CS6845 - Pseudorandomness
Assignment 2

Due date: Feb 15, in class (11 AM)

Instructions

- For each day's delay in submission, you will lose 50% marks.
- Discussing the questions on the assignment with your classmates is allowed, but **no** collaboration is allowed while writing up the solution. If you discuss an assignment problem with someone, please acknowledge that while writing the solution. This will **not** affect your grade.
- Checking online sources for answers is **strongly discouraged**. If you need any clarification, you are welcome to discuss with me or Dinesh. Any academic dishonesty will result in zero marks for the assignment.
- Please be concise in stating your answers/proofs.

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1. (6 points) (**Randomized Max-3SAT**) Let S be a set of clauses of the form $\ell_i \vee \ell_j \vee \ell_k$, where each clause has exactly 3 literals (A literal is a variable or its negation).
   (a) (1 point) Show that there is an assignment to the variables that satisfies at least $7|S|/8$ clauses.
   (b) (3 points) Give a randomized algorithm that assigns values to the variables such that at least $7|S|/8$ clauses are satisfied. What is the probability that your algorithm fails to achieve this? What is the running time of the algorithm if you want the error probability to be at most $1/100$?
   (c) (2 points) Give a deterministic algorithm that gives an assignment to the variables that satisfies at least $7|S|/8$ clauses. What is the time complexity of the deterministic algorithm?

2. (4 points) (**Expander mixing lemma via matrix decomposition**) In the class, we showed that every random walk matrix A for a graph with spectral expansion $\gamma = 1 - \lambda$ can be written as $A = \gamma J + \lambda E$, where J is the $n \times n$ matrix with all its entries $1/n$ and E is an $n \times n$ matrix such that $\|E\| \leq 1$. Here, $\|E\| = \max_{x \in \mathbb{R}^n} \|xE\|/\|x\|$. In this exercise, we will prove a slightly weaker version of the expander mixing lemma using this matrix decomposition.

   Let G be a d-regular digraph on n vertices that has spectral expansion λ. Let S, T be two sets such that $|S| = \alpha n$ and $|T| = \beta n$ and let $\mathcal{E}(S,T)$ denote the set of edges between S and T. Show that

$$\left| \frac{|\mathcal{E}(S,T)|}{nd} - \alpha\beta \right| \leq \lambda\sqrt{\alpha\beta} + \lambda\alpha\beta.$$

3. (10 points) (**Hitting time of a random walk**) For a digraph G, we define its hitting time as

$$\mathrm{hit}(G) = \max_{i,j \in V} \min \{t \mid \Pr[\text{ a random walk of length } t \text{ from } i \text{ visits } j\ ] \geq 1/2\}$$

   In class, we saw that for any undirected graph G, the hitting time is $\mathrm{poly}(n)$. In this exercise we will study some properties of directed graphs.

(a) (3 points) Show that for every n, there exists a digraph G with n vertices, outdegree 2, and $\mathrm{hit}(G) = 2^{\Omega(n)}$.

(b) (4 points) Let G be a regular digraph with random walk matrix M. Show that $\lambda(G) = \sqrt{\lambda(G')}$ where G' is the undirected graph whose random-walk matrix is MM^T.

(c) (3 points) A digraph G is called *Eulerian* if it is connected and every vertex has the same number of outgoing edges as incoming edges. Show that if G is n-vertex Eulerian digraph of maximum degree d, then $\mathrm{hit}(G) = \mathrm{poly}(n, d)$.

4. (5 points) (**Expander from finite fields**) Let $\mathbb{F}$ be a finite field. Consider a graph G with vertex set $\mathbb{F} \times \mathbb{F}$ and edge set $\{((a, b), (c, d)) \mid ac = b + d\}$. In other words, we connect vertex (a, b) to all points on the line $y = ax - b$. Prove that G is $|\mathbb{F}|$-regular and $\lambda(G) \leq \frac{1}{\sqrt{|\mathbb{F}|}}$.

   (**Hint**: Consider the graph G^2)