7. Expander graphs

7.1 Introduction

We will continue with our discussion of expander graphs in this lecture.

7.2 Vertex expansion

Definition 7.1. *A digraph is a (K, A) vertex expander if for all subsets S of the vertices such that $|S| \leq K$, the neighborhood of S defined as $N(S) = \{u \mid \exists v \ s.t \ (u, v) \in E\}$ is of size at least $A \cdot |S|$.*

We saw in the last class that a random graph is a good expander.

Lemma 7.2. *For every constant $D \geq 32$, there exists a constant $n_0 > 0$ such that for all $N > n_0$, there exists a $(\frac{N}{10D}, \frac{5D}{8})$ vertex expander.*

We will now see how to perform error reduction using such expanders.
Error reduction using expanders 1: Let $\mathcal{A}$ be a randomized algorith computing a function f such that:

- If $f(I) = 1$, then $\Pr[\mathcal{A}(I) = 1] = 1$.
- If $f(I) = 0$, then $\Pr[\mathcal{A}(I) = 1] \leq \frac{1}{20}$.

If $\mathcal{A}$ has access to k bits of randomness, then it means that on $2^k/16$ strings the algorithm gets a wrong answer on the input I. Suppose we have a bipartite expander $G(L, R, E)$ with $|L| = |R| = 2^k$ such that for each set $S \subseteq L$ with $|S| \leq \frac{2^k}{10D}$, $N(S) \geq \frac{5D}{8}|S|$. The new algorithm $\mathcal{A}'$ works as follows: Choose a vertex $v \in L$ at random. Let $r_1, r_2, \ldots, r_m$ be the neighbors of v. If $\mathcal{A}(I, r_j)$ returns 1 for each j, then $\mathcal{A}'$ returns 1, else it returns 0. Notice that the algorithm answers correctly when $f(I) = 1$. For an I such that $f(I) = 0$, let $B = \{r \mid \mathcal{A}(I, r) = 1\}$. We know that $|B| \leq 2^k/16$. The algorithm $\mathcal{A}'$ answers incorrectly only when it chooses a $v \in L$ such that $N(v) \subseteq B$. Let $B' \subseteq L$ be the set of vertices such that for each $v \in B'$, $N(v) \subseteq B$. The probability that $\mathcal{A}'$ will give a wrong answer is the fraction of vertices in B'. If $|B'| > \frac{2^k}{11D}$, then $N(B') > \frac{5D}{8}\frac{2^k}{11D} > \frac{2^k}{20}$. But this is a contradition since $N(B') \subseteq B$ and $|B| \leq \frac{2^k}{20}$. Therefore, the probability that $\mathcal{A}'$ will err is at most $\frac{1}{11D}$.

But is this efficient? It is, if we can find an explicit expander family such that for each vertex v, I can get the neighbors of v in time poly(k).

7.3 Spectral expansion

Now we move on to another notion of expansion, spectral expansion. For this, we need to look at the graph in a more algebraic way.

7.3.1 Graphs and their spectra

For a d-regular graph G, we can associate the *normalized adjacency matrix* A which has n rows and n columns indexed by the vertices of G. If G has an edge (u, v), then we set $A[u, v] = 1/d$. Otherwise, we set it to 0. If the graph is undirected then we know that the matrix A is a real symmetric matrix. For a real symmetric matrix A, an eigenvector is a vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$ for real λ.

Theorem 7.3 (Spectral theorem for symmetric matrices)**.** *Let A be a real symmetric $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding eigenspaces $W_1, W_2, \ldots, W_k$, where $W_i = \{v \mid Av = \lambda_i v\}$. Then, the eigenspaces $W_1, W_2, \ldots, W_k$ are orthogonal and span $\mathbb{R}^n$. Moreover the dimension of the eigenspace W_i is the multiplicity of the eigenvalue λ_i.*

The set of n eigenvalues of A (included with their multiplicities) is known as the spectrum of the graph and reveal important properties of the graph. We will refer to the eigenvalues as $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ as the n eigenvalues of A in decreasing order. It can be shown that the eigenvalues all have absolute value less than 1.

Lemma 7.4. *A d-regular graph G is connected if and only if the multiplicity of λ_1 is 1.*

Proof. We will first show that if the graph is disconnected then the multiplicity is greater than 1.Let G_1 and G_2 be two connected components of the graph G. Let U_{G_1} be the vector such that for each $v \in G_1$, $U_{G_1}[v] = 1/d$ and 0 otherwise. Then U_{G_1} is an eigenvector of 1. Similary U_{G_2} is also an eigenvector of 1 and orthogonal to U_{G_1}.

Now, we will show that if the multiplicity of λ_1 is greater than 1, then the graph is disconnected. We know that the vector $(1, 1, \ldots, 1)$ is an eigenvector with eigenvalue 1. Let v be another vector with eigenvalue 1. Then, we know that $v \perp u$. Let $v_i = \max_j v_j$ and let $G_v = \{v_k \mid v_k = v_i\}$. Now, $\sum_{j=1}^{n} A[i, j]v_j = v_i$. This can be rewritten as $\sum_{(i,j)\in E} A[i, j]v_j = v_i$. Thus, v_i which is the element with the maximum value is a convex sum of v_js. Therefore, $A[i, j] \neq 0$ implies $v_i = v_j$. Taking the contrapositive of this, G_v is a connected component. $\square$

Exercise 7.5. *Show that a d-regular graph G has k connected components iff the eigenspace corresponding to the eigenvalue 1 has dimension k.*

Lemma 7.6. *If a d-regular connected graph G is bipartite then $\lambda_n = -1$.*

Proof. If G is bipartite and connected, then G^2 is a multigraph with two connected components where G^2 is the graph obtained by putting an edge from u to v if there is a path of length 2 from u to v in G. If there are k paths of length 2 from u to v, then we put k parallel edges. Now, G^2 is a d^2-regular multigraph, and it is easy to see that the normalized adjaceny matrix of G^2 is A^2. The eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ with the same eigenvectors as A. $\square$