CS6845: Pseudorandomness 8 Feb 2018

13. Zig-zag product of graphs

13.1 Introduction

In this lecture, we will study the zig-zag product of graphs and use it to construct fully explicit constant-degree expanders.

13.2 Zig-zag product

Let G_1 be an (n_1, d_1, λ_1)-expander, and let G_2 be a (d_1, d_2, λ_2)-expander. The zig-zag product of G_1 and G_2, denoted by $G_1 ⓩ G_2$ is a graph on $n_1 d_1$ vertice with degree d_2^2 defined in the following way. Let $(i_i, i_2) \in [d_2] \times [d_2]$, and let $v[i]$ denote the i^{th} neighbor of a vertex v. For a vertex $(u, j) \in G_1 ⓩ G_2$, we obtain the $(i_1, i_2)^{th}$ neighbor as follows: Let $j' = j[i_1]$, and let $v = u[j']$, and let $j'' = j'[i_2]$. Then $(u, j)[i_1, i_2] = (u, j'')$.

Let Rot_{G_1} and Rot_{G_2} denote the rotation maps corresponding to the graphs G_1 and G_2. Let us first calculate the rotation map corresponding to $G_1 ⓩ G_2$. Let $(v, i) \in G_1 ⓩ G_2$, and $(j, k) \in [d_2] \times [d_2]$; then $\mathrm{Rot}_{G_1 ⓩ G_2}((v, i), (j, k))$ is obtained as follows:

1. Let $\mathrm{Rot}_{G_2}(i, j) = (i_1, j_1)$.
2. Let $\mathrm{Rot}_{G_1}(v, i_1) = (w, i_2)$.
3. Let $\mathrm{Rot}_{G_2}(i_2, k) = (i_3, k_1)$.

Then $\mathrm{Rot}_{G_1 ⓩ G_2}((v, i), (j, k)) = (w, i_3)$. So, taking a random step from a vertex (v, i) in $G_1 ⓩ G_2$ corresponds to taking a random step in the graph G_2 in the "cloud" correspnding to v (zig step), then taking a step to the "cloud" w based on the vertex you ended up in the zig step, and then finally taking a random step in the "cloud" corresponding to w (zag step).

Notice that the step going across the cloud from v to w depends on the random zig step. The zig step is a random step in an expander. So, if that is mixed well, then the step going from v to w is indeed like a random step on the larger graph G_1. We will analyze the random walks on $G_1 ⓩ G_2$ and prove the following theorem.

Theorem 13.1. *Let G_1 be an (n_1, d_1, λ_1)-expander and let G_2 be a (d_1, d_2, λ_2)-expander. Then, $G_1 ⓩ G_2$ is an $(n_1 d_1, d_2^2, f(\lambda_1, \lambda_2))$-expander where $f(\lambda_1, \lambda_2) \leq \lambda_1 + \lambda_2 + \lambda_2^2$ such that $f(\lambda_1, \lambda_2) < 1$ when $\lambda_1, \lambda_2 < 1$.*

Before we analyze the zig-zag product, let us first see how to obtain the explicit family of constant-degree expanders using the theorem above.

13.3 Constant-degree expanders using the zig-zag product

Let H be a $(d^4, d, 1/8)$-expander. Definte the following family of graphs inductively.

Theorem 13.2. *For $t \in \mathbb{N}$ consider the following family:*

$$\begin{aligned} G_1 &= H^2 \\ G_t &= G_{t-1}^2 ⓩ H \end{aligned}$$

The following statements are true for $\{G_t\}$.

1. *For $t \in \mathbb{N}$, G_t is an $(d^{4t}, d^2, 1/2)$-expander.*
2. *The time to compute Rot_{G_t} is poly$(|G_t|)$.*

Proof. We prove (1) by induction on t. For $t = 1$, G_1 is a $(d^4, d^2, 1/64)$-expander, and thus the base case is proved. For $t > 1$, $G_t = G_{t-1}ⓩH$. Let G_{t-1} be a $(d^{4(t-1)}, d^2, 1/2)$-expander. Now, G_{t-1}^2 is a $(d^{4(t-1)}, d^4, 1/4)$-expander and H is a $(d^4, d, 1/8)$-expander. Therefore, $G_{t-1}^2ⓩH$ is a (d^{4t}, d^2, λ)-expander, where $\lambda \leq 1/4 + 1/8 + 1/64 < 1/2$. This completes the induction.

To prove (2), let $T(G_t)$ denote the time to compute the rotation map of G_t. For this we need two oracle queries to the rotation maps of G_{t-1} and poly$(\log |G_{t-1}|)$ to compute using the oracle queries. Therefore, we have the following recurrence.

$$T(G_t) = 2T(G_{t-1}) + \mathrm{poly}(\log |G_{t-1}|).$$

This shows that we need 2^tpoly$(\log |G_{t-1}|)$ time to compute the rotation maps which is poly$(|G_t|)$. □

Now to obtain the fully explicit family we are after, we will use the tensor product to make the size of the graph grow faster.

Theorem 13.3. *Let H be a $(d^8, d, 1/8)$-expander. For $t \in \mathbb{N}$, consider the following family:*

$$\begin{aligned} G_1 &= H^2, \\ G_2 &= H \otimes H, \\ G_t &= \left(G_{\lceil \frac{t-1}{2} \rceil} \otimes G_{\lfloor \frac{t-1}{2} \rfloor} \right)^2 ⓩ H. \end{aligned}$$

The following statements are true for $\{G_t\}$.

1. *For $t \in \mathbb{N}$, G_t is an $(d^{8t}, d^2, 1/2)$-expander.*
2. *The time to compute Rot_{G_t} is poly$(\log |G_t|)$.*

Proof. The proof of (1) is again by induction on t. The graph H^2 is a $(d^8, d^2, 1/2)$-expander, so the statement is true for G_1. Similarly, $H \otimes H$ is a $(d^{16}, d^2, 1/2)$-expander, so the statement is true for G_2. Consider G_t for $t > 2$. Now, $G_{\lceil \frac{t-1}{2} \rceil} \otimes G_{\lfloor \frac{t-1}{2} \rfloor}$ is a $(d^{8(t-1)}, d^4, 1/2)$-expander. Therefore, $\left(G_{\lceil \frac{t-1}{2} \rceil} \otimes G_{\lfloor \frac{t-1}{2} \rfloor}\right)^2$ is a $(d^{8(t-1}, d^8, 1/4)$-expander. Therefore G_t is a (d^{8t}, d^2, λ)-expander where $\lambda \leq 1/4 + 1/8 + 1/64 < 1/2$.

Now to prove (2) we again write the recurrence for $T(G_t)$. Now the recurrence can be rewritten as follows:

$$T(G_t) = 2T(G_{(t-1)/2}) + \text{poly}(\log |G_{t-1}|, \log |H|).$$

Solving this we get that $T(G_t) = 2^{\log t}\text{poly}(\log |G_{t-1}|, \log |H|)$. Therefore $\{G_t\}$ is a fully explicit family of expanders. $\square$

Now we move to the analysis of zig-zag products.