

Extremely efficient generation of Gamma random variables for $\alpha \geq 1$

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Abstract

The Gamma distribution is well-known and widely used in many signal processing and communications applications. In this letter, a simple and extremely efficient accept/reject algorithm is introduced for the generation of independent random variables from a Gamma distribution with any shape parameter $\alpha \geq 1$. The proposed method uses another Gamma distribution with integer $\alpha_p \leq \alpha$, from which samples can be easily drawn, as proposal function. For this reason, the new technique attains a higher acceptance rate (AR) for $\alpha \geq 3$ than all the methods currently available in the literature, with $\text{AR} \rightarrow 1$ as $\alpha \rightarrow \infty$.

I. INTRODUCTION

The Gamma probability density function (PDF) is given by $p_o(x) = C_1 p(x)$, with $C_1 = \frac{\beta^\alpha}{\Gamma(\alpha)}$, and

$$p(x) = x^{\alpha-1} \exp(-\beta x), \quad x \geq 0, \quad (1)$$

where $\alpha > 0$ is the shape parameter, $\beta > 0$ is the rate parameter and $\Gamma(\alpha)$ is the Gamma function [6, Chapter 4]. The Gamma distribution is well-known and widely used in different fields, such as Bayesian inference [10], signal processing [11] and digital communications [14]. In particular, in communications it has been recently applied in the simulation of fading/shadowing channels using the Weibull-Gamma model [4, 2] or the effects of the turbulent atmosphere in free-space optical links with the Gamma-Gamma approach, which requires two independent Gamma random variables [9, 13].

All of the aforementioned applications need the generation of independent Gamma random variables (RVs), X , with arbitrary values of α and β , i.e., $X \sim G(\alpha, \beta)$. When α is an integer, an exact sampler is available. Indeed, if $\alpha = n \in \mathbb{N}^+$ the Gamma PDF becomes an *Erlang* PDF [6], and X can be

generated as the sum of $n = \alpha$ independent exponential RVs, i.e., $X = \sum_{i=1}^n E_i$, where each E_i follows an exponential PDF with parameter β . These exponentials can be easily obtained through the inversion method [6, Chapter 3], allowing us to express X as

$$X = -\frac{1}{\beta} \sum_{i=1}^{\alpha} \ln(U_i) = -\frac{1}{\beta} \ln \left(\prod_{i=1}^{\alpha} U_i \right), \quad (2)$$

where the U_i are uniform RVs, i.e., $U_i \sim \mathcal{U}([0, 1])$. For $\alpha \notin \mathbb{N}^+$, the problem of generating a Gamma RV X is usually divided in two subcases: $\alpha < 1$ and $\alpha \geq 1$. Focusing on the second case, which is the one addressed in this paper, we note that an exact sampler does not exist, but several accept/reject methods have been introduced (see [6, 8, 7] for a review of the approaches proposed). Most of these methods consider only $\beta = 1$, since, given $\tilde{X} \sim G(\alpha, 1)$, it can be easily shown that $X = \frac{1}{\beta} \tilde{X} \sim G(\alpha, \beta)$.

In this letter, we develop an extremely efficient rejection sampler to draw independent samples from a Gamma PDF with $\alpha \geq 1$ and any value of β . The proposed method outperforms all the alternative techniques reported in the literature in terms of acceptance rate (i.e., the key performance measure of a accept/reject methods) for $\alpha \geq 3$. The main idea is using a suitable Gamma PDF with an integer $\alpha_p \leq \alpha$ as a proposal density, from which samples can be easily drawn using Eq. (2). Since the proposal is itself another Gamma PDF, it provides a very good fit of the target, thus attaining very high acceptance rates that tend to 100% for $\alpha \rightarrow +\infty$, i.e., virtually providing exact sampling.

II. BACKGROUND: ACCEPT/REJECT ALGORITHM

Rejection sampling (RS) is a classical technique for generating independent samples from an arbitrary target PDF, $p_o(x) = C_1 p(x)$ with $x \in \mathcal{D}$ and $C_1 = [\int_{\mathcal{D}} p(x) dx]^{-1}$, using an alternative simpler proposal PDF, $\pi_o(x) = C_2 \pi(x)$ with $x \in \mathcal{D}$ and $C_2 = [\int_{\mathcal{D}} \pi(x) dx]^{-1}$, such that $\pi(x) \geq p(x)$, i.e., $\pi(x)$ is a hat function w.r.t. $p(x)$. RS works by generating samples from the proposal density, $x' \sim \pi_o(x)$, accepting them when $u' \leq p(x')/\pi(x')$, with $u' \sim \mathcal{U}([0, 1])$, and rejecting them otherwise. The key performance measure for RS is the average acceptance rate (AR), $a_R = \int_{\mathcal{D}} \frac{p(x)}{\pi(x)} \pi_o(x) dx = \frac{C_2}{C_1} \leq 1$. The value of a_R depends on how close the proposal is to the target, and determines the efficiency of the approach. Hence, the main difficulty when designing an RS algorithm is finding a good hat function, $\pi(x) \geq p(x)$, such that $\pi(x)$ and $p(x)$ are as close as possible and drawing samples from $\pi_o(x) = C_2 \pi(x)$ can be done easily and efficiently.

III. NOVEL TECHNIQUE

In this letter, we consider as target density the PDF given in Eq. (1) with $\alpha \geq 1$ and any $\beta > 0$. As proposal PDF, we suggest using another Gamma density with different parameters, namely,

$$\pi_o(x) \propto \pi(x) = K_p x^{\alpha_p-1} \exp(-\beta_p x), \quad x \geq 0, \quad (3)$$

where $\alpha_p = \lfloor \alpha \rfloor \leq \alpha$, with $\lfloor \alpha \rfloor$ denoting the integer part of $\alpha \in [1, +\infty)$, and the remaining parameters (β_p and K_p) adjusted to obtain the same location and value of the maximum for the proposal and the target. On the one hand, for $\alpha \geq 2$,

$$\beta_p = \frac{\alpha_p - 1}{x_{\max}} = \beta \frac{\alpha_p - 1}{\alpha - 1}, \quad (4)$$

$$\begin{aligned} K_p &= \frac{p(x_{\max})}{x^{\alpha_p-1} \exp(-\beta_p x)} \\ &= \exp(\alpha_p - \alpha) \left(\frac{\alpha - 1}{\beta} \right)^{(\alpha - \alpha_p)}, \end{aligned} \quad (5)$$

where $x_{\max} = \frac{\alpha-1}{\beta}$ is the location of the single maximum of the Gamma PDF, obtained solving $\frac{dp(x)}{dx} = 0$.

For $1 \leq \alpha < 2$, $\alpha_p = 1$, we set

$$\beta_p = \frac{\beta}{\alpha}, \quad (6)$$

$$K_p = \exp(1 - \alpha) \left(\frac{\alpha}{\beta} \right)^{\alpha-1}. \quad (7)$$

In this case, they are obtained finding the exponential function tangent to $p(x)$ at the *optimal point*

$$x^* = \frac{\alpha}{\beta}.$$

This is an optimal value since it maximizes the acceptance rate, as shown in the Appendix A. When $\alpha = 1$, the parameters are $\alpha_p = \alpha$, $\beta_p = \beta$ and $K_p = 1$. In this case the unique intersection point between $\pi(x)$ and $p(x)$ is x^* . The values in Eqs. (6)-(7) can be easily obtained analytically computing the tangent straight line to the function $\log[p(x)]$ at x^* (note that the Gamma pdf is a log-concave density).

Thanks to this choice of α_p and the parameters derived in Eqs. (4)-(5)-(6) and (7), we can ensure that: **(a)** we can draw samples exactly from $\pi_o(x) \propto \pi(x)$ [6]; **(b)** $\pi(x) \geq p(x)$ for all $x \geq 0$, as proved in the following section. Figure 1 depicts some examples of envelope and target functions with different values of the parameters.

Our algorithm can be summarized in the following three steps: **(1)** calculate the parameters of the proposal PDF, $\pi_o(x) \propto \pi(x)$; **(2)** draw a sample x' from $\pi_o(x)$ using the direct approach described in

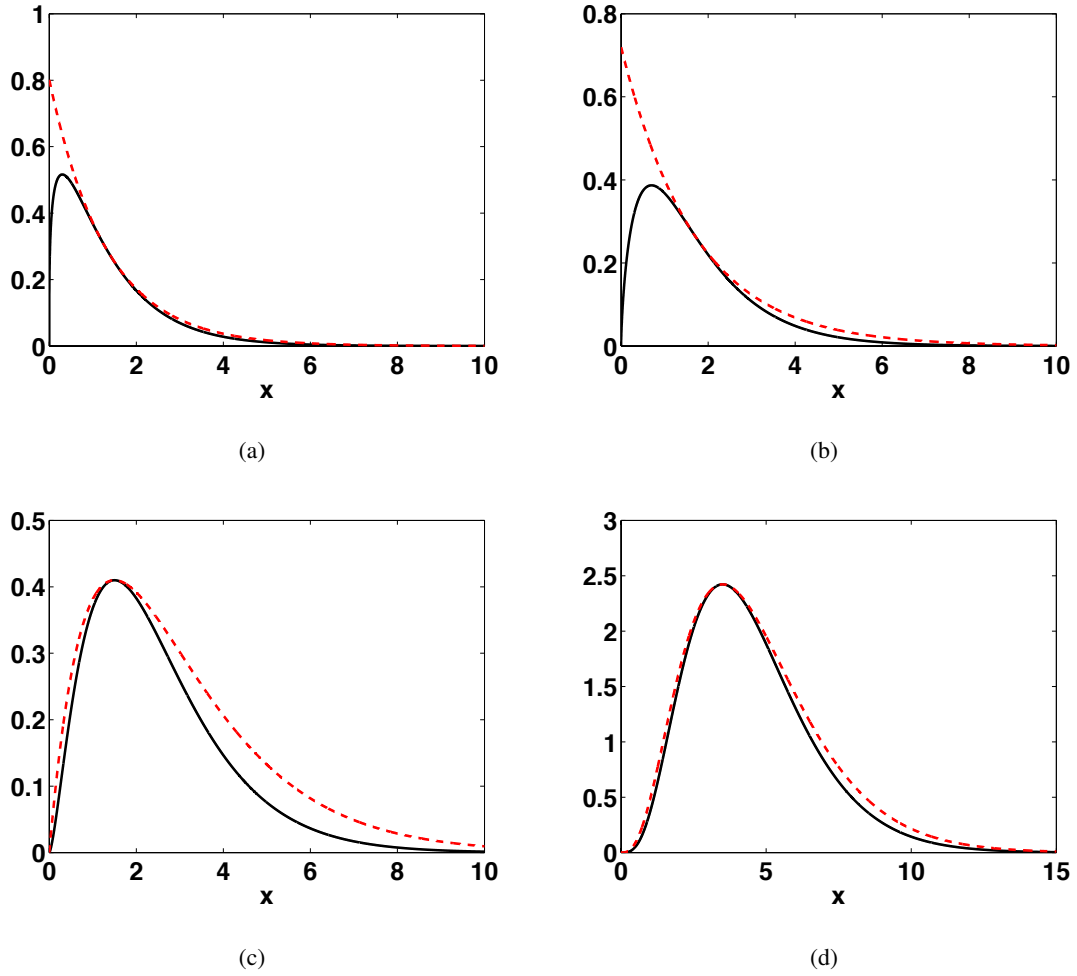


Fig. 1. The target function $p(x)$ (solid line) and the proposed envelope function $\pi(x)$ (dashed line) with different values of the parameters: **(a)** $\alpha = 1.3$, $\beta = 1$, **(b)** $\alpha = 1.7$, $\beta = 1$, **(c)** $\alpha = 2.5$, $\beta = 1$ and **(d)** $\alpha = 4.5$, $\beta = 1$.

Eq. (2), i.e., generate α_p independent uniform RVs, $u_i \sim \mathcal{U}([0, 1])$ with $i = 1, \dots, \alpha_p$, and set

$$x' = -\frac{1}{\beta_p} \ln \left(\prod_{i=1}^{\alpha_p} u_i \right); \quad (8)$$

(3) accept x' with probability $p(x')/\pi(x')$, discarding it otherwise. Repeat steps (2) and (3) until the desired number of samples have been obtained.

IV. PROOF OF THE RS INEQUALITY

A. Case $\alpha \geq 2$

Consider first the proposal pdf for $\alpha \geq 2$. In order to apply the RS technique, we need to ensure that $\pi(x) \geq p(x)$, i.e.,

$$K_p x^{\alpha_p - 1} \exp(-\beta_p x) \geq x^{\alpha - 1} \exp(-\beta x), \quad \forall x \geq 0. \quad (9)$$

For $x > 0$, Eq. (9) can be rewritten alternatively as

$$K_p \exp(\Omega x) \geq x^{\alpha - \alpha_p}, \quad (10)$$

where $\Omega \triangleq \beta - \beta_p$ and $x^{\alpha - \alpha_p}$ presents a sub-linear growth, since $0 \leq \alpha - \alpha_p < 1$. Finally, taking the logarithm on both sides of (10),

$$\ln(K_p) + \Omega x \geq (\alpha - \alpha_p) \ln(x). \quad (11)$$

Now, since $\alpha \geq \alpha_p$ and β_p is given by (4), we note that

$$\Omega = \beta - \beta_p = \beta \left(1 - \frac{\alpha_p - 1}{\alpha - 1} \right) \geq 0. \quad (12)$$

Hence, the linear function on the left hand side of (11) is increasing. Moreover, since $\alpha \geq \alpha_p$, the logarithmic function on the right hand side of (11) is increasing and . Consequently, since both functions are increasing and concave for $x > 0$ (i.e., their second derivatives are lower or equal than zero), they can have at most two intersection points. Indeed, in the sequel we show that they are tangent at $x = x_{\max}$, which is the only contact point between both curves for $x > 0$. In order to prove this, we need to show:

(1) that both functions are equal at $x = x_{\max}$, i.e.,

$$\ln(K_p) + \Omega x_{\max} = (\alpha - \alpha_p) \ln(x_{\max}) = (\alpha - \alpha_p) \ln \left(\frac{\alpha - 1}{\beta} \right), \quad (13)$$

which is fulfilled by construction of the proposal, as K_p and β_p are set to achieve $\pi(x_{\max}) = p(x_{\max})$;

(2) that their first derivatives are equal, i.e.,

$$\left. \frac{d(\ln(K_p) + \Omega x)}{dx} \right|_{x=x_{\max}} = \left. \frac{d((\alpha - \alpha_p) \ln x)}{dx} \right|_{x=x_{\max}}, \quad (14)$$

and taking the derivatives we obtain

$$\Omega = \frac{\alpha - \alpha_p}{x_{\max}} = \frac{\beta(\alpha - \alpha_p)}{\alpha - 1} = \beta \left(1 - \frac{\alpha_p - 1}{\alpha - 1} \right), \quad (15)$$

which is the result given by Eq. (12).

Consequently, since x grows faster than $\ln(x)$, we can guarantee that $\ln(K_p) + \Omega x \geq (\alpha - \alpha_p) \ln(x)$ for $x > 0$, with equality only at $x = x_{\max}$. Hence, the RS inequality in (9) is satisfied and the proposal is indeed a hat function for $p(x)$, i.e., $\pi(x) \geq p(x)$, with equality only at $x = 0$, $x = x_{\max}$ and $x \rightarrow +\infty$.

B. Case $1 \leq \alpha < 2$

For, $1 \leq \alpha < 2$, we have $\alpha_p = 1$ and the proposal pdf is built computing the tangent straight line $r(x) = -\beta_p x + \log[K_p]$ tangent to $\log[p(x)]$ at a generic point x^* (for the optimal choice of this point, see the Appendix A). Then, the envelope function is

$$\pi(x) = \exp(r(x)) = K_p \exp(-\beta_p x).$$

For the log-concavity of $p(x)$, i.e.,

$$\frac{d^2 \log[p(x)]}{dx^2} \geq 0,$$

we have

$$r(x) \geq \log[p(x)],$$

and since the exponential function is a monotonically increasing transformation, we have $\pi(x) \geq p(x)$, that is the needed condition to apply the RS technique.

V. ACCEPTANCE RATE OF THE NOVEL RS SCHEME

The acceptance rate of the novel scheme can be calculated analytically. Indeed, in general, we have

$$a_R = \frac{\int_0^{+\infty} p(x) dx}{\int_0^{+\infty} \pi(x) dx},$$

so that

$$a_R = \begin{cases} \frac{\Gamma(\alpha)}{\alpha^\alpha e^{1-\alpha}}, & \text{for } 1 \leq \alpha < 2, \\ \left(\frac{e}{\alpha-1}\right)^{(\alpha-\alpha_p)} \frac{\Gamma(\alpha)}{\Gamma(\alpha_p)}, & \text{for } \alpha \geq 2, \end{cases} \quad (16)$$

that is independent from the parameter β .

VI. RESULTS

Gamma generators in the literature are usually designed and compared for $\beta = 1$, without loss of generality. Hence, in order to obtain a fair comparison we only consider $\beta = 1$, although our approach is valid for any value of β . We have compared the acceptance rate (AR), a_R , of the different algorithms described below [6, 8, 7]:

- **Our method (M1):** For $1 \leq \alpha < 2$, the AR is indicated as a_{R1} and is given in Eq. (16). Note that, for $\alpha \rightarrow +\infty$, we obtain $a_{R1} \rightarrow 1$, i.e., our approach provides exact sampling asymptotically.
- **Log-logistic method (M2) [5]:** The proposal PDF is $\pi(x) = K_1 \frac{x^{\lambda-1}}{(\mu+x^\lambda)^\lambda}$, with $\lambda = \sqrt{2\alpha-1}$, $\mu = \alpha^\lambda$ and $K_1 = 4\alpha^{(\alpha+\lambda)}e^{-\alpha}$. The theoretical AR is $a_{R2} = \frac{\Gamma(\alpha)\mu\lambda}{K_1}$. For $\alpha \rightarrow +\infty$, $a_{R2} \rightarrow \frac{\sqrt{\pi}}{2} \approx 0.88$.

- **Cauchy method (M3) [1]:** The proposal is $\pi(x) = K_2 \frac{\lambda}{\lambda^2 + (x-a)^2}$, with $\lambda = \sqrt{2\alpha - 1}$, $a = \alpha - 1$ and $K_2 = \frac{c}{2\pi} \lambda (a-1)^{(a-1)} e^{1-a}$, where $c = \pi + 2 \arctan(a/\lambda)$. The AR is $a_{R3} = \frac{\Gamma(\alpha)}{\pi K_2}$ and we have $a_{R3} \rightarrow \frac{1}{\sqrt{\pi}} \approx 0.56$, for $\alpha \rightarrow +\infty$.
- **T-student method (M4) [3]:** The proposal is $\pi(x) = K_3 (1 + 0.5(\frac{x-\alpha+1}{\eta})^2)^{-3/2}$ with $\eta = \sqrt{\frac{3\alpha-0.75}{2}}$ and $K_3 = (\alpha - 1)^{\alpha-1} e^{1-\alpha}$. It can be shown that $a_{R4} \rightarrow \sqrt{\frac{\pi}{6}} \approx 0.72$ for $\alpha \rightarrow +\infty$.
- **Modified Ratio-of-Uniforms (RoU) (M5) [12]:** It is a variant of the RoU scheme [6, 8], relocating the mode of $p(x)$ at $x = 0$. The asymptotic AR is $a_{R4} \rightarrow \frac{\sqrt{e\pi}}{4} \approx 0.73$, for $\alpha \rightarrow +\infty$. The AR, a_{R4} , is almost constant for all values of $\alpha > 1$, as shown in Figure 2.

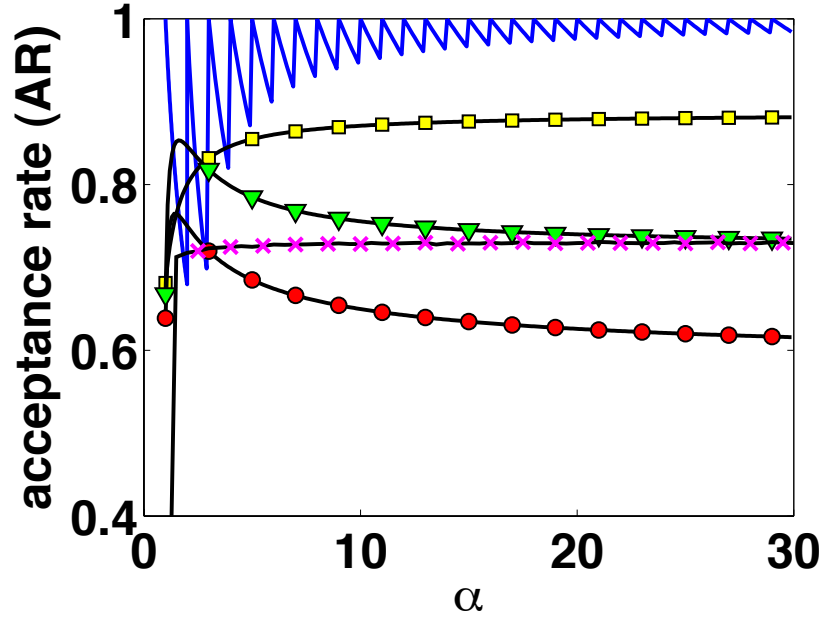


Fig. 2. Acceptance rate (AR) using our method M1 (continuous line), M2 (squares), M3 (circles), M4 (triangles) and M5 (x-marks), for $1 \leq \alpha \leq 30$ and $\beta = 1$.

Figure 2 shows the ARs of all the techniques described above, obtained empirically after drawing $N = 6 \cdot 10^5$ independent samples, for different values of α . For $1 \leq \alpha \leq 1.35$ our the technique M1 provide the best results whereas for $1.35 \leq \alpha \leq 2$ the best technique is M4. For $2 \leq \alpha \leq 2.37$ the newly proposed approach (M1) provides the highest AR while for $2.37 < \alpha < 3$, the best method is M2 or M4 depending of the value of α . For $\alpha \geq 3$, our technique (M1) is extremely efficient, outperforming (except for $3.87 \leq \alpha < 4$ where M2 is slightly better) the rest of the methods and providing the best results ever reported in the literature. The minimum AR obtained with M1 is ≈ 0.68 for $\alpha \approx 1.99$. Furthermore, our

technique provides exact sampling (i.e., $a_{R1} \rightarrow 1$) asymptotically as $\alpha \rightarrow +\infty$.

VII. CONCLUSIONS

We have developed a rejection sampling (RS) scheme for generating Gamma random variables, with arbitrary values of $\alpha \geq 1$ and β , where the proposal PDF is itself another Gamma density. The proposed algorithm is simple and extremely efficient, providing the best acceptance rates ever reported in the literature for $\alpha \geq 3$.

VIII. ACKNOWLEDGEMENT

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APPENDIX

A. Optimal choice of the tangent point for $1 \leq \alpha < 2$

First, we recall the notation $p_o(x) = C_1 p(x)$ with

$$C_1 = \left[\int_{\mathcal{D}} p(x) dx \right]^{-1},$$

and $\pi_o(x) = C_2 \pi(x)$ with

$$C_2 = \frac{1}{I_\pi} = \left[\int_{\mathcal{D}} \pi(x) dx \right]^{-1}.$$

The acceptance rate (AR) is

$$\begin{aligned} a_R &= \int_{\mathcal{D}} \frac{p(x)}{\pi(x)} \pi_o(x) dx \\ &= \frac{\int_{\mathcal{D}} p(x) dx}{\int_{\mathcal{D}} \pi(x) dx} = \frac{C_2}{C_1} = \frac{1}{C_1 I_\pi} \leq 1. \end{aligned} \tag{17}$$

Since the area below the target $p(x)$ is given (then fixed), the only way to increase the AR is diminishing the area I_π below the envelope function, i.e., we desire to build a function $\pi(x)$ such that satisfies jointly both conditions

$$\begin{cases} I_\pi^* = \min_{\pi(x)} \int_{\mathcal{D}} \pi(x) dx, \\ \pi(x) \geq p(x). \end{cases} \tag{18}$$

For $1 \leq \alpha < 2$, the novel technique uses as proposal function of the form in Eq. (3) with $\alpha_p = 1$, and the other parameters, as function of a generic tangent point θ , are

$$\begin{aligned}\beta_p(\theta) &= -\frac{\alpha-1}{\theta} + \beta, \quad \theta \neq 0 \\ K_p(\theta) &= \theta^{\alpha-1} e^{1-\alpha}.\end{aligned}$$

Therefore, our proposal for $1 \leq \alpha < 2$ has the following form

$$\pi(x) = K_p(\theta) \exp(-\beta_p(\theta)x), \quad x \in \mathcal{D} = \mathbb{R}^+, \quad \theta \in \mathbb{R}^+ \setminus \{0\},$$

then

$$\begin{aligned}I_\pi(\theta) &= \int_0^{+\infty} K_p(\theta) \exp(-\beta_p(\theta)x) dx = \frac{K_p(\theta)}{\beta_p(\theta)} \\ &= \frac{\theta^{\alpha-1} e^{1-\alpha}}{-\frac{\alpha-1}{\theta} + \beta}, \quad \theta \neq 0.\end{aligned}\tag{19}$$

The value of θ that minimizes $I_\pi(\theta)$ is a solution of the equation

$$\begin{aligned}\frac{dI_\pi(\theta)}{d\theta} &= 0, \\ \frac{\theta^{\alpha-1}(\alpha - \beta\theta)(1 - \alpha)e^{1-\alpha}}{(\beta\theta - \alpha + 1)^2} &= 0.\end{aligned}\tag{20}$$

The solution are $\theta^* = 0$ (that is not admissible) and

$$x^* = \theta^* = \frac{\alpha}{\beta}.$$

Choosing this value x^* as tangent point to construct the envelope function,

$$\pi(x) = \left(\frac{\alpha}{\beta}\right)^{\alpha-1} e^{1-\alpha} \exp\left(-\frac{\beta}{\alpha}x\right),$$

we maximize the AR of the RS scheme. Since $C_1 = \frac{\beta^\alpha}{\Gamma(\alpha)}$, the AR in this case is

$$\begin{aligned}a_R &= \frac{\Gamma(\alpha)\beta_p(x^*)}{\beta^\alpha K_p(x^*)} \\ &= \frac{\Gamma(\alpha)\frac{\beta}{\alpha}}{\beta^\alpha \left(\frac{\alpha}{\beta}\right)^{\alpha-1} e^{1-\alpha}} = \frac{\Gamma(\alpha)}{\beta^\alpha \left(\frac{\alpha}{\beta}\right)^\alpha e^{1-\alpha}} \\ &= \frac{\Gamma(\alpha)}{\alpha^\alpha e^{1-\alpha}},\end{aligned}\tag{21}$$

independent from the parameter β .

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