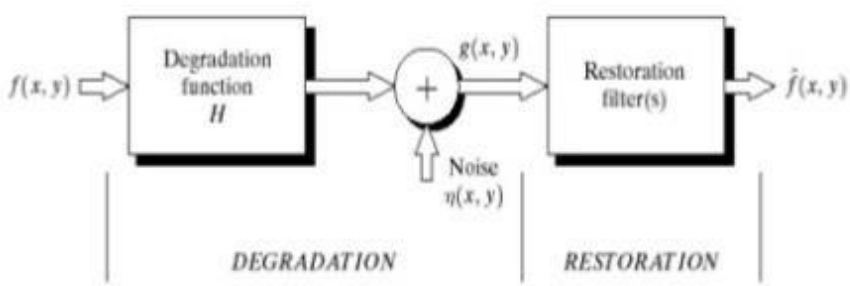


Fig: A model of the image degradation / restoration process

Continuous degradation model

- Motion blur. It occurs when there is relative motion between the object and the camera during exposure.

$$h(i)=\begin{cases}\frac{1}{L}, & \text{if } -\frac{L}{2}\le i\le\frac{L}{2}\\ 0, & \text{otherwise}\end{cases}$$

- Atmospheric turbulence. It is due to random variations in the reflective index of the medium between the object and the imaging system and it occurs in the imaging of astronomical objects.

$$h(i,j)=K\exp\left(-\frac{i^2+j^2}{2\sigma^2}\right)$$

- Uniform out of focus blur

$$h(i,j)=\begin{cases}\frac{1}{\pi R}, & \text{if } \sqrt{i^2+j^2}\le R\\ 0, & \text{otherwise}\end{cases}$$

- Uniform 2-D blur

$$h(i,j)=\begin{cases}\frac{1}{(L)^2}, & \text{if } -\frac{L}{2}\le i,j\le\frac{L}{2}\\ 0, & \text{otherwise}\end{cases}$$

Two dimensional discrete degradation model. Circular convolution

Suppose we have a two-dimensional discrete signal $f(i,j)$ of size $A\times B$ samples which is due to a degradation process. The degradation can now be modeled by a two dimensional discrete impulse response $h(i,j)$ of size $C\times D$ samples.

We form the extended versions of $f(i,j)$ and $h(i,j)$, both of size $M\times N$, where $M\ge A+C-1$ and $N\ge B+D-1$, and periodic with period $M\times N$. These can be denoted as $f_e(i,j)$ and $h_e(i,j)$. For a space invariant degradation process we obtain

$$y_e(i,j)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_e(m,n)h_e(i-m,j-n)+n_e(i,j)$$

Using matrix notation we can write the following form

$$\mathbf{y}=\mathbf{Hf}+\mathbf{n}$$

Where,

$\mathbf{f}$ and $\mathbf{y}$ are MN-dimensional column vectors that represent the lexicographic ordering of images $f_e(i,j)$ and $h_e(i,j)$ respectively.

$$\mathbf{H}=\begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_{M-1} & \cdots & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \cdots & \mathbf{H}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{M-1} & \mathbf{H}_{M-2} & \cdots & \mathbf{H}_0 \end{bmatrix}$$

$$\mathbf{H}_j=\begin{bmatrix} h_e(j,0) & h_e(j,N-1) & \cdots & h_e(j,1) \\ h_e(j,1) & h_e(j,0) & \cdots & h_e(j,2) \\ \vdots & \vdots & \ddots & \vdots \\ h_e(j,N-1) & h_e(j,N-2) & \cdots & h_e(j,0) \end{bmatrix}$$

The analysis of the diagonalisation of $\mathbf{H}$ is a straightforward extension of the one-dimensional case.

In that case we end up with the following set of $M \times N$ scalar problems.

$$Y(u,v)=MNH(u,v)F(u,v)(+N(u,v))$$

$$u=0,1,\ldots,M-1,\ v=0,1,\ldots,N-1$$

Unconstrained & Constrained Restoration

Explain algebraic approach to (i) unconstrained restoration and (ii) constrained restoration.

unconstrained restoration

In the absence of any knowledge about the noise 'n', a meaningful criterion function is to seek an f^ such that H f^ approximates of in a least square sense by assuming the noise term is as small as possible.

It is also known as least square error approach.

n = g-Hf

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Where,

H = system operator.
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(ii) constrained restoration

The set-based approach described previously can be generalized so that any number of prior constraints can be imposed as long as the constraint sets are closed convex.

If the constraint sets have a non-empty intersection, then a solution that belongs to the intersection set can be found by the method of POCS. Any solution in the intersection set is consistent with the a priori constraints and therefore it is a feasible solution. Let $Q_1, Q_2, \ldots, Q_m$ be closed convex sets in a finite dimensional vector space, with $P_1, P_2, \ldots, P_m$ their respective projectors.

The iterative procedure

$$\mathbf{f_{k+1}} = P_1 P_2, \ldots P_m \mathbf{f_k}$$

converges to a vector that belongs to the intersection of the sets $Q_i, i = 1, 2, \ldots, m$, for any starting vector $\mathbf{f_0}$. An iteration of the form $\mathbf{f_{k+1}} = P_1 P_2 \mathbf{f_k}$ can be applied in the problem described previously.

where we seek for an image which lies in the intersection of the two ellipsoids defined by

$$Q_{\mathbf{f|y}} = \{\mathbf{f} \mid \|\mathbf{y} - \mathbf{Hf}\|^2 \le E^2\} \text{ and } Q_{\mathbf{f}} = \{\mathbf{f} \mid \|\mathbf{Cf}\|^2 \le \varepsilon^2\}$$

The respective projections $P_1\mathbf{f}$ and $P_2\mathbf{f}$ are defined by

$$P_1\mathbf{f} = \mathbf{f} + \lambda_1 \left(\mathbf{I} + \lambda_1 \mathbf{H^T H}\right)^{-1} \mathbf{H^T}(\mathbf{y} - \mathbf{Hf})$$

$$P_2\mathbf{f} = [\mathbf{I} - \lambda_2 \left(\mathbf{I} + \lambda_2 \mathbf{C^T C}\right)^{-1} \mathbf{C^T C}]\mathbf{f}$$

Constrained Least Square Restoration

What is constrained least squares restoration? Explain.

Only the mean and variance of the noise is required

The degradation model in vector-matrix form

$$\mathbf{g}_{MN\times 1} = \mathbf{H}_{MN\times MN}\mathbf{f}_{MN\times 1} + \boldsymbol{\eta}_{MN\times 1}$$

The objective function

$$\min C = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}[\nabla^2 f(x,y)]^2$$

$$\textit{subject to } \|\mathbf{g}-\mathbf{Hf}\|^2 = \|\boldsymbol{\eta}\|^2$$

The solution

$$\hat{F}(u,v) = \frac{H^*(u,v)}{|H(u,v)|^2 + \gamma P(u,v)} G(u,v)$$

$$p(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

In that case we seek for a solution that minimizes the function

$$M(\mathbf{f}) = \|\mathbf{y}-\mathbf{Hf}\|^2$$

A necessary condition for $M(\mathbf{f})$ to have a minimum is that its gradient with respect to $\mathbf{f}$ is equal to zero. This gradient is given below

$$\frac{\partial M(\mathbf{f})}{\partial \mathbf{f}} = \nabla_{\mathbf{f}} M(\mathbf{f}) = 2(-\mathbf{H}^{\mathbf{T}}\mathbf{y} + \mathbf{H}^{\mathbf{T}}\mathbf{Hf})$$

And by using the steepest descent type of optimization we can formulate an iterative rule as follows:

$$\mathbf{f}_{\mathbf{0}} = \beta\mathbf{H}^{\mathbf{T}}\mathbf{y}$$

$$\mathbf{f}_{\mathbf{k+1}} = \mathbf{f}_{\mathbf{k}} - \mu\frac{\partial M(\mathbf{f}_{\mathbf{k}})}{\partial \mathbf{f}_{\mathbf{k}}} = \mathbf{f}_{\mathbf{k}} + \beta\mathbf{H}^{\mathbf{T}}(\mathbf{y}-\mathbf{Hf}_{\mathbf{k}}) = \beta\mathbf{H}^{\mathbf{T}}\mathbf{y} + (\mathbf{I}-\beta\mathbf{H}^{\mathbf{T}}\mathbf{H})\mathbf{f}_{\mathbf{k}}$$

Constrained least squares iteration

In this method we attempt to solve the problem of constrained restoration iteratively. As already mentioned the following functional is minimized

$$M(\mathbf{f},\alpha)=\|\mathbf{y}-\mathbf{Hf}\|^2+\alpha\|\mathbf{Cf}\|^2$$

The necessary condition for a minimum is that the gradient of $M(\mathbf{f},\alpha)$ is equal to zero. That gradient is

$$\Phi(\mathbf{f})=\nabla_{\mathbf{f}}M(\mathbf{f},\alpha)=2[(\mathbf{H}^{\mathbf{T}}\mathbf{H}+\alpha\mathbf{C}^{\mathbf{T}}\mathbf{C})\mathbf{f}-\mathbf{H}^{\mathbf{T}}\mathbf{y}]$$

The initial estimate and the updating rule for obtaining the restored image are now given by

$$\mathbf{f}_{\mathbf{0}}=\beta\mathbf{H}^{\mathbf{T}}\mathbf{y}$$

$$\mathbf{f}_{\mathbf{k+1}}=\mathbf{f}_{\mathbf{k}}+\beta[\mathbf{H}^{\mathbf{T}}\mathbf{y}-(\mathbf{H}^{\mathbf{T}}\mathbf{H}+\alpha\mathbf{C}^{\mathbf{T}}\mathbf{C})\mathbf{f}_{\mathbf{k}}]$$

It can be proved that the above iteration (known as **Iterative CLS** or **Tikhonov-Miller Method**) converges if

$$0<\beta<\frac{2}{|\lambda_{\max}|}$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix

$$(\mathbf{H}^{\mathbf{T}}\mathbf{H}+\alpha\mathbf{C}^{\mathbf{T}}\mathbf{C})$$

If the matrices $\mathbf{H}$ and $\mathbf{C}$ are block-circulant the iteration can be implemented in the frequency domain.

Geometric Transformations

We consider image transformations such as rotation, scaling and distortion of images. Such transformations are frequently used as pre-processing steps in applications such as document understanding, where the scanned image may be mis-aligned.

There are two basic steps in geometric transformations:

A spatial transformation of the physical rearrangement of pixels in the image
A grey level interpolation, which assigns grey levels to the transformed image

Spatial transformation

Pixel coordinates (x,y) undergo geometric distortion to produce an image with coordinates (x',y'):

$$\begin{aligned} x' &= r(x,y) \\ y' &= s(x,y), \end{aligned}$$

where r and s are functions depending on x and y.

Examples:

1. Suppose $r(x,y) = \frac{x}{2}$, $s(x,y) = \frac{y}{2}$. This halves the size of the image. This transformation can be represented using a matrix equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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2. Rotation about the origin by an angle θ is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Remember the origin of the image is usually the top left hand corner. To rotate about the centre one needs to do a transformation of the origin to the centre of the image first.

Tie points

Often the spatial transformation needed to correct an image is determined through tie points. These are points in the distorted image for which we know their corrected positions in the final image. Such tie points are often known for satellite images and aerial photos. We will illustrate this concept with the example of correcting a distorted quadrilateral region in an image.

We model such a distortion using a pair of bilinear equations:

$$x' = c_1x + c_2y + c_3xy + c_4$$

$$y' = c_5x + c_6y + c_7xy + c_8.$$

We have 4 pairs of tie point coordinates. This enables us to solve for the 8 coefficients $c_1 \ldots c_8$.

We can set up the matrix equation using the coordinates of the 4 tie points:

$$\begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & x_1y_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & y_1 & x_1y_1 & 1 \\ x_2 & y_2 & x_2y_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2 & y_2 & x_2y_2 & 1 \\ x_3 & y_3 & x_3y_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & y_3 & x_3y_3 & 1 \\ x_4 & y_4 & x_4y_4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_4 & y_4 & x_4y_4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix}$$

In shorthand we can write this equation as

$$[XY] = [M][C],$$

which implies

$$[C] = [M]^{-1}\,[XY].$$

Having solved for the coefficients $c_1, \ldots, c_8$ we can use them in our original bilinear equations above to obtain the corrected pixel coordinates (x',y') for all pixels (x,y) in the original image within (or near to) the quadrilateral being considered.

To correct for more complex forms of distortion, for example lens distortion, one can use higher order polynomials plus more tie points to generate distortion correction coefficients $c_1, \ldots, c_n$.

Grey Level Interpolation

The problem we have to consider here is that, in general, the distortion correction equations will produce values x' and y' that are not integers. We end up with a set of grey levels for non integer positions in the image. We want to determine what grey levels should be assigned to the integer pixel locations in the output image.

The simplest approach is to assign the grey value for $F(x,y)$ to the pixel having closest integer coordinates to $\hat{F}(x',y')$. The problem with this is that some pixels may be assigned two grey values, and some may not be assigned a grey level at all - depending on how the integer rounding turns out.

The way to solve this is to look at the problem the other way round. Consider integer pixel locations in the output image and calculate where they must have come from in the input image. That is, work out the inverse image transformation.

These locations in the input image will not (in general) have integer coordinates. However, we do know the grey levels of the 4 surrounding integer pixel positions. All we have to do is interpolate across these known intensities to determine the correct grey level of the position when the output pixel came from.

Various interpolation schemes can be used. A common one is bilinear interpolation, given by

$$v(x,y) = c_1x + c_2y + c_3xy + c_4,$$

where $v(x,y)$ is the grey value at position (x,y).

Thus we have four coefficients to solve for. We use the known grey values of the 4 pixels surrounding the `come from' location to solve for the coefficients.

We need to solve the equation

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & x_1y_1 & 1 \\ x_2 & y_2 & x_2y_2 & 1 \\ x_3 & y_3 & x_3y_3 & 1 \\ x_4 & y_4 & x_4y_4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

or, in short,

$$[V] = [M][C],$$

which implies

$$[C] = [M]^{-1}[V].$$

This has to be done for every pixel location in the output image and is thus a lot of computation! Alternatively one could simply use the integer pixel position closest to the `come from location'. This is adequate for most cases.