CS6845: Pseudorandomness 2 Feb 2018

11. Expander mixing lemma, random walks on expanders

11.1 Introduction

In this lecture, we will complete the proof of expander mixing lemma and prove bounds on random walks on expanders.

11.2 Expander mixing lemma

Lemma 11.1 (Expander mixing lemma)**.** *Let G be a d-regular n-vertex digraph with spectral expansion $1-\lambda$. Then for all sets of vertices S and T of densities $\alpha = |S|/n$ and $\beta = |T|/n$, we have*

$$\left| \frac{|E(S,T)|}{nd} - \alpha\beta \right| \le \lambda\sqrt{\alpha(1-\alpha)\beta(1-\beta)} \le \lambda\sqrt{\alpha\beta} \le \lambda.$$

In the last lecture we saw that

$$\begin{aligned} \left| \frac{|E(S,T)|}{nd} - \alpha\beta \right| &= \frac{1}{n}\left| \chi_S^{\perp} A (\chi_T^{\perp})^t \right| \\ &\le \frac{1}{n}\|\chi_S^{\perp} A\|\|\chi_T^{\perp}\| \le \frac{1}{n}\lambda\|\chi_S^{\perp}\|\|\chi_T^{\perp}\|. \end{aligned}$$

So it is sufficient to bound the norms $\|\chi_S^{\perp}\|$ and $\|\chi_T^{\perp}\|$. Now, $\|\chi_S\|^2 = \|\chi_S^{\parallel}\|^2 + \|\chi_S^{\perp}\|^2 = \|\alpha n u\|^2 + \|\chi_S^{\perp}\|^2 = \alpha^2 n + \|\chi_S^{\perp}\|^2$. Since $\|\chi_S\|^2 = \alpha n$, we have $\|\chi_S^{\perp}\| = \sqrt{n\alpha(1-\alpha)}$. Similarly, $\|\chi_T^{\perp}\| = \sqrt{n\beta(1-\beta)}$. This completes the proof of the expander mixing lemma.

11.3 Random walks on expanders

We will now show that random walks on expanders never get concentrated on any small set of vertices. This gives a natural way to reduce the error of one-sided error randomized algorithms. The theorem we will prove is the following.

Theorem 11.2. *Let G be a regular digraph on n vertices, and let B be a set of vertices of cardinality μn. Let $v_1, v_2, \ldots, v_t$ be a random walk on G where v_1 is chosen uniformly from G. Then we have the following.*

$$\Pr\left[\bigwedge_{i=1}^{t} v_i \in B\right] \le (\mu + \lambda(1-\mu))^t$$

Let P be an $n \times n$ diagonal matrix defined as follows: $P(i,i) = 1$ if $i \in B$, else $P(i,i) = 0$. The probability that a random vertex chosen from G is in B is given by $|uP|_1$ where u is the uniform distribution vector and $|v|_1 = \sum |v_i|$. Conditioned on the first vertex being in B the probability that one step of the random walk stays in B is given by $|uPAP|_1$. Using induction we can prove the following.

Lemma 11.3. *Let G be a regular digraph on n vertices, and let B be a set of vertices of cardinality μn. Let $v_1, v_2, \ldots, v_t$ be a random walk on G where v_1 is chosen uniformly from G. Then we have the following.*

$$\Pr\left[\bigwedge_{i=1}^{t} v_i \in B\right] \le |uP(AP)^t|_1$$

To prove an upper bound on the ℓ_1 norm, we are going to use a matrix decomposition of the random walk matrix A. First we need the notion of spectral norm.

Definition 11.4. *Let A be any matrix over $\mathbb{R}$. The spectral norm of A is defined as*

$$\|A\| = \max_{x \in \mathbb{R}^n} \frac{\|xA\|}{\|x\|}$$

We use the following lemma for the matrix decomposition.

Lemma 11.5. *Let G be a regular digraph on n vertices with a random walk matrix A. Then G has spectral expansion $\gamma = 1 - \lambda$ iff $A = \gamma J + \lambda E$, where J is the $n \times n$ matrix with all entries $1/n$ and $\|E\| \le 1$.*

Proof. Let $E = (A - \gamma J)/\lambda$. In the forward direction, suppose G has spectral expansion $\gamma = 1 - \lambda$. Then, we need to show that $\|E\| \le 1$. For u, we have $uE = \frac{1}{\lambda}uA - \frac{\gamma}{\lambda}uJ$. Now, both uA and uJ is u since they are both random walk matrices. Therefore, $\|uE\| = \|u\|$. Now for any $v \perp u$, we have $vE = \frac{1}{\lambda}vA - \frac{\gamma}{\lambda}vJ$. Since $v \perp u$, $vJ = 0$. Therefore $\|vE\| = \frac{1}{\lambda}\|vA\| \le \|v\|$.

To prove the other direction, let $v \perp u$ be any vector. Then $vA = v(\gamma J + \lambda E)$. Since $v \perp u$, $vJ = 0$. Therefore, $\|vA\| = \lambda \|vE\| \le \lambda \|v\|$. □

Since P is a diagonal matrix, $P^2 = P$, therefore, we can rewrite $uP(AP)^t$ as $u(PAP)^t$. Also since it is easier to work with ℓ_2 norm, we will bound the spectral norm of PAP first.

Claim 11.6. *$\|PAP\| \le \mu + \lambda(1 - \mu)$.*

Proof. Write $PAP = P(\gamma J + \lambda E)P = \gamma PJP + \lambda PEP$. Now, $\|PAP\| \le \|\gamma PJP\| + \|\lambda PEP\| \le \gamma \|PJP\| + \lambda$.

Let $x \in \mathbb{R}^n$. Then we have $xPJP = yJP$ where $y = xP$. Now, $\|xPJP\| = \|yJP\|$. This can be rewritten as $\|xPJP\| = \|(\sum y_i)uP\| \le |\sum y_i|\, \|uP\|$. Since y has at most $|B|$ many non-zero entries, we can use the Cauchy-Schwarz inequality to write $\|xPJP\| \le \sqrt{\mu n}\|y\|\sqrt{\mu/n}$. Therefore $\|xPJP\| \le \mu\|x\|$. Therefore, $\|PAP\| \le \gamma\mu + \lambda = \mu + \lambda(1 - \mu)$. □