2: Restricted 3-Body Problem *(70 points)*

From high school physics classes or the dynamics related classes at the university, we are familiar with the equations when an object (such as a satellite) moves under the gravitational field of a celestial body (such as a planet or a star) or rather both objects orbit around their common center of mass. 2-body problem has an analytic solution as we are already familiar with the Keppler's laws of planetary motion and the related equations.

In this problem, you will investigate a problem of celestial mechanics called the *restricted three-body problem.* We consider two space objects of masses $m_1 = \mu$ and $m_2 = 1 - \mu$ circulating in a plane and a third body of a much smaller (negligible) mass moving in their gravitational field.

We can think of these two space objects as the earth and the moon, where the moon moves around the earth with distance 1 (this is a normalized distance). A third object, which is relatively much smaller than the first two, such that it would not make any change in the orbits of the first two, is also moving in the space. You can think of this as a spacecraft.

When we restrict the problem to a planar motion (hence the name *restricted*), the position of the spacecraft (the lightest body) is determined by the vector $(y_1(t), y_2(t))$ which is the position vector with respect to the system of coordinates in which the center of mass of Earth-Moon system is at the origin. We normalize all physical units in such a way that the mass of Moon is $m_1 = \mu = 0.012277471$ and the mass of Earth is $m_2 = 1 - m_1 = 1 - \mu = \mu^*$. In this system of coordinates the equations of motions can be written as

$$
\begin{aligned}
y_1''(t) &= y_1(t) + 2y_2'(t) - \mu^* \frac{y_1(t) + \mu}{D_1} - \mu \frac{y_1(t) - \mu^*}{D_2} \\
y_2''(t) &= y_2(t) - 2y_1'(t) - \mu^* \frac{y_2(t)}{D_1} - \mu \frac{y_2(t)}{D_2} \\
D_1 &= ((y_1 + \mu)^2 + y_2^2)^{3/2} \\
D_2 &= ((y_1 - \mu^*)^2 + y_2^2)^{3/2} \\
\mu^* &= 1 - \mu
\end{aligned}
$$

with the initial conditions

$$
\begin{aligned}
y_1(0) &= 0.994 \\
y_1'(0) &= 0 \\
y_2(0) &= 0 \\
y_2'(0) &= -2.0015851063790825
\end{aligned}
$$

for

$$0 \le t \le T = 17.06521656015796$$

here T represents the time required for the spacecraft to complete one full round of orbit.

(a) Write the equations of motion as a system of 1st-order equations.

(b) Implement Euler method for this problem (use $h = T/24000$).

(c) Implement RK4 method for this problem (use $h = T/6000$).

(d) Explore functionality of ode45 in MATLAB and implement the problem using this solver.

(e) What can you conclude about the properties of this trajectory (you may want to run the simulation for a slightly longer time than T).

(f) Explain, which of the method would you see as the most suitable for analyzing the trajectories, both from the point of view of accuracy and efficiency.

REFERENCES

http://www.math.udel.edu/ plechac/M371/Labs/lab010.pdf

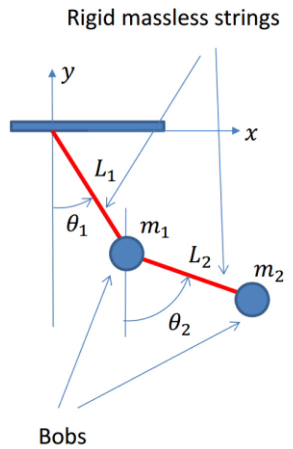
https://www.math.psu.edu/shen_w/451/NoteWeb/HW9-F12.pdf

http://www.math.utah.edu/ wright/courses/5620/homework/hw05.pdf

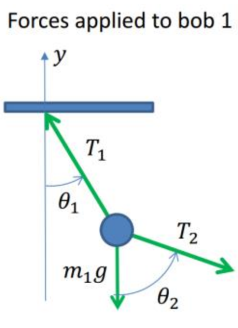
http://www4.ncsu.edu/ dpapp/427-16fall/hw6.pdf

3: Double Pendulum Motion *(80 points)*

In class, I showed you the matlab file exchange site where people share their matlab codes. In one example, we went over an animation of the double pendulum motion. In this problem, you will investigate the double pendulum motion. Consider the below system: There are two bobs with masses m_1 and m_2. The mass m_1 is connected to the wall with a rigid massless string of length L_1 and mass m_2 is connected to the mass m_1 with a rigid massless string of length L_2.



The forces acting on mass m_1 is as seen in the following figure along with the corresponding equation of motion:

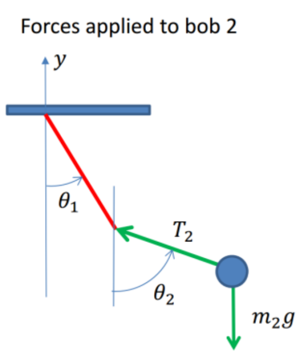


Newton's second law of motion

$$m_1 \ddot{x}_1 = -T_1 \sin\theta_1 + T_2 \sin\theta_2$$

$$m_1 \ddot{y}_1 = T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g$$

Similarly, the forces acting on mass m_2 is as seen in the following figure along with the corresponding equation of motion:



Newton's second law of motion

$$m_2 \ddot{x}_2 = -T_2 \sin\theta_2$$

$$m_2 \ddot{y}_2 = T_2 \cos\theta_2 - m_2 g$$

We have 4 equations of motion. However, we can reduce these 4 equations to 2 equations, because the lengths of the strings are constant.

(1) $m_2 \ddot{x}_2 = -T_2 \sin\theta_2$ $\times \cos\theta_2$

(2) $m_2 \ddot{y}_2 = T_2 \cos\theta_2 - m_2 g$ $+$ $\times \sin\theta_2$

$$\Rightarrow \boxed{\cos\theta_2 \; \ddot{x}_2 + \sin\theta_2 \, \ddot{y}_2 = -g \sin\theta_2}$$

(3) $m_1 \ddot{x}_1 = -T_1 \sin\theta_1 + T_2 \sin\theta_2$

(4) $m_1 \ddot{y}_1 = T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g$

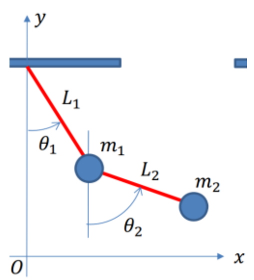
When we add Eq.(1) to Eq.(3), and Eq.(2) to Eq.(4), we obtain:

$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = -T_1 \sin\theta_1$ $\times \cos\theta_1$

$m_1 \ddot{y}_1 + m_2 \ddot{y}_2 = T_1 \cos\theta_1 - (m_1 + m_2) g$ $+$ $\times \sin\theta_1$

$$\Rightarrow \boxed{\begin{aligned} &\cos\theta_1 \, (m_1 \ddot{x}_1 + m_2 \, \ddot{x}_2) \\ &\qquad + \sin\theta_1 \, (m_1 \ddot{y}_1 + m_2 \ddot{y}_2) \\ &\qquad = -g(m_1 + m_2) \sin\theta_1 \end{aligned}}$$

When we relate the Cartesian coordinates of the masses to angles θ_1 and θ_2



$$\begin{aligned} x_1 &= L_1 \sin\theta_1 \\ y_1 &= L_2 + L_1 - L_1 \cos\theta_1 \\ x_2 &= L_1 \sin\theta_1 + L_2 \sin\theta_2 \\ y_2 &= L_2 + L_1 - L_1 \cos\theta_1 - L_2 \cos\theta_2 \end{aligned}$$

After replacing the Cartesian coordinates (x_1, y_1, x_2, y_2) with appropriate forms in terms of θ_1 and θ_2 we obtain the equations of motion in θ_1 and θ_2.

$$\boxed{\begin{aligned} L_1 \ddot{\theta}_1 + \mu_2 L_2 \cos(\theta_1 - \theta_2) \; \ddot{\theta}_2 &= -g \sin\theta_1 - \mu_2 L_2 \sin(\theta_1 - \theta_2) \, \dot{\theta}_2^2 \\ L_1 \cos(\theta_1 - \theta_2) \, \ddot{\theta}_1 + L_2 \; \ddot{\theta}_2 &= -g \sin\theta_2 + L_1 \sin(\theta_1 - \theta_2) \, \dot{\theta}_1^2 \end{aligned}}$$

Here μ2 is the reduced mass of the second bob.

$$\mu_2 = \frac{m_2}{m_2 + m_1}$$

Hence we have coupled differential equations for the angles θ_1 and θ_2

$$L_1 \ddot{\theta}_1 + \mu_2 L_2 \cos(\theta_1 - \theta_2) \; \ddot{\theta}_2 = -g \sin\theta_1 - \mu_2 L_2 \sin(\theta_1 - \theta_2) \, \dot{\theta}_2^2$$

$$L_1 \cos(\theta_1 - \theta_2) \, \ddot{\theta}_1 + L_2 \; \ddot{\theta}_2 = -g \sin\theta_2 + L_1 \sin(\theta_1 - \theta_2) \, \dot{\theta}_1^2$$

In order to solve for θ_1 and θ_2 using one of the numerical methods we learned in this class, we might want to write the differential equations for θ_1 and θ_2 explicitly. The above two equations can be written in matrix form as follows:

$$\mathbf{A}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix}=\mathbf{B}$$

$$\mathbf{A}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}=\begin{pmatrix}L_1 & \mu_2 L_2\cos(\theta_1-\theta_2)\\ L_1\cos(\theta_1-\theta_2) & L_2\end{pmatrix}$$

$$\mathbf{B}=\begin{pmatrix}B_1\\ B_2\end{pmatrix}=\begin{pmatrix}-g\sin\theta_1-\mu_2 L_2\sin(\theta_1-\theta_2)\,\dot{\theta}_2^2\\ -g\sin\theta_2+L_1\sin(\theta_1-\theta_2)\,\dot{\theta}_1^2\end{pmatrix}$$

This system of linear equation can be solved for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ by multiplying both sides with the inverse of the **A** matrix:

$$\boxed{\ddot{\theta}_1=\frac{A_{22}B_1-A_{12}B_2}{A_{11}A_{22}-A_{12}A_{21}},\qquad \ddot{\theta}_2=\frac{-A_{21}B_1+A_{11}B_2}{A_{11}A_{22}-A_{12}A_{21}}}$$

In order to solve this set of differential equations numerically using Runge-Kutta 4 method or matlab's ode45 function, we need to write it as a system of first order differential equations.

$$z_1=\theta_1,\qquad z_2=\dot{\theta}_1,\qquad z_3=\theta_2,\qquad z_4=\dot{\theta}_2$$

Thus, the coupled first order differential equations to be solved are:

$$\boxed{\begin{aligned}\dot{z}_1&=z_1\\ \dot{z}_2&=\frac{A_{22}B_1-A_{12}B_2}{A_{11}A_{22}-A_{12}A_{21}}\\ \dot{z}_3&=z_4\\ \dot{z}_4&=\frac{-A_{21}B_1+A_{11}B_2}{A_{11}A_{22}-A_{12}A_{21}}\end{aligned}}$$

$$\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}=\begin{pmatrix}L_1 & \mu_2 L_2\cos(z_1-z_3)\\ L_1\cos(z_1-z_3) & L_2\end{pmatrix}$$

$$\begin{pmatrix}B_1\\ B_2\end{pmatrix}=\begin{pmatrix}-g\sin z_1-\mu_2 L_2\sin(z_1-z_3)\,z_4^2\\ -g\sin z_3+L_1\sin(z_1-z_3)\,z_2^2\end{pmatrix}$$

Use the following mass and length values along with the given initial conditions to solve this problem

$m_1=m_2=1$ kg, $\qquad L_1=L_2=10$ cm

Initial conditions:

$$\theta_1(0)=\theta_2(0)=90^0$$
$$\dot{\theta}_1(0)=\dot{\theta}_2(0)=0$$

(a) Implement RK4 method to solve this problem
(b) Use ode45 function of MATLAB to solve this problem
(c) Present your results. Be creative.
(BONUS) Create an animation of the double pendulum system. For this part you can use and modify to your needs any available matlab code online, such as ones on matlab file-exchange (https://www.mathworks.com/matlabcentral/fileexchange/) or similar sites.

Reference: http://volkov.eng.ua.edu/ME349/2017-Fall-ME349-04-NumAnalysis2.pdf