CS6845: Pseudorandomness 29 Jan 2018

8. Graph spectra, mixing and random walks

8.1 Introduction

We will continue with our discussion of expander graphs in this lecture, and show that good spectral expansion implies that random walks will mix well.

8.2 Graphs and their spectra

For a d-regular graph G, we can associate the *normalized adjacency matrix* A which has n rows and n columns indexed by the vertices of G. If G has an edge (u, v), then we set $A[u, v] = 1/d$. Otherwise, we set it to 0. If the graph is undirected then we know that the matrix A is a real symmetric matrix. For a real symmetric matrix A, an eigenvector is a vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$ for real λ.

Theorem 8.1 (Spectral theorem for symmetric matrices)**.** *Let A be a real symmetric $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding eigenspaces $W_1, W_2, \ldots, W_k$, where $W_i = \{v \mid Av = \lambda_i v\}$. Then, the eigenspaces $W_1, W_2, \ldots, W_k$ are orthogonal and span $\mathbb{R}^n$. Moreover the dimension of the eigenspace W_i is the multiplicity of the eigenvalue λ_i.*

The set of n eigenvalues of A (included with their multiplicities) is known as the spectrum of the graph and reveal important properties of the graph. We will refer to the eigenvalues as $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ as the n eigenvalues of A in decreasing order.

Lemma 8.2. *Let G be a d-regular graph and A be its normalized adjacency matrix. Then $|\lambda| \leq 1$ for each eigenvalue of A.*

Proof. Let v be an eigenvector with eigenvalue λ. For each $i \in \{0, 1\}$, $\sum_j A[i, j]v_j = \lambda v_i$. Now, let i be the index such that $|v_i| = \max_{j \in \{1,2,\ldots,n\}} |v_j|$. Then $|\lambda v_i| = |\sum_j A[i, j]v_j| \leq \sum_j |A[i, j]||v_j| \leq |v_i| \sum_j A[i, j] = |v_i|$. □

Lemma 8.3. *Show that a d-regular graph G has k connected components iff the eigenspace corresponding to the eigenvalue 1 has dimension k.*

Lemma 8.4. *If a d-regular connected graph G is bipartite then $\lambda_n = -1$.*

We will see that the second largest eigenvalue of A is shows how well-connected the graph G, and the difference $1 - \lambda_2$ will give us a measure of expansion. We will define spectral expansion in the more general case of directed graphs with in-degree and out-degree d.

8.2.1 Mixing of random walks

Let G be a d-regular directed graph (possibly with parallel edges) and let A be the normalized adjacency matrix of G. Notice that in this more general case, the matrix A is no longer symmetric (even though it is doubly stochastic) and hence the spectral theorem is no longer applicable. We will define the spectral expansion in terms of how well random walks on these graphs mix.

A random walk on a graph G is the following: If the walk is currently at a vertex $v \in G$, it chooses a nieghbor uniformly at random (i.e./ with probability $1/d$) from its set of neighbors and moves to that vertex. Formally, if the initial probability distribution on the set of vertices is π, then the new probability distribution after one step of the random walk is πA. With this in mind, we define the spectral expansion of G as

$$\lambda(G) = \max_{\pi} \frac{\|\pi A - u\|}{\|\pi - u\|},$$

where π is a probability distribution and u is the uniform distribution over the set of vertices. An alternate way to look at spectral expansion is given in the lemma below.

Lemma 8.5. *For a d-regular directed multigraph G with normalized adjacency matrix A, we have*

$$\max_{\pi} \frac{\|\pi A - u\|}{\|\pi - u\|} = \max_{x \perp u} \frac{\|xA\|}{\|x\|}$$

Proof. For any distribution π and the vector u, the vector $\pi - u$ is orthogonal to u since

$$\langle \pi - u, u\rangle = \langle \pi, u\rangle - \langle u, u\rangle = 0,$$

and $(\pi - u)A = \pi A - uA = \pi A - u$. Thus, we know that $\max_{\pi} \frac{\|\pi A - u\|}{\|\pi - u\|} \leq \max_{x \perp u} \frac{\|xA\|}{\|x\|}$.

Now for an $x \perp u$, $u + \alpha x$ is a probability distribution for a small enough α, and this shows that $\max_{\pi} \frac{\|\pi A - u\|}{\|\pi - u\|} \geq \max_{x \perp u} \frac{\|xA\|}{\|x\|}$. □

We will refer to $\gamma(G) = 1 - \lambda(G)$ as the spectral gap for the graph G and it measures how well random walks in the graph mix.

Lemma 8.6. *Let G be a d-regular graph and A be its normalized adjacency matrix. Then $\lambda(G) = |\lambda_2|$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the eigenvectors corresponding to A with eigenvalues $\lambda_1 = 1, \lambda_2, \ldots, \lambda_n$. Any vector $x \perp u$ can be written as $x = \alpha_2 v_2 \lambda_3 v_3 + \cdots + \alpha_n v_n$. Now, $xA = \alpha_2\lambda_2 v_2 + \alpha_3\lambda_3 v_3 + \ldots + \alpha_n \lambda_n v_n$. Therefore, $\|xA\|^2 = \sum_i \alpha_2^2 \lambda_2^2 \|v_i\|^2 \leq \lambda_2^2 \|x\|^2$. Therefore, $\max_{x \perp u} \frac{\|xA\|}{\|x\|} \leq \lambda_2$, and the equality is achieved for $x = v_2$. □

Theorem 8.7 (spectral expansion ⇒ mixing)**.** *Let G be a d-regular directed graph and let A be its normalized adjacency matrix. Then, for any $t > 0$ and any probability distribution π on the set of vertices the following holds:*

$$\|\pi A^t - u\| \leq \lambda(G)^t \|\pi - u\| \leq \lambda(G)^t.$$

Proof. First note that

$$\begin{aligned}
\|\pi - u\|^2 &= \langle \pi - u, \pi - u \rangle \\
&= \|\pi\|^2 + \|u\|^2 - 2\langle \pi, u \rangle \\
&\leq \sum_i \pi(i) + \frac{1}{n} - \frac{2}{n} \sum_i \pi(i) \\
&= 1 - \frac{1}{n} \leq 1.
\end{aligned}$$

The lemma follows by induction on t. □

8.3 Undirected reachability with logarithmic space

Now we will see an algorithm to check if an undirected graph G has a path from a vertex s to t that uses only logarithmic space.

- **Input**: Graph G with degree d, nodes s and t.
- Perform a random walk of length $k = \text{poly}(n, d)$ starting from s.
- Accept if the random walk ends in t, else reject.

In the next lecture we will see how to prove the correctness of this algorithm by giving a bound on $\lambda(G)$.