1 Lagrangian and Eulerian formulation

In terms of solving momentum equations, it is necessary to characterize the velocity field (and of course its time derivatives). There are two ways of characterizing the field.

The Eulerian form defines that the field is a function of time. In the coordinate frame used, space, which is given using (x, y, z), is fixed in the inertial frame. (x, y, z) are independent of time. In other words, sensors are placed at fixed points to measure the fields. When the location of a fluid element can be described as $\boldsymbol{X}$, the velocity field is given as

$$\boldsymbol{V}_E(t) = \frac{d\boldsymbol{X}}{dt}. \tag{1}$$

The Lagrangian form, on the other hand, defines that the field is a function of space and time. The coordinate frame moves with a considered element and thus depends on space and time. In other words, sensors are attached to the element all the time. When the location of a fluid element is $\boldsymbol{v}$, the velocity field is defined as

$$\boldsymbol{V}_L(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{X}}{\partial t}. \tag{2}$$

For a time derivative of some quantity field, f, these two forms provide the following relationship:

$$\frac{df_E(t)}{dt} = \frac{\partial f_L(\boldsymbol{X}, t)}{\partial t} + \frac{\partial \boldsymbol{X}}{\partial t} \cdot \frac{\partial f_L(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}. \tag{3}$$

2 Strong and weak forms of problems

We discuss this section, referring to Hughes (2000). Let f be defined as $f : \bar{\Omega} \rightarrow \mathbf{R}$. We define the domain as $[0, 1]$. Given g and h, which are constant, a strong form is given as the following statement: Find $u : \bar{\Omega} \rightarrow \mathbf{R}$ such that

$$\begin{aligned} u_{xx} + f &= 0, & (4)\\ u(1) &= g, & (5)\\ u_x(0) &= -h. & (6) \end{aligned}$$

This form is called the strong form. In this form, subscripts x mean partial derivatives. Because there are two boundaries, it is also called a two-point boundary value problem.

To discuss the weak form, we need two collections. The first collection is a set of trial solutions (candidates of solutions). The trial solutions are obtained based on the first boundary condition, $u(1) = g$, without the secondary boundary condition. However, we add the following condition:

$$\int_0^1 u_x^2 dx < \infty. \tag{7}$$

This form may be called a H^1-function. A set of the trial solutions are any solutions that satisfy these conditions, which is given as

$$\mathcal{U} = \{u | u \in H^1, u(1) = g\}. \tag{8}$$

Another collection is a set of the weighting function, which is defined as

$$\mathcal{W} = \{w | w \in H^1, w(1) = 0\}. \tag{9}$$

The reason $w(1) = 0$ matters is given below.

Using these collections, the weak form is given as the following statement: Find $u \in \mathcal{U}$ such that for all $w \in \mathcal{W}$,

$$\int_0^1 w_x u_x dx = \int_0^1 w f dx + w(0)h. \tag{10}$$

The reason of $w(1) = 0$ is related to the proof that the weak form is the necessary-sufficient condition, or theoretically equal to, the strong form. The detailed discussion are given in Hughes (2000). Here, the basic concept is provided. Let's multiply w, which satisfies the second collection, with the first equation in the strong form and integrate the derived equation over the defined domain. This yields

$$\int_0^1 w u_{xx} dx + \int_0^1 w f dx = 0. \tag{11}$$

Considering the partial derivative of the first term on the left-hand side, we obtain

$$\int_0^1 w u_{xx} dx = -\int_0^1 w_x u_x dx + w(0) u_x. \tag{12}$$

Substituting Equation (12) to Equation (11) yields

$$\int_0^1 w_x u_x dx = \int_0^1 w f dx + w(0) u_x. \tag{13}$$

Obviously, this is identical to Equation (10).

3 3-dimensional topographic diffusion

3.1 Theoretical formulation

In this problem, there are three nodes that make a polyhedron shape model. Therefore, each element, j, has nodes of $\boldsymbol{x}_1^j$, $\boldsymbol{x}_2^j$, and $\boldsymbol{x}_3^j$. Note that these nodes are ordered in the right-hand role way. We describe it as z at a given location and denote the area, S^j, that is enclosed by the line, l^j. Let's define $\boldsymbol{t}$ as a unit vector that is normal to the defined line (outward from the area positive).

$$\iint_{S^j} \frac{\partial h^j}{\partial t} dS^j = -\int_{l^j} \boldsymbol{t}^j \cdot k^j \tan\theta^j \boldsymbol{u}^j dl^j \tag{14}$$

where $\boldsymbol{t}^j$ is a unit vector pointing outward on an element, θ^j is the slope, $\boldsymbol{u}^j$ is the horizontal component of an acceleration vector. k^j is the degradation parameter. We note that the negative sign on the right hand side comes from our definition of $\boldsymbol{t}^j$, which, again, is defined as being outward. Thus, this equation describes that if an outward flow is dominant, the material height change should be negative. To determine $\boldsymbol{u}^j$, do the following process. Define the unit acceleration vector as $\boldsymbol{a}$ and

$$\boldsymbol{v}^j = \frac{\boldsymbol{n}^j \times \boldsymbol{a}^j}{|\boldsymbol{n}^j \times \boldsymbol{a}^j|} \tag{15}$$

where $\boldsymbol{n}^j$ is the normal unit vector. Then, compute

$$\boldsymbol{u}^j = \boldsymbol{v}^j \times \boldsymbol{n}^j \tag{16}$$

Importantly, this equation gives the following constraint:

$$\iint_S \frac{\partial h}{\partial t} dS = 0 \tag{17}$$

This equation gives a surface integral over the entire surface of the body. Note that the right hand side becomes zero because there is no longer a clear definition of a line integral.

Consider $\tan\theta$, which defines the slope on an element. Assume that we obtain a unit acceleration vector, $\boldsymbol{a}^j$, for element j. Using $\boldsymbol{n}^j$, we obtain

$$\cos\theta^j = -\boldsymbol{n}^j \cdot \boldsymbol{a}^j \tag{18}$$

Because our interest is in $\tan\theta^j$, we use the following condition

$$1 + \tan^2\theta^j = \frac{1}{\cos^2\theta^j} \tag{19}$$

Therefore, we obtain

$$\tan\theta^j = \begin{cases} +\left(\frac{1}{\cos^2\theta^j} - 1\right)^{\frac{1}{2}} & \text{if } \cos\theta^j > 0 \\ -\left(\frac{1}{\cos^2\theta^j} - 1\right)^{\frac{1}{2}} & \text{if } \cos\theta^j < 0 \end{cases} \tag{20}$$

Considering the divergence role, we obtain

$$\iint_{S^j} \frac{\partial h^j}{\partial t} dS^j = -\iint_{S^j} k^j \nabla \cdot \tan\theta \boldsymbol{u}^j dS^j \tag{21}$$

We note that k^j is constant over a given triangular element. Also, h^j is scalar although it is necessary to define the change of the altitude in 3-dimensional space. In this problem, we define the change along the direction of $-\boldsymbol{a}_f$.

Let's use the Galerkin method, we add a function ϕ to the equation derived and consider when $\phi = \boldsymbol{N}^j$.

$$\iint_{S^j} \boldsymbol{N}^j \frac{\partial h^j}{\partial t} dS^j = -\iint_{S^j} k^j \boldsymbol{N}^j (\nabla \cdot \tan\theta^j \boldsymbol{u}^j) dS^j \tag{22}$$

Now, we define

$$h^j = \begin{bmatrix} N_1^j & N_2^j & N_3^j \end{bmatrix} \begin{bmatrix} h_1^j \\ h_2^j \\ h_3^j \end{bmatrix} \quad = \quad \boldsymbol{N}^{jT} \bar{\boldsymbol{h}}^j \tag{23}$$

Similarly,

$$\begin{aligned} \tan\theta^j &= f^j = \begin{bmatrix} N_1^j & N_2^j & N_3^j \end{bmatrix} \begin{bmatrix} f_1^j \\ f_2^j \\ f_3^j \end{bmatrix} = \boldsymbol{N}^{jT} \bar{\boldsymbol{f}}^j &(24)\\ \boldsymbol{u}^j &= \begin{bmatrix} \boldsymbol{u}_1^j & \boldsymbol{u}_2^j & \boldsymbol{u}_3^j \end{bmatrix} \begin{bmatrix} N_1^j \\ N_2^j \\ N_3^j \end{bmatrix} = \bar{\boldsymbol{U}}^j \boldsymbol{N}^j &(25) \end{aligned}$$

Based on these definitions, we obtain

$$\boldsymbol{N}^j (\nabla \cdot \tan\theta^j \boldsymbol{t}^j) = \boldsymbol{N}^j (\nabla \cdot \bar{\boldsymbol{U}}^j \boldsymbol{N}^j \boldsymbol{N}^{jT} \bar{\boldsymbol{f}}^j) \tag{26}$$

Finally, we obtain

$$\iint_{S^j} \boldsymbol{N}^j \boldsymbol{N}^{jT} dS^j \dot{\bar{\boldsymbol{h}}}^j = - \iint_{S^j} k^j \boldsymbol{N}^j (\nabla \cdot \bar{\boldsymbol{U}}^j \boldsymbol{N}^j \boldsymbol{N}^{jT} \bar{\boldsymbol{f}}^j) dS^j \tag{27}$$

Then, without dot products,

$$\iint_{S^j} \boldsymbol{N}^j \boldsymbol{N}^{jT} dS^j \dot{\bar{\boldsymbol{h}}}^j = - \iint_{S^j} k^j \boldsymbol{N}^j \{\nabla^T (\bar{\boldsymbol{U}}^j \boldsymbol{N}^j \boldsymbol{N}^{jT} \bar{\boldsymbol{f}}^j)\} dS^j \tag{28}$$

3.2 Numerical modeling

Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ be the nodes of each polygon. Considering variables at these three nodes, we describes the following quantities

$$\begin{aligned} \boldsymbol{x}^j &= s(\boldsymbol{x}_2^j - \boldsymbol{x}_1^j) + t(\boldsymbol{x}_3^j - \boldsymbol{x}_1^j) + \boldsymbol{x}_1^j &(29)\\ h^j &= s(h_2^j - h_1^j) + t(h_3^j - h_1^j) + h_1^j &(30)\\ f^j &= s(f_2^j - f_1^j) + t(f_3^j - f_1^j) + f_1^j &(31) \end{aligned}$$

Using these expressions, we obtain $\boldsymbol{N}^{jT}$ as

$$\boldsymbol{N}^{jT} = \begin{bmatrix} 1 - s - t & s & t \end{bmatrix} \tag{32}$$

This condition yields

$$\begin{aligned} \iint_{S^j} \boldsymbol{N}^j \boldsymbol{N}^{jT} dS^j &= \int_0^1 \int_0^{1-t} \boldsymbol{N}^j \boldsymbol{N}^{jT} ds dt |\boldsymbol{e}_{21} \times \boldsymbol{e}_{31}| \\ &= \frac{1}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} |\boldsymbol{e}_{21} \times \boldsymbol{e}_{31}| &(33) \end{aligned}$$

where $\boldsymbol{e}_{21} = (\boldsymbol{x}_2^j - \boldsymbol{x}_1^j)/|\boldsymbol{x}_2^j - \boldsymbol{x}_1^j|$ and $\boldsymbol{e}_{31} = (\boldsymbol{x}_3^j - \boldsymbol{x}_1^j)/|\boldsymbol{x}_3^j - \boldsymbol{x}_1^j|$.

Then, we consider the ∇ operator. $\boldsymbol{x} = (x, y, z)^T$ are functions of (s, t). This yields

$$\frac{\partial}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial}{\partial t} \tag{34}$$

$$\frac{\partial}{\partial y} = \frac{\partial s}{\partial y}\frac{\partial}{\partial s} + \frac{\partial t}{\partial y}\frac{\partial}{\partial t} \tag{35}$$

$$\frac{\partial}{\partial z} = \frac{\partial s}{\partial z}\frac{\partial}{\partial s} + \frac{\partial t}{\partial z}\frac{\partial}{\partial t} \tag{36}$$

Let's define $\nabla = [\partial/\partial x, \partial/\partial y, \partial/\partial z]^T$, and this is described as

$$\begin{aligned}
\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} &= \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \\ \frac{\partial s}{\partial z} & \frac{\partial t}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{bmatrix} \\
&= \frac{1}{\Delta} \begin{bmatrix} (y_3 z_1 - y_1 z_3) & (y_1 z_2 - y_2 z_1) \\ (x_1 z_3 - x_3 z_1) & (x_2 z_1 - x_1 z_2) \\ (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{bmatrix}
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\Delta &= \det \begin{bmatrix} \boldsymbol{x}_2 - \boldsymbol{x}_1 & \boldsymbol{x}_3 - \boldsymbol{x}_1 & \boldsymbol{x}_1 \end{bmatrix} \\
&= -x_3 y_2 z_1 + x_2 y_3 z_1 + x_3 y_1 z_2 - x_1 y_3 z_2 - x_2 y_1 z_3 + x_1 y_2 z_3
\end{aligned} \tag{38}$$

Based on these formulation, by combining Equation (28) for all the elements, we obtain the following matrix form.

$$\mathbf{A}\dot{\bar{\boldsymbol{h}}} = \boldsymbol{b} \tag{39}$$

We do not numerically integrate this equation over a considered time. More specifically, while we will use a regular numeric integrator, we do not compute $\bar{\boldsymbol{h}}$ over multiple time steps. We compute it over one single time step and covert it to the position change as discussed above.

4 Heat equation

4.1 Galerkin's formulation

Galerkin's formulation is a way of discretizing continuous expressions by adding some weights functions. The weight functions can be determined arbitrarily because they are just the weight function (i.e., we can define them as we wish). The integral form of the heat equation is given as

$$\iiint_V \rho c \frac{\partial T}{\partial t} dV = -\iint_S \boldsymbol{n} \cdot \boldsymbol{q} dS - \iint_S \boldsymbol{Q} \cdot \boldsymbol{n} dS. \tag{40}$$

In this equation, $\boldsymbol{Q}$ is at the boundary condition, and $\boldsymbol{n}$ is the normal vector on the surface (outward positive). Therefore, Galerkin's form of this expression is given as

$$\iiint_V \left(\rho c \frac{\partial T}{\partial t}\phi + \boldsymbol{\nabla}\cdot\boldsymbol{q}\phi\right) dV + \iint_S \boldsymbol{Q}\cdot\boldsymbol{n}\phi dS = R. \tag{41}$$

Here, R is a residual of this equation. By discretizing this form with multiple elements $j = 1, ..., m$, we obtain

$$\sum_{j=1}^{m}\iiint_{V^j} \left(\rho c \frac{\partial T}{\partial t}\phi^j + \boldsymbol{\nabla}\cdot\boldsymbol{q}\phi^j\right) dV^j + \sum_{j=1}^{m}\iint_{S^j} \boldsymbol{Q}\cdot\boldsymbol{n}\phi^j dS^j = R. \tag{42}$$

4.2 FEM formulation

Now consider that T^j is characterized as the temperature in element j by using an approximation function,

$$T^j = \sum_{i=1}^{n} N_i^j T_i^j = \boldsymbol{N}^{jT}\bar{\boldsymbol{T}}^j. \tag{43}$$

where T_i^j is the node temperature in element j. Given this, $\boldsymbol{q}^j$ is given as

$$\boldsymbol{q}^j = -k^j \nabla T^j = -k^j(\nabla \boldsymbol{N}^j)^T\bar{\boldsymbol{T}}^j = -k^j \mathbf{P}^{jT}\bar{\boldsymbol{T}}^j. \tag{44}$$

Importantly, the direction of heat is always opposite to that of the temperature gradient. In other words, heat should move from high temperature to low temperature. Now use Equation (42). We will consider n different ϕ^js. In other words, we will consider ϕ_{ji} where $i = 1, ..., n$.

$$\sum_{j=1}^{m}\iiint_{V^j} \left(\rho^j c^j \frac{\partial T^j}{\partial t}\phi_{ji} + \boldsymbol{\nabla}\cdot\boldsymbol{q}^j\phi_{ji}\right) dV^j + \sum_{j=1}^{m}\iint_{S^j} \boldsymbol{Q}\cdot\boldsymbol{n}^j\phi_{ji} dS^j = R_i. \tag{45}$$

Now assume that $\phi_{ij} = N_i^j$, and thus

$$\sum_{j=1}^{m}\iiint_{V^j} \left(\rho^j c^j \frac{\partial T^j}{\partial t}N_i^j + \boldsymbol{\nabla}\cdot\boldsymbol{q}^j N_i^j\right) dV_j + \sum_{j=1}^{m}\iint_{S^j} \boldsymbol{Q}^j\cdot\boldsymbol{n}^j N_i^j dS^j = R_i. \tag{46}$$

For the second term of the left hand side, we use a partial integral:

$$-\sum_{j=1}^{m}\iiint_{V^j} \boldsymbol{\nabla}\cdot\boldsymbol{q}^j N_i^j dV_j = \sum_{j=1}^{m}\iiint_{V^j} \boldsymbol{q}^j\cdot\nabla N_i^j dV_j - \sum_{j=1}^{m}\iint_{S^j} \boldsymbol{n}^j\cdot\boldsymbol{q}^j N_i^j dS_j \tag{47}$$

In our discussion, we do not add additional heat in this form. In other words, when we sum all the elements, the second surface integral term disappears. Thus,

$$-\sum_{j=1}^{m}\iiint_{V^j} \boldsymbol{\nabla}\cdot\boldsymbol{q}^j N_i^j dV_j = \sum_{j=1}^{m}\iiint_{V^j} \boldsymbol{q}\cdot\nabla N_i^j dV_j. \tag{48}$$

Then,

$$\sum_{j=1}^{m}\iiint_{V^j}\left(\rho^j c^j \frac{\partial T^j}{\partial t} N_i^j - \boldsymbol{q}^j \cdot \nabla N_i^j\right) dV_j + \sum_{j=1}^{m}\iint_{S^j} \boldsymbol{Q}^j \cdot \boldsymbol{n}^j N_i^j dS^j = R_i. \tag{49}$$

Next, rewrite this equation:

$$\sum_{j=1}^{m}\iiint_{V^j}\left(\rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT} \dot{\bar{\boldsymbol{T}}}^j + k^j \mathbf{P}^j \mathbf{P}^{jT} \bar{\boldsymbol{T}}^j\right) dV_j + \sum_{j=1}^{m}\iint_{S^j} \boldsymbol{Q}^j \cdot \boldsymbol{n}^j N_i^j dS^j = R_i. \tag{50}$$

$$\sum_{j=1}^{m}\left(\mathbf{A}^j \dot{\bar{\boldsymbol{T}}}^j + \mathbf{B}^j \bar{\boldsymbol{T}}^j + \boldsymbol{C}^j\right) = \boldsymbol{R} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix} \tag{51}$$

where

$$\begin{aligned} \mathbf{A}^j &= \iiint_{V^j} \rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT} dV_j, & (52)\\ \mathbf{B}^j &= \iiint_{V^j} k^j \mathbf{P}^j \mathbf{P}^{jT} dV^j, & (53)\\ \boldsymbol{C}^j &= \iint_{S^j} \boldsymbol{Q}^j \cdot \boldsymbol{n}^j \boldsymbol{N}^{jT} dS^j. & (54) \end{aligned}$$

Now consider the global elements. Define a vector of the temperature at all nodes:

$$\bar{\boldsymbol{T}} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}. \tag{55}$$

Given this vector, develop $\mathbf{A}$, $\mathbf{B}$, and $\boldsymbol{C}$ for the global nodes:

$$\mathbf{A}\dot{\bar{\boldsymbol{T}}} + \mathbf{B}\bar{\boldsymbol{T}} + \boldsymbol{C} = \boldsymbol{R}. \tag{56}$$

This should be done based on the boundary condition. By setting $\boldsymbol{R}$ to be zero, we obtain the equation:

$$\mathbf{A}\dot{\bar{\boldsymbol{T}}} + \mathbf{B}\bar{\boldsymbol{T}} + \boldsymbol{C} = \mathbf{0}. \tag{57}$$

Now, if an equilibrium state is considered, $\dot{\bar{\boldsymbol{T}}} = \mathbf{0}$; therefore,

$$\bar{\boldsymbol{T}} = -\mathbf{B}^{-1}\boldsymbol{C}. \tag{58}$$

4.3 Definition of matrices

4.3.1 2-dimensional case

Consider that an element is triangular, and we will consider three nodes. We now assume

$$T = ax + by + c. \tag{59}$$

Consider (x_1, y_1), (x_2, y_2), and (x_3, y_3) to determine

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{60}$$

We can solve

$$a = -\frac{T_1(y_3 - y_2) + T_2(y_1 - y_3) + T_3(y_2 - y_1)}{\Delta}, \tag{61}$$

$$b = -\frac{T_1(x_2 - x_3) + T_2(x_3 - x_1) + T_3(x_1 - x_2)}{\Delta}, \tag{62}$$

$$c = -\frac{T_1(x_3 y_2 - x_2 y_3) + T_2(x_1 y_3 - x_3 y_1) + T_3(x_2 y_1 - x_1 y_2)}{\Delta}. \tag{63}$$

where

$$\Delta = \det \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1). \tag{64}$$

$$T = N_1 T_1 + N_2 T_2 + N_3 T_3 = \boldsymbol{N}^{jT} \bar{\boldsymbol{N}}^j. \tag{65}$$

where

$$N_1 = -\frac{1}{\Delta}\{(y_3 - y_2)x + (x_2 - x_3)y + (x_3 y_2 - x_2 y_3)\}, \tag{66}$$

$$N_2 = -\frac{1}{\Delta}\{(y_1 - y_3)x + (x_3 - x_1)y + (x_1 y_3 - x_3 y_1)\}, \tag{67}$$

$$N_3 = -\frac{1}{\Delta}\{(y_2 - y_1)x + (x_1 - x_2)y + (x_2 y_1 - x_1 y_2)\}. \tag{68}$$

Also, using this expression, we obtain

$$\mathbf{P}^{jT} = -\frac{1}{\Delta} \begin{bmatrix} (y_3 - y_2) & (y_1 - y_3) & (y_2 - y_1) \\ (x_2 - x_3) & (x_3 - x_1) & (x_1 - x_2) \end{bmatrix}. \tag{69}$$

For the planar case, it is necessary to integrate these parameters over the plate. In this case, $\mathbf{A}^j$, $\mathbf{B}^j$, and $\boldsymbol{C}^j$ are rewritten as

$$\mathbf{A}^j = \iint_{S^j} \rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT} dS_j, \tag{70}$$

$$\mathbf{B}^j = \iint_{S^j} k^j \mathbf{P}^j \mathbf{P}^{jT} dS^j, \tag{71}$$

$$\boldsymbol{C}^j = \int_{L^j} \boldsymbol{Q}^j \cdot \boldsymbol{n}^j \boldsymbol{N}^{jT} dL^j. \tag{72}$$

We now consider the following parameter operation,

$$x = s(x_2 - x_1) + t(x_3 - x_1) + x_1, \tag{73}$$
$$y = s(y_2 - y_1) + t(y_3 - y_1) + y_1, \tag{74}$$

where $0 \le (s,t) \le 1$ and $0 \le s+t \le 1$. Using this yields

$$\iint_{S^j} dS^j = \int_0^1 \int_0^{1-s} \Delta dt ds = \int_0^1 \int_0^{1-t} \Delta ds dt = \frac{\Delta}{2}, \tag{75}$$

$$\begin{aligned}\iint_{S^j} x dS^j &= \int_0^1 \int_0^{1-s} (s(x_2 - x_1) + t(x_3 - x_1) + x_1)\Delta dt ds, \\ &= \frac{\Delta(x_2 + x_3 + x_1)}{6},\end{aligned} \tag{76}$$

$$\begin{aligned}\iint_{S^j} y dS^j &= \int_0^1 \int_0^{1-s} (s(y_2 - y_1) + t(y_3 - y_1) + y_1)\Delta dt ds, \\ &= \frac{\Delta(y_2 + y_3 + y_1)}{6},\end{aligned} \tag{77}$$

$$\iint_{S^j} x^2 dS^j = \frac{\Delta}{12}(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3), \tag{78}$$

$$\iint_{S^j} y^2 dS^j = \frac{\Delta}{12}(y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_1y_3 + y_2y_3), \tag{79}$$

$$\iint_{S^j} xy dS^j = \frac{\Delta}{12}\{x_1(2y_1 + y_2 + y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(y_1 + y_2 + 2y_3)\}. \tag{80}$$

Here we note that

$$\int_0^1 \int_0^{1-s} t dt ds = \int_0^1 \int_0^{1-t} s ds dt = \frac{1}{6}, \tag{81}$$

$$\int_0^1 \int_0^{1-s} t^2 dt ds = \int_0^1 \int_0^{1-t} s^2 ds dt = \frac{1}{12}, \tag{82}$$

$$\int_0^1 \int_0^{1-s} st dt ds = \frac{1}{24}. \tag{83}$$

Using these integrals, we obtain

$$\mathbf{A}^j = \frac{\rho^j c^j \Delta}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \tag{84}$$

$$\mathbf{B}^j = \frac{k^j}{\Delta} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{23} & b_{33} \end{bmatrix}. \tag{85}$$

where

$$
\begin{aligned}
b_{11} &= (x_2 - x_3)^2 + (y_2 - y_3)^2, & (86)\\
b_{12} &= (x_2 - x_3)(-x_1 + x_3) + (y_1 - y_3)(-y_2 + y_3), & (87)\\
b_{13} &= (x_1 - x_2)(x_2 - x_3) + (y_1 - y_2)(y_2 - y_3), & (88)\\
b_{21} &= (x_2 - x_3)(-x_1 + x_3) + (y_1 - y_3)(-y_2 + y_3), & (89)\\
b_{22} &= (x_1 - x_3)^2 + (y_1 - y_3)^2, & (90)\\
b_{23} &= (x_1 - x_2)(-x_1 + x_3) + (-y_1 + y_2)(y_1 - y_3), & (91)\\
b_{31} &= (x_1 - x_2)(x_2 - x_3) + (y_1 - y_2)(y_2 - y_3), & (92)\\
b_{32} &= (x_1 - x_2)(-x_1 + x_3) + (-y_1 + y_2)(y_1 - y_3), & (93)\\
b_{33} &= (x_1 - x_2)^2 + (y_1 - y_2)^2. & (94)
\end{aligned}
$$

4.3.2 3-dimensional cases

Consider that an element is a tetrahedron, and we will consider four nodes of this:

$$T = ax + by + cz + d. \tag{95}$$

Considering the temperature at T_i, we obtain

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}. \tag{96}$$

Using this form, we obtain

$$
\begin{aligned}
a &= \frac{1}{\Delta}\{T_1(-y_3z_2 + y_4z_2 + y_2z_3 - y_4z_3 - y_2z_4 + y_3z_4) \\
&\quad +T_2(y_3z_1 - y_4z_1 - y_1z_3 + y_4z_3 + y_1z_4 - y_3z_4) \\
&\quad +T_3(y_4z_1 + y_1z_2 - y_4z_2 - y_1z_4 - y_2z_1 + y_2z_4) \\
&\quad +T_4(y_2z_1 - y_3z_1 - y_1z_2 + y_3z_2 + y_1z_3 - y_2z_3)\} \qquad (97)
\end{aligned}
$$

$$
\begin{aligned}
b &= \frac{1}{\Delta}\{T_1(x_3z_2 - x_4z_2 - x_2z_3 + x_4z_3 + x_2z_4 - x_3z_4) \\
&\quad +T_2(-x_3z_1 + x_4z_1 + x_1z_3 - x_4z_3 - x_1z_4 + x_3z_4) \\
&\quad +T_3(-x_4z_1 - x_1z_2 + x_4z_2 + x_2z_1 - x_2z_4 + x_1z_4) \\
&\quad +T_4(-x_2z_1 + x_3z_1 + x_1z_2 - x_3z_2 - x_1z_3 + x_2z_3)\} \qquad (98)
\end{aligned}
$$

$$
\begin{aligned}
c &= \frac{1}{\Delta}\{T_1(-x_3y_2 + x_4y_2 + x_2y_3 - x_4y_3 - x_2y_4 + x_3y_4) \\
&\quad +T_2(x_3y_1 - x_4y_1 - x_1y_3 + x_4y_3 + x_1y_4 - x_3y_4) \\
&\quad +T_3(x_4y_1 + x_1y_2 - x_4y_2 - x_1y_4 - x_2y_1 + x_2y_4) \\
&\quad +T_4(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3)\} \qquad (99)
\end{aligned}
$$

$$
\begin{aligned}
d &= \frac{1}{\Delta}\{T_1(x_4y_3z_2 - x_3y_4z_2 - x_4y_2z_3 + x_2y_4z_3 + x_3y_2z_4 - x_2y_3z_4) \\
&\quad +T_2(-x_4y_3z_1 + x_3y_4z_1 + x_4y_1z_3 - x_1y_4z_3 - x_3y_1z_4 + x_1y_3z_4) \\
&\quad +T_3(x_4y_2z_1 - x_2y_4z_1 - x_4y_1z_2 + x_1y_4z_2 + x_2y_1z_4 - x_1y_2z_4) \\
&\quad +T_4(-x_3y_2z_1 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 + x_1y_2z_3)\} \qquad (100)
\end{aligned}
$$

We consider the following conditions:

$$
\begin{aligned}
\mathbf{A}^j &= \iiint_{V^j} \rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT} dV^j, \qquad (101) \\
\mathbf{B}^j &= \iiint_{V^j} k^j \mathbf{P}^j \mathbf{P}^{jT} dV^j, \qquad (102) \\
\boldsymbol{C}^j &= \iint_{S^j} \boldsymbol{Q}^j \cdot \boldsymbol{n}^j \boldsymbol{N}^{jT} dS^j. \qquad (103)
\end{aligned}
$$

For the 3-dimensional integral, we consider the following form:

$$\boldsymbol{x} = \boldsymbol{x}_1 + s(\boldsymbol{x}_2 - \boldsymbol{x}_1) + t(\boldsymbol{x}_3 - \boldsymbol{x}_1) + u(\boldsymbol{x}_4 - \boldsymbol{x}_1). \qquad (104)$$

where $0 \le s \le 1$, $0 \le t \le 1$, $0 \le u \le 1$, and $0 \le s + t + u \le 1$. For this,

$$\int_0^1 \int_0^{1-t} \int_0^{1-u-t} ds dt du = \frac{1}{6}. \qquad (105)$$

Using these relationships, we obtain

$$\mathbf{A}^j = \frac{\rho^j c^j \Delta}{120} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad (106)$$

For $\mathbf{P}^{jT}$, we obtain

$$\mathbf{P}^{jT} = \frac{1}{\Delta}\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix} \tag{107}$$

where

$$b_{11} = -y_3z_2 + y_4z_2 + y_2z_3 - y_4z_3 - y_2z_4 + y_3z_4 \tag{108}$$
$$b_{21} = x_3z_2 - x_4z_2 - x_2z_3 + x_4z_3 + x_2z_4 - x_3z_4 \tag{109}$$
$$b_{31} = -x_3y_2 + x_4y_2 + x_2y_3 - x_4y_3 - x_2y_4 + x_3y_4 \tag{110}$$
$$b_{12} = y_3z_1 - y_4z_1 - y_1z_3 + y_4z_3 + y_1z_4 - y_3z_4 \tag{111}$$
$$b_{22} = -x_3z_1 + x_4z1 + x_1z_3 - x_4z_3 - x_1z_4 + x_3z_4 \tag{112}$$
$$b_{32} = x_3y_1 - x_4y1 - x_1y_3 + x_4y_3 + x_1y_4 - x_3y_4 \tag{113}$$
$$b_{13} = y_4z_1 + y_1z_2 - y_4z_2 - y_1z_4 - y_2z_1 + y_2z_4 \tag{114}$$
$$b_{23} = -x_4z_1 - x_1z_2 + x_4z_2 + x_2z_1 - x_2z_4 + x_1z_4 \tag{115}$$
$$b_{33} = x_4y_1 + x_1y_2 - x_4y_2 - x_1y_4 - x_2y_1 + x_2y_4 \tag{116}$$
$$b_{14} = y_2z_1 - y_3z_1 - y_1z_2 + y_3z_2 + y_1z_3 - y_2z_3 \tag{117}$$
$$b_{24} = -x_2z_1 + x_3z_1 + x1z_2 - x_3z_2 - x_1z_3 + x_2z_3 \tag{118}$$
$$b_{34} = x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3 \tag{119}$$

Using this, we obtain $\mathbf{B}^j$ as

$$\mathbf{B}^j = \frac{k^j}{6\Delta}\mathbf{P}^j\mathbf{P}^{jT} \tag{120}$$

4.4 Boundary condition for radiation

In this section, we discuss the boundary condition for radiation. The surface boundary condition is given as

$$\sum_{k=1}^{m_s}\iint_{S^k}\boldsymbol{Q}^k\cdot\boldsymbol{n}^k dS^k = \sum_{k=1}^{m_s}\iint_{S^k}\boldsymbol{f}^k\cdot\boldsymbol{n}^k dS^k + \sum_{k=1}^{m_s}\iint_{S^k}\epsilon_0\sigma T^{k4}\boldsymbol{n}^k\cdot\boldsymbol{n}^k dS^k \tag{121}$$

Here, k indicates surface elements. We note that $\boldsymbol{n}^k$ is a unit vector (outward positive) on the surface element. For radiation, we assume that radiation occurs in the normal direction. Therefore, the left hand side should be negative if heat comes into an element. Considering Galerkin's formulation, the boundary condition is given

$$\iint_{S^k}\boldsymbol{Q}^k\cdot\boldsymbol{n}^k N^k dS^k = \iint_{S^k}\boldsymbol{f}^k\cdot\boldsymbol{n}^k N^k dS^k + \iint_{S^k}\epsilon_0\sigma T^{k4}N^k dS^k. \tag{122}$$

Let's consider the characterization of the governing equation derived above. Let's consider a surface element, which includes the boundary condition. In this case,

$$\iiint_{V^j}\left(\rho^j c^j\boldsymbol{N}^j\boldsymbol{N}^{jT}\dot{\bar{\boldsymbol{T}}}^j + k^j\mathbf{P}^j\mathbf{P}^{jT}\bar{\boldsymbol{T}}^j\right)dV_j + \iint_{S^j}\epsilon_0\sigma T^{j4}N^j dS^j \tag{123}$$

$$+\iint_{S^j}\boldsymbol{f}^j\cdot\boldsymbol{n}^j N^j dS^j = \boldsymbol{R}^j. \tag{124}$$

Note that the last term on the left-hand side is rewritten to represent the boundary condition in this problem. On the other hand, for elements beneath the surface elements, we do not have consider the boundary condition, and thus

$$\iiint_{V^j} \left(\rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT} \dot{\bar{\boldsymbol{T}}}^j + k^j \mathbf{P}^j \mathbf{P}^{jT} \bar{\boldsymbol{T}}^j\right) dV_j = \boldsymbol{R}^j. \tag{125}$$

The goal of this formulation is to minimize the residuals, $\boldsymbol{R}^j$, by considering all the elements.

These formulations allow us to describe

$$\mathbf{A}^j \dot{\bar{\boldsymbol{T}}}^j + \mathbf{B}^j \bar{\boldsymbol{T}}^j + \iint_{S^j} \boldsymbol{f}^j \cdot \boldsymbol{n}^j \boldsymbol{N}^j dS^j + \iint_{S^j} \epsilon_0 \sigma T^{j4} \boldsymbol{N}^j dS^j = \boldsymbol{R}^j. \tag{126}$$

4.4.1 1-D case

In this case we assume that $j = 1$ is the only surface element. For this case, $m_s = 1$. Also, we assume that $\boldsymbol{f}^j \cdot \boldsymbol{n}^j = -f^j$, and $dS^j = 1$. In this case, we describe the equation for element 1.

$$(\mathbf{A}^1 \dot{\bar{\boldsymbol{T}}}^1 + \mathbf{B}^1 \bar{\boldsymbol{T}}^1) - \{f^1 - \epsilon_0 \sigma (\boldsymbol{N}^{1T} \bar{\boldsymbol{T}}^1)^4\} \boldsymbol{N}^1 = \boldsymbol{R}^1. \tag{127}$$

For any other elements, it can be rewritten as

$$\mathbf{A}^j \dot{\bar{\boldsymbol{T}}}^j + \mathbf{B}^j \bar{\boldsymbol{T}}^j = \boldsymbol{R}^j. \tag{128}$$

Because there is only one element on the surface, $\boldsymbol{N}^{1T} \bar{\boldsymbol{T}}^1 = \bar{T}_1^1$. Also, $\boldsymbol{N}^1 = [1, 0]^T$. Thus, we can simplify this equation

$$(\mathbf{A}^1 \dot{\bar{\boldsymbol{T}}}^1 + \mathbf{B}^1 \bar{\boldsymbol{T}}^1) - \{f^1 - \epsilon_0 \sigma (\bar{T}_1^1)^4\} \boldsymbol{N}^1 = \boldsymbol{R}^1. \tag{129}$$

Considering $\boldsymbol{R}^j = 0$, We obtain

$$\mathbf{A} \dot{\bar{\boldsymbol{T}}} + \mathbf{B} \bar{\boldsymbol{T}} = \{f^1 - \epsilon_0 \sigma (\bar{T}_1^1)^4\} \boldsymbol{N}. \tag{130}$$

where $\boldsymbol{N} = [1, 0, \cdots, 0]^T$.

4.4.2 2-D case

Now, the equation for the 2-dimensional case can be describe as

$$\iint_{S^j} \left(\rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT} \dot{\bar{\boldsymbol{T}}}^j + k^j \mathbf{P}^j \mathbf{P}^{jT} \bar{\boldsymbol{T}}^j\right) dS_j + \int_{l^j} (\epsilon_0 \sigma T^{j4} \boldsymbol{N}^j + \boldsymbol{f}^j \cdot \boldsymbol{n}^j \boldsymbol{N}^j) dl^j = \boldsymbol{R}^j. \tag{131}$$

Consider the normal component of the heat flux to be f_t^j. Then, we can describe $\boldsymbol{f}^j \cdot \boldsymbol{n}^j = -f_t^j$. We assume that over each element of l^j, this component is constant. It is not technically true because of its variation and the surface conditions. Then,

$$\iint_{S^j} \left(\rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT} \dot{\bar{\boldsymbol{T}}}^j + k^j \mathbf{P}^j \mathbf{P}^{jT} \bar{\boldsymbol{T}}^j\right) dS_j + \int_{l^j} (\epsilon_0 \sigma T^{j4} \boldsymbol{N}^j - f_t^j \boldsymbol{N}^j) dl^j = \boldsymbol{R}^j. \tag{132}$$

We skip the discussions about the surface integral (the first term) on the left-hand side because the formulation is already given. For the boundary condition, which is given as a form of a surface integral, we pick two nodes, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, to describe the surface nodes. Note that even if other pairs are taken, the formulation does not change. The surface element is given as

$$\boldsymbol{x} = u\boldsymbol{x}_1 + (1-u)\boldsymbol{x}_2. \tag{133}$$

Then, the integral can be given as

$$\int_{l^j} dl^j = \int_0^1 \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| du. \tag{134}$$

Using this integration process, we obtain

$$\begin{aligned} \int_{l^j} N_1^j dl^j &= \int_0^1 \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| N_1^j du, \\ &= \frac{1}{2}\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \end{aligned} \tag{135}$$

$$\begin{aligned} \int_{l^j} N_2^j dl^j &= \int_0^1 \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| N_2^j du, \\ &= \frac{1}{2}\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \end{aligned} \tag{136}$$

$$\int_{l^j} N_3^j dl^j = \int_0^1 \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| N_3^j du = 0 \tag{137}$$

Equations (147) and (148) are identical. Let Δ be $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$ and χ be

$$\chi = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \tag{138}$$

Using these notations, we obtain

$$\int_{l^j} Y^{j4} N_1^j dl^j = \frac{\chi}{30}(5T_1^4 + 4T_1^3 T_2 + 3T_1^2 T_2^2 + 2T_1 T_2^3 + T_2^4), \tag{139}$$

$$\int_{l^j} Y^{j4} N_2^j dl^j = \frac{\chi}{30}(T_1^4 + 2T_1^3 T_2 + 3T_1^2 T_2^2 + 4T_1 T_2^3 + 5T_2^4), \tag{140}$$

$$\int_{l^j} Y^{j4} N_3^j dl^j = 0. \tag{141}$$

4.4.3 3-D case

Now, the equation for the 3-dimensional case can be describe as

$$\iiint_{V^j} \left(\rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT} \dot{\bar{\boldsymbol{T}}}^j + k^j \mathbf{P}^j \mathbf{P}^{jT} \bar{\boldsymbol{T}}^j\right) dV^j + \iint_{S^j} (\epsilon_0 \sigma T^{j4} \boldsymbol{N}^j + \boldsymbol{f}^j \cdot \boldsymbol{n}^j \boldsymbol{N}^j) dS^j = \mathbf{0}. \tag{142}$$

Similar to the 2-dimensional case, let's consider the normal component of the heat flux to be f_t^j. Then, we can describe $\boldsymbol{f}^j \cdot \boldsymbol{n}^j = -f_t^j$. Again, we assume that over each surface element of l^j, this component is constant, while it is not technically true because of its variation and the surface conditions. Then,

$$\iiint_{V^j}\left(\rho^j c^j \boldsymbol{N}^j \boldsymbol{N}^{jT}\dot{\bar{\boldsymbol{T}}}^j + k^j \mathbf{P}^j \mathbf{P}^{jT}\bar{\boldsymbol{T}}^j\right) dV^j + \iint_{S^j}(\epsilon_0 \sigma T^{j4}\boldsymbol{N}^j - f_t^j \boldsymbol{N}^j) dS^j = \mathbf{0}. \quad (143)$$

For the boundary condition, we pick three nodes, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$, to describe the surface nodes. Note that even if other pairs are taken, the formulation does not change. The surface element is given as

$$\boldsymbol{x} = \boldsymbol{x}_1 + s(\boldsymbol{x}_2 - \boldsymbol{x}_1) + t(\boldsymbol{x}_3 - \boldsymbol{x}_1) \quad (144)$$

Then, the integral can be given as

$$\int_{S^j} dS^j = \int_0^1 \int_0^{1-t} ds dt S^j \quad (145)$$

where

$$S^j = \frac{1}{2}\|(\boldsymbol{x}_2 - \boldsymbol{x}_1) \times (\boldsymbol{x}_3 - \boldsymbol{x}_1)\| \quad (146)$$

Using this integration process, we obtain

$$\begin{aligned}\int_{S^j} N_1^j dS^j &= \int_0^1 \int_0^{1-t} S^j N_1^j ds dt, \\ &= \frac{S^j}{6} \quad (147)\\ \int_{l^j} N_2^j dl^j &= \int_0^1 \int_0^{1-t} S^j N_2^j du, \\ &= \frac{S^j}{6} \quad (148)\\ \int_{l^j} N_2^j dl^j &= \int_0^1 \int_0^{1-t} S^j N_2^j du, \\ &= \frac{S^j}{6} \quad (149)\\ \int_{l^j} N_4^j dl^j &= \int_0^1 \int_0^{1-t} S^j N_3^j du = 0 \quad (150)\end{aligned}$$

Using these notations, we obtain

$$\begin{aligned}
\iint_{S^j} Y^{j4} N_1^j dS^j &= \frac{S^j}{210}(5T_1^4 + 4T_1^3T_2 + 3T_1^2T_2^2 + 2T_1T_2^3 + T_2^4 \\
&\quad +4T_1^3T_3 + 3T_1^2T_2T_3 + 2T_1T_2^2T_3 + T_2^3T_3 \\
&\quad +3T_1^2T_3^2 + 2T_1T_2T_3^2 + T_2^2T_3^2 + 2T_1T_3^3 \\
&\quad +T_2T_3^3 + T_3^4) \qquad (151) \\
\iint_{S^j} Y^{j4} N_2^j dS^j &= \frac{S^j}{210}(T_1^4 + 2T_1^3T_2 + 3T_1^2T_2^2 + 4T_1T_2^3 + 5T_2^4 + T_1^3T_3 \\
&\quad +2T_1^2T_2T_3 + 3T_1T_2^2T_3 + 4T_2^3T_3 + T_1^2T_3^2 + 2T_1T_2T_3^2 \\
&\quad +3T_2^2T_3^2 + T_1T_3^3 + 2T_2T_3^3 + T_3^4) \qquad (152) \\
\iint_{S^j} Y^{j4} N_3^j dS^j &= \frac{S^j}{210}(T_1^4 + T_1^3T_2 + T_1^2T_2^2 + T_1T_2^3 + T_2^4 + 2T_1^3T_3 \\
&\quad +2T_1^2T_2T_3 + 2T_1T_2^2T_3 + 2T_2^3T_3 + 3T_1^2T_3^2 \\
&\quad +3T_1T_2T_3^2 + 3T_2^2T_3^2 + 4T_1T_3^3 + 4T_2T_3^3 + 5T_3^4) \qquad (153) \\
\iint_{S^j} Y^{j4} N_4^j dS^j &= 0 \qquad (154)
\end{aligned}$$

5 Structure

5.1 Linear Elasticity

We define the displacement (Slaughter, 2002; Shearer, 2009). Consider the location of a particle at time, t_0, is given as $\boldsymbol{x}(t_0)$, while that at time, t, is given as $\boldsymbol{x}(t)$. Then, the displacement of this particle is described as

$$\boldsymbol{u}(t, \boldsymbol{x}(t_0)) = \boldsymbol{x}(t) - \boldsymbol{x}(t_0). \qquad (155)$$

The material derivative of this particle is given as

$$\boldsymbol{a} = \frac{D^2\boldsymbol{x}}{Dt^2}. \qquad (156)$$

Considering that $\boldsymbol{x}(t_0)$ is constant, we obtain that

$$\rho\frac{D^2\boldsymbol{x}}{Dt^2} = \rho\frac{D^2\boldsymbol{u}}{Dt^2} = \nabla\boldsymbol{\sigma} + \rho\boldsymbol{b}. \qquad (157)$$

Considering the Lagrangian motion, we obtain that

$$\frac{D^2\boldsymbol{u}}{Dt^2} \approx \frac{\partial^2\boldsymbol{u}}{\partial t^2}, \qquad (158)$$

assuming that the spatial derivative is negligible compared to the time derivative. However, this needs to be tested. Considering this assumption, we obtain

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} = \nabla\boldsymbol{\sigma} + \rho\boldsymbol{b}. \qquad (159)$$

5.2 Theoretical formulation

5.2.1 2-dimensional case

We first discuss the elastic energy term, which consists of the stress and the strain. Because the stress tensor is assumed to be symmetric, this tensor is defined as a 3 by 1 vector. Therefore, the strain vector is given as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{1}{2}\frac{\partial}{\partial y} & \frac{1}{2}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A}\boldsymbol{u}. \tag{160}$$

Similarly, the stress tensor is also symmetric, so we consider a 3 by 1 vector to describe the stress components.

$$\begin{aligned} \boldsymbol{\sigma} &= \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E\frac{\partial u}{\partial x} \\ E\frac{\partial v}{\partial y} \\ \frac{G}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{bmatrix} \\ &= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{1}{2}\frac{\partial}{\partial y} & \frac{1}{2}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \boldsymbol{E}\boldsymbol{\epsilon}. \end{aligned} \tag{161}$$

We start from the elastic energy:

$$\frac{1}{2}\int_V \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} dV = \frac{1}{2}\int_V (\boldsymbol{EHU})^T \boldsymbol{HU} dV = \frac{1}{2}\boldsymbol{U}^T \int_0^1 \int_0^{1-t} \boldsymbol{H}^T \boldsymbol{EH} \left|J\right| dsdt\boldsymbol{U}. \tag{162}$$

Next, the work is rewritten as

$$-\int_V \rho \boldsymbol{g} \cdot \boldsymbol{u} dV = -\rho \int_0^1 \int_0^{1-t} \boldsymbol{g}^T \boldsymbol{X} \left|J\right| dsdt\boldsymbol{U}, \tag{163}$$

where $\boldsymbol{g} = [g_x, g_y]^T$, and J is the Jacobian matrix.

Finally, we consider the kinetic energy.

$$\frac{1}{2}\int_V \rho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} dV = \frac{1}{2}\int_V \rho(\boldsymbol{XU}')^T \boldsymbol{XU}' dV = \frac{\rho}{2}\boldsymbol{U}'^T \int_0^1 \int_0^{1-t} \boldsymbol{X}^T \boldsymbol{X} \left|J\right| dsdt\boldsymbol{U}'. \tag{164}$$

From these formulations, we finally write the equation of motion (note we used the Lagrangian form defined above).

$$\int_0^1 \int_0^{1-t} \boldsymbol{X}^T \boldsymbol{X} \left|J\right| dsdt\boldsymbol{U}'' = \int_0^1 \int_0^{1-t} \boldsymbol{X}^T \boldsymbol{g} \left|J\right| dsdt - \frac{1}{\rho}\int_0^1 \int_0^{1-t} \boldsymbol{H}^T \boldsymbol{EH} \left|J\right| dsdt\boldsymbol{U}. \tag{165}$$

5.2.2 3-dimensional case

This section focuses on theoretical development for FEM to discuss structural conditions. We target 3-dimensional cases; however, by doing so, it is easier to imagine processes for our applications. Let's consider small deformation, $\boldsymbol{u}$. The equation of motion is given as

(Elastic deformation)
(Hooke's Law) →

$$\rho \frac{d^2 \boldsymbol{u}}{dt^2} = \nabla \boldsymbol{\sigma} + \rho \boldsymbol{b}. \tag{166}$$

✓

In this equation, σ is usually described as a 3 by 3 matrix. However, in this problem, we describe this quantity as a 6-dimensional vector. Considering this vector, we define a matrix that describes partial derivatives, which is denoted as ∇.

$$\nabla \boldsymbol{\sigma} = \begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \tag{167}$$

Let's define the weak form of this equation.

$$\iiint_V \phi \rho \frac{d^2 \boldsymbol{u}}{dt^2} dV = \iiint_V \left(\phi \nabla \boldsymbol{\sigma} + \phi \rho \boldsymbol{b} \right) dV \tag{168}$$

Let's consider that we consider four-node tetrahedron mesh elements whose nodes are $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, and $\boldsymbol{x}_4$. In this case, we consider the deformation solution at each node as $\boldsymbol{u}_i$, where $i = 1, ..., 4$. The idea here is to describe

$$\boldsymbol{u}^j = \boldsymbol{N}^{jT} \bar{\boldsymbol{u}}^j \tag{169}$$
$$\boldsymbol{x}^j = \boldsymbol{N}^{jT} \bar{\boldsymbol{x}}^j \tag{170}$$
$$\boldsymbol{b}^j = \boldsymbol{N}^{jT} \bar{\boldsymbol{b}}^j \tag{171}$$
$$\boldsymbol{\sigma}^j = \boldsymbol{R}^{jT} \bar{\boldsymbol{u}}^j \tag{172}$$

$\boldsymbol{u}^j$, $\boldsymbol{x}^j$, and $\boldsymbol{b}^j$ are 3-dimensional vectors. Because we consider four nodes, $\bar{\boldsymbol{u}}^j$, $\bar{\boldsymbol{x}}^j$, and $\bar{\boldsymbol{b}}^j$ are 12-dimensional vectors for each element. Therefore, $\boldsymbol{N}^{jT}$ is a 3 by 12 matrix. For $\boldsymbol{\sigma}^j$ is a 6-dimensional vector, so $\boldsymbol{R}^{jT}$ is a 6-by-12 matrix.

Let's discuss $\boldsymbol{R}^j$. Here, we consider the linear elastic model:

$$\boldsymbol{\sigma} = \mathbf{E} \epsilon$$

$\rightarrow \sigma = E\epsilon + \eta \frac{d\epsilon}{dt}$

$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xx} \\ 2\gamma_{xy} \\ 2\gamma_{xz} \\ 2\gamma_{yz} \end{bmatrix} \tag{173}$$

Also, the strain is given as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\gamma_{xy} \\ 2\gamma_{xz} \\ 2\gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \nabla^T \boldsymbol{u}. \tag{174}$$

Considering $\boldsymbol{\sigma}^j = \mathbf{E}\boldsymbol{\epsilon}^j = \mathbf{E}\nabla^T \boldsymbol{u}^j$, we give $\boldsymbol{\sigma}^j = \mathbf{E}\nabla^T \boldsymbol{N}^{jT}\bar{\boldsymbol{u}}^j$. Therefore, $\boldsymbol{R}^{jT} = \mathbf{E}\nabla^T \boldsymbol{N}^{jT}$.

Substituting these equations into Equation (178) yields

$$\iiint_V \phi\rho \boldsymbol{N}^{jT}\ddot{\boldsymbol{u}}^j dV = \iiint_V \left(\phi\nabla \boldsymbol{R}^j \bar{\boldsymbol{u}}^j + \phi\rho\boldsymbol{b}\right) dV \tag{175}$$

Consider ϕ to be $\boldsymbol{N}^j$, we obtain

$$\iiint_V \rho \boldsymbol{N}^j \boldsymbol{N}^{jT}\ddot{\boldsymbol{u}}^j dV = \iiint_V \left(\boldsymbol{N}^j\nabla \boldsymbol{R}^j \bar{\boldsymbol{u}}^j + \rho\boldsymbol{N}^j \boldsymbol{N}^{jT}\boldsymbol{b}^j\right) dV \tag{176}$$

In this equation, by using the partial integral law, we obtain

$$\iiint_V \boldsymbol{N}^j\nabla \boldsymbol{R}^j \bar{\boldsymbol{u}}^j dV = -\iiint_V \nabla\boldsymbol{N}^j \boldsymbol{R}^j \bar{\boldsymbol{u}}^j dV \tag{177}$$

Therefore, considering $\boldsymbol{R}^{jT} = \mathbf{E}\nabla^T \boldsymbol{N}^{jT}$, we can rewrite the equation as

$$\iiint_V \rho \boldsymbol{N}^j \boldsymbol{N}^{jT}\ddot{\boldsymbol{u}}^j dV = -\iiint_V \nabla\boldsymbol{N}^j \mathbf{E}\nabla^T \boldsymbol{N}^{jT}\bar{\boldsymbol{u}}^j dV + \iiint_V \rho\boldsymbol{N}^j \boldsymbol{N}^{jT}\boldsymbol{b}^j dV \tag{178}$$

In numerical implementations, we consider the multiplication of ρ multiple times may be redundant, so we divide this equation by ρ to obtain

$$\iiint_V \boldsymbol{N}^j \boldsymbol{N}^{jT}\ddot{\boldsymbol{u}}^j dV = -\iiint_V \frac{1}{\rho}\nabla\boldsymbol{N}^j \mathbf{E}\nabla^T \boldsymbol{N}^{jT}\bar{\boldsymbol{u}}^j dV + \iiint_V \boldsymbol{N}^j \boldsymbol{N}^{jT}\boldsymbol{b}^j dV \tag{179}$$

6 Applications

6.1 2-dimensional case

Consider that $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{X}(\boldsymbol{x})\boldsymbol{U}(t)$, where $\boldsymbol{x} = [x, y]^T$. Here, we consider a triangular element, and each edge is located at (x_i, y_i), where $i = 1, 2, 3$. So, $\boldsymbol{U}(t) = [u_1, v_1, u_2, v_2, u_3, v_3]$. In this analysis, a linear function is used to describe $\boldsymbol{X}$ and $\boldsymbol{U}$, which are written as

$$\begin{aligned} \boldsymbol{u} &= s(\boldsymbol{u}_2 - \boldsymbol{u}_1) + t(\boldsymbol{u}_3 - \boldsymbol{u}_1) + \boldsymbol{u}_1 = s\boldsymbol{u}_2 + t\boldsymbol{u}_3 + (1 - s - t)\boldsymbol{u}_1, && (180) \\ \boldsymbol{x} &= s(\boldsymbol{x}_2 - \boldsymbol{x}_1) + t(\boldsymbol{x}_3 - \boldsymbol{x}_1) + \boldsymbol{x}_1 = s\boldsymbol{x}_2 + t\boldsymbol{x}_3 + (1 - s - t)\boldsymbol{x}_1. && (181) \end{aligned}$$

Note that s and t are parameters that satisfy $0 \leq s \leq 1$, $0 \leq t \leq 1$, and $0 \leq s+t \leq 1$. We define a vector, $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}) = [s(\boldsymbol{x}), t(\boldsymbol{x})]^T$.

$$\boldsymbol{u} \;=\; \begin{bmatrix} 1-(s+t) & 0 & s & 0 & t & 0 \\ 0 & 1-(s+t) & 0 & s & 0 & t \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}, \tag{182}$$

$$\boldsymbol{x} \;=\; \begin{bmatrix} 1-(s+t) & 0 & s & 0 & t & 0 \\ 0 & 1-(s+t) & 0 & s & 0 & t \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}. \tag{183}$$

Then, we define $\boldsymbol{X}(\boldsymbol{x}) = \boldsymbol{X}(\boldsymbol{\xi})$, which is given as

$$\boldsymbol{X}(\boldsymbol{x}) = \boldsymbol{X}(\boldsymbol{\xi}) = \begin{bmatrix} 1-(s+t) & 0 & s & 0 & t & 0 \\ 0 & 1-(s+t) & 0 & s & 0 & t \end{bmatrix}. \tag{184}$$

From these definitions, we can describe the stress and strain

$$\boldsymbol{\epsilon} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{1}{2}\frac{\partial}{\partial y} & \frac{1}{2}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{1}{2}\frac{\partial}{\partial y} & \frac{1}{2}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} 1-(s+t) & 0 & s & 0 & t & 0 \\ 0 & 1-(s+t) & 0 & s & 0 & t \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}. \tag{185}$$

and the stress can be expressed using $\boldsymbol{E}$ and $\boldsymbol{\epsilon}$.

From the definition of $\boldsymbol{x}$,

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \Delta x_2 & \Delta x_3 \\ \Delta y_2 & \Delta y_3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} + \boldsymbol{x}_1. \tag{186}$$

where

$$\Delta x_2 \;=\; x_2 - x_1, \tag{187}$$
$$\Delta x_3 \;=\; x_3 - x_1, \tag{188}$$
$$\Delta y_2 \;=\; y_2 - y_1, \tag{189}$$
$$\Delta y_3 \;=\; y_3 - y_1. \tag{190}$$

From this expression, we also obtain

$$\boldsymbol{\xi} = \begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{\Delta x_2 \Delta y_3 - \Delta y_2 \Delta x_3} \begin{bmatrix} \Delta y_3 & -\Delta x_3 \\ -\Delta y_2 & \Delta x_2 \end{bmatrix} \left\{ \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right\} \tag{191}$$

Taking a partial derivative of $\boldsymbol{\xi}$ with respect to $\boldsymbol{x}$, we calcualte

$$\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{bmatrix} = \frac{1}{\Delta x_2 \Delta y_3 - \Delta y_2 \Delta x_3} \begin{bmatrix} \Delta y_3 & -\Delta x_3 \\ -\Delta y_2 & \Delta x_2 \end{bmatrix}. \tag{192}$$

With these expressions, we finally compute $\boldsymbol{\epsilon}$ by using vector and matrix expressions.

$$\begin{aligned} \boldsymbol{\epsilon} &= \frac{1}{\Delta x_2 \Delta y_3 - \Delta y_2 \Delta x_3} \qquad (193) \\ & \begin{bmatrix} -\Delta y_3 + \Delta y_2 & 0 & \Delta y_3 & 0 & -\Delta y_2 & 0 \\ 0 & \Delta x_3 - \Delta x_2 & 0 & -\Delta x_3 & 0 & \Delta x_2 \\ \frac{1}{2}(\Delta x_3 - \Delta x_2) & \frac{1}{2}(-\Delta y_3 + \Delta y_2) & -\frac{1}{2}\Delta x_3 & \frac{1}{2}\Delta y_3 & \frac{1}{2}\Delta x_2 & -\frac{1}{2}\Delta y_2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}, \\ &= \boldsymbol{HU}. \qquad (194) \end{aligned}$$

Also, the stress vector is simply written as

$$\boldsymbol{\sigma} = \boldsymbol{EHU}. \tag{195}$$

Next, we convert the integral form from three-dimensional space to the parameter space. Using the magnitude of the determinant, $\|J\|$, we obtain

$$\int_V f(x,y) dV = t \int_A f(x,y) dx dy = t \int_0^1 \int_0^{1-t} f(s,t) \left| J \right| ds dt, \tag{196}$$

where

$$|J| = \left| \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} \right| = \left| \Delta x_2 \Delta y_3 - \Delta y_2 \Delta x_3 \right|. \tag{197}$$

Here we assume a uniform thickness of the test material.

In this form, $\boldsymbol{H}^T \boldsymbol{EH}$ becomes constant. Also, $|J|$ is eliminated. Simply, calculating these integral forms, we obtain

$$\begin{aligned} \int_0^1 \int_0^{1-t} \boldsymbol{X}^T \boldsymbol{X} ds dt &= \frac{1}{24} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}, \qquad (198) \\ \int_0^1 \int_0^{1-t} \boldsymbol{X}^T \boldsymbol{g} ds dt &= \frac{1}{6} \begin{bmatrix} g_x \\ g_y \\ g_x \\ g_y \\ g_x \\ g_y \end{bmatrix}, \qquad (199) \\ \int_0^1 \int_0^{1-t} ds dt &= \frac{1}{2}. \qquad (200) \end{aligned}$$

6.2 3-dimensional formulation

We define the following functions:

$$\begin{aligned}
\boldsymbol{u}^j &= l(\bar{\boldsymbol{u}}_2^j - \bar{\boldsymbol{u}}_1^j) + m(\bar{\boldsymbol{u}}_3^j - \bar{\boldsymbol{u}}_1^j) + n(\bar{\boldsymbol{u}}_4^j - \bar{\boldsymbol{u}}_1^j) + \bar{\boldsymbol{u}}_1^j && (201)\\
\boldsymbol{x}^j &= l(\bar{\boldsymbol{x}}_2^j - \bar{\boldsymbol{x}}_1^j) + m(\bar{\boldsymbol{x}}_3^j - \bar{\boldsymbol{x}}_1^j) + n(\bar{\boldsymbol{x}}_4^j - \bar{\boldsymbol{x}}_1^j) + \bar{\boldsymbol{x}}_1^j && (202)\\
\boldsymbol{b}^j &= l(\bar{\boldsymbol{b}}_2^j - \bar{\boldsymbol{b}}_1^j) + m(\bar{\boldsymbol{b}}_3^j - \bar{\boldsymbol{b}}_1^j) + n(\bar{\boldsymbol{b}}_4^j - \bar{\boldsymbol{b}}_1^j) + \bar{\boldsymbol{b}}_1^j && (203)
\end{aligned}$$

From Equation (202), we obtain

$$(\boldsymbol{x}^j - \bar{\boldsymbol{x}}^j) = \mathbf{M}\boldsymbol{l} \tag{204}$$

where $\boldsymbol{l} = [l, m, n]^T$, and $\mathbf{M}$ is the matrix obtained from the equations above. From this,

$$\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial l}{\partial x} & \frac{\partial l}{\partial y} & \frac{\partial l}{\partial z} \\ \frac{\partial m}{\partial x} & \frac{\partial m}{\partial y} & \frac{\partial m}{\partial z} \\ \frac{\partial n}{\partial x} & \frac{\partial n}{\partial y} & \frac{\partial n}{\partial z} \end{bmatrix} = \mathbf{M}^{-1} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}. \tag{205}$$

This expression is used to convert ∇ to an expression with partials with respect to l, m, and n. In other words,

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial l}{\partial x}\frac{\partial}{\partial l} + \frac{\partial m}{\partial x}\frac{\partial}{\partial m} + \frac{\partial n}{\partial x}\frac{\partial}{\partial n} && (206)\\
\frac{\partial}{\partial y} &= \frac{\partial l}{\partial y}\frac{\partial}{\partial l} + \frac{\partial m}{\partial y}\frac{\partial}{\partial m} + \frac{\partial n}{\partial y}\frac{\partial}{\partial n} && (207)\\
\frac{\partial}{\partial z} &= \frac{\partial l}{\partial z}\frac{\partial}{\partial l} + \frac{\partial m}{\partial z}\frac{\partial}{\partial m} + \frac{\partial n}{\partial z}\frac{\partial}{\partial n} && (208)
\end{aligned}$$

From this expression, we obtain

$$\begin{aligned}
\nabla^T \boldsymbol{N}^{jT} &= \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n & 0 & 0 \\ 0 & 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n & 0 \\ 0 & 0 & 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n \end{bmatrix} \\
&= \begin{bmatrix}
-\frac{\partial \chi}{\partial x} & 0 & 0 & \frac{\partial l}{\partial x} & 0 & 0 & \frac{\partial m}{\partial x} & 0 & 0 & \frac{\partial n}{\partial x} & 0 & 0 \\
0 & -\frac{\partial \chi}{\partial y} & 0 & 0 & \frac{\partial l}{\partial y} & 0 & 0 & \frac{\partial m}{\partial y} & 0 & 0 & \frac{\partial n}{\partial y} & 0 \\
0 & 0 & -\frac{\partial \chi}{\partial z} & 0 & 0 & \frac{\partial l}{\partial z} & 0 & 0 & \frac{\partial m}{\partial z} & 0 & 0 & \frac{\partial n}{\partial z} \\
-\frac{\partial \chi}{\partial y} & -\frac{\partial \chi}{\partial x} & 0 & \frac{\partial l}{\partial y} & \frac{\partial l}{\partial x} & 0 & \frac{\partial m}{\partial y} & \frac{\partial m}{\partial x} & 0 & \frac{\partial n}{\partial y} & \frac{\partial n}{\partial x} & 0 \\
-\frac{\partial \chi}{\partial x} & 0 & -\frac{\partial \chi}{\partial z} & \frac{\partial l}{\partial x} & 0 & \frac{\partial l}{\partial z} & \frac{\partial m}{\partial x} & 0 & \frac{\partial m}{\partial z} & \frac{\partial n}{\partial x} & 0 & \frac{\partial n}{\partial z} \\
0 & -\frac{\partial \chi}{\partial z} & -\frac{\partial \chi}{\partial y} & 0 & \frac{\partial l}{\partial z} & \frac{\partial l}{\partial y} & 0 & \frac{\partial m}{\partial z} & \frac{\partial m}{\partial y} & 0 & \frac{\partial n}{\partial z} & \frac{\partial n}{\partial y}
\end{bmatrix} && (209)
\end{aligned}$$

Each element can be computed by using $\mathbf{M}$ that was defined above.

Now, define each variable, $\boldsymbol{u}^j$, $\boldsymbol{x}^j$, and $\boldsymbol{b}^j$.

$$\begin{aligned}
\boldsymbol{u}^j &= \begin{bmatrix} 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n & 0 & 0 \\ 0 & 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n & 0 \\ 0 & 0 & 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{bmatrix} \\
&= \mathbf{N}^{jT} \bar{\boldsymbol{u}}^j
\end{aligned} \tag{210}$$

$$\begin{aligned}
\boldsymbol{x}^j &= \begin{bmatrix} 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n & 0 & 0 \\ 0 & 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n & 0 \\ 0 & 0 & 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \\ x_3 \\ y_3 \\ z_3 \\ x_4 \\ y_4 \\ z_4 \end{bmatrix} \\
&= \mathbf{N}^{jT} \bar{\boldsymbol{x}}^j
\end{aligned} \tag{211}$$

$$\begin{aligned}
\boldsymbol{b}^j &= \begin{bmatrix} 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n & 0 & 0 \\ 0 & 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n & 0 \\ 0 & 0 & 1-\chi & 0 & 0 & l & 0 & 0 & m & 0 & 0 & n \end{bmatrix} \begin{bmatrix} b_{x1} \\ b_{y1} \\ b_{z1} \\ b_{x2} \\ b_{y2} \\ b_{z2} \\ b_{x3} \\ b_{y3} \\ b_{z3} \\ b_{x4} \\ b_{y4} \\ b_{z4} \end{bmatrix} \\
&= \mathbf{N}^{jT} \bar{\boldsymbol{b}}^j
\end{aligned} \tag{212}$$

where $\chi = l + m + n$.

$$\int_0^1 \int_0^{1-n} \int_0^{1-m-n} \boldsymbol{N}^j \boldsymbol{N}^{jT} dldmdn = \frac{1}{120} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \tag{213}$$

$$\int_0^1 \int_0^{1-n} \int_0^{1-m-n} \boldsymbol{X}^T dldmdn = \frac{1}{24} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{214}$$

$$\int_0^1 \int_0^{1-n} \int_0^{1-m-n} dldmdn = \frac{1}{6} \tag{215}$$

6.2.1 Calculation of stress components

Importantly, the discussions above are mainly focused on computing deformation at nodes. We sometimes (always) want stress components. For this case, the stress components are computed for each element. In other words, given the deformation of four nodes of an element, the stress components of that element is given as

$$\begin{aligned} \boldsymbol{\sigma}^j &= \mathbf{E} \nabla^T \boldsymbol{N}^{jT} \\ &= \mathbf{E} \begin{bmatrix} -\frac{\partial \chi}{\partial x} & 0 & 0 & \frac{\partial l}{\partial x} & 0 & 0 & \frac{\partial m}{\partial x} & 0 & 0 & \frac{\partial n}{\partial x} & 0 & 0 \\ 0 & -\frac{\partial \chi}{\partial y} & 0 & 0 & \frac{\partial l}{\partial y} & 0 & 0 & \frac{\partial m}{\partial y} & 0 & 0 & \frac{\partial n}{\partial y} & 0 \\ 0 & 0 & -\frac{\partial \chi}{\partial z} & 0 & 0 & \frac{\partial l}{\partial z} & 0 & 0 & \frac{\partial m}{\partial z} & 0 & 0 & \frac{\partial n}{\partial z} \\ -\frac{\partial \chi}{\partial y} & -\frac{\partial \chi}{\partial x} & 0 & \frac{\partial l}{\partial y} & \frac{\partial l}{\partial x} & 0 & \frac{\partial m}{\partial y} & \frac{\partial m}{\partial x} & 0 & \frac{\partial n}{\partial y} & \frac{\partial n}{\partial x} & 0 \\ -\frac{\partial \chi}{\partial x} & 0 & -\frac{\partial \chi}{\partial z} & \frac{\partial l}{\partial x} & 0 & \frac{\partial l}{\partial z} & \frac{\partial m}{\partial x} & 0 & \frac{\partial m}{\partial z} & \frac{\partial n}{\partial x} & 0 & \frac{\partial n}{\partial z} \\ 0 & -\frac{\partial \chi}{\partial z} & -\frac{\partial \chi}{\partial y} & 0 & \frac{\partial l}{\partial z} & \frac{\partial l}{\partial y} & 0 & \frac{\partial m}{\partial z} & \frac{\partial m}{\partial y} & 0 & \frac{\partial n}{\partial z} & \frac{\partial n}{\partial y} \end{bmatrix} \begin{bmatrix} u_1^j \\ v_1^j \\ w_1^j \\ u_2^j \\ v_2^j \\ w_2^j \\ u_3^j \\ v_3^j \\ w_3^j \\ u_4^j \\ v_4^j \\ w_4^j \end{bmatrix} \end{aligned} \tag{216}$$

6.3 Semi 3-dimensional case

In this case, we consider an idea case that a plate deforms in the out-of-plane direction. For this, we consider $u = v = 0$. Therefore, the strain and stress vectors can be simplified.

Therefore, the strain vector is given as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{zz} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial z} \\ \frac{1}{2}\frac{\partial w}{\partial x} \\ \frac{1}{2}\frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial z} \\ \frac{1}{2}\frac{\partial}{\partial x} \\ \frac{1}{2}\frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} w \end{bmatrix} \tag{217}$$

Similarly, the stress tensor is also symmetric, so we consider a 3 by 1 vector to describe the stress components.

$$\begin{aligned} \boldsymbol{\sigma} &= \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \\ &= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} \nu & 0 & 0 \\ \nu & 0 & 0 \\ 1-\nu & 0 & 0 \\ 0 & 1-2\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \epsilon_{zz} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} && (218) \\ &= \boldsymbol{E}\boldsymbol{\epsilon}. && (219) \end{aligned}$$

We start from the elastic energy:

$$\frac{1}{2}\int_V \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon} dV = \frac{1}{2}\int_V (\boldsymbol{E}\boldsymbol{H}\boldsymbol{U})^T \boldsymbol{H}\boldsymbol{U} dV = \frac{1}{2}\boldsymbol{U}^T \int_0^1 \int_0^{1-t} \boldsymbol{H}^T \boldsymbol{E}\boldsymbol{H} \left|J\right| dsdt \boldsymbol{U}. \tag{220}$$

Next, the work is rewritten as

$$-\int_V \rho \boldsymbol{g}\cdot\boldsymbol{u} dV = -\rho \int_0^1 \int_0^{1-t} \boldsymbol{g}^T \boldsymbol{X} \left|J\right| dsdt \boldsymbol{U}, \tag{221}$$

where $\boldsymbol{g} = [g_x, g_y]^T$, and J is the Jacobian matrix.

Finally, we consider the kinetic energy.

$$\frac{1}{2}\int_V \rho \dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}} dV = \frac{1}{2}\int_V \rho(\boldsymbol{X}\boldsymbol{U}')^T \boldsymbol{X}\boldsymbol{U}' dV = \frac{\rho}{2}\boldsymbol{U}'^T \int_0^1 \int_0^{1-t} \boldsymbol{X}^T \boldsymbol{X} \left|J\right| dsdt \boldsymbol{U}'. \tag{222}$$

From these formulations, we finally write the equation of motion (note we used the Lagrangian form defined above).

$$\int_0^1 \int_0^{1-t} \boldsymbol{X}^T \boldsymbol{X} \left|J\right| dsdt \boldsymbol{U}'' = \int_0^1 \int_0^{1-t} \boldsymbol{X}^T \boldsymbol{g} \left|J\right| dsdt - \frac{1}{\rho}\int_0^1 \int_0^{1-t} \boldsymbol{H}^T \boldsymbol{E}\boldsymbol{H} \left|J\right| dsdt \boldsymbol{U}. \tag{223}$$

7 Application of FEMs to structure

Consideration of the Bernoulli-Euler beam model. The work is described as

$$W = \int_0^l q(x)^e v(x)^e dx. \tag{224}$$

The node displacement vector is given as $\boldsymbol{u} = [v_1, \theta_1, v_2, \theta_2]^T$. Then, the displacement, $v(\xi)^e$, is given as

$$\begin{aligned} v^e(\xi) &= N_{v1}v_1 + N_{\theta 1}\theta_1 + N_{v2}v_2 + N_{\theta 2}\theta_2, &(225)\\ &= [N_{v1},\ N_{\theta 1},\ N_{v2},\ N_{\theta 2}] \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \boldsymbol{N}^{eT}\boldsymbol{u}^e. &(226)\end{aligned}$$

where $\xi = 2x/l - 1$, and

$$\begin{aligned} N^e_{v1} &= \frac{1}{4}(1-\xi)^2(2+\xi) = \frac{1}{4}(2-3\xi+\xi^3), &(227)\\ N^e_{\theta 1} &= \frac{l}{8}(1-\xi)^2(1+\xi) = \frac{l}{8}(1-\xi-\xi^2+\xi^3), &(228)\\ N^e_{v2} &= \frac{1}{4}(1+\xi)^2(2-\xi) = \frac{1}{4}(2+3\xi-\xi^3), &(229)\\ N^e_{\theta 2} &= -\frac{l}{8}(1+\xi)^2(1-\xi) = -\frac{l}{8}(1+\xi-\xi^2-\xi^3) &(230)\end{aligned}$$

Then,

$$[N''^e_{v1},\ N''^e_{\theta 1},\ N''^e_{v2},\ N''^e_{\theta 2}] = \left[\frac{3}{2}\xi,\ \frac{l}{4}(3\xi-1),\ -\frac{3}{2}\xi,\ \frac{l}{4}(3\xi+1)\right]. \tag{231}$$

And, we consider that double primes mean $N'' = d^2N/d\xi^2$.

If the loading only acts at $x = l$, $W = q_2 v_2$. If the loading acts at $x = 0$ and $x = l$, then $W = q_1 v_1 + q_2 v_2$. If the loading constantly acts over the element,

$$\begin{aligned} W &= q\int_0^l v(x)^e dx, \\ &= \frac{ql}{2}\int_{-1}^{1}(N_{v1}v_1 + N_{\theta 1}\theta_1 + N_{v2}v_2 + N_{\theta 2}\theta_2)d\xi. &(232)\end{aligned}$$

Consider the deformation of a Bernoulli-Euler beam, $\boldsymbol{u}(\boldsymbol{x}) = [u(x,y), v(x,y)]^T$. The strain is ϵ, while the stress is given as $\sigma = E\epsilon$. In this theory, it is assumed that

$$u(x,y) = -y\frac{\partial v(x)}{\partial x}. \tag{233}$$

Therefore,

$$\epsilon = \frac{\partial u(x,y)}{\partial x} = -y\frac{\partial^2 v(x)}{\partial x^2}. \tag{234}$$

and

$$\sigma = E\epsilon = E\frac{\partial u(x,y)}{\partial x} = -Ey\frac{\partial^2 v(x)}{\partial x^2}. \tag{235}$$

Considering the potential, we obtain

$$U = \frac{1}{2}\int_V \sigma\epsilon dV = \frac{1}{2}\int_0^L \int_A E\epsilon^2 dAdx = \frac{1}{2}\int_0^L \int_A Ey^2 \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dAdx \tag{236}$$

Defining the moment as I, where

$$I = \int_A y^2 dA, \tag{237}$$

we rewrite

$$U = \frac{1}{2}\int_0^L EI \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx = \frac{1}{2}\left(\frac{d\xi}{dx}\right)^4 \int_0^L EI\boldsymbol{v}^{eT}\boldsymbol{N}''^{e}\boldsymbol{N}''^{eT}\boldsymbol{v}^e dx. \tag{238}$$

Also, the work is given as

$$W = \int_0^L qv dx = \int_0^L q\boldsymbol{v}^{eT}\boldsymbol{N}^e dx \tag{239}$$

where q is the loading.

Here, we obtain

$$\mathbf{A} = \int_0^L \boldsymbol{N}''^{e}\boldsymbol{N}''^{eT} dx = \frac{l}{2}\int_{-1}^1 \boldsymbol{N}''^{e}\boldsymbol{N}''^{eT} d\xi. \tag{240}$$

and

$$\boldsymbol{B} = \int_0^L \boldsymbol{N}^{eT} dx = \int_{-1}^1 \boldsymbol{N}^{eT}\frac{dx}{d\xi}d\xi == \frac{l}{2}\int_{-1}^1 \boldsymbol{N}^{eT} d\xi \tag{241}$$

Given this definition, we derive

$$\Pi = U - W = \frac{1}{2}EI\boldsymbol{v}^{eT}\mathbf{A}\boldsymbol{v}^e - q\boldsymbol{v}^{eT}\mathbf{B}. \tag{242}$$

To minimize the potential, we consider that

$$\frac{\partial \Pi}{\partial \boldsymbol{v}^{eT}} = EI\mathbf{A}\boldsymbol{v}^e - q\boldsymbol{B} = \mathbf{0}. \tag{243}$$

8 Practical exercise

A bar is attached with the ceiling under a uniform gravity field, g. The length of the bar is $2L$, and the area of the bar is uniform and defined as A. The attachment of the bar with the ceiling is $x = 0$. The resistance force acting on the bar, or the normal stress times the area, σ at x follows the differential equation:

$$\frac{\partial \sigma(x)}{\partial x} + b = 0, \tag{244}$$

where $b = Ag$ is the force acting at x.

Now, we consider that there are three nodes: $x_0 = 0$, $x_1 = L$, and $x_2 = 2L$. Our question is to compute the resistance force at these points $[\sigma_0, \sigma_1, \sigma_2]$ under the boundary condition that σ_2 at $x_2 = 2L$ is zero.

In this problem, in each segment, i.e., $x_0 - x_1$ and $x_1 - x_2$, we assume that we can write $\sigma(x)$ as follows. For the $x_0 - x_1$ segment,

$$\sigma(x) = \frac{\sigma_1 - \sigma_0}{L} x + \sigma_0. \tag{245}$$

On the other hand, for the $x_1 - x_2$ segment,

$$\sigma(x) = \frac{\sigma_2 - \sigma_1}{L} x + \sigma_1. \tag{246}$$

Substituting these equations into the equilibrium equation, we obtain

$$\frac{1}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \end{bmatrix} = -b \tag{247}$$

$$\frac{1}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = -b \tag{248}$$

$$\tag{249}$$

Combining these equations yields,

$$\frac{1}{L} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} -b \\ -b \end{bmatrix} \tag{250}$$

Giving the boundary condition gives

$$\frac{1}{L} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \end{bmatrix} = \begin{bmatrix} -b \\ -b \end{bmatrix} \tag{251}$$

Therefore,

$$\begin{bmatrix} \sigma_0 \\ \sigma_1 \end{bmatrix} = \begin{bmatrix} 2Lb \\ Lb \end{bmatrix} \tag{252}$$

9 Formulation

It is not necessary to use a parabolic function to describe $\boldsymbol{X}(\xi)$ because our variational equation only includes $\boldsymbol{X}(\xi)$ and $\boldsymbol{X}'(\xi)$. For a linear function, a useful expression may be

$$u(\xi, t) = \boldsymbol{X}(\xi)\boldsymbol{U}(t) = [f_1, f_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{253}$$

where

$$f_1 = \frac{1-x}{l}, \tag{254}$$

$$f_2 = \frac{x}{l}. \tag{255}$$

Integrating these functions over 0 to l, we obtain

$$\int_0^l \boldsymbol{X}(\xi)\boldsymbol{X}(\xi)^T d\xi = \frac{l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{256}$$

$$\int_0^l \frac{E}{\rho}\boldsymbol{X}'(\xi)\boldsymbol{X}'(\xi)^T d\xi = \frac{E}{l\rho}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{257}$$

$$\int_0^l \boldsymbol{X}(\xi) d\xi = \frac{l}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{258}$$

9.1 Parabolic function

The interpolation function is defined by using three points (x_1, x_2, and x_3). Here, we define that the distance between x_1 and x_2 and that between x_2 and x_3 are both l. Therefore, the equation derived above is given as

$$\int_{x_1}^{x_3} \boldsymbol{X}(x)\boldsymbol{X}(x)^T dx \boldsymbol{U}(t)'' = -\int_{x_1}^{x_3} \frac{E}{\rho}\boldsymbol{X}'(x)\boldsymbol{X}'(x)^T dx \boldsymbol{U}(t) + \int_{x_1}^{x_3} g\boldsymbol{X}(x)dx. \tag{259}$$

Now, we convert x to be $x_1 + \xi$. Then, we obtain

$$\int_0^{2l} \boldsymbol{X}(\xi)\boldsymbol{X}(\xi)^T d\xi \boldsymbol{U}(t)'' = -\int_0^{2l} \frac{E}{\rho}\boldsymbol{X}'(\xi)\boldsymbol{X}'(\xi)^T d\xi \boldsymbol{U}(t) + \int_0^{2l} g\boldsymbol{X}(\xi)d\xi. \tag{260}$$

We define the interpolation function as

$$u(\xi, t) = a(t)\xi^2 + b(t)\xi + c(t), \tag{261}$$

where $a(t)$, $b(t)$, and $c(t)$ are time-dependent. Considering that the deformation at $\xi = 0$, $\xi = l$, and $\xi = 2l$ are u_1, u_2, and u_3, respectively, we obtain

$$a = \frac{u_1 - 2u_2 + u_3}{2l^2}, \tag{262}$$

$$b = \frac{-3u_1 + 4u_2 - u_3}{2l}, \tag{263}$$

$$c = u_1. \tag{264}$$

Using these expressions, we obtain

$$u(\xi, t) = \boldsymbol{X}(\xi)\boldsymbol{U}(t) = [f_1, f_2, f_3]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \tag{265}$$

where

$$
\begin{aligned}
f_1 &= \frac{1}{2l^2}x^2 - \frac{3}{2l}x + 1, && (266)\\
f_2 &= -\frac{1}{l^2}x^2 + \frac{2}{l}x, && (267)\\
f_3 &= \frac{1}{2l^2}x^2 - \frac{1}{2l}x. && (268)
\end{aligned}
$$

Integrating these functions over 0 to $2l$, we obtain

$$
\int_0^{2l} \boldsymbol{X}(\xi)\boldsymbol{X}(\xi)^T d\xi = \frac{l}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{269}
$$

$$
\int_0^{2l} \frac{E}{\rho}\boldsymbol{X}'(\xi)\boldsymbol{X}'(\xi)^T d\xi = \frac{E}{6l\rho}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{270}
$$

$$
\int_0^{2l} \boldsymbol{X}(\xi) d\xi = \frac{l}{3}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \tag{271}
$$

References

Hughes, T.J., 2000. The Finite Element Method: Linear Static and Dynamics Finite Element Analaysis. Dover.

Shearer, P.M., 2009. Introduction to Seismology. 2 ed., Cambridge University Press.

Slaughter, W.S., 2002. The Linearized Theory of Elasticity. Birkhäuser.