Logical Deduction in AI

Inferencing by Resolution Refutation

Predicate Logic

Wherever Mary goes, so does the lamb. Mary goes to school. So the lamb goes to school.

No contractors are dependable. Some engineers are contractors. Therefore some engineers are not dependable.

All dancers are graceful. Ayesha is a student. Ayesha is a dancer. Therefore some student is graceful.

Every passenger is either in first class or second class. Each passenger is in second class if and only if he or she is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore some passengers are in second class.

New Additions in Proposition (First Order Logic)

Function Symbols

Variables, Constants, Predicate Symbols and

New Connectors: ∃ (there exists), ∀(for all)

Wherever Mary goes, so does the Lamb. Mary goes to School. So the Lamb goes to School.

Predicate: goes(x,y) to represent x goes to y

New Connectors: ∃ (there exists), ∀(for all)

F1: ∀x(goes(Mary, x) → goes(Lamb, x))

F2: goes(Mary, School) ✓ ground instance

G: goes(Lamb, School) ✓

To prove: (F1 ∧ F2) → G) is always true VALID

Inferencing in Predicate Logic

Domain: D

Constant Symbols: M, N, O, P, ….

Variable Symbols: x,y,z,….

Function Symbols: F(x), G(x,y), H(x,y,z)

Predicate Symbols: p(x), q(x,y), r(x,y,z),

Connectors: ~, Λ, V, →, ∃, ∀

Terms:

Well-formed Formula:

Free and Bound Variables:

Interpretation, Valid, Non-Valid, Satisfiable, Unsatisfiable

What is an Interpretation? Assign a domain set D, map constants, functions, predicates suitably. The formula will now have a truth value

Example:

F1: $\forall x(g(M, x) \rightarrow g(L, x))$

F2: $g(M, S)$

G: $g(L, S)$

∀x

M →, L →, S →

Γ g(x, y)

Interpretation 1: D = {Akash, Baby, Home, Play, Ratan, Swim}, etc.,

Interpretation 2: D = Set of Integers, etc.,

INFINITE INTERPRETATIONS

How many interpretations can there be?

To prove **Validity**, means $\boxed{(F1 \land F2) \rightarrow G)}$ is true under all interpretations

To prove **Satisfiability** means $(F1 \land F2) \rightarrow G)$ is true under at least one interpretation

n 2^n Truth

Propositional logic

Resolution Refutation for Propositional Logic

To prove validity of

$F = ((F1 \wedge F2 \wedge \ldots \wedge Fn) \rightarrow G)$

we shall attempt to prove that

$\sim F = (F1 \wedge F2 \wedge \ldots \wedge Fn \wedge \sim G)$

is unsatisfiable

→ FALSE under all interpretations

Steps for Proof by Resolution Refutation:

1. Convert of Clausal Form / Conjunctive Normal Form (CNF, Product of Sums).
2. Generate new clauses using the resolution rule.
3. At the end, either False will be derived if the formula ~F is unsatisfiable implying F is valid.

If Asha is elected VP then Rajat is chosen as G-Sec and Bharati is chosen as Treasurer. Rajat is not chosen as G-Sec. Therefore Asha is not elected VP.

F1: $(a \rightarrow (b \wedge c)) = (\sim a \vee b) \wedge (\sim a \vee c)$

F2: ~b, G: ~a, ~G: a

Clauses of Clause Form: $\sim F = (C1 \wedge C2 \wedge C3 \wedge C4)$

where: C1: $(\sim a \vee b)$

C2: $(\sim a \vee c)$

C3: $\sim b$

C4: a

To prove that ~F is False

Let $C1 = a \vee b$ and $C2 = \sim a \vee c$

then a new clause $C3 = b \vee c$ can be derived.

RESOLUTION RULE

(Proof by showing that $((C1 \wedge C2) \rightarrow C3)$ is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form $C1 = a$ and $C2 = \sim a$ from which False can be derived.

If the propositional formula is satisfiable then we will not reach a contradiction and eventually no new clauses will be derivable.

For propositional logic the procedure terminates.

Resolution Rule is Sound and Complete

Applying Resolution Refutation

Let C1 = a V b and C2 = ~a V c then a new clause C3 = b V c can be derived. c V c

(Proof by showing that ((C1 ∧ C2) → C3) is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form C1 = a and C2 = ~a from which False can be derived.

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Resolution Rule is Sound and Complete

If Asha is elected VP then Rajat is chosen as G-Sec and Bharati is chosen as Treasurer. Rajat is not chosen as G-Sec. Therefore Asha is not elected VP.

F1: (a → (b ∧ c)) = (~a V b) ∧ (~a V c)

$(F1 \wedge F2) \rightarrow G$

F2: ~b

$\boxed{F1 \wedge F2 \wedge \neg G}$

G: ~a

~G: a

Clauses of Clause Form: ~F = (C1 ∧ C2 ∧ C3 ∧ C4)

where: C1: (~a V b)

C2: (~a V c)

C3: ~b

C4: a

To prove that ~F is False

New Clauses Derived

C5: ~a (Using C1 and C3)

C6: False (using C4 and C5)

C5 : ¬a

FALSE

Example

Let C1 = a V b and C2 = ~a V c then a new clause C3 = b V c can be derived.

(Proof by showing that ((C1 ∧ C2) → C3) is a valid formula).

To prove unsatisfiability use the Resolution Rule repeatedly to reach a situation where we have two contradictory clauses of the form C1 = a and C2 = ~a from which False can be derived.

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For propositional logic the procedure terminates.

Resolution Rule is Sound and Complete

Rajesh either took the bus or came by cycle to class. If he came by cycle or walked to class he arrived late. Rajesh did not arrive late. Therefore he took the bus to class.

bus cycle walk late

F1: $(bus \wedge \neg cycle) \vee (\neg bus \wedge cycle)$

incl OR $(bus \vee cycle)$

F2: $(cycle \vee walk) \rightarrow late$

F3: $\neg late$

G: bus c1

C2

F1: $(bus \vee cycle) \wedge (\neg bus \vee \neg cycle)$

F2: $\neg(cycle \vee walk) \vee late = (\neg cycle \wedge \neg walk) \vee late$

C5

$(late \vee \neg cycle) \wedge (late \vee \neg walk)$

C3 C4

F3: $\neg late$

¬G: $\neg bus$ C6

C6 & C1: C7: cycle

C7 & C3: C8: late

C8 & C5: FALSE

Resolution Refutation for Predicate Logic

Given a formula F which we wish to check for validity, we first check if there are any free variables. We then quantify all free variables universally.

Create F' = ~F and check for unsatisfiability of F'

STEPS:

Conversion to Clausal (CNF) Form:

- Handling of Variables and Quantifiers, Ground Instances

Applying the Resolution Rule:

- Concept of Unification
- Principle of Most General Unifier (mgu)
- Repeated application of Resolution Rule using mgu

F1: ~~∀~~x(goes(Mary, x) → goes(Lamb, x))

F2: goes(Mary, School)

G: goes(Lamb, School)

To prove: (F1 Λ F2) → G) is valid

constants
variables
predicates
functions
∀ ∃

CONVERSION TO CLAUSAL FORM IN PREDICATE LOGIC

1. Remove implications and other Boolean symbols converting to equivalent forms using ~, V, Λ
2. Move negates (~) inwards as close as possible
3. Standardize (Rename) variables to make them unambiguous
4. Remove Existential Quantifiers by an appropriate new function /constant symbol taking into account the variables dependent on the quantifier (Skolemization)
5. Drop Universal Quantifiers
6. Distribute V over Λ and convert to CNF

$\boxed{F1 \wedge F2 \wedge \neg G}$ $\forall x \{ \neg goes(Mary, x) \vee goes(Lamb(x)) \}$

C1: ¬goes(Mary, x) V goes (Lamb(x))

C2: goes(Mary, School) C3: ¬goes(Lamb, School)

C4: goes(Lamb, School) → FALSE

Conversion to Clausal Form

1. Remove implications and other Boolean symbols converting to equivalent forms using ~, V, Λ
2. Move negates (~) inwards as close as possible
3. Standardize (Rename) variables to make them unambiguous
4. Remove Existential Quantifiers by an appropriate new function /constant symbol taking into account the variables dependent on the quantifier (Skolemization)
5. Drop Universal Quantifiers
6. Distribute V over Λ and convert to CNF

$\forall x(\forall y(\text{student}(y) \rightarrow \text{likes}(x, y)) \rightarrow (\exists z(\text{likes}(z,x)))$

$\forall x \{ \forall y (\text{student}(y) \rightarrow \text{likes}(x,y)) \rightarrow \exists z (\text{likes}(z,x)) \}$

$\forall x \{ \neg (\forall y (\text{student}(y) \rightarrow \text{likes}(x,y))) \vee \exists z (\text{likes}(z,x)) \}$

$\forall x \{ \neg (\forall y (\neg \text{student}(y) \vee \text{likes}(x,y))) \vee \exists z (\text{likes}(z,x)) \}$

$\forall x \{ \exists y ((\text{student}(y) \wedge \neg \text{likes}(x,y))) \vee \exists z (\text{likes}(z,x)) \}$

$F(x), G(x)$

$\cancel{\forall x} \{ (\text{student}(F(x)) \wedge \neg \text{likes}(x, F(x))) \vee \text{likes}(G(x), x) \}$

Substitution, Unification, Resolution

Consider clauses:

- C1: ~studies(x,y) V passes(x,y)
- C2: studies(Madan,z)
- C3: ~passes(Chetan, Physics)
- C4: ~passes(w, Mechanics)

What new clauses can we derive by the resolution principle?

Ground Clause and a more general clause

Concept of substitution / unification and the Most General Unifier (mgu)

Resolution Rule for Predicate Calculus: Repeated Application of Resolution using mgu

$\exists x \forall y\, p(x,y) \Rightarrow \forall y\, p(A,y) \Rightarrow p(A,y)$

$\forall x \exists y\, p(x,y) \Rightarrow \forall x\, p(x,F(x))$

SKOLEM FN $\Rightarrow p(x,F(x))$

C1 & C2 : C′ passes(Madan, z)

C1 & C2 C″ passes(Madan, Physics)

C′ is more general than C″

C1 & C4 : y/Mechanics

¬ studies (x, Mechanics) C‴

C2 & C‴ : FALSE

most general unifier

Examples

F1: ∀x(contractor(x) → ~dependable(x)) c(x)

F2: ∃x(engineer(x) ∧ contractor(x)) d(x)

G: ∃x(engineer(x) ∧ ~dependable(x)) e(x)

F1: $\forall x\{\neg c(x) \vee \neg d(x)\}$ c1

F2: $e(A) \wedge c(A)$ c2 c3

¬G: $\neg \exists x (e(x) \wedge \neg d(x))$

$\forall x (\neg e(x) \vee d(x))$ c4

$d(A)$ c5

$\neg c(A)$: c6

FALSE

F1: ∀x(dancer(x) → graceful (x))

F2: student(Ayesha), F3: dancer(Ayesha)

G: ∃x(student (x) ∧ graceful(x))

F1: $\forall x\{\neg d(x) \vee g(x)\}$ C1

F2: $s(A)$: C2 F3: $d(A)$: C3

¬G: $\neg \exists x\{s(x) \wedge g(x)\}$

$\forall x\{\neg s(x) \vee \neg g(x)\}$ C4

x/A $\neg g(A)$

x/A $\neg d(A)$

FALSE

0. F1 ∧ F2 ∧ … ∧ ¬G

1. Convert to Clausal Form

2. Resolution Rule

UNSATISFIABILITY

Thank you