CS6845: Pseudorandomness 1 Feb 2018

10. Equivalence of various notions of expansion

10.1 Introduction

In this lecture, we will compare spectral expansion, vertex expansion and edge expansion. We will then state and prove the expander mixing lemma.

10.2 Comparing vertex and spectral expansion

Now we will show that the two notions of vertex and spectral expansion are equivalent up to some parameters.

Theorem 10.1 (Spectral expansion $\Rightarrow$ vertex expansion)**.** *If G is a regular digraph with spectral expansion (spectral gap) $\gamma = 1-\lambda$ for some $\lambda \in [0,1]$, then for any $\alpha \in [0,1]$, G is an $\left(\alpha n, \frac{1}{(1-\alpha)\lambda^2+\alpha}\right)$-vertex expander.*

Before we proceed with the proof of this statement, we need some definitions. For a probability distribution π over some domain, the collision probability $\mathrm{Coll}(\pi)$ is the probability that two random samples drawn according to π returns the same value. I.e. $\mathrm{Coll}(\pi) = \sum \pi_i^2$. Similarly the support of π, $\mathrm{Supp}(\pi) = \{i \mid \pi_i \neq 0\}$.

Observation 10.2. *For any probability distribution π,* $\mathrm{Coll}(\pi) = \|\pi\|^2 = \frac{1}{n} + \|\pi - u\|^2$.

Proof. $\mathrm{Coll}(\pi) = \|\pi\|^2$ from the definition of collision probability. Rewrite π as $\pi - u + u$. Then, $\|\pi\|^2 = \|(\pi-u)+u\|^2 = \|\pi-u\|^2 + \frac{1}{n}$. □

Observation 10.3. *For any probability distribution π,* $\mathrm{Coll}(\pi) \geq \frac{1}{|\mathrm{Supp}(\pi)|}$.

Proof. Using Cauchy-Schwartz, we can say that $1 = \sum_{i \in \mathrm{Supp}(\pi)} \pi_i \leq \sqrt{|\mathrm{Supp}\,\pi|}\sqrt{\sum_{i \in \mathrm{Supp}(i)} \pi_i^2}$. □

Proof of Theorem 10.1. We know that for every probability distribution π, $\|\pi A - u\|^2 \leq \lambda^2 \|\pi - u\|^2$. From the first observation above, this can be written as $\mathrm{Coll}(\pi A) - \frac{1}{n} \leq \lambda^2 \left(\mathrm{Coll}(\pi) - \frac{1}{n}\right)$. Let π be the probability distribution that is uniform over a set S such that $|S| \leq \alpha n$. Then, $\mathrm{Coll}(\pi) = \frac{1}{|S|}$. Also the support of πA is the set of vertices in $N(S)$. Therefore, $\mathrm{Supp}(\pi A) = N(S)$. Substituting in the equation above, we get $\frac{1}{|N(S)|} - \frac{1}{n} \leq \lambda^2 \left(\frac{1}{|S|} - \frac{1}{n}\right)$. Using $n \geq |S|/\alpha$, we get the theorem. □

Now, the converse of this statement.

Theorem 10.4 (vertex expansion $\Rightarrow$ spectral expansion)**.** *For every $\delta > 0$, and $d > 0$, there exists a $\gamma > 0$ such that if G is a d-regular $\left(\frac{n}{2}, 1+\delta\right)$-vertex expander, then it also has spectral expansion γ (which depends on δ and d).*

10.2.1 Remarks on the equivalence

A few points to note about the equivalence between vertex and spectral expansion.

- It is necessary to start with an $\left(\frac{n}{2}, 1+\delta\right)$-vertex expander in Theorem 10.4 since if you take two disconnected $(n/2, 1+\delta)$-vertex expander then it is is an $(\alpha n, 1+\delta)$ expander, but not a spectral expander.
- Also, it is possible to add edges to a graph and reduce its spectral expansion. How does adding self-loops change the spectral expansion?
- For every constant d, any d-regular n-vertex multigraph G satisfies $\lambda(G) \geq \frac{2\sqrt{d-1}}{d} - o(1)$. Also, there are explicit constructions that achieve this bound. These are the "best" spectral expanders, whereas the theorem does not give you tight bounds.

10.3 Edge expansion

A d-regular digraph G is a (K, ε)-edge expander if for all sets S of at most k vertices, the cut size $|E(S, \bar{S})|$ is at least $\varepsilon|S|d$. In other words, a d-regular digraph G is a (K, ε)-edge expander if for all sets S of size at most k, a random walk of one step will take you out of the set S if you initially had uniform distribution on S. We now show that it is equivalent to spectral expansion.

Theorem 10.5. *The following two statements are hold.*

1. *[Spectral expansion $\Rightarrow$ edge expansion] If a d-regular, n-vertex digraph G has spectral expansion γ, then G is an $(n/2, \gamma/2)$-edge expander.*
2. *[Edge expansion $\Rightarrow$ spectral expansion] If a d-regular, n-vertex digraph G is an $(n/2, \varepsilon)$ edge expander and at least α fraction of edges leaving each vertex are self-loops for some $\alpha \in [0, 1]$, then G has spectral expansion $\alpha\varepsilon^2/2$.*

The self loops are needed to avoid the case of bipartite expanders. Bipartite expanders are good vertex and edge expanders, but poor spectral expanders. Before we prove the "spectral expansion $\Rightarrow$ edge expansion" part of the theorem, we will prove a more general statement known as the *expander mixing lemma.*

Lemma 10.6 (Expander mixing lemma)**.** *Let G be a d-regular n-vertex digraph with spectral expansion $1-\lambda$. Then for all sets of vertices S and T of densities $\alpha = |S|/n$ and $\beta = |T|/n$, we have*

$$\left|\frac{|E(S,T)|}{nd} - \alpha\beta\right| \leq \lambda\sqrt{\alpha(1-\alpha)\beta(1-\beta)} \leq \lambda\sqrt{\alpha\beta} \leq \lambda.$$

Proof. Let χ_S and χ_T denote the characteristic vectors corresponding to the set S and T. The edge set $|E(S,T)| = \chi_S(dA)\chi_T$. Now, let $\chi_S = \chi_S^{\parallel} + \chi_S^{\perp}$, and $\chi_T = \chi_T^{\parallel} + \chi_T^{\perp}$, where $\chi_S^{\parallel}$ is the component of χ_S along the uniform distribution u, and $\chi_S^{\perp}$ is the component of χ_S

that is perpendicular to u. We can write $\chi_S^{\|} = \frac{\langle \chi_S, u\rangle}{\langle u, u\rangle} u = \alpha n$. Similarly, $\chi_T^{\|} = \beta n$. We can write

$$\begin{aligned}
\frac{|E(S,T)|}{nd} &= \frac{1}{n}\left(\alpha n u + \chi_S^{\perp}\right) A \left(\beta n u + \chi_T^{\perp}\right)^t \\
&= \frac{1}{n}\left(\alpha\beta n^2 u A u^t + \beta n \chi_S^{\perp} A u^t + \alpha n u A \chi_T^{\perp} + \chi_S^{\perp} A (\chi_T^{\perp})^t\right) \\
&= \alpha\beta + \frac{1}{n}\chi_S^{\perp} A (\chi_T^{\perp})^t
\end{aligned}$$

To prove the lemma we need to bound the norm of $\chi_S^{\perp} A (\chi_T^{\perp})^t$. $\square$