EE 457 HW 1

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Question 1-

Defining $v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, we can determine linear independency of this set of vectors by showing $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \iff \alpha_1 = \alpha_2 = \alpha_3 = 0$. Let's assume $v_1 = c_1 v_2 + c_2 v_3$ where $c_1 = -\frac{\alpha_2}{\alpha_1}$ and $c_2 = -\frac{\alpha_3}{\alpha_1}$. Then, $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3c_1 + 2c_2 \\ c_1 - c_2 \\ 2c_1 + c_2 \end{bmatrix}$. By using second and third rows, $c_1 = -\frac{1}{3}$, and inserting c_1 into first row yields $c_2 = 1$. Putting this values into the second row results in equality $-3 = -\frac{1}{3} - 1$ which is false. Therefore, we conclude that v_1 is linearly independent of v_2 and v_3. In the same manner, by assuming $v_2 = c_3 v_3$ we can show that v_2 and v_3 are linearly independet as well. Finally, it is concluded that this set of vectors are linearly independent. ■

Question 2-

Defining $x = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \nu_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \nu_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, we can start finding the closest point to x by calculating orthogonal vectors which span subspace W with the use of Gram- Schmid method. Then, $u_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $u_2 = u_1 - \frac{\langle \nu_2, u_1 \rangle}{\langle \nu_2, \nu_2 \rangle} \nu_2 = \begin{bmatrix} 1/2 \\ -3/2 \\ 0 \end{bmatrix}$ are orthogonal vectors. Since x is a three dimensional vector and we only have two linearly independent orthogonal vectors which span a two dimensional subspace W, we can represent x with an additional residual vector, r, which is orthogonal to both u_1 and u_2 as follows, $x = \lambda_1 u_1 + \lambda_2 u_2 + r$. By taking inner product of x with u_1 and u_2 seperately, values of λ_1 and λ_2 can be determined, respectively.

$$\begin{aligned} \langle x, u_1 \rangle =& \lambda_1 \langle u_1, u_1 \rangle + \lambda_2 \langle u_2, u_1 \rangle + \lambda_1 \langle r, u_1 \rangle \\ 4 =& \lambda_1 14 + 0 + 0 \\ \lambda_1 =& 2/7 \end{aligned}$$

$$\begin{aligned} \langle x, u_2 \rangle =& \lambda_1 \langle u_1, u_2 \rangle + \lambda_2 \langle u_2, u_2 \rangle + \lambda_1 \langle r, u_2 \rangle \\ 5 =& 0 + \lambda_2 10/4 + 0 \\ \lambda_2 =& 2 \end{aligned}$$

In the above equations, note that inner product of orthogonal vectors is zero. Then, we define the closest vector p to x as $p = \lambda_1 u_1 + \lambda_2 u_2$ and it is $p = \begin{bmatrix} 13/7 \\ -19/7 \\ 4/7 \end{bmatrix}$.

Question 3-

(i) Positivity:

$$(\sum_{k=1}^{n} |x_i|^p)^{\frac{1}{p}} = (|x_1|^p + |x_2|^p + .. + |x_n|^p)^{\frac{1}{p}} \geq 0 \tag{1}$$

$$x_i = 0, \ \forall i = 1, 2, .., n \implies (|0|^p + |0|^p + .. + |0|^p)^{\frac{1}{p}} = 0 \tag{2}$$

(ii) Triangle Inequality: Let $x, y \in \mathbb{C}^n$ and x_i, y_i indicate i^{th} elements.

$$|x_i{}^\star y_i|^p = |x_i{}^\star y_i||x_i{}^\star y_i|^{p-1} \leq (|x_i| + |y_i|)|x_i{}^\star y_i|^{p-1} \tag{3}$$

$$|x_i{}^\star y_i|^p \leq |x_i||x_i{}^\star y_i|^{p-1} + |y_i||x_i{}^\star y_i|^{p-1} \tag{4}$$

Inequality argument given in (3) is actually triangle inequality itself and it can be proven as follows,

$$\begin{aligned} x_i^2 + y_i^2 + 2|x_i||y_i| &\geq x_i^2 + y_i^2 + 2x_i y_i \\ (|x_i| + |y_i|)^2 &\geq (x_i + y_i)^2 \\ ||x_i| + |y_i|| &\geq |x_i + y_i| \\ |x_i| + |y_i| &\geq |x_i + y_i| \end{aligned} \tag{5}$$

Since inequality (4) holds for every elements of x and y, following inequality also holds,

$$\sum_{k=1}^{n} |x_i^\star y_i|^p \leq \sum_{k=1}^{n} |x_i||x_i{}^\star y_i|^{p-1} + \sum_{k=1}^{n} |y_i||x_i{}^\star y_i|^{p-1} \tag{6}$$

By defining $\frac{1}{p} + \frac{1}{q} = 1$ and using Hölder inequality $\|\alpha\beta\|_1 \leq \|\alpha\|_p \|\beta\|_q$, elements of the right hand side of (6) can be represented as follows,

$$\sum_{k=1}^{n} |x_i||x_i{}^\star y_i|^{p-1} \leq (\sum_{k=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{k=1}^{n} |x_i^\star y_i|^{(p-1)q})^{\frac{1}{q}} \tag{7}$$

$$\sum_{k=1}^{n} |y_i||x_i{}^\star y_i|^{p-1} \leq (\sum_{k=1}^{n} |y_i|^p)^{\frac{1}{p}} (\sum_{k=1}^{n} |x_i^\star y_i|^{(p-1)q})^{\frac{1}{q}} \tag{8}$$

Summing up the right hand sides of (7) and (8), inserting it into (6) yields,

$$\sum_{k=1}^{n} |x_i^\star y_i|^p \leq ((\sum_{k=1}^{n} |x_i|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |y_i|^p)^{\frac{1}{p}}) (\sum_{k=1}^{n} |x_i^\star y_i|^{(p-1)q})^{\frac{1}{q}} \tag{9}$$

Using $(p-1)q = p$ and dividing both sides with $(\sum_{k=1}^{n} |x_i^\star y_i|^{(p-1)q})^{\frac{1}{q}}$ yields,

$$(\sum_{k=1}^{n} |x_i^\star y_i|^p)^{1-\frac{1}{q}} \leq (\sum_{k=1}^{n} |x_i|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |y_i|^p)^{\frac{1}{p}} \tag{10}$$

and since $\frac{1}{p} + \frac{1}{q} = 1$,

$$(\sum_{k=1}^{n} |x_i^\star y_i|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{n} |x_i|^p)^{\frac{1}{p}} + (\sum_{k=1}^{n} |y_i|^p)^{\frac{1}{p}} \tag{11}$$

(iii) Scaling:

$$\begin{aligned} \|\alpha x\|_p &= (\sum_{k=1}^{n} |\alpha x_i|^p)^{\frac{1}{p}} \\ &= (|\alpha x_1|^p + |\alpha x_2|^p + .. + |\alpha x_n|^p)^{\frac{1}{p}} \\ &= (\alpha^p |x_1|^p + \alpha^p |x_2|^p + .. + \alpha^p |x_n|^p)^{\frac{1}{p}} \\ &= (\alpha^p)^{\frac{1}{p}} (|x_1|^p + |x_2|^p + .. + |x_n|^p)^{\frac{1}{p}} \\ &= \alpha (|x_1|^p + |x_2|^p + .. + |x_n|^p)^{\frac{1}{p}} \\ &= \alpha \|x\|_p \end{aligned} \tag{12}$$

Since $\|.\|_p$ satisfies all three conditions, it is a valid norm. ■

Question 4-

$A \in \mathbb{R}^{m\times n}$, $a_{ij} \in A$ $i = 1, 2, .., m$ $j = 1, 2, .., n$

(i) Positivity:

$$\|A\|_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{1/2} = (|a_{11}| + |a_{12}| + .. + |a_{1n}| + |a_{21}| + |a_{22}| + |a_{2n}| + |a_{mn}|)^{1/2} \geq 0$$

$$\forall\ i, \forall\ j,\ a_{ij} = 0 \implies \|A\|_F = (|0| + |0| + .. + |0| + |0| + |0| + |0| + |0|)^{1/2} = 0 \tag{13}$$

(ii) Triangle inequality:

Let $B \in \mathbb{R}^{m\times n}$, $b_{ij} \in B$ $i = 1, 2, .., m$ $j = 1, 2, .., n$

$$\begin{aligned}
\|A + B\|_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij} + b_{ij}|^2)^{1/2} &\leq (\sum_{i=1}^{m}\sum_{j=1}^{n}(|a_{ij}|^2 + |b_{ij}|^2))^{1/2} && (14)\\
&= (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n}|b_{ij}|^2)^{1/2} && (15)\\
&\leq (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{1/2} + (\sum_{i=1}^{m}\sum_{j=1}^{n}|b_{ij}|^2)^{1/2} && (16)\\
&= (\|A\|_F^2)^{1/2} + (\|B\|_F^2)^{1/2}\\
&= \|A\|_F + \|B\|_F && (17)
\end{aligned}$$

It is possible to get inequality given in (14) through using inequality proven in (5). By similar argument it is also possible to get the inequality shown in (16).

(iii) Scalability:

Let $\alpha \in \mathbb{R}$

$$\begin{aligned}
\|\alpha A\|_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}|\alpha a_{ij}|^2)^{1/2} = (\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha^2|a_{ij}|^2)^{1/2} &= (\alpha^2\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{1/2}\\
&= |\alpha|(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{1/2}\\
&= |\alpha|\|A\|_F && (18)
\end{aligned}$$

Since $\|.\|_F$ satisfies all three conditions, it is a valid norm. ■

Question 5-

a) Nullspace of matrix A can be found by calculating $S = \{x : x \in \mathbb{R}^3, Ax = 0\}$. Elements of the set S can be found by performing row operations on A and solving the resulting equations as follows,

$$\begin{bmatrix}4 & 2 & 6\\2 & 5 & 3\\6 & 3 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \implies \begin{bmatrix}1/2 & 0 & 0\\-1/2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-3/2 & 0 & 1\end{bmatrix}\begin{bmatrix}4 & 2 & 6\\2 & 5 & 3\\6 & 3 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \implies \begin{bmatrix}2 & 1 & 3\\0 & 4 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \implies$$

$$\begin{aligned}
2x_1 + x_2 + 3x_3 &= 0\\
x_2 &= 0
\end{aligned}$$

$$x = x_3\begin{bmatrix}-3/2\\0\\1\end{bmatrix} = \alpha\begin{bmatrix}-3/\sqrt{14}\\0\\2/\sqrt{14}\end{bmatrix}$$

where $\alpha \in \mathbb{R}$. Then, nullspace of A, $N(A) = \{\alpha\begin{bmatrix}-0.80 & 0 & 0.53\end{bmatrix}^T : \alpha \in \mathbb{R}\}$.

b) In order to find range space of A (column space) we need to determine the linearly independent columns of A and it can be achieved through calculating reduced row echelon form of A^T. Additionally, since A is symmetric, i.e. $A^T = A$, for this specific example, we can continue with row reducing process shown above as the following,

$$\begin{bmatrix} 1/2 & -1/8 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, range space of A is $R(A) = \{\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 3/2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : \alpha_1, \alpha_2 \in \mathbb{R}\}$

c) Since b is in the range space of A, solution exists, i.e. $\alpha_1 = 14$ and $\alpha_2 = 15$. $\exists x, Ax = b \iff b \in R(A)$. Additionally, $Rank([A : b]) = Rank(A)$

d) Solution set to $Ax = b$ can be found by row reducing augmented matrix $[A : b]$ with the use of left nonsingular matrices shown in a) and b), as following,

$$\begin{bmatrix} 4 & 2 & 6 & 14 \\ 2 & 5 & 3 & 15 \\ 6 & 3 & 9 & 21 \end{bmatrix} \sim \sim \begin{bmatrix} 1 & 0 & 3/2 & 5/2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$\begin{aligned} x_1 =& \frac{5}{2} - \frac{3}{2}x_3, \\ x_2 =& 2, \end{aligned}$$

and Solution set S is given by,

$$S = \{x = \begin{bmatrix} 5/2 \\ 2 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} : \alpha_1 \in \mathbb{R}\}$$

Question 6-

a) Eigenvalues of A can be calculated by solving characteristic equation $det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)$$

Therefore, eigenvalues of A are $\lambda_1, \lambda_2 = 1, \lambda_3 = 2$.

b) $\lambda_1 = 1$ $d_1 = 2$ and $\lambda_2 = 2$ $d_2 = 1$. We need to check geometric multiplicities for each λ.
To find the nullspace for $\lambda_1 = 1$:

$$\begin{aligned} (A - \lambda_1 I)x =& 0 \\ =& \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} x \implies x_1 = x_1,\ x_2 = x_2,\ x_3 = 0 \end{aligned}$$

Therefore, $N(A - \lambda_1 I) = \{\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : \alpha_1,\ \alpha_2 \in \mathbb{R}\}$. Consequently, $\dim(N(A - \lambda_1 I)) = 2$, $\eta_1 = 2$. Eigenvectors of λ_1 are actually vectors which consist a base for the null space. Therefore eigenvectors for $\lambda_1 = 1$ are $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

By following the same procedure for $\lambda_2 = 2$,

$$\begin{aligned} (A - \lambda_1 I)x =& 0 \\ =& \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} x \implies x_1 = -x_3,\ x_2 = -x_3,\ x_3 = x_3 \end{aligned}$$

As a result, $N(A - \lambda_2 I) = \{\alpha_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : \alpha_1 \in \mathbb{R}\}$ and $\dim(N(A - \lambda_2 I)) = 1,\ \eta_1 = 1$. The eigenvector for λ_2 is $v_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Eigenvalue decomposition of matrix A is,

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Question 7-

a) Characteristic equation $det(Q_1 - \lambda I) = 0$ yields $\lambda^3 - 8\lambda^2 + 8\lambda + 1 = 0$. Its roots are eigenvalues of Q_1 and found with the use of Newton-Raphson method. Thus, eigenvalues are $\lambda_1 = -0.112, \lambda_2 = 1.31$ and $\lambda_3 = 6.80$. From the observation of negative and positive eigenvalues, it is concluded that Q_1 is **indefinite**.

b) Characteristic equation $det(Q_2 - \lambda I) = 0$ yields $\lambda^3 - 7\lambda^2 + 13\lambda - 3 = 0$. Thus, eigenvalues are $\lambda_1 = 0.267, \lambda_2 = 3.00$ and $\lambda_3 = 3.73$. Since all eigenvalues are positive, it is concluded that Q_1 is **positive definite**.

c) Characteristic equation $det((Q_1 + Q_2) - \lambda I) = 0$ yields $\lambda^3 - 15\lambda^2 + 60\lambda - 56 = 0$. Thus, eigenvalues are $\lambda_1 = 1.34, \lambda_2 = 4.59$ and $\lambda_3 = 9.05$. Since all eigenvalues are positive, it is concluded that $Q_1 + Q_2$ is **positive definite**.

d) Characteristic equation $det((Q_1 Q_2) - \lambda I) = 0$ yields $\lambda^3 - 17\lambda^2 + 69\lambda + 3 = 0$. Thus, eigenvalues are $\lambda_1 = -0.43, \lambda_2 = 6.82$ and $\lambda_3 = 10.22$. Since there are both negative and positive eigenvalues, it is concluded that $Q_1 + Q_2$ is **indefinite**.

e) Characteristic equation $det((Q_2 Q_1) - \lambda I) = 0$ yields $\lambda^3 - 17\lambda^2 + 69\lambda + 3 = 0$. Thus, eigenvalues are $\lambda_1 = -0.43, \lambda_2 = 6.82$ and $\lambda_3 = 10.22$. Since there are both negative and positive eigenvalues, it is concluded that $Q_1 + Q_2$ is **indefinite**.

Notes:
i) Eigenvalues of $Q_1 Q_2$ and $Q_2 Q_1$ are equal.
ii) Positive, negative definiteness can also be calculated numerically. Using power method and inverse power method to find maximum and minimum eigenvalues, respectively, one can determine whether all eigenvalues have the same sign or not.

Question 8-

Picking up two elements x_1, x_2 from the set $S = \{x \in \mathbb{R}^n : Ax = b\}$, we can show $\{\alpha x_1 + (1-\alpha)x_2, \alpha \in [0,1]\} \subseteq S$ as following,

$$\begin{aligned} Ax_1 &= b \\ Ax_2 &= b \\ A(\alpha x_1 + (1-\alpha)x_2) &= \alpha A x_1 + (1-\alpha)Ax_2 = \alpha b + (1-\alpha)b = b \end{aligned}$$

Therefore, $\alpha x_1 + (1-\alpha)x_2$ is also a solution for $Ax = b$ and it is in the set S. This proves that $\{\alpha x_1 + (1-\alpha)x_2, \alpha \in [0,1]\} \subseteq S$. ■