

Function and Limits

- $\lim_{x \rightarrow \pm \infty} \frac{Ax^\alpha}{Bx^\beta} \begin{cases} 0 & \text{if } \alpha < \beta \\ \frac{A}{B} & \text{if } \alpha = \beta \\ \pm \infty & \text{if } \alpha > \beta \end{cases}$
- $\lim_{x \rightarrow c} \frac{\sin(g(x))}{g(x)} = 1 \big( \lim_{x \rightarrow c} g(x) = 0 \big)$
- $\lim_{x \rightarrow c} \frac{\tan(g(x))}{g(x)} = 1$
- $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{\tan(x)}{x} = 1$
- $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$

Differentiation

parametric differentiaton:  $\frac{d^2y}{dx^2} = \frac{d}{dx} \big( \frac{dy}{dx} \big) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$a^{f(x)}$	$\ln a \cdot f'(x) a^{f(x)}$
$\log_a f(x)$	$\log_a e \cdot \frac{f'(x)}{f(x)}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad  f(x)  < 1$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad  f(x)  < 1$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\cot^{-1} f(x)$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{ f(x) \sqrt{[f(x)]^2-1}}$
$\csc^{-1} f(x)$	$-\frac{f'(x)}{ f(x) \sqrt{[f(x)]^2-1}}$

**Second Derivative Test:**  $f'(c) = 0, f''(c) < 0$  then local max,  $f''(c) > 0$  local min.

**L'Hopital's Rule:** Given  $\lim_{x \rightarrow c} f(x)$  and  $g(x) = 0 / \pm \infty$

$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$

- Use for  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$

Trigo Identities

- $\sec^2 x - 1 = \tan^2 x$
- $\csc^2 x - 1 = \cot^2 x$
- $\sin A \cos A = \frac{1}{2} \sin 2A$
- $\cos^2 A = \frac{1}{2} (1 + \cos 2A)$
- $\sin^2 A = \frac{1}{2} (1 - \cos 2A)$
- $\sin A \cos B = \frac{1}{2} (\sin(A+B) + \sin(A-B))$
- $\cos A \sin B = \frac{1}{2} (\sin(A+B) - \sin(A-B))$
- $\cos A \cos B = \frac{1}{2} (\cos(A+B) + \cos(A-B))$
- $\sin A \sin B = \frac{1}{2} (\cos(A+B) - \cos(A-B))$

Integration

$f(x)$	$\int f(x)$
$\tan ax$	$\frac{1}{a} \ln  \sec(ax) $
$\cot ax$	$\frac{1}{a} \ln  \cot(ax) $
$\sec ax$	$\frac{1}{a} \ln  \sec(ax) + \tan(ax) $
$\csc ax$	$\frac{1}{a} \ln  \csc(ax) + \cot(ax) $
$\frac{1}{a^2 + (x+b)^2}$	$\frac{1}{a} \tan^{-1} \big( \frac{x+b}{a} \big)$
$\frac{1}{\sqrt{a^2 - (x+b)^2}}$	$\sin^{-1} \big( \frac{x+b}{a} \big)$
$\frac{1}{a^2 - (x+b)^2}$	$\frac{1}{2a} \ln \left  \frac{x+b+a}{x+b-a} \right $
$\frac{1}{(x+b)^2 - a^2}$	$\frac{1}{2a} \ln \left  \frac{x+b-a}{x+b+a} \right $

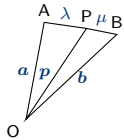
**Substitution**  $\int f(g(x)) \cdot g'(x) dx = \int f(u) du, u = g(x)$   
**By Parts**  $\int uv' dx = uv - \int u'v dx$ , order: LIATE:  
Differentiate to integrate

Application of Integration

- about x axis
- Vol Disk:  $V = \pi \int_a^b f(x)^2 - g(x)^2 dx$
  - Vol Shell:  $V = 2\pi \int_a^b x|f(x) - g(x)| dx$  (absolute!!)
  - Length of curve:  $\int_a^b \sqrt{1 + f'(x)^2} dx$

Vectors

unit vector:  $\hat{p} = \frac{p}{|p|}, \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$



**ratio theorem**  
 $p = \frac{\mu a + \lambda b}{\lambda + \mu}$

**midpoint theorem**  
 $p = \frac{a+b}{2}$

Dot Product

- $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = |a||b| \cos \theta$
- $a \perp b \Rightarrow a \cdot b = 0$
- $a \parallel b \Rightarrow a \cdot b = |a||b|$

Cross Product

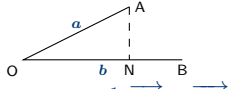
$$a \times b = \begin{vmatrix} i & j & k \\ a_i & a_2 & a_3 \\ b_i & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} (a_2b_3 - a_3b_2) \\ -(a_1b_3 - a_3b_1) \\ (a_1b_2 - a_2b_1) \end{pmatrix}$$

$$|a \times b| = |a||b| \sin \theta$$
$$a \perp b \Rightarrow a \times b = |a||b|$$

$$a \parallel b \Rightarrow a \times b = 0$$

Parallelogram =  $|a \times b|$

Projection



$$\text{comp}_b a = |b| \cos \theta = \frac{a \cdot b}{|a|}$$
$$\text{proj}_b a = \text{comp}_b a \cdot \frac{b}{|b|} = \frac{a \cdot b}{|a|^2} b$$

$$\overrightarrow{ON} = \frac{a \cdot b}{a \cdot a} a = \frac{a \cdot b}{|a|^2} b$$

$$\triangle ANO = \frac{1}{2} |\overrightarrow{OA} \times \overrightarrow{ON}|$$

Lines

$r = r_0 + tv = \langle x, y, z \rangle$   
 $\langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

$$\begin{pmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{pmatrix}$$

Planes

$n = \langle a, b, c \rangle, r = \langle x, y, z \rangle, r_0 \langle x_0, y_0, c_0 \rangle$   
Scalar:  $n \cdot r = n \cdot r_0$   
Cartesian:  $ax + by + cz = d$

Distance from Point to Plane

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Partial Derivatives

Chain Rule

For  $z(t) = f(x(t), y(t))$ ,  
 $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

For  $z(s, t) = f(x(s, t), y(s, t))$ ,  
 $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$   
 $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$

Arc Length of  $r(t)$ :  $\int_a^b |r'(t)| dt$

Implicit Differentiation

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Directional Derivative

Gradient vector at  $f(x, y) : \triangle f = f_x i + f_y j$   
 $D_u f(x, y) = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = \langle f_x, f_y \rangle \cdot \hat{u} = \triangle f \cdot \hat{u}$  (Unit Vector)  
Tangent Plane:  $\triangle f \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

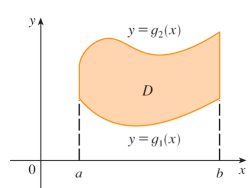
Critical Points

$$D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$$

$D$	$f_{xx}(a, b)$	local
+	+	min
+	-	max
-	any	saddle point
0	any	no conclusion

Double Integrals

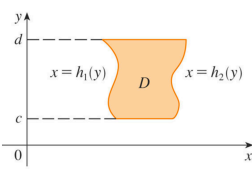
Type I



$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

$$D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

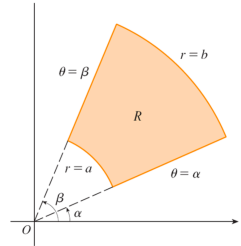
Type II



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

$$D = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

Polar Coordinates



$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$
$$\int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

Surface Area

$$S = \iint_R \sqrt{f_x^2 + f_y^2 + 1} dA$$

ODE

form	change of variable
$\frac{dy}{dx} = f(x)g(y)$	$\int \frac{1}{g(y)} dy = \int f(x) dx + C$
$y' = g(\frac{y}{x})$	Set $v = \frac{y}{x}$ $\Rightarrow y' = v + xv'$
$y' = f(ax + by + c)$ $\Rightarrow y' = \frac{ax+by+c}{\alpha x + \beta y + \gamma}$	Set $v = ax + by$
$y' + P(x)y = Q(x)$	$R = e^{\int P(x) dx}$ $\Rightarrow y \cdot R = \int Q \cdot R dx$ $z = y^{1-n}$ $\Rightarrow$ sub in Z solve linear
$y' + P(x)y = Q(x)y^n$	

Population Models

$$N_\infty = \frac{B}{s}, \hat{N} =$$

Population Now

**Malthus**  
 $N(t) = \hat{N} e^{kt}$   
 $k = B - D$

**Logistic**

$$\frac{1}{N_\infty} + \left( \frac{1}{\hat{N}} - \frac{1}{N_\infty} \right) e^{-Bt}$$
$$N' = \frac{N(N_\infty - N)}{1 + \left( \frac{N_\infty}{\hat{N}} - 1 \right) e^{-Bt}}$$

Uranium Decay into Thorium

$U(t) = U_0 e^{-k_u t}, k = \frac{\ln 2}{\text{halflife}}, \frac{dU}{dt} = -k_u U$   
Thorium:  
 $T(t) = \frac{K_u U_0}{K_t - K_u} (e^{-k_u t} - e^{-k_t t}), \frac{dT}{dt} = k_u U - k_T T$

Series

Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1}, a \neq 0$  converges to  $\frac{a}{1-r}$  when  $|r| < 1$ , diverges otherwise

If series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$

Tests

Decreasing function -> differentiate and see the range where  $x < 0$

Test	Method
$n^{th}$ term	$\lim_{n \rightarrow \infty} a_n \neq 0$ or does not exist, then divergent
Integral	$f(n) = a_n$ is continuous, positive, decreasing function $\forall x \geq 1$ and $\int_1^{\infty} f(x)dx$ converges else divergent
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent $\leftrightarrow p > 1$
Harmonic Series	$\sum_{n=1}^{\infty} \frac{1}{n}$ divergent
Ratio If Factorial	$0 \leq \lim_{n \rightarrow \infty} \left  \frac{a_{n+1}}{a_n} \right  = L < 1$ abs. convergent, $> 1$ divergent, $= 1$ inconclusive
Root If nth power	$0 \leq \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L < 1$ abs. convergent, $> 1$ divergent, $= 1$ inconclusive
Alternating series	$b_n$ decreasing, $\lim_{n \rightarrow \infty} b_n = 0$ , then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 \dots$ is convergent
Power Series	$b_n$ decreasing, $\lim_{n \rightarrow \infty} b_n = 0$ , then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 \dots$ is convergent
Comparison Test	Compare to well known series such as p-series, harmonic

Power Series

- $\sum_{n=0}^{\infty} c_n(x-a)^n$  converges at **ONE OF**
- $x = a$
  - For all  $x$
  - converges if  $|x-a| < R$  and diverges if  $|x-a| > R$  (R is radius of convergence)
- If  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = L$  or  $\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} = L, L \in \mathbb{R}$  or  $\infty$ , then  $R = \frac{1}{L}$

Taylor and Maclaurin Series

If f has power series repr @  $f = a$ ,  
 $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, |x-a| < R, R > 0$ , then  
 $c_n = \frac{f^{(n)}(a)}{n!}$ .

Maclaurin Series:  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$  For  $-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For  $-1 < x < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$
$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$$
$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$
$$\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$
$$= 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots$$