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Unit – 1 ⇝ Sets and Functions

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Introduction

→ Set theory was developed by German mathematician Georg Cantor (1845 - 1918).

→ He first encountered sets while working on problems on trigonometric series.

→ The concept of set serves as a fundamental part of the present-day mathematics. Now a days, this concept is being used in almost every branch of mathematics.

→ Set is used to define the concepts of relation, function, geometry, sequences, probability, etc.

1.1 – Set and Its Representation

Set

→ Collection of well-defined objects is known as a Set.

Here, well-defined means there is no confusion regarding inclusion and exclusion of an object.

→ Set is denoted by upper case letters $A, B, C,$ etc.

→ For example:

(1) The collection of all the uppercase alphabets up to G is a Set.

(2) The collection of rivers of India is a Set.

(3) The collection of five most renowned mathematician is **not** a Set.

(4) The collection of beautiful songs is **not** a Set.

Element of Set

→ Each object in the set is known as an element or member of the set.

→ Elements of set is denoted by lower case letter a, b, c, etc.

→ If any object is present in a set, we use symbol $\in$ (belongs to).

→ If any object is **not** present in a set, we use symbol $\notin$ (does not belong to).

→ For Example:

(1) Apple $\in$ The collection of fruits

(2) Apple $\notin$ The collection of vegetables

Methods to Represent a Set

→ There are two methods to represent any set.

(1) Listing Method

(2) Property Method

→ **Listing Method**

- In Listing Method, elements of the set are

    (1) Written as a list,

    (2) Separated by the comma,

    (3) Enclosed within the curly braces { }.

- This method is also known as Tabular Form or Roster From.

- It may be noted that while writing the set in roster form an element is not generally repeated.

- Sequence of elements is not important while arranging elements of set.

- For example:

    (1) Statement: The set of prime numbers less than 10.

    $$A = \{2, 3, 5, 7\}$$

    (2) Statement: The set of letters forming the word "mathematics".

    $$B = \{m, a, t, h, e, i, c, s\}$$
    $$= \{a, c, e, h, i, m, s, t\}$$

→ **Property Method**

- In Property Method, elements of the set are

    (1) Follow a single common property,

    (2) Enclosed within the curly braces { }.

- This method is also known as Set-Builder from.

- For example:

  (1) Statement: The set of all vowels in English alphabets.

  Using Listing Method : $B = \{ a,\ e,\ i,\ o,\ u \}$

  Using Property Method : $B = \{ x : x \text{ is a vowel in English alphabets} \}$

  (2) Using Property Method : $A = \{x \mid x < 6\}$

$$\{\ x\ \mid\ x < 6\ \}$$

the set of　all x　such that　number less than 6

Standard Sets

→ $\mathbb{N}$ = The set of natural numbers

$= \{ 1,\ 2,\ 3,\ \ldots \}$

→ $\mathbb{Z}$ = The set of integers

$= \{ \ldots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots \}$

→ $\mathbb{Q}$ = The set of rational numbers

$$= \left\{ \frac{p}{q} : p,\ q \in \mathbb{Z},\ q \neq 0 \right\}$$

→ $\mathbb{T}$ = The set of irrational numbers

$= \{ \ldots,\ \sqrt{2},\ \ldots,\ \sqrt[3]{2},\ \ldots,\ \sqrt[4]{3},\ \ldots,\ e,\ \pi,\ \ldots \}$

→ $\mathbb{R}$ = The set of real numbers

→ $\mathbb{Z}^+$ = The set of positive integers

$= \{ 1,\ 2,\ 3,\ \ldots \} = \mathbb{N}$

→ $\mathbb{Q}^+$ = The set of positive rational numbers

→ $\mathbb{R}^+$ = The set of positive real numbers

Examples of Method-1.1: Set and Its Representation

H	1	Let $A = \{1, 2, 3, 4, a, x, 6\}$. Fill the appropriate symbol $\in$ or $\notin$ in the blank spaces: **(1)** 3 ___ A **(2)** 5 ___ A **(3)** 1 ___ A **(4)** a ___ A **(5)** c ___ A **(6)** y ___ A **(7)** 2 ___ A **(8)** 4 ___ A **Answer**: **(1)** $\in$, **(2)** $\notin$, **(3)** $\in$, **(4)** $\in$, **(5)** $\notin$, **(6)** $\notin$, **(7)** $\in$, **(8)** $\in$.
C	2	Write the following sets in set-builder form: $A = \{2, 4, 6, 8, \ldots\}$ **Answer**: $\mathbf{A = \{x : x \text{ is an even number} \;\&\; x \in \mathbb{N}\} = \{2x : x \in \mathbb{N}\}}$
H	3	Give another description of the following sets: $A = \{6, 12, 18, 24, 30, 36, 42, 48\}$ $B = \{1, 3, 5, 7, 9\}$ $C = \{1, 8, 27, 64, 125, 216\}$ **Answer**: $\mathbf{A = \{x : x \text{ is multiple of six less than } 50\}}$ $\mathbf{B = \{x : x \text{ is an odd natural number less than } 10\}}$ $\mathbf{C = \{x^3 : x \in \mathbb{N} \;\&\; x \le 6\}}$
T	4	Give another description of the following sets: $A = \{\text{red, orange, yellow, green, blue, indigo, violet}\}$ $B = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$ **Answer**: $\mathbf{A = \{x : x \text{ is color in a rainbow}\}}$ $\mathbf{B = \{(x, x^2) : x = 1, 2, 3, 4, 5\}}$

C	5	Write the following sets in roster form: $C = \{ (x, y) : x, y \in \mathbb{N}, x \text{ divides } y \ \& \ y \leq 6 \}$ **Answer:** $\mathbf{C} = \left\{ \begin{array}{c} (1,1), (1,2), (1,3), (1,4), (1,5), \\ (1,6), (2,2), (2,4), (2,6), (3,3), \\ (3,6), (4,4), (5,5), (6,6) \end{array} \right\}$
H	6	Give another description of the following sets: $A = \{ x : x \text{ is a letter in the word “TRIGONOMETRY”} \}$ $B = \{ x : x \text{ is a prime number between 50 and 70} \}$ $C = \{ a : a \in \mathbb{N}, a \text{ is perfect square number}, a \leq 100 \}$ **Answer:** $\mathbf{A} = \{ \mathbf{T, R, I, G, O, N, M, E, Y} \}$ $\mathbf{B} = \{ \mathbf{53, 59, 61, 67} \}$ $\mathbf{C} = \{ \mathbf{1, 4, 9, 16, 25, 36, 49, 64, 81, 100} \}$
T	7	Give another description of the following sets: $A = \{ x : x \text{ is an integer } \& \ 5 \leq x \leq 12 \}$ $B = \{ x : x \text{ is a divisor of 36 } \& \ x \in \mathbb{N} \}$ $C = \{ (x, y) : x, y \in \mathbb{N}, \lvert x - y \rvert \text{ is even number } \& \ x, y \leq 6 \}$ **Answer:** $\mathbf{A} = \{ \mathbf{5, 6, 7, 8, 9, 10, 11, 12} \}$ $\mathbf{B} = \{ \mathbf{1, 2, 3, 4, 6, 9, 12, 18, 36} \}$ $\mathbf{C} = \left\{ \begin{array}{c} (1,1), (1,3), (1,5), (2,2), (2,4), (2,6), \\ (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), \\ (5,1), (5,3), (5,5), (6,2), (6,4), (6,6) \end{array} \right\}$

T	8	Give another description of the following sets: $A = \{ x : x \in \mathbb{N},\ 4 < x^2 < 40 \}$ $B = \{ a : a \in \mathbb{N},\ a \text{ is a multiple of } 7\ \&\ a \le 49 \}$ $C = \{ (x,\ y) : x,\ y \in \mathbb{N}, \text{both are even numbers and } x + y = 18 \}$ **Answer**: $\mathbf{A = \{ 3,\ 4,\ 5,\ 6 \}}$ $\mathbf{B = \{ 7,\ 14,\ 21,\ 28,\ 35,\ 42,\ 49 \}}$ $\mathbf{C = \left\{ \begin{matrix} (2,\ 16),\ (4,\ 14),\ (6,\ 12),\ (8,\ 10), \\ (10,\ 8),\ (12,\ 6),\ (14,\ 4),\ (16,\ 2) \end{matrix} \right\}}$
C	9	Write the following sets in roster form: $A = \{ x : x \in \mathbb{N},\ x \text{ is divisor of } 70 \}$ **Answer**: $\mathbf{A = \{ 1,\ 2,\ 5,\ 7,\ 10,\ 14,\ 35,\ 70 \}}$
H	10	Write the following sets in roster form: $A = \{ x : x \in \mathbb{N},\ x \text{ is divisor of } 50 \}$ **Answer**: $\mathbf{A = \{ 1,\ 2,\ 5,\ 10,\ 25,\ 50 \}}$
T	11	Write the following sets in roster form: $A = \{x : x \in \mathbb{Z}, x \text{ is divisor of } 45\}$ **Answer**: $\mathbf{A = \{ -45,\ -15,\ -9,\ -5,\ -3\ , -1,\ 1,\ 3,\ 5\ , 9,\ 15\ , 45 \}}$
C	12	Write the following sets in roster form: $B = \{ x : x^2 - 60x - 256 = 0,\ x \in \mathbb{Z} \}$ **Answer**: $\mathbf{B = \{ -4,\ 64 \}}$
H	13	Write the following sets in roster form: $B = \{ x : x^3 - x = 0,\ x \in \mathbb{Z} \}$ **Answer**: $\mathbf{B = \{ -1,\ 0,\ 1 \}}$
T	14	Write the following sets in roster form: $B = \{ x : x^3 + x - 2 = 0,\ x \in \mathbb{Z} \}$ **Answer**: $\mathbf{B = \{ -2, 1 \}}$

T	15	Give another description of the following sets: $A = \{ x : x \in \mathbb{R},\ x^2 - 1 = 0 \}$ $B = \{ x : x \in \mathbb{N},\ (x-1)(x+2) = 0 \}$ $C = \{ (x, y) : x, y \in \mathbb{N},\ x + y = 4 \}$ **Answer**: $\mathbf{A = \{-1,\ 1\}}$ $\mathbf{B = \{1\}}$ $\mathbf{C = \{(1, 3),\ (2, 2),\ (3, 1)\}}$

1.2 – Types of Sets

→ **Finite Set**

- A set which contains finite number of elements is known as a finite set.
- For example:

  $A = \{ -2,\ -1,\ 0 \} = \{ x : -3 < x < 1,\ x \in \mathbb{Z} \}$

→ **Infinite Set**

- A set which contains infinite number of elements is known as an infinite set.
- For example:

  $A = \{ \ldots,\ -2,\ -1,\ 0 \} = \{ x : x < 1,\ x \in \mathbb{Z} \}$,

  $\mathbb{N},\ \mathbb{Z},\ \mathbb{Q},\ \mathbb{R}\ \&\ \mathbb{C}$.

→ **Empty Set or Null Set**

- A set which contains no element is known as an empty set.
- It is denoted as $\{\ \}$ or $\boldsymbol{\phi}$.
- For example:

  $A = \{ x : x < 1,\ x \in \mathbb{N} \} = \phi$.

→ **Non-empty Set**

- A set which contains at least one element is known as a non-empty set.
- For example:

  $A = \{ \ldots,\ -2,\ -1,\ 0 \} = \{ x : x < 1,\ x \in \mathbb{Z} \}$.

→ **Singleton Set**

- A set which contains exactly one element is known as a singleton set.
- For example:

  $A = \{ 1 \} = \{ x : x \leq 1, x \in \mathbb{N} \}$.

→ **Equal Sets**

- If elements of set A and set B are exactly same then set A and set B are known as equal sets.
- It is denoted by $\mathbf{A = B}$ and read as "A equal B".
- If elements of set A and B are not exactly same then set A and set B are known as unequal sets.
- It is denoted by $\mathbf{A \neq B}$ and read as "A is not equal B".
- For example:

  For, $A = \{ x : x \leq 1, x \in \mathbb{N} \}$, $B = \{ x : x \leq 1, x \in \mathbb{Z} \}$, $C = \{ 1 \}$

  $A = C$ and $A \neq B$.

→ **Universal Set**

- A set which has elements of all the related sets, without any repetition of elements is known as a universal set.
- It is denoted by **U**.
- For example:

  (1) For the set of all integers $\mathbb{Z}$, universal set can be $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$.

  (2) In human population studies, the universal set consists of all the people in the world.

1.3 – Subset

→ If all elements of set A are present in set B, then set A is known as subset of B.

→ It is denoted by $\mathbf{A \subseteq B}$ and read as "A is subset of B".

→ If set A is **not** subset of set B, then it is denoted by $\mathbf{A \nsubseteq B}$ and read as "A is not subset of B".

→ For example:

Let $X = \{ 2, 1, 0 \}$, $Y = \{ 0, 1, 2, 3, 4, 5 \}$ & $Z = \{ 0, 3, 5 \}$

For above sets, $X \subseteq Y$, $Z \subseteq Y$ & $X \nsubseteq Z$.

→ For any set A,

$\phi \subseteq A$,

$A \subseteq A$.

→ If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

→ If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

→ Number of subsets of the **empty set** = **1** (itself).

→ Number of subsets of a **non-empty set** with n elements = $\mathbf{2^n}$; $n \in \mathbb{N}$.

→ For example:

Let $X = \{\,2,\ 1,\ 0\,\}$

Number of subsets of $X = 2^n = 2^3 = 8$.

Subsets of X are: ϕ, $\{\,2\,\}$, $\{\,1\,\}$, $\{\,0\,\}$, $\{\,2,\ 1\,\}$, $\{\,2,\ 0\,\}$, $\{\,1,\ 0\,\}$, X.

Types of Subsets

→ Let a non-empty set A with n elements.

- Improper Subset
  - Subset **A** (set itself) is known as an improper subset of set A.
  - Number of improper subsets of the **empty set** = 1 (itself).
  - Number of improper subsets of a **non-empty set** with n elements = 1; $n \in \mathbb{N}$.

- Proper Subset
  - Subsets **other** than set **A** (set itself) are known as proper subsets of set A.
  - Let X be a proper subset of A, then

    It is denoted by $\mathbf{X \subset A}$ and read as "X is proper subset of A",
  - Number of proper subsets of the **empty set** $= 0$.
  - Number of proper subsets of a **non-empty set** with n elements $= 2^n - 1$; $n \in \mathbb{N}$.
  - For example:

    Let $X = \{2, \ 1, \ 0\}$

    Improper subsets of X are : X

    Number of proper subsets of X $= 2^n - 1$

    $= 2^3 - 1 = 7$

    Proper subsets of X are as follow:

    $\phi, \ \{2\}, \ \{1\}, \ \{0\}, \ \{2, 1\}, \ \{2, 0\}, \ \{1, 0\}$

Super Set

→ If A is a subset of B, then B is known as super set of A.

→ It is denoted by $\mathbf{B \supset A}$ and read as "B is super set of A".

→ For example:

Let $X = \{2, \ 1, \ 0\}$, $\quad Y = \{0, \ 1, \ 2, \ 3, \ 4, \ 5\}$ $\quad$ & $\quad Z = \{0, \ 3, \ 5\}$

For above sets, $Y \supset X$, $\ Y \supset Z$, $\ Y \supseteq Y$ $\ \&\ $ $Z \not\supset X$.

→ For example:

Let $W = \{0, \ 3, \ 5\}$, $\quad X = \{2, \ 1, \ 0\}$, $\quad Y = \{0, \ 1, \ 2, \ 3, \ 4, \ 5\}$

& $\ Z = \{3, \ 5, \ 0\}$

For above sets, $X \subset Y$, $\ Z \subset Y$, $\ X \not\subset Z$ & $Y \supset W$.

$Z \subseteq W$ and $Z = W$ are true, but $Z \subset W$ is not true.

Examples of Method-1.2: Subset

C	1	Write down all the subsets of the set $B = \{ a,\ \{ b,\ c \},\ d \}$. Also find number of proper subsets. **Answer:** $\boldsymbol{\phi,\ \{ a \},\ \{ \{ b,\ c \} \},\ \{ d \},\ \{ a, \{ b,\ c \} \},\ \{ a,\ d \},}$ $\boldsymbol{\{ \{ b,\ c \},\ d \},\ B}$ **Number of proper subsets = 7**
C	2	Find number of elements of set which has 1023 proper subsets, if possible. **Answer: 10**
H	3	Find number of elements of set which has 514 proper subsets, if possible. **Answer: Such set is not possible.**
T	4	Find number of elements of set which has 2047 proper subsets, if possible. **Answer: 11**
C	5	Let $A = \{ 1,\ 2,\ \{ 3,\ 4 \},\ 5 \}$. Which of the following are incorrect? **(1)** $\{ 3,\ 4 \} \subset A$ **(2)** $\{ 3,\ 4 \} \in A$ **(3)** $\{ \{3,\ 4\} \} \subset A$ **(4)** $1 \in A$ **(5)** $1 \subset A$ **(6)** $\{ 1,\ 2,\ 5 \} \in A$ **(7)** $\{ 1,\ 2,\ 5 \} \subset A$ **(8)** $\{ 1,\ 2,\ 3 \} \subset A$ **(9)** $\phi \in A$ **(10)** $\phi \subset A$ **(11)** $\{ \phi \} \subset A$ **(12)** $A \supset \{ 1,\ 5 \}$ **Answer: (1), (5), (6), (8), (9), (11).**
H	6	Consider the sets ϕ, $A = \{1, 3\}$, $B = \{1, 5, 9\}$, $C = \{1, 3, 5, 7, 9\}$. Insert the symbol $\subset$ or $\not\subset$ between each of the following pair of set. (i) ϕ ____ B (ii) A ____ B (iii) A ____ C (iv) B ____ C **Answer: (i) $\subset$ (ii) $\not\subset$ (iii) $\subset$ (iv) $\subset$**
T	7	Let $A = \{ a,\ e,\ i,\ o,\ u \}$ and $B = \{ a,\ b,\ c,\ d \}$. Is A a subset of B? If not why? **Answer: No; Reason: $\boldsymbol{i \notin B}$**

Method – 2 ⇝ Set Operations and its Properties

2.1 – Union of Sets

→ A set that contains all elements of set A and set B, in which common elements are written once only, is known as union of sets A and B.

→ It is denoted by $\mathbf{A \cup B}$ and read as "A union B".

→ For example:

For, $A = \{1, 2, 3, 4, 5\}$ & $B = \{3, 4, 5, 6, 7\}$

$\mathbf{A \cup B} = \{1, 2, 3, 4, 5, 6, 7\}$

→ Using Property Method:

$A \cup B = \{x : x \in A$ **or** $x \in B\}$.

→ Properties of union of sets:

(1) Commutative Law : $A \cup B = B \cup A$

(2) Associative Law : $(A \cup B) \cup C = A \cup (B \cup C)$

(3) Identity Law : $A \cup \phi = A$

(4) Idempotent Law : $A \cup A = A$

(5) Domination Law : $A \cup U = U$

Examples of Method-2.1: Union of Sets

C	1	Let $A = \{x : x \in \mathbb{R}\,;\, x^2 - 3x - 4 = 0\}$, $B = \{x : x \in \mathbb{Z}\,;\, x^2 = x\}$. Find $A \cup B$. **Answer**: $\mathbf{A \cup B = \{-1, 0, 1, 4\}}$
H	2	Let $A = \{2, 3, 4, 5, 6, 9\}$, $B = \{1, 4, 5, 6, 7, 8\}$. Find $A \cup B$. **Answer**: $\mathbf{A \cup B = \{1, 2, 3, \ldots, 9\}}$

H	3	Find the union of the following sets: $X = \{1, 3, 5\}$, $Y = \{1, 2, 3\}$ **Answer:** $\mathbf{X \cup Y = \{1, 2, 3, 5\}}$
H	4	Find the union of the following sets: $A = \{a, e, i, o, u\}$, $B = \{a, b, c\}$ **Answer:** $\mathbf{A \cup B = \{a, b, c, e, i, o, u\}}$
T	5	Find the union of the following sets: $A = \{1, 2, 3\}$, $B = \phi$ **Answer:** $\mathbf{A \cup B = \{1, 2, 3\}}$
T	6	Find the union of the following sets: $A = \{x : x \text{ is a natural number and multiple of } 3\}$ $B = \{x : x \text{ is a natural number less than } 6\}$ **Answer:** $\mathbf{A \cup B = \{x : x = 1, 2, 4, 5 \text{ or a multiple of } 3\}}$
T	7	Find the union of the following sets: $A = \{x : x \text{ is a natural number and } 1 < x \leq 6\}$ $B = \{x : x \text{ is a natural number and } 6 < x < 10\}$ **Answer:** $\mathbf{A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9\}}$
T	8	Let $A = \{x : x \text{ is a divisor of } 24\}$, $B = \{x : x \text{ is a divisor of } 18\}$, $C = \{x : x \text{ is a divisor of } 6\}$. Verify the identity: $(A \cup B) \cup C = A \cup (B \cup C)$ **Hint:** $\mathbf{A \cup (B \cup C) = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}}$
T	9	Let $A = \{1, 3, 5, 7, 9\}$ $B = \{1, 5, 6, 8\}$, $C = \{1, 4, 6, 7\}$. Verify the identity: $(A \cup B) \cup C = A \cup (B \cup C)$ **Hint:** $\mathbf{(A \cup B) \cup C = \{1, 3, 4, 5, 6, 7, 8, 9\}}$

2.2 – Intersection of Sets

→ A set that contains all common elements of set A and set B, is known as intersection of sets A and B.

→ It is denoted by **$A \cap B$** and read as "A intersection B".

→ For example:

For, $A = \{1, 2, 3, 4, 5\}$ & $B = \{3, 4, 5, 6, 7\}$

$\mathbf{A \cap B} = \{3, 4, 5\}$

→ Using Property Method:

$A \cap B = \{x : x \in A$ **and** $x \in B\}$.

Disjoint Sets

→ Let A and B be two distinct sets.

If **$A \cap B = \phi$**, then set A and B are known as disjoint sets.

→ For example:

For, $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, f\}$

$A \cap B = \phi$

Here, A & B are disjoint sets.

→ Properties of intersection of sets:

(1)	Commutative Law	:	$A \cap B = B \cap A$
(2)	Associative Law	:	$(A \cap B) \cap C = A \cap (B \cap C)$
(3)	Identity Law	:	$A \cap U = A$
(4)	Idempotent Law	:	$A \cap A = A$
(5)	Domination Law	:	$A \cap \phi = \phi$
(6)	Distributive Laws	:	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
		:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Examples of Method-2.2: Intersection of Sets

C	1	Let $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 5, 6, 8\}$, $C = \{1, 4, 6, 7\}$. Which of the following pairs of sets are disjoint? **(1)** A and B **(2)** B and C **(3)** A and C **Answer: None of the pairs is disjoint.**
H	2	Let $A = \{x : x \in \mathbb{R};\ x^2 - 4 = 0\}$, $B = \{x : x \in \mathbb{Z};\ x^3 = -x\}$. **Show that A and B are disjoint sets.**
T	3	Which of the following pairs of sets are disjoint? **(1)** $X = \{1, 2, 3, 4\}$, $Y = \{x : x \text{ is a natural number and } 4 < x \leq 6\}$ **(2)** $A = \{a, e, i, o, u\}$, $B = \{c, d, e, f\}$ **(3)** $P = \{x : x \text{ is an even integer}\}$, $Q = \{x : x \text{ is an odd integer}\}$ **Answer: (1), (3)**
H	4	Let $A = \{2, 3, 4, 5, 6, 9\}$, $B = \{1, 4, 5, 6, 7, 8\}$. Find $A \cap B$. **Answer:** $A \cap B = \{4, 5, 6\}$
C	5	Let $A = \{x : x \text{ is a divisor of } 24\}$ $B = \{x : x \text{ is a divisor of } 18\}$ $C = \{x : x \text{ is a divisor of } 6\}$ Verify the following identities: **(1)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **(2)** $(A \cap B) \cap C = A \cap (B \cap C)$ **Hint:** (1) $A \cap (B \cup C) = \{1, 2, 3, 6\}$ (2) $(A \cap B) \cap C = \{1, 2, 3, 6\}$

H	6	Let $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 5, 6, 8\}$, $C = \{1, 4, 6, 7\}$. Verify the following identities: **(1)** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **(2)** $(A \cap B) \cap C = A \cap (B \cap C)$ **Hint:** (1) $\mathbf{A \cup (B \cap C) = \{1, 3, 5, 6, 7, 9\}}$ (2) $\mathbf{A \cap (B \cap C) = \{1\}}$
T	7	If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, then find the following: **(1)** $A \cap B$ **(2)** $B \cap C$ **(3)** $A \cap C \cap D$ **Answer:** (1) $\mathbf{A \cap B = \{7, 9, 11\}}$ (2) $\mathbf{B \cap C = \{11, 13\}}$ (3) $\mathbf{A \cap C \cap D = \phi}$
T	8	If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, then find the following: **(1)** $A \cap C$ **(2)** $B \cap D$ **(3)** $A \cap (B \cup C)$ **Answer:** (1) $\mathbf{A \cap C = \{11\}}$ (2) $\mathbf{B \cap D = \phi}$ (3) $\mathbf{A \cap (B \cup C) = \{7, 9, 11\}}$

T	9	If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, then find the following: **(1)** $A \cap (B \cup D)$ **(2)** $C \cap D$ **(3)** $A \cap B \cap C$ **(4)** $(A \cup D) \cap (B \cup C)$ **Answer**: (1) $\mathbf{A \cap (B \cup D) = \{7, 9, 11\}}$ (2) $\mathbf{C \cap D = \{15\}}$ (3) $\mathbf{A \cap B \cap C = \{11\}}$ (4) $\mathbf{(A \cup D) \cap (B \cup C) = \{7, 9, 11, 15\}}$
T	10	Let $A = \{x : x \text{ is a divisor of } 24\}$ $B = \{x : x \text{ is a divisor of } 18\}$ $C = \{x : x \text{ is a divisor of } 6\}$ Verify the following identities: **(3)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ **(4)** $(A \cap B) \cap C = A \cap (B \cap C)$ **Hint**: (1) $\mathbf{A \cap (B \cup C) = \{1, 2, 3, 6\}}$ (2) $\mathbf{(A \cap B) \cap C = \{1, 2, 3, 6\}}$

2.3 – Difference of Sets

→ A set that contains elements of set A which are not present in B, is known as difference of sets A and B.

→ It is denoted by **A − B** and read as "A minus B".

→ For example:

For, $A = \{1, \ 2, \ 3, \ 4, \ 5\}$ & $B = \{3, \ 4, \ 5, \ 6, \ 7\}$

$\mathbf{A - B} = \{1, \ 2\}$

$\mathbf{B - A} = \{6, \ 7\}$

→ Using Property Method:

$A - B = \{x : x \in A \textbf{ and } x \notin B\}$

$B - A = \{x : x \in B \textbf{ and } x \notin A\}$

→ $A - B \neq B - A$.

→ Properties of difference of sets:

(1) $A - B \neq B - A$

(2) $A - B = A - (A \cap B)$

(3) $B - A = B - (A \cap B)$

(4) $A - (B \cup C) = (A - B) \cap (A - C)$

(5) $A - (B \cap C) = (A - B) \cup (A - C)$

Examples of Method-2.3: Difference of Sets

C	1	Let $A = \{2, \ 4, \ 6, \ 7, \ 8, \ 9\}$, $B = \{1, \ 3, \ 4, \ 5, \ 7\}$, $C = \{1, \ 4, \ 7, \ 9, \ 10\}$. Find $B - C$ & $A - (B \cup C)$ **Answer**: $\mathbf{A - B = \{2, \ 6, \ 8, \ 9\}}$, $\mathbf{B - C = \{3, \ 5\}}$, $\mathbf{A - (B \cup C) = \{2, \ 6, \ 8\}}$.

H	2	Let $A = \{2, 3, 4, 5, 6, 9\}$, $B = \{1, 4, 5, 6, 7, 8\}$. Find $A - B$, $B - A$. **Answer**: $\mathbf{A - B = \{2, 3, 9\}}$, $\mathbf{B - A = \{1, 7, 8\}}$
H	3	Let $A = \{x : x \in \mathbb{R};\ x^2 - 7x + 10 = 0\}$, $B = \{x : x \in \mathbb{Z};\ x^2 - 3x - 10 = 0\}$. Find $A - B$, $B - A$. **Answer**: $\mathbf{A - B = \{2\}}$, $\mathbf{B - A = \{-2\}}$
H	4	Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 6\}$, $C = \{1, 2, 3\}$. Verify the following identities: **(1)** $A - (B \cup C) = (A - B) \cap (A - C)$ **(2)** $A - (B \cap C) = (A - B) \cup (A - C)$ **(3)** $A - (B - C) = (A - B) \cup (A \cap C)$ **Hint**: (1) $\mathbf{A - (B \cup C) = \{4\}}$ (2) $\mathbf{A - (B \cap C) = \{2, 4, 5\}}$ (3) $\mathbf{A - (B - C) = \{1, 2, 3, 4\}}$
H	5	If $A = \{3, 6, 9, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$, $D = \{5, 10, 15, 20\}$, then find **(1)** $A - B$ **(2)** $A - C$ **(3)** $A - D$ **Answer**: (1) $\mathbf{A - B = \{3, 6, 9, 15, 18, 21\}}$ (2) $\mathbf{A - C = \{3, 9, 15, 18, 21\}}$ (3) $\mathbf{A - D = \{3, 6, 9, 12, 18, 21\}}$

T	6	If $A = \{3, 6, 9, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$, $D = \{5, 10, 15, 20\}$, then find **(1)** $B - A$ **(2)** $C - A$ **(3)** $D - A$ **Answer:** (1) $\mathbf{B - A = \{4, 8, 16, 20\}}$ (2) $\mathbf{C - A = \{2, 4, 8, 10, 14, 16\}}$ (3) $\mathbf{D - A = \{5, 10, 20\}}$
T	7	If $A = \{3, 6, 9, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$, $D = \{5, 10, 15, 20\}$, then find **(1)** $B - C$ **(2)** $B - D$ **(3)** $C - B$ **Answer:** (1) $\mathbf{B - C = \{20\}}$ (2) $\mathbf{B - D = \{4, 8, 12, 16\}}$ (3) $\mathbf{C - B = \{2, 6, 10, 14\}}$
T	8	If $A = \{3, 6, 9, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$, $D = \{5, 10, 15, 20\}$, then find **(1)** $D - B$ **(2)** $C - D$ **(3)** $D - C$ **Answer:** (1) $\mathbf{D - B = \{5, 10, 15\}}$ (2) $\mathbf{C - D = \{2, 4, 6, 8, 12, 14, 16\}}$ (3) $\mathbf{D - C = \{5, 15, 20\}}$

2.4 – Symmetric Difference of Sets

→ A set that contains elements of set $A \cup B$ which are not present in $A \cap B$, is known as symmetric difference of sets A and B.

→ It is denoted by **$A \Delta B$** and read as "A delta B".

→ $A \Delta B = (\mathbf{A} \cup \mathbf{B}) - (\mathbf{A} \cap \mathbf{B})$

or

$A \Delta B = (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A})$

→ For example:

For, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$

$$\begin{aligned} A \Delta B &= (A \cup B) - (A \cap B) \\ &= \{1, 2, 3, 4, 5, 6, 7\} - \{3, 4, 5\} \\ &= \{\mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{7}\} \end{aligned}$$

$$\begin{aligned} A \Delta B &= (A - B) \cup (B - A) \\ &= \{1, 2\} \cup \{6, 7\} \\ &= \{\mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{7}\} \end{aligned}$$

→ Using Property Method:

$A \Delta B = \{x : x \in A \cup B$ **and** $x \notin A \cap B\}$

$A \Delta B = \{x : x \in A - B$ **or** $x \in B - A\}$

→ Properties of symmetric difference of sets:

(1) Commutative Law : $A \Delta B = B \Delta A$

(2) Associative Law : $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

(3) $A \Delta A = \phi$

(4) $A \Delta \phi = A$

Examples of Method-2.4: Symmetric Difference of Sets

C	1	Let $A = \{1, 2, 3, 4, 5\}$. Verify the identity: $A \Delta \phi = A$.
H	2	Let $A = \{2, 3, 4, 5, 6, 9\}$, $B = \{1, 4, 5, 6, 7, 8\}$. Find $A \Delta B$. **Answer**: $\mathbf{A \Delta B = \{1, 2, 3, 7, 8, 9\}}$
H	3	Let $A = \{x : x \in \mathbb{R}; x^2 - 3x - 4 = 0\}$, $B = \{x : x \in \mathbb{Z}; x^2 = x\}$. Find $A \Delta B$. **Answer**: $\mathbf{A \Delta B = \{-1, 0, 1, 4\}}$
T	4	Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 6\}$, $C = \{1, 2, 3\}$. Verify the following identities: **(1)** $A \Delta B = B \Delta A$ **(2)** $(A \Delta B) \Delta C = A \Delta (B \Delta C)$ **Hint**: (1) $\mathbf{A \Delta B = \{2, 4, 6\}}$ (2) $\mathbf{(A \Delta B) \Delta C = \{1, 3, 4, 6\}}$

2.5 – Complement of Set

→ A set that contains elements of universal set U which are not present in A, is known as complement of set A.

→ It is denoted by **A'** or **$\overline{A}$** and read as "A complement" or "A bar" respectively.

→ For example:

For, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{3, 4, 5, 6, 7\}$.

$A' = \{1, 2, 8, 9, 10\}$

→ Using Property Method:

$A' = \{x : x \in U$ **and** $x \notin A\} = U - A$

→ Properties of complement of set:

(1) $(A')' = A$

(2) $(U)' = \phi$

(3) $(\phi)' = U$

(4) $A \cap A' = \phi$

(5) $A \cup A' = U$

(6) $A \, \Delta \, U = A'$

(7) $A - B = A - (A \cap B) = A \cap B'$

(8) $B - A = B - (A \cap B) = B \cap A'$

(9) De Morgen's Law: $(A \cup B)' = A' \cap B'$

(10) De Morgen's Law: $(A \cap B)' = A' \cup B'$

Examples of Method-2.5: Complement of Sets

C	1	Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $A = \{1, 3, 5, 7, 9\}$ $B = \{1, 5, 6, 8\}$ Verify the following identities: **(1)** $(A \cup B)' = A' \cap B'$ **(2)** $(A \cap B)' = A' \cup B'$ **Hint:** (1) $(\mathbf{A} \cup \mathbf{B})' = \{\mathbf{2}, \mathbf{4}, \mathbf{10}\}$ (2) $(\mathbf{A} \cap \mathbf{B})' = \{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \mathbf{10}\}$
H	2	Let $U = \{2, 4, 6, 8, 10\}$, $A = \{2, 4, 6\}$, $B = \{8, 10\}$. Verify De Morgan's laws for given sets.
H	3	Let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3\}$, $B = \{3, 4, 5\}$. Find A', B', $A' \cap B'$. **Answer:** $\mathbf{A}' = \{\mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$, $\mathbf{B}' = \{\mathbf{1}, \mathbf{2}, \mathbf{6}\}$, $\mathbf{A}' \cap \mathbf{B}' = \{\mathbf{1}, \mathbf{6}\}$
T	4	Let $U = \{1, 2, 3, \ldots, 20\}$ $A = \{x : x \text{ is a multiple of } 3, x \leq 20\}$ $B = \{x : x \text{ is a multiple of } 2, x \leq 20\}$ Verify the following identities: **(1)** $(A \cup B)' = A' \cap B'$ **(2)** $(A \cap B)' = A' \cup B'$ **(3)** $A - B = A \cap B'$ **(4)** $B - A = B \cap A'$ **Hint:** (1) $(\mathbf{A} \cup \mathbf{B})' = \{\mathbf{1}, \mathbf{5}, \mathbf{7}, \mathbf{11}, \mathbf{13}, \mathbf{17}, \mathbf{19}\}$ (2) $(\mathbf{A} \cap \mathbf{B})' = \{\mathbf{x} : \mathbf{x} \textbf{ is not a multiple of } \mathbf{6}, \mathbf{x} \leq \mathbf{20}\}$ (3) $\mathbf{A} - \mathbf{B} = \{\mathbf{3}, \mathbf{9}, \mathbf{15}\}$ (4) $\mathbf{B} - \mathbf{A} = \{\mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{10}, \mathbf{14}, \mathbf{16}, \mathbf{20}\}$

T	5	Let $U = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}$, $A = \{1,\ 2,\ 3\}$, $B = \{3,\ 4,\ 5\}$ $C = \{1,\ 3,\ 5,\ 6,\ 7\}$. Find $A',\ B',\ C',\ A \cap B'$ **Answer**: $\mathbf{A' = \{4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}}$, $\mathbf{B' = \{1,\ 2,\ 6,\ 7,\ 8,\ 9,\ 10\}}$, $\mathbf{C' = \{2,\ 4,\ 8,\ 9,\ 10\}}$, $\mathbf{A \cap B' = \{1,\ 2\}}$

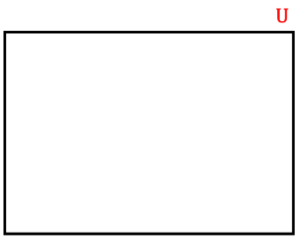
Method – 3 ⇝ Venn Diagram

Venn Diagram

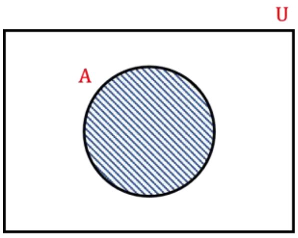
→ The relationship between sets can be represented using diagrams are known as Venn diagram.

→ Venn diagram is named after the English logician, John Venn.

→ In Venn diagram, the universal set is represented by rectangle and its subsets usually by circles.



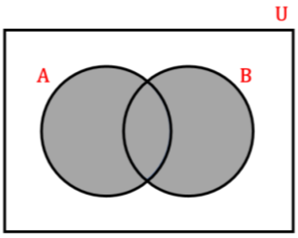
U



A

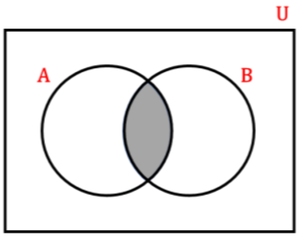
→ Venn diagram of some fundamental operations between **two** sets are drawn below:

(1) **Union of Sets**



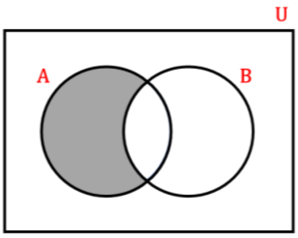
$A \cup B$

(2) **Intersection of Sets**

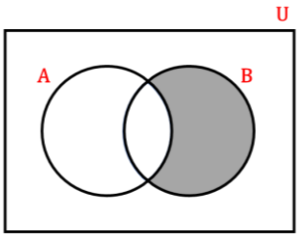


$A \cap B$

(3) **Difference of Sets**

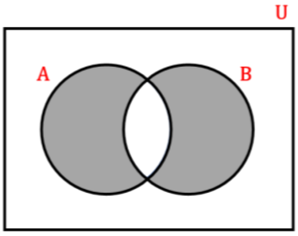


$A - B$



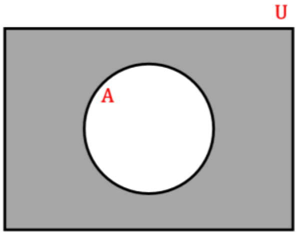
$B - A$

(4) Symmetric Difference of Sets



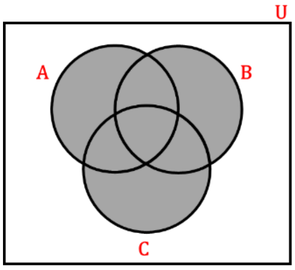
A Δ B

(5) Complement of Set

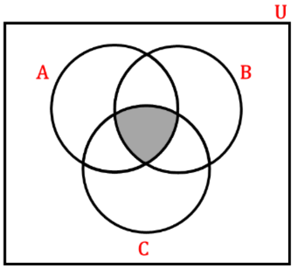


A′

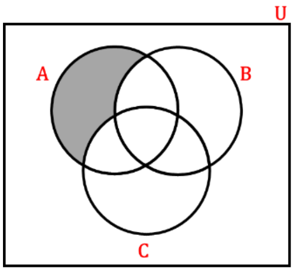
→ Venn diagram of some fundamental operations between **three** sets are drawn below:



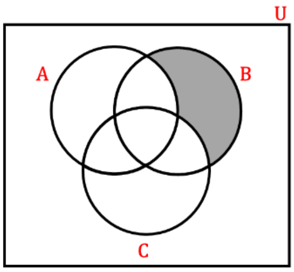
A ∪ B ∪ C



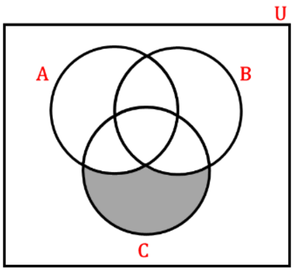
A ∩ B ∩ C



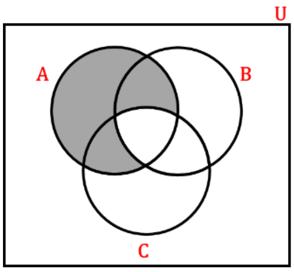
A – B – C = A – (B ∪ C)



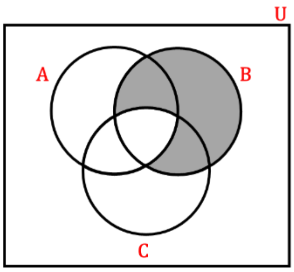
B – A – C = B – (A ∪ C)



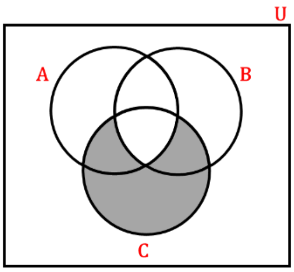
C – A – B = C – (A ∪ B)



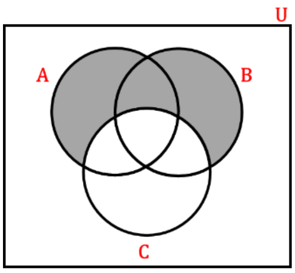
A – (B ∩ C)



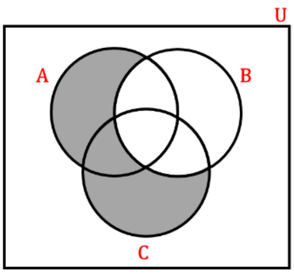
B – (A ∩ C)



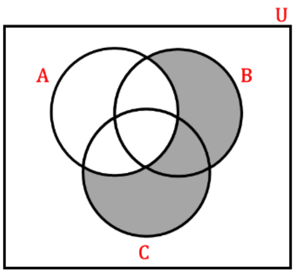
C – (A ∩ B)



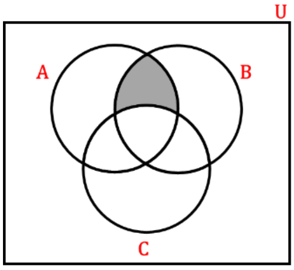
(A ∪ B) – C



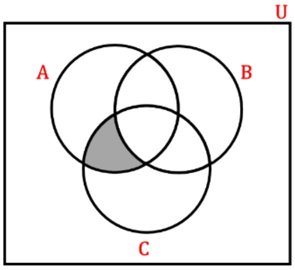
(A ∪ C) – B



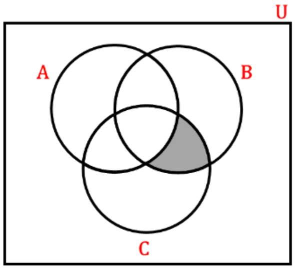
(B ∪ C) – A



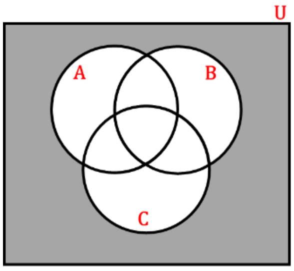
$(A \cap B) - C$



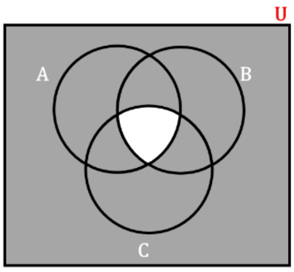
$(A \cap C) - B$



$(B \cap C) - A$

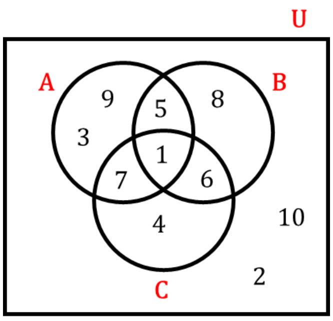


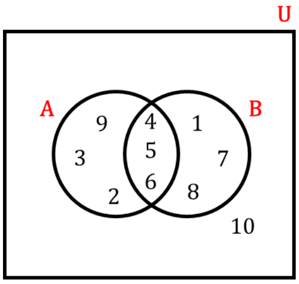
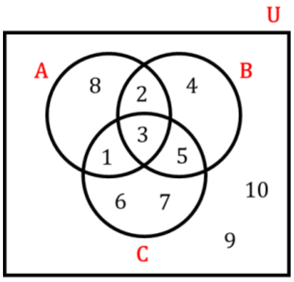
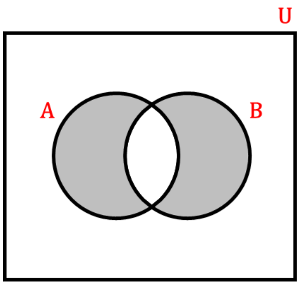
$A' \cap B' \cap C'$

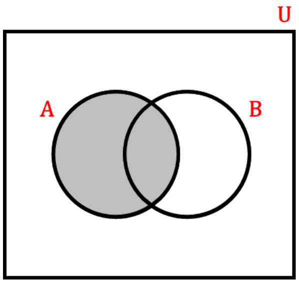
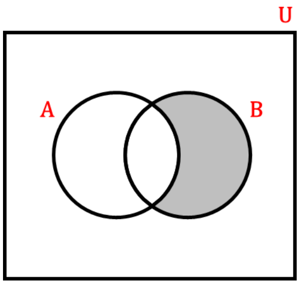
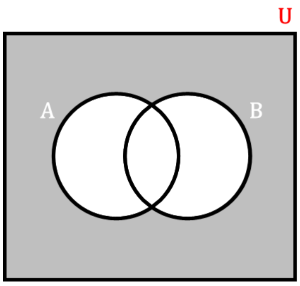
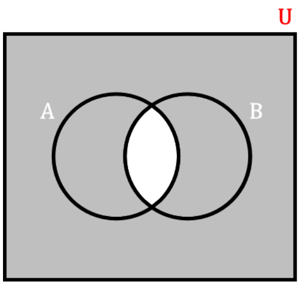


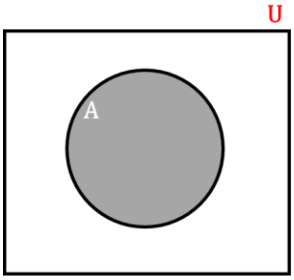
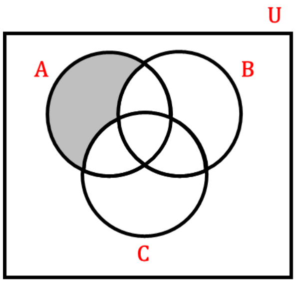
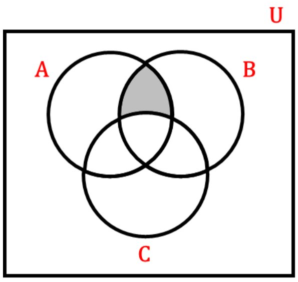
$A' \cup B' \cup C'$

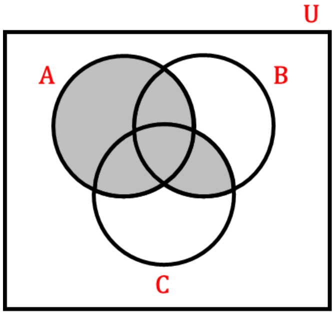
Examples of Method-3: Venn Diagram

C	1	Draw a combine Venn diagram for $U = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}$ $A = \{1,\ 3,\ 5,\ 7,\ 9\}$ $B = \{1,\ 5,\ 6,\ 8\}$ $C = \{1,\ 4,\ 6,\ 7\}$ **Answer**: 

H	2	Draw a combine Venn diagram for U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 } A = { 2, 3, 4, 5, 6, 9 } B = { 1, 4, 5, 6, 7, 8 } **Answer**: 
T	3	Draw a combine Venn diagram for U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }, A = { 1, 2, 3, 8 }, B = { 2, 3, 4, 5 }, C = { 1, 3, 5, 6, 7 }. **Answer**: 
C	4	Prove below identity using Venn diagram. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ **Answer**: 

H	5	Prove $(A \cap B) \cup (A - B) = A$ using Venn diagram. **Answer**: 
T	6	Prove $B - A = B \cap A'$ using Venn diagram. **Answer**: 
C	7	Prove $(A \cup B)' = A' \cap B'$ using Venn diagram. **Answer**: 
H	8	Prove $(A \cap B)' = A' \cup B'$ using Venn diagram. **Answer**: 

T	9	Prove $\phi' \cap A = A$ using Venn diagram. **Answer**: 
C	10	Prove below identity using Venn diagram. $A - (B \cup C) = (A - B) \cap (A - C)$ **Answer**: 
H	11	Prove below identity using Venn diagram. $A \cap (B - C) = (A \cap B) - (A \cap C)$ **Answer**: 

T	12	Prove below identity using Venn diagram. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **Answer**: 

Method – 4 ⇝ Cardinality of a Set

Cardinality of Set

→ The number of elements in a finite set A is known as cardinality of set A.

→ Cardinality of set A is denoted by **| A |** or **n(A)** and read as "cardinality of A".

→ For example:

Let $A = \{1, 2, 3, 5, x\}$

No. of elements in set A are 5.

Therefore, $|A| = 5$.

→ Cardinality of set A is also known as Cardinal number of set A.

Important Results of Cardinality

→ Let A, B and C be finite sets in a finite universal set U.

(1) The Inclusion-Exclusion Principle for two sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(2) The Inclusion-Exclusion Principle for three sets A, B and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

(3) $|A - B| = |A| - |A \cap B|$

(4) $|A \Delta B| = |A| + |B| - 2|A \cap B|$

(5) $|A'| = |U| - |A|$

(6) $|A' \cap B'| = |(A \cup B)'|$ $\{\because$ De Morgen's Law $\}$

$= |U| - |A \cup B|$

(7) $|A' \cup B'| = |(A \cap B)'|$ $\{\because$ De Morgen's Law $\}$

$= |U| - |A \cap B|$

(8) $|A' \cap B' \cap C'| = |(A \cup B \cup C)'|$ $\{\because$ De Morgen's Law $\}$

$= |U| - |A \cup B \cup C|$

(9) $|A' \cup B' \cup C'| = |(A \cap B \cap C)'|$ $\{\because$ De Morgen's Law $\}$

$= |U| - |A \cap B \cap C|$

Examples of Method-4: Cardinality of Set

C	1	Let A and B are two subsets of universal set U, such that $\mid A \mid = 20,\ \ \mid B \mid = 30,\ \ \mid U \mid = 80,\ \ \mid A \cap B \mid = 10$. Find $\mid A' \cap B' \mid$. **Answer**: $\mid \mathbf{A' \cap B'} \mid = \mathbf{40}$
H	2	Let A and B be sets, such that $\mid A \mid = 50,\ \mid B \mid = 50,\ \mid A \cup B \mid = 75$. Find $\mid A \cap B \mid$. **Answer**: $\mid \mathbf{A \cap B} \mid = \mathbf{25}$
T	3	Let A and B are sets, such that $\mid A \mid = 12,\ \mid A \cup B \mid = 36,\ \mid A \cap B \mid = 8$. Find $\mid B \mid$. **Answer**: $\mid \mathbf{B} \mid = \mathbf{32}$
C	4	Among 50 patients admitted to a hospital, 25 are diagnosed with pneumonia, 30 with bronchitis and 10 with both pneumonia and bronchitis. **(1)** Calculate the number of patients diagnosed with pneumonia or bronchitis or both. **(2)** Calculate the number of patients not diagnosed with pneumonia or bronchitis. **Answer**: (1) **45 patients diagnosed with pneumonia or bronchitis or both.** (2) **5 patients not diagnosed with pneumonia or bronchitis**
H	5	In a school there are 20 teachers who teach mathematics or physics. From which 12 teach mathematics and 4 teach both physics and mathematics. How many teach physics? **Answer**: **12 teachers teach physics**
T	6	In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football? **Answer**: **5 students like to play both cricket and football**

T	7	In a survey of 600 students in a school, 150 students were found to be taking tea and 225 taking coffee, 100 were taking both tea and coffee. Find how many students were taking neither tea nor coffee? **Answer: 325 students were taking neither tea nor coffee**
C	8	In a survey it was found that 21 people liked product A, 26 liked product B and 29 liked product C. If 14 people liked products A and B, 12 people liked products C and A, 14 people liked products B and C and 8 liked all the three products. Find how many liked product C only. **Answer: 11 people liked product C only**
H	9	Among 18 students in a room, 7 study mathematics, 10 study science and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many students study none of the three subjects? **Answer: 2 students study none of the three subjects**
H	10	A survey in a year 1986 asked households whether they had a VCR, a CD player or cable TV. 40 had a VCR, 60 had a CD player and 50 had cable TV. 25 owned VCR and CD player. 30 owned a CD player and had cable TV. 35 owned a VCR and had cable TV. 25 households had all three. How many households had at least one of the three? **Answer: 85 households had at least one of the three**

T	11	It was found that in the first-year computer science class of 80 students, 50 knew COBOL, 55 C language and 46 PASCAL. It was also known that 37 knew C language & COBOL, 28 C language & PASCAL and 25 PASCAL & COBOL. 7 students however knew none of the languages. **(1)** How many knew all the three languages? **(2)** How many knew exactly two languages? **(3)** How many knew exactly one language? **Answer**: (1) **12 students all the three languages** (2) **54 students knew exactly two languages** (3) **7 students exactly one language**
T	12	An advertising agency finds that among its 170 clients, 115 use television, 110 use radio and 130 use magazines. Also 85 use television and magazines, 75 use television and radio, 95 use radio and magazines, 70 use all the three. **(1)** How many uses only radio? **(2)** How many uses only television? **(3)** How many uses television and magazine but not radio? **Answer**: (1) **10 clients uses only radio** (2) **25 clients uses only television** (3) **15 clients uses television and magazine but not radio**

Method – 5 ⇝ Power Set and Cartesian Product

Power Set

→ Given a set A, the set of all the subsets of a set A is known as power set of set A.

→ It is denoted by **$P(A)$** and read as “Power set of A”.

i. e., $P(A) = \{ X : X \subseteq A \}$.

→ For example:

Let $A = \{ a, \ b, \ c \}$, then

$$P(A) = \left\{ \phi, \ A, \ \{a\}, \ \{b\}, \ \{c\}, \ \{a, \ b\}, \ \{b, \ c\}, \ \{a, \ c\} \right\}$$

→ Properties of power set:

- For any set A,

  (1) $\phi \in P(A) \ \& \ A \in P(A)$.

  (2) $X \in P(A) \Leftrightarrow X \subseteq A$.

  (3) If set A has **n** elements, then $|P(A)| = \mathbf{2^n}$

  and $\left|P\left(P(A)\right)\right| = \mathbf{2^{(2^n)}}$

- For example:

  Let $A = \{ a, \ b, \ c \}$.

  Here, set A has 3 elements.

  $\Rightarrow \quad |P(A)| = 2^3 = \mathbf{8}$

  $\Rightarrow \left|P\left(P(A)\right)\right| = 2^{(2^3)} = 2^{\mathbf{8}} = \mathbf{256}$

Cartesian product

→ Cartesian product of sets is set of ordered pair.

→ Cartesian product of sets A and B is denoted by **$A \times B$** which is read as “A cross B” and defined as follow:

$A \times B = \{ (\mathbf{a}, \mathbf{b}) : \mathbf{a} \in \mathbf{A} \text{ and } \mathbf{b} \in \mathbf{B} \}$

→ Cartesian product of sets B and A is denoted by **$B \times A$** which is read as “B cross A” and defined as follow:

$B \times A = \{ (\mathbf{b}, \mathbf{a}) : \mathbf{b} \in \mathbf{B} \text{ and } \mathbf{a} \in \mathbf{A} \}$

→ For example:

Let $A = \{ a, \ b \}$ and $B = \{ 1, \ 2 \}$.

- Cartesian product of set A and B is

$$A \times B = \{ a, \ b \} \times \{ 1, \ 2 \}$$

$$= \{ (a, 1), \ (a, 2), \ (b, 1), \ (b, 2) \}$$

- Similarly, cartesian product of set B and A is

$$B \times A = \{ 1, \ 2 \} \times \{ a, \ b \}$$

$$= \{ (1, a), \ (1, b), \ (2, a), \ (2, b) \}$$

→ Properties of Cartesian Product:

(1) If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.

(2) If $| A | = m$ and $| B | = n$, then $| A \times B | = m \cdot n$.

(3) $A \times B \neq B \times A$.

(4) $A \times B = B \times A$ if and only if $A = B$.

(5) Distributive properties:

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$
- $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Examples of Method-5: Power Set and Cartesian Product

C	1	Give the power sets of following set: $B = \{ x : x \text{ is a prime number}, \ x \in \mathbb{N} \text{ and } x < 8 \}$ **Answer**: $\mathbf{B = \{ 2, \ 3, \ 5, \ 7 \}}$, $\mathbf{P(B) = \left\{ \begin{array}{c} \phi, \ \{2\}, \ \{3\}, \ \{5\}, \ \{7\}, \ \{2, 3\}, \ \{2, 5\}, \\ \{2, 7\}, \ \{3, 5\}, \ \{3, 7\}, \ \{5, 7\}, \ \{2, 3, 5\} \\ \{3, 5, 7\}, \ \{5, 7, 2\}, \ \{7, 2, 5\}, \ \{2, 3, 5, 7\} \end{array} \right\}}$
H	2	Give the power set of following set: $A = \{ x : x \text{ is multiple of } 3, \ x \in \mathbb{N} \text{ and } x \leq 12 \}$ **Answer**: $\mathbf{A = \{ 3, \ 6, \ 9, \ 12 \}}$ $\mathbf{P(A) = \left\{ \begin{array}{c} \phi, \ \{3\}, \ \{6\}, \ \{9\}, \ \{12\}, \{3, 6\}, \ \{3, 9\}, \\ \{3, 12\}, \ \{6, 9\}, \ \{6, 12\}, \ \{9, 12\}, \ \{3, 6, 9\}, \\ \{6, 9, 12\}, \ \{9, 12, 3\}, \{12, 3, 6\}, \ \{3, 6, 9, 12\} \end{array} \right\}}$
T	3	Give the power set of $A = \{ 1, \ 2, \ 3 \}$. **Answer**: $\mathbf{P(A) = \left\{ \phi, \ A, \ \{1\}, \ \{2\}, \ \{3\}, \ \{1, \ 2\}, \ \{2, \ 3\}, \ \{1, \ 3\} \right\}}$
C	4	Let $A = \{ a, b, c, d, e \}$ be a set. What is cardinality of $P(A)$ and $P\big(P(A)\big)$? **Answer**: $\mathbf{\mid P(A) \mid = 2^5 = 32}$, $\mathbf{\big\mid P\big(P(A)\big) \big\mid = 2^{32} = 4294967296}$
H	5	Let $A = \{ \alpha \mid \alpha \text{ is consonant of alphbets} \}$ be a set. What is cardinality of $P(A')$ and $P\big(P(A')\big)$? **Answer**: $\mathbf{\mid P(A') \mid = 2^5 = 32}$, $\mathbf{\big\mid P\big(P(A')\big) \big\mid = 2^{32} = 4294967296}$
T	6	Let $A = \{ 1, \ 2, \ 3, \ 4 \}$ be a set. What is cardinality of $P(A)$ and $P\big(P(A)\big)$? **Answer**: $\mathbf{\mid P(A) \mid = 2^4 = 16}$, $\mathbf{\big\mid P\big(P(A)\big) \big\mid = 2^{16} = 65,536}$

C	7	Consider the sets $A = \{a, b, c\}$, $B = \{1, 2\}$ and $C = \{b, c, d, e\}$. Write $A \times B$, $(A \times B) \cap (C \times B)$. **Answer**: $\mathbf{A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}}$ $\mathbf{(A \times B) \cap (C \times B) = \{(b, 1), (b, 2), (c, 1), (c, 2)\}}$
H	8	Let $A = \{x : x \text{ is vowel in English alphabates}\}$ be a set. Write $A \times A$. **Answer**: $\mathbf{A = \{a, e, i, o, u\}}$, $\mathbf{A \times A = \{(x, y) : x \in A, y \in A\}}$ **OR** $\mathbf{A \times A = \left\{ \begin{matrix} (a, a), & (a, e), & (a, i), & (a, o), & (a, u), \\ (e, a), & (e, e), & (e, i), & (e, o), & (e, u), \\ (i, a), & (i, e), & (i, i), & (i, o), & (i, u), \\ (o, a), & (o, e), & (o, i), & (o, o), & (o, u), \\ (u, a), & (u, e), & (u, i), & (u, o), & (u, u) \end{matrix} \right\}}$
T	9	Let $A = \{\alpha, \beta\}$ and $B = \{1, 2, 3\}$. Write $B \times B$, $A \times A$ and $(A \times B) \cap (B \times A)$. **Answer**: $\mathbf{B \times B = \left\{ \begin{matrix} (1, 1), & (1, 2), & (1, 3), \\ (2, 1), & (2, 2), & (2, 3), \\ (3, 1), & (3, 2), & (3, 3), \end{matrix} \right\}}$ $\mathbf{A \times A = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}}$, $\mathbf{(A \times B) \cap (B \times A) = \phi}$

T	10	Consider the following sets: $A = \{1,\ 3,\ 5\}$, $B = \{2,\ 4,\ 6,\ 8\}$, $C = \{b,\ c\}$. Verify the following statements. **(1)** $A \times (B \cap C) = (A \times B) \cap (A \times C)$ **(2)** $A \times (B \cup C) = (A \times B) \cup (A \times C)$ **(3)** $(A \cap B) \times C = (A \times C) \cap (B \times C)$ **(4)** $(A \cup B) \times C = (A \times C) \cup (B \times C)$ **Hint :** (1) $\mathbf{A \times (B \cap C) = \phi}$ (2) $\mathbf{A \times (B \cup C) = \left\{ \begin{array}{c} (1,\ 2),\ (1,\ 4),\ (1,\ 6),\ (1,\ 8),\ (1,\ b), \\ (1,\ c),\ (3,\ 2),\ (3,\ 4),\ (3,\ 6),\ (3,\ 8), \\ (3,\ b),\ (3,\ c),\ (5,\ 2),\ (5,\ 4),\ (5,\ 6), \\ (5,\ 8),\ (5,\ b),\ (5,\ c) \end{array} \right\}}$ (3) $\mathbf{(A \cap B) \times C = \phi}$ (4) $\mathbf{(A \cup B) \times C = \left\{ \begin{array}{c} (1,\ b),\ (2,\ b),\ (3,\ b),\ (4,\ b), \\ (5,\ b),\ (6,\ b),\ (8,\ b), \\ (1,\ c),\ (2,\ c),\ (3,\ c),\ (4,\ c), \\ (5,\ c),\ (6,\ c),\ (8,\ c) \end{array} \right\}}$

Part - II ⇝ Function

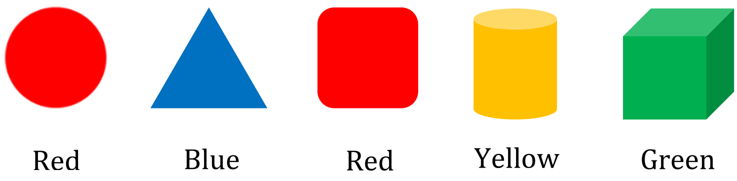
Introduction

→ One of the most important concepts in mathematics is function.

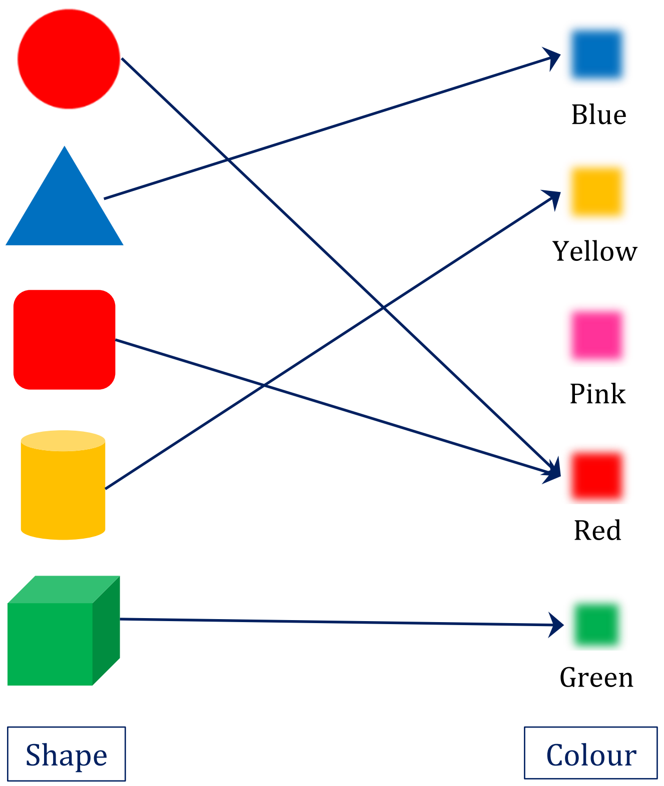
→ Function is a relation from one set to another set.

→ Let understand with it an example.

→ Consider different shapes with different color as follow:

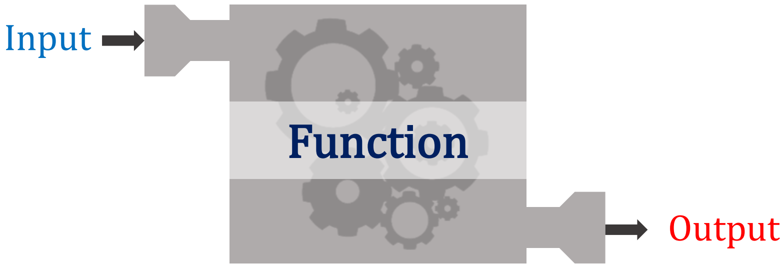


→ Now, making pair of shape with its appropriate color.

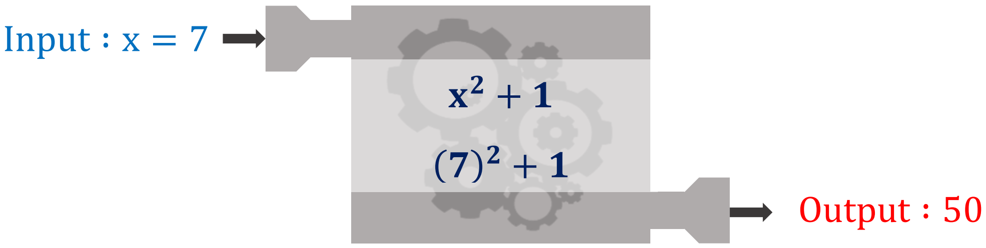


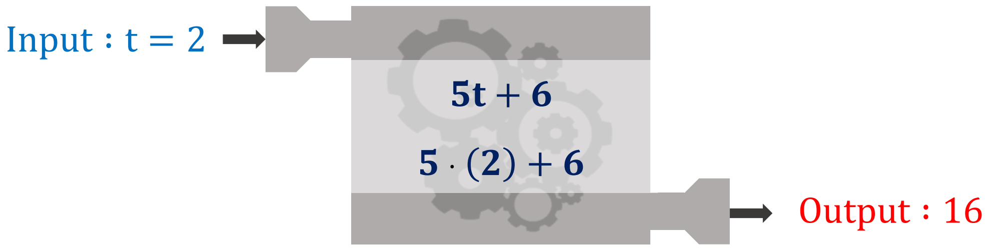
→ Above example illustrates function that associate colored shape to its color.

→ We can view a function as machine that can take an object (Input) and turn it into a different object (Output).



→ For example:





Method – 6 ⇝ Definitions

Function or Mapping

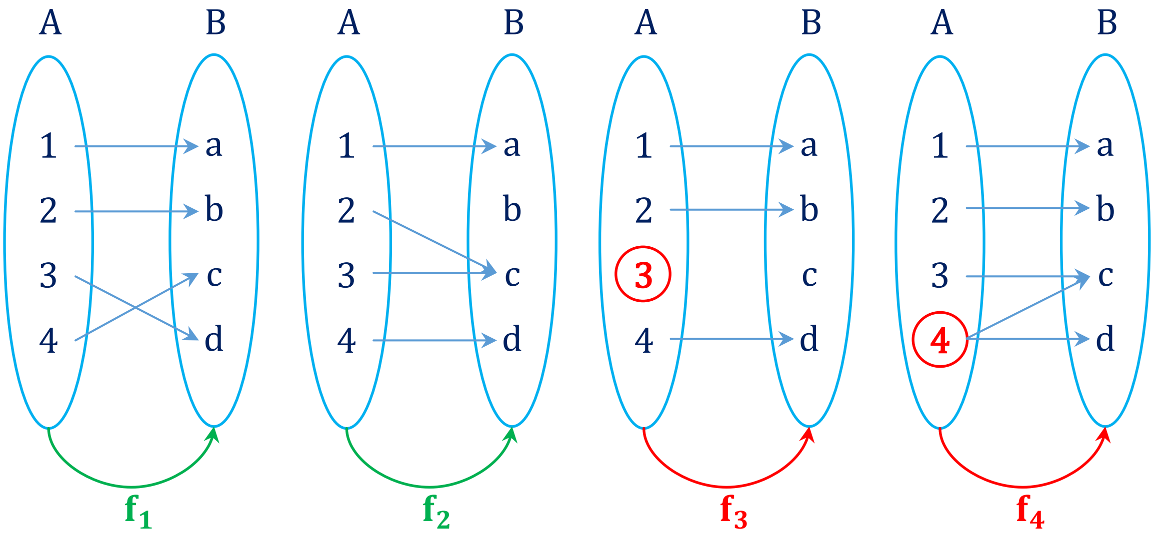
→ Let A and B be two nonempty sets.

→ If **every element** of A is assigned **to unique** element of set B, then such assignment is known as function from A to B.

→ Function is written as $\mathbf{f : A \to B}$ and read as "function f from A to B."

→ For example:

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$ be two sets, consider following relations from set A to set B:



- Relation f_1 is a function because every element of set A are related to unique element of B.
- Relation f_2 is a function because every element of set A are related to unique element of B.
- Relation f_3 is **not** a function because one element of set A '3' is **not related** to any element of B.
- Relation f_4 is **not** a function because one element of set A '4' is related to **more than one** element of B.

→ Let's operate function f on an element $a \in A$, it is denoted as $f(a)$ and read as "f of a".

→ If $f(a) = b$ then it is read as "function f maps a to b".

→ Let $f : A \to B$.

If $|A| = m$ and $|B| = n$, then we can create n^m different functions from set A to B.

- For example:
  - Let $f : A \to B$ be a function such that $|A| = 4$ and $|B| = 5$.

    Number of different functions can be generated is $(5)^4 = 625$.
  - Let $f : B \to A$ be a function such that $|A| = 4$ and $|B| = 5$.

    Number of different functions can be generated is $(4)^5 = 1024$.

Domain of a Function

→ Let $f : A \to B$ be a function.

→ Set A is known as **domain** of function f.

→ Domain of function f is denoted by D_f and read as "domain of function f".

$$f : A \to B$$

Domain (D_f)

Codomain of a Function

→ Let $f : A \to B$ be a function.

→ Set B is known as **codomain** of function f.

→ Codomain of function f is denoted by C_f and read as "codomain of function f".

$$f : A \to B$$

Codomain (C_f)

Image of an element under function

→ Let $f : A \to B$ be function from set A to B.

→ Let $f(a) = b$, where $a \in A$ and $b \in B$.

→ Here, the element "**b**" is known as **image of** an element "**a**" under the function f.

→ Furthermore, an element "**a**" is **preimage** of the element "**b**".

Range of function or Image of function

→ The set of all images of the elements of set A under function $f : A \to B$ is known as range or image of function f.

→ Range or image of function f is denoted by $\mathbf{R_f}$ or $\mathbf{Im(f)}$ or $\mathbf{f(A)}$ and read as "range of function f" or "image of A" or "function of A".

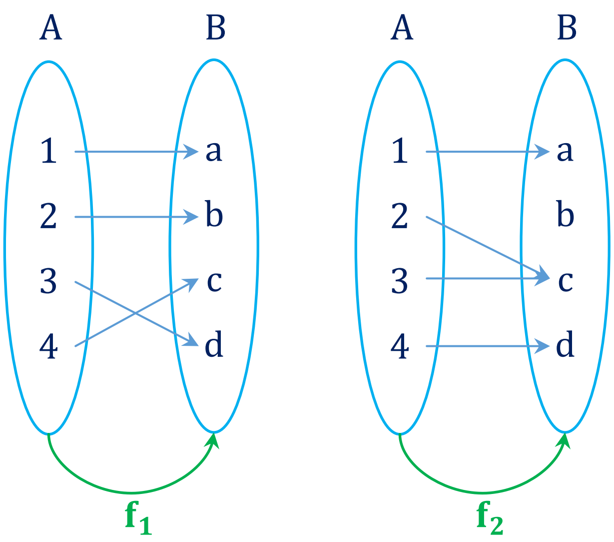
→ i.e., Range of function f is $R_f = \{ b : b \in B \text{ such that } f(a) = b \text{ for some } a \in A \}$.

→ For every function $f : A \to B$,

$\mathbf{R_f \subseteq B}$ **(Codomain $\mathbf{C_f}$).**

→ For example:

Consider function $f_1 : A \to B$ and $f_2 : A \to B$.



- For function f_1:
  - Function f_1 can be written as follow:

    $f_1(1) = a, \ f_1(2) = b, \ f_1(3) = d, \ f_1(4) = c$

    or

    $f_1 = \{ (1, a), (2, b), (3, d), (4, c) \}$
  - Domain of function f_1 : $D_{f_1} = \{ 1, 2, 3, 4 \}$
  - Codomain of function f_1 : $C_{f_1} = \{ a, b, c, d \}$
  - Range of function f_1 : $R_{f_1} = \{ a, b, c, d \}$

- For function f_2:
  - Function f_2 can be written as follow:

    $f_2(1) = a, \;\; f_2(2) = c, \;\; f_2(3) = c, \;\; f_2(4) = d$

    or

    $f_2 = \{ (1, a), \;\; (2, c), \;\; (3, c), \;\; (4, d) \}$
  - Domain of function f_2 : $D_{f_2} = \{ 1, \;\; 2, \;\; 3, \;\; 4 \}$
  - Codomain of function f_2 : $C_{f_2} = \{ a, \;\; b, \;\; c, \;\; d \}$
  - Range of function f_2 : $R_{f_2} = \{ a, \;\; c, \;\; d \}$

→ Following are some examples of function.

- $f : \mathbb{R} \to \mathbb{R}, \;\; f(x) = x^2 + 1$
- $f : \mathbb{R} \to \mathbb{R}, \;\; f(x) = \sin x$
- $f : \mathbb{R}^+ \to \mathbb{R}, \;\; f(x) = \log x$

Examples of Method-6: Definitions

C	1	Which of the following relations are function? If it is, determine its domain and range. **(1)** $f = \{ (4, 2), \; (6, 3), \; (8, 4), \; (10, 5), \; (12, 6), \; (14, 7) \}$ **(2)** $g = \{ (2, 1), \; (5, 1), \; (8, 1), \; (11, 1), \; (14, 1), \; (17, 1) \}$ **(3)** $h = \{ (1, 3), \; (1, 5), \; (2, 5) \}$ **Answer:** (1) **f is a function,** **Domain** $D_f = \{ 4, \; 6, \; 8, \; 10, \; 12, \; 14 \}$, **Range** $R_f = \{ 2, \; 3, \; 4, \; 5, \; 6, \; 7 \}$ (2) **g is a function,** **Domain** $D_g = \{ 2, \; 5, \; 8, \; 11, \; 14, \; 17 \}$, **Range** $R_g = \{ 1 \}$ (3) **h is not a function.**

H	2	Which of the following relations are function? If it is, determine its domain and range. **(1)** $f = \{(2,1),\ (5,1),\ (8,1),\ (11,1),\ (14,1),\ (17,1)\}$ **(2)** $g = \{(2,1),\ (4,2),\ (6,3),\ (8,4),\ (10,5),\ (12,6), (14,7)\}$ **(3)** $h = \{(1,3),\ (1,5),\ (2,5)\}$ **Answer: (1) f is a function,** **Domain** $D_f = \{2,\ 5,\ 8,\ 11,\ 14, 17\}$, **Range** $R_f = \{1\}$ **(2) g is a function,** **Domain** $D_g = \{2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14\}$, **Range** $R_g = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$ **(3) h is not a function.**
T	3	Which of the following relations are function? If it is, determine its domain and range. **(1)** $f = \{(1,2),\ (3,2),\ (5,2),\ (7,2),\ (9,2)\}$ **(2)** $g = \{(1,1),\ (2,7),\ (3,3),\ (4,5),\ (5,4),\ (6,6),\ (7,2)\}$ **(3)** $h = \{(2,3),\ (2,5),\ (3,5)\}$ **Answer: (1) f is a function,** **Domain** $D_f = \{1,\ 3,\ 5,\ 7,\ 9\}$, **Range** $R_f = \{2\}$ **(2) g is a function,** **Domain** $D_g = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$, **Range** $R_g = \{1,\ 7,\ 3,\ 5,\ 4,\ 6,\ 2\}$ **(3) h is not a function.**

Method – 7 ⇝ Types of Functions

(1) **Real Function**

→ If **range** of any function is either $\mathbb{R}$ or a subset of $\mathbb{R}$, then it is known as a **real valued function**.

→ For example:

$f : \mathbb{N} \to \mathbb{N},\ f(x) = x^2$ is real valued function as $R_f = \mathbb{N} \subset \mathbb{R}$

→ Further, if **domain** of any function f is either $\mathbb{R}$ or a subset of $\mathbb{R}$, it is known as a **real function**.

→ For example:

$f : \mathbb{N} \to \mathbb{N},\ f(x) = x^2$ is real function as $D_f = \mathbb{N} \subset \mathbb{R}$

(2) **Identity function**

→ Let A be any nonempty set.

→ The function $f : A \to A,\ f(\mathbf{x}) = \mathbf{x},\ \forall\, x \in A$ is known as identity function on set A.

→ Identity function on set A is denoted by $\mathbf{I_A}$ and read as "identity function on set A".

→ This function maps any element of A to itself.

→ Range of identity function f is its codomain.

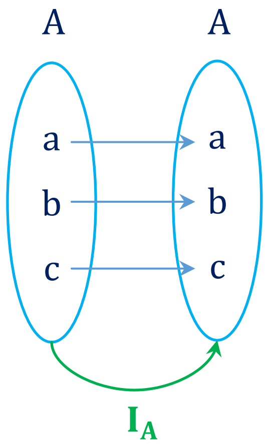
i.e. $R_f = \text{Codomain}$

→ For example:

Let $A = \{\, a,\ b,\ c \,\}$.

Identity function on A is $I_A : A \to A$, which is defined as follow:

$f(a) = a, \quad f(b) = b, \quad f(c) = c.$



(3) **Constant function**

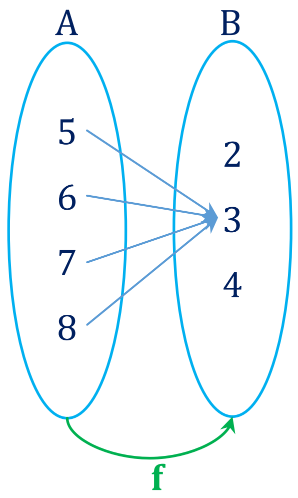
→ A function whose range is singleton set is known as constant function.

→ Thus, a function $f : A \to B$, $f(x) = c$, $\forall\, x \in A$ where $c \in B$, is known as a constant function.

→ For example:

Let $A = \{5, 6, 7, 8\}$ and $B = \{2, 3, 4\}$.

A function $f : A \to B$, which is defined as $f(x) = 3$



Range of function $f : R_f = \{3\}$

(4) **Even function**

→ A function $f : A \to B$ is known as an even function if $f(-x) = f(x), \forall\, x \in A$.

→ For example:

Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$.

Now, $f(-x) = (-x)^2 = x^2 = f(x)$

So, $f(-x) = f(x)$

Thus, $f(x) = x^2$ is an even function.

(5) **Odd function**

→ A function $f : A \to B$ is known as an odd function if $f(-x) = -f(x), \forall\, x \in A$.

→ For example:

Let $f : \mathbb{R} \to \mathbb{R}\ ;\ f(x) = x^3$.

Now, $f(-x) = (-x)^3 = -x^3 = -f(x)$

So, $f(-x) = -f(x)$

Thus, $f(x) = x^3$ is an odd function.

(6) **Modulus function**

→ A function $f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} x\ ;\ x \geq 0 \\ -x\ ;\ x < 0 \end{cases}$ is known as a modulus function.

→ It is denoted by $|\ x\ |$ and read as "modulus of x" or "absolute value of x".

→ For example:

- Let $x = 5$,

  $|\ 5\ | = 5 \qquad \{\because\ 5 > 0\}$

- Let $x = -2.5$

  $|-2.5\ | = -(-2.5) \qquad \{\because\ -2.5 < 0\}$

  $= 2.5$

→ The range of modulus function is $R_f = \mathbb{R}^+ \cup \{\ 0\ \}$.

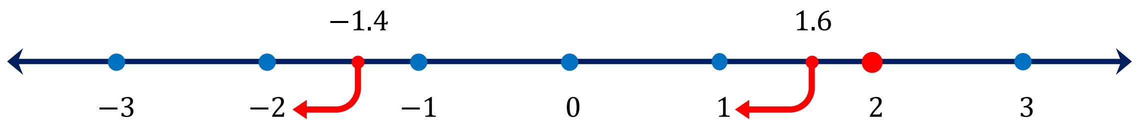
(7) **Greatest integer function or Floor function**

→ A function $f : \mathbb{R} \to \mathbb{R},\ f(x) = \max\{\ n \in \mathbb{Z} : n \leq x\ \},\ \forall\ x \in \mathbb{R}$ is known as a greatest integer function or floor function.

→ It is denoted by $\lfloor\ \mathbf{x}\ \rfloor$ and read as "floor value of x".

→ For example:

$\lfloor 1.6 \rfloor = 1, \qquad \lfloor 2 \rfloor = 2, \qquad \lfloor -1.4 \rfloor = -2$



→ The range of floor function is $R_f = \mathbb{Z}$.

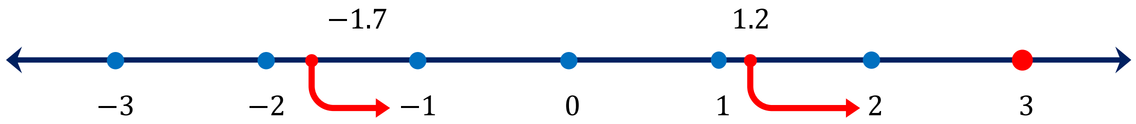
(8) **Least integer function or Ceiling function**

→ A function $f : \mathbb{R} \to \mathbb{R},\ f(x) = \min\{\ n \in \mathbb{Z} : n \geq x\ \},\ \forall\ x \in \mathbb{R}$ is known as a least integer function or ceiling function.

→ It is denoted by $\lceil\ \mathbf{x}\ \rceil$ and read as "ceiling value of x".

→ For example:

→ $\lceil 1.2 \rceil = 2,\ \lceil 3 \rceil = 3,\ \lceil -1.7 \rceil = -1$



→ The range of this function is $R_f = \mathbb{Z}$.

(9) **Integer value function**

→ A function $f : \mathbb{R} \to \mathbb{R},\ f(x) = \text{Integer value of } x,\ \forall\, x \in \mathbb{R}$ is known as integer value function.

→ It is denoted by $\mathbf{INT(x)}$ and read as "integer value of x".

→ This function converts any real number x into an integer by deleting the fractional part of the number.

→ For example:

$\text{INT}(\pi) = \text{INT}(\mathbf{3}.14159) = \mathbf{3},\qquad \text{INT}(\mathbf{7}) = \mathbf{7},$

$\text{INT}(\sqrt{5}) = \text{INT}(\mathbf{2}.2361) = \mathbf{2},\qquad \text{INT}(\mathbf{-8}.5) = \mathbf{-8},$

Examples of Method-7: Types of Functions

C	1	Determine whether given real valued functions are even or odd. $f(x) = 2x^2$ and $h(x) = 2x - 1$. **Answer: $f(x)$ is even function and** **$h(x)$ is neither even nor odd function..**
H	2	Determine whether given real valued functions are even or odd. $f(x) = \dfrac{2x^2}{3}$ and $h(x) = \dfrac{2x-1}{3x+1}$. **Answer: $f(x)$ is even function and** **$h(x)$ is neither even nor odd function..**

T	3	Determine whether given real valued functions are even or odd. $f(x) = 3x^4$ and $g(x) = 5x - 1$. **Answer:** $\mathbf{f(x)}$ **is even function and** $\mathbf{g(x)}$ **is neither even nor odd function.**
C	4	Find value of $\lceil 2.4 \rceil$, $\lceil -8.71 \rceil$, $\lfloor 4.5 \rfloor$, $\lfloor -7.23 \rfloor$ & $\text{INT}(3.2)$. **Answer:** $\lceil \mathbf{2.4} \rceil = \mathbf{3}$, $\lceil \mathbf{-8.71} \rceil = \mathbf{-8}$, $\lfloor \mathbf{4.5} \rfloor = \mathbf{4}$, $\lfloor \mathbf{-7.23} \rfloor = \mathbf{-8}$, $\mathbf{INT(3.2) = 3}$
H	5	Find value of $\lceil 3.6 \rceil$, $\lceil -9.22 \rceil$, $\lfloor 6.6 \rfloor$, $\lfloor -7.23 \rfloor$ & $\text{INT}(6.6)$. **Answer:** $\lceil \mathbf{3.6} \rceil = \mathbf{4}$, $\lceil \mathbf{-9.22} \rceil = \mathbf{-9}$, $\lfloor \mathbf{6.6} \rfloor = \mathbf{6}$, $\lfloor \mathbf{-9.23} \rfloor = \mathbf{-10}$, $\mathbf{INT(3.2) = 6}$
T	6	Find value of $\lceil 4.56 \rceil$, $\lceil -3.7 \rceil$, $\lfloor 6.4 \rfloor$, $\lfloor -9.10 \rfloor$ & $\text{INT}(4.5)$. **Answer:** $\lceil \mathbf{4.56} \rceil = \mathbf{5}$, $\lceil \mathbf{-3.7} \rceil = \mathbf{-3}$, $\lfloor \mathbf{6.4} \rfloor = \mathbf{6}$, $\lfloor \mathbf{-9.1} \rfloor = \mathbf{-10}$ $\mathbf{INT(4.5) = 4}$
C	7	Let $X = \{-1, 0, 2, 4, 7\}$. Find $f(X)$ if $f(x) = \left\lfloor \frac{x^2+1}{3} \right\rfloor$ for all $x \in X$. **Answer:** $\mathbf{f(X) = \{0, 1, 5, 16\}}$
H	8	Let $X = \{-3, -2, -1, 0, 1\}$. Find $f(X)$ if $f(x) = \left\lfloor \frac{x^3-1}{2} \right\rfloor$ for all $x \in X$. **Answer:** $\mathbf{f(X) = \{-14, -5, -1, 0\}}$
T	9	Let $S = \{0, 1, 3, 4, 5\}$. Find $f(S)$ if $f(x) = \left\lceil \frac{x^2}{2} \right\rceil$ for all $x \in S$ **Answer:** $\mathbf{f(S) = \{0, 1, 5, 8, 13\}}$
C	10	Let $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2x + 1$, $g(x) = \lvert 2x^3 - 3x \rvert$, then find $f(1)$ & $g(-3)$. **Answer:** $\mathbf{f(1) = 0}$, $\mathbf{g(-3) = 45}$

H	11	If $f:\mathbb{R}\to\mathbb{R},\ f(x)=2\lvert x\rvert-\lvert -x\rvert$, then find $f(3),\ f\left(\frac{1}{2}\right)$ & $f(-3)$. **Answer**: $\mathbf{f(3)=3,\quad f\left(\frac{1}{2}\right)=\frac{1}{2},\quad f(-3)=3}$
T	12	For the function $f(x)=\frac{\lvert 3x^3+5x-6\rvert}{5x^2+1}$. Evaluate $f(-1)$ & $f(2)$. **Answer**: $\mathbf{f(-1)=\frac{7}{3},\quad f(2)=\frac{4}{3}}$

Method – 8 ⇝ Algebra of Real Functions

Addition of Functions

→ Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are two real functions with $A \cap B \neq \phi$.

→ Then, addition of functions is

$(f + g) : (A \cap B) \to \mathbb{R}$ and defined as

$(f + g)(x) = f(x) + g(x), \quad \forall\, x \in A \cap B$

→ For example:

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two real functions defined as

$f(x) = x^2 + 2x + 1, \qquad g(x) = x + 2.$

$$
\begin{aligned}
(f + g)(x) &= f(x) + g(x) \\
&= (x^2 + 2x + 1) + (x + 2) \\
&= x^2 + 2x + 1 + x + 2 \\
&= x^2 + 3x + 3
\end{aligned}
$$

Subtraction of Functions

→ Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are two real functions with $A \cap B \neq \phi$.

→ Then, subtraction of functions is

$(f - g) : (A \cap B) \to \mathbb{R}$ and defined as

$(f - g)(x) = f(x) - g(x), \quad \forall\, x \in A \cap B$

→ For example:

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two real functions defined as

$f(x) = x^2 + 2x + 1, \qquad g(x) = x + 2.$

$$
\begin{aligned}
(f - g)(x) &= f(x) - g(x) \\
&= (x^2 + 2x + 1) - (x + 2) \\
&= x^2 + 2x + 1 - x - 2 \\
&= x^2 + x - 1
\end{aligned}
$$

Multiplication of Functions

→ Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are two real functions with $A \cap B \neq \phi$.

→ Then, multiplication of functions is

$(f \cdot g) : (A \cap B) \to \mathbb{R}$ and defined as

$(f \cdot g)(x) = f(x) \cdot g(x), \quad \forall\, x \in A \cap B$

→ For example:

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two real functions defined as

$f(x) = 2x + 1, \qquad g(x) = x - 2.$

$$
\begin{aligned}
(f \cdot g)(x) &= f(x) \cdot g(x) \\
&= (2x + 1) \cdot (x - 2) \\
&= 2x^2 - 4x + x - 2 \\
&= 2x^2 - 3x - 2
\end{aligned}
$$

Division of Functions

→ Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are two real functions with $A \cap B \neq \phi$. Then,

→ Division of functions is

$\left(\frac{f}{g}\right) : (A \cap B) - \{x : g(x) = 0\} \to \mathbb{R}$ and defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \forall\, x \in A \cap B$$

→ For example:

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two real functions defined as

$f(x) = x^2 + 1, \qquad g(x) = x^3 - 1.$

$$
\begin{aligned}
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \\
&= \frac{x^2 + 1}{x^3 - 1}
\end{aligned}
$$

Examples of Method-8: Algebra of Real Functions

C	1	Let $f(x) = x^2 + 2x$ and $g(x) = 5x - 7$ be real functions, then find $(f+g)(x)$, $(f \cdot g)(x)$, $(f-g)(x)$, $\left(\frac{f}{g}\right)(x)$. **Answer:** $(f+g)(x) = x^2 + 7x - 7$, $(f \cdot g)(x) = 5x^3 + 3x^2 - 14x$, $(f-g)(x) = x^2 - 3x + 7$, $\left(\frac{f}{g}\right)(x) = \frac{x^2 + 2x}{5x - 7}$
H	2	Let $f(x) = x^2$ & $g(x) = 2x + 1$ be the real functions, then find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$ & $\left(\frac{f}{g}\right)(x)$. **Answer:** $(f+g)(x) = x^2 + 2x + 1$, $(f-g)(x) = x^2 - 2x - 1$, $(f \cdot g)(x) = 2x^3 + x^2$, $\left(\frac{f}{g}\right)(x) = \frac{x^2}{2x + 1}$
H	3	Let $f(x) = \sqrt{x}$ & $g(x) = x$ be the real functions, then find $(f+g)(x)$, $(f-g)(x)$, $(g \cdot f)(x)$ & $\left(\frac{g}{f}\right)(x)$. **Answer:** $(f+g)(x) = \sqrt{x} + x$, $(f-g)(x) = \sqrt{x} - x$, $(g \cdot f)(x) = x \cdot \sqrt{x} = x^{\frac{3}{2}}$, $\left(\frac{g}{f}\right)(x) = \sqrt{x}$
H	4	Let $f(x) = x^3 - 3x^2 + 3x - 1$ & $g(x) = x^2 + 4x + 4$ be the real functions, then find $(f+g)(x)$, $(f-g)(x)$ & $(g-f)(x)$. **Answer:** $(f+g)(x) = x^3 - 2x^2 + 7x + 3$ $(f-g)(x) = x^3 - 4x^2 - x - 5$ $(g-f)(x) = -x^3 + 4x^2 + x + 5$

T	5	Let $f(x) = (x+4)(x-5)$ & $g(x) = (x-5)(x+2)$ be the real functions, then find the following: $(f+g)(x)$, $(f-g)(x)$ & $\left(\frac{f}{g}\right)(x)$. **Answer**: $\mathbf{(f+g)(x) = 2x^2 - 4x - 30}$ $\mathbf{(f-g)(x) = 2x - 10}$ $\mathbf{\left(\frac{f}{g}\right)(x) = \frac{x+4}{x+2}}$
T	6	Find $(f+g)(x)$, $(f \cdot g)(x)$, $(h \cdot g)(x)$ & $(f-h)(x)$ for the following functions: $f(x) = x^2 + 3$, $g(x) = \frac{1}{x}$ & $h(x) = \frac{2x+3}{x}$. **Answer**: $\mathbf{(f+g)(x) = \frac{x^3+3x+1}{x}}$ $\mathbf{(f \cdot g)(x) = \frac{x^2+3}{x}}$ $\mathbf{(h \cdot g)(x) = \frac{2x+3}{x^2}}$ $\mathbf{(f-h)(x) = \frac{x^3+x-3}{x}}$
T	7	Let $f(x) = \frac{x+1}{x}$, $g(x) = x+1$ & $h(x) = \frac{x}{x+1}$ be the real functions, then find $(f+h)(x)$, $(g \cdot h)(x)$, $(f-g)(x)$, $\left(\frac{f}{h}\right)(x)$. **Answer**: $\mathbf{(f+h)(x) = \frac{2x^2+2x+1}{x^2+x}}$, $\mathbf{(g \cdot h)(x) = x}$, $\mathbf{(f-g)(x) = \frac{-x^2+1}{x}}$, $\mathbf{\left(\frac{f}{h}\right)(x) = \frac{x^2+2x+1}{x^2}}$

Method – 9 ⇝ One–One, Onto & Bijective Functions

One–One Function

→ Let $f : X \to Y$ be a function. If distinct elements of set X have distinct image in set Y, then, function f is known as a one–one function.

i.e., for every $x_1,\ x_2 \in X$, $\mathbf{x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)}$

or

$$\mathbf{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$$

→ A one – one function is also known as **injection**.

→ If function f is **not** one–one then it is known as **many–one** function.

→ For example:

- Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = x - 3$ is one–one or not.

  Solution:

  Let $x_1,\ x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2)$

  We have,

  $$f(x_1) = f(x_2)$$

  $$\Rightarrow x_1 - 3 = x_2 - 3 \qquad \{\because f(x) = x - 3\}$$

  $$\Rightarrow \quad x_1 = x_2 - 3 + 3$$

  $$\Rightarrow \quad x_1 = x_2$$

  Here, $f(x_1) = f(x_2) \implies x_1 = x_2$.

  Hence, $f(x) = x - 3$ is one–one function.

- Determine whether the function $f : \mathbb{Z} \to \mathbb{Z}^+$ be a function defined as $f(x) = x^2$ is one–one or not.

  Solution:

  $f(-1) = (-1)^2 = 1$ and

  $f(1) = (1)^2 = 1$

  Here, $f(-1) = f(1)$

  but $\quad -1 \neq 1$

  Hence, $f(x) = x^2$ is not one–one function.

Onto Function

→ Let $f : X \rightarrow Y$ be a function. If every element of set Y has preimage in set X under function f, then function f is known as an onto function.

i.e., $\mathbf{y \in Y}$ (codomain) there is $\mathbf{x \in X}$ (domain) such that $\mathbf{f(x) = y}$.

→ An onto function is also known as **surjection**.

→ If a given function is **not** an onto function, then it is known as an **into** function.

→ For example:

- Determine whether the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined as

  $f(x) = 2x + 3$ is onto or not.

  Solution:

  Let $f(\mathbf{x}) = \mathbf{y}$, such that $y \in \mathbb{R}$ (codomain).

  Consider,

  $$f(\mathbf{x}) = \mathbf{y}$$

  $$\Rightarrow\ 2\mathbf{x} + 3 = \mathbf{y} \qquad \{\because f(x) = 2x + 3\}$$

  $$\Rightarrow\ \mathbf{x} = \frac{y - 3}{2}$$

  Since, $y \in \mathbb{R}$

  Then $x = \dfrac{y - 3}{2} \in \mathbb{R}$ (domian).

  Therefore, for every $y \in \mathbb{R}$ there is $x \in \mathbb{R}$ such that $f(x) = y$.

  Hence, $f(x) = 2x + 3$ is onto function.

- Determine whether the function $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function defined as

  $f(x) = 9x - 2$ is onto or not.

  Solution:

  Let $f(\mathbf{x}) = \mathbf{y}$, such that $y \in \mathbb{N}$ (codomain)

  Consider,

  $$f(\mathbf{x}) = \mathbf{y}$$

  $$\Rightarrow\ 9\mathbf{x} - 2 = \mathbf{y} \qquad \{\because f(x) = 9x - 2\}$$

  $$\Rightarrow\ \mathbf{x} = \frac{y + 2}{9}$$

Take $y = 5 \Rightarrow x = \frac{7}{9} \notin \mathbb{N}$

Therefore, $y = 5$ has no preimage x in domain such that $f(x) = y$.

Hence, $f(x) = 9x - 2$ is not onto function.

Bijective Function

→ Let $f : X \to Y$ be a function. If function f is both one–one and onto, then function f is known as bijective function.

→ A bijective function f is also known as **bijection**.

→ For example:

- Determine whether the function $f : \mathbb{Z} \to \mathbb{Z}^+$ be a function defined as $f(x) = x^2$ is bijective or not.

  Solution:

  - one–one:

    $f(-1) = (-1)^2 = 1$ and

    $f(1) = (1)^2 = 1$

    Here, $f(-1) = f(1)$

    but $\quad -1 \neq 1$

    Hence, $f(x) = x^2$ is not one–one function.

    Therefore, f is not bijective function.

- Determine whether the function $f : \mathbb{N} \to \mathbb{N}$ be a function defined as $f(x) = 2x$ is bijective or not.

  Solution:

  - one–one:

    Let $a, b \in \mathbb{N}$ with $f(a) = f(b)$

    We have,

    $f(\mathbf{x_1}) = f(\mathbf{x_2})$

    $\Rightarrow \quad 2\mathbf{x_1} = 2\mathbf{x_2} \qquad \{\because f(x) = 2x\}$

    $\Rightarrow \quad x_1 = x_2$

    Here, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

    Hence, $f(x) = 2x$ is one–one function.

- <u>onto</u>:

  Let $f(\mathbf{x}) = \mathbf{y}$ such that $b \in \mathbb{N}$ (codomain).

  Consider,

  $$f(\mathbf{x}) = \mathbf{y}$$

  $$\Rightarrow \quad 2\mathbf{x} = \mathbf{y}\ \{\because f(x) = 2x\}$$

  $$\Rightarrow \quad x = \frac{y}{2}$$

  Take $y = 1 \Rightarrow x = \frac{1}{2} \notin \mathbb{N}$ (domian)

  Therefore, for $y = 1 \in \mathbb{N}$ there does not exist $x \in \mathbb{N}$ such that $f(x) = y$.

  Hence, $f(x) = 2x$ is not onto function from $\mathbb{N}$ to $\mathbb{N}$.

  Here, function f is one–one but not onto.

  Therefore, f is not bijective function.

<u>Examples of Method-9: One-One, Onto & Bijective Function</u>

C	1	Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one–one. **Hint: Check images of distinct element.**
H	2	Let $f : \mathbb{R} \to \mathbb{R}$ defiend as $f(x) = 2x$ for all $x \in \mathbb{R}$. State whether the function f is one–one or not. **Answer: function f is one – one.**
T	3	Let $f : \mathbb{R} \to \mathbb{R}$ defiend as $f(x) = \frac{x+2}{4}$ for all $x \in \mathbb{R}$. State whether the function f is one–one or not. **Answer: function f is one – one.**

C	4	Determine whether the function is one–one, onto or bijective. Justify your answer. **(1)** $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3 - 4x$ **(2)** $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 1 + x^2$ **Answer: (1) f is bijective, (2) g is neither one – one nor onto**
C	5	Prove that the function $f: \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ defined by $f(x) = \frac{1}{x}$ is one–one and onto. **Hint: Apply definition of one – one and onto.**
H	6	Let $f: \mathbb{R} - \{4\} \to \mathbb{R} - \{0\}$ defiend as $f(x) = \frac{2}{x-4}$ for all $x \in \mathbb{R} - \{4\}$. State whether the function f is onto or not. **Answer: function f is onto.**
T	7	Let $f: \mathbb{R} \to \mathbb{R}$ defiend as $f(x) = 2x - 5$ for all $x \in \mathbb{R}$. State whether the function f is one–one or onto or both. **Answer: function f is one – one and onto both.**
T	8	Let $f: \mathbb{R} \to \mathbb{R}$ defiend as $f(x) = x^3$ for all $x \in \mathbb{R}$. State whether the function f is bijection or not. **Answer: function f is bijection.**
T	9	Verify whether the function $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^3 - 3$ is bijection or not. **Answer: g is bijection.**

Method – 10 ⇝ Composition of Functions

Introduction

→ Let us understand composition of functions with an example.

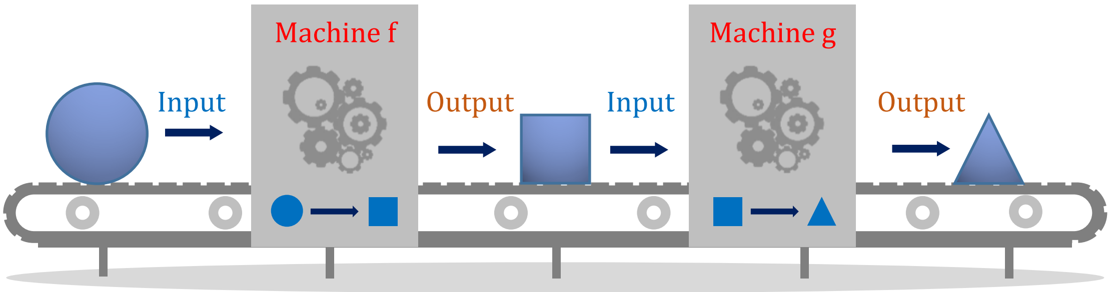
→ We want to convert circular shape into triangular shape.

→ We have two machines f and g.

→ Machine f convert shapes to a square and machine g convert shapes to a triangle.

→ Note that, f only takes circle as an input and g only takes square as an input.

→ So, the conversion process will be as follow:



→ Here, we put two machines together to get desired output. Where we used output of machine f as an input of machine g.

Composition of Two Functions

→ Let $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ and $\mathbf{g} : \mathbf{C} \to \mathbf{D}$ be two functions.

→ Function $\mathbf{g} \circ \mathbf{f} : \mathbf{A} \to \mathbf{D}$ is defined if $\mathbf{f}(\mathbf{A}) \subseteq \mathbf{C}$ and defined as

$(g \circ f)(x) = \mathbf{g}(\mathbf{f}(x)),\ \forall\, x \in A$

is known as composition of function g with f.

→ A function $\mathbf{g} \circ \mathbf{f}$ is read as “g of f”.

→ Function $\mathbf{f} \circ \mathbf{g} : \mathbf{C} \to \mathbf{B}$ is defined if $\mathbf{g}(\mathbf{C}) \subseteq \mathbf{A}$ and defined as

$(f \circ g)(x) = \mathbf{f}(\mathbf{g}(x)),\ \forall\, x \in C$

is known as composition of function f with g.

→ A function $\mathbf{f} \circ \mathbf{g}$ is read as “f of g”.

→ For example:

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 2$ and

  $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x - 3$ then

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(x+2) && \{\because f(x) = x+2\} \\
&= 2(x+2)-3 && \{\because g(x) = 2x-3\} \\
&= 2x+4-3 \\
&= 2x+1
\end{aligned}
$$

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(2x-3) && \{\because g(x) = 2x-3\} \\
&= (2x-3)+2 && \{\because f(x) = x+2\} \\
&= 2x-1
\end{aligned}
$$

Properties of Composition of Functions

→ Let $f : A \to B$, then $f \circ I_A = f = I_B \circ f$.

→ In general, $f \circ g \neq g \circ f$.

→ If f, g and h are three functions with suitably chosen domain and codomain, then

(1) If f and g are one – one function, then $f \circ g$ and $g \circ f$ are one – one function.

(2) If f and g are onto function, then $f \circ g$ and $g \circ f$ are onto function.

(3) If f and g are bijective function, then $f \circ g$ and $g \circ f$ are bijective function.

(4) $f^n(x) = \underbrace{(f \circ f \circ \ldots \circ f)}_{n\text{ times}}(x)$

(5) $(f \circ h \circ g) = (f \circ h) \circ g = f \circ (h \circ g)$

Examples of Method-10: Composition of Function

C	1	Find $g \circ f$ if $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3, \quad f(3) = 4, \quad f(4) = 5, \quad f(5) = 5,$ $g(3) = 7, \quad g(4) = 7, \quad g(5) = 11, \quad g(9) = 11.$ **Answer:** $g \circ f = \{(2, 7), (3, 7), (4, 11), (5, 11)\}$

H	2	Let $X = \{1, 2, 3\}$. Let $f, g, h, s : X \to X$ as given below. $f = \{(1, 2), (2, 3), (3, 1)\}$, $g = \{(1, 2), (2, 1), (3, 3)\}$, $h = \{(1, 1), (2, 2), (3, 1)\}$, $s = \{(1, 1), (2, 2), (3, 3)\}$. Then find $(f \circ g)$, $(f \circ h \circ g)$, $(s \circ s)$ and $(f \circ s)$. **Answer:** $\mathbf{f \circ g = \{(1, 3), (2, 2), (3, 1)\}}$, $\mathbf{s \circ s = \{(1, 1), (2, 2), (3, 3)\}}$, $\mathbf{f \circ s = \{(1, 2), (2, 3), (3, 1)\}}$, $\mathbf{f \circ h \circ g = \{(1, 3), (2, 2), (3, 2)\}}$
T	3	Let $X = \{1, 2, 3\}$. Let $f, g, h, s : X \to X$ as given below. $f = \{(1, 2), (2, 3), (3, 1)\}$, $g = \{(1, 2), (2, 1), (3, 3)\}$, $h = \{(1, 1), (2, 2), (3, 1)\}$, $s = \{(1, 1), (2, 2), (3, 3)\}$. Find $(g \circ f)$, $(g \circ s)$, $(s \circ g)$ & $(f \circ g \circ h)$. **Answer:** $\mathbf{g \circ f = \{(1, 1), (2, 3), (3, 2)\}}$, $\mathbf{g \circ s = \{(1, 2), (2, 1), (3, 3)\}}$, $\mathbf{s \circ g = \{(1, 2), (2, 1), (3, 3)\}}$, $\mathbf{f \circ g \circ h = \{(1, 3), (2, 2), (3, 3)\}}$
C	4	Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x^2$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$, $g(x) = \sqrt{x}$. If possible, find $(f \circ g)(x)$ and $(g \circ f)(x)$. **Answer:** $\mathbf{(f \circ g)(x) = -x}$ & $\mathbf{g \circ f}$ **is not possible.**
H	5	Let $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$, where $x \in \mathbb{R}$. Find $(g \circ f)(x)$, $(g \circ g)(x)$, $(f \circ h)(x)$ and $(h \circ f)(x)$. **Answer:** $\mathbf{(g \circ f)(x) = x}$, $\mathbf{(g \circ g)(x) = x - 4}$, $\mathbf{(f \circ h)(x) = 3x + 2}$, $\mathbf{(h \circ f)(x) = 3x + 6}$
H	6	Let functions f and g be defined by $f(x) = 3x + 2$ and $g(x) = x^2 + 1$. Find $(g \circ f)(5)$ and $(f \circ g)(3)$. **Answer:** $\mathbf{(g \circ f)(5) = 290}$, $\mathbf{(f \circ g)(3) = 32}$

T	7	If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given by $f(x) = \cos x$ & $g(x) = 3x^2$, then show that $g \circ f \neq f \circ g$.
T	8	Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ & $g(x) = 3x + 2$. **(1)** What is the composition of f and g? **(2)** What is the composition of g and f? **Answer:** $(g \circ f)(x) = 6x + 11$, $(f \circ g)(x) = 6x + 7$
T	9	Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 4$. Find $f \circ g$ and $g \circ f$. **Answer:** $(f \circ g)(x) = x^2 + 8x + 14$, $(g \circ f)(x) = x^2 + 2$
T	10	Let f and g be real valued functions defined as follow: $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find $(g \circ f)(4)$ and $(f \circ g)(4)$. **Answer:** $(g \circ f)(4) = 79$, $(f \circ g)(4) = 29$

Method – 11 ⇝ Inverse Function

Invertible Function

→ A function $f : X \to Y$ is invertible function if and only if it is bijection.

Inverse Function

→ If a function $f : X \to Y$ is invertible, then there exists a function $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$.

→ Here, function g is known as inverse function of function f.

→ Inverse function of function f is denoted by $\mathbf{f^{-1}}$ and read as "f inverse".

Properties of Inverse Functions

→ Let f and g be one–one and onto. Also, consider that composition of these functions is well defined.

(1) $(f^{-1})^{-1} = f$

(2) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

(3) $f^{-1} \circ f = f \circ f^{-1} = I$

i.e., $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$

(4) In usual notation, $(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$.

→ For example:

- Let $f : \{1, 2, 3\} \to \{1, 4, 9\}$ defined as follow:

  $f(1) = 1,\ f(2) = 4,\ f(3) = 9$ **or** $f = \{(1, 1), (2, 4), (3, 9)\}$

  Solution:

  Since, each element of domain has distinct image. So, function f is one–one.

  Also, all the elements of codomain have distinct preimage. So, function f is onto.

  Therefore, f^{-1} exist.

  $f^{-1}(1) = 1,\ f^{-1}(4) = 2,\ f^{-1}(9) = 3$ **or** $f^{-1} = \{(1, 1), (4, 2), (9, 3)\}$

- Find inverse of a function $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2x - 5$ if possible.

  Solution:

  To find inverse first we have to check that function f is one–one and onto.

  - <u>one–one</u>:

    Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$

    We have,

    $$f(\mathbf{x_1}) = f(\mathbf{x_2})$$

    $\Rightarrow\ 2\mathbf{x_1} - 5 = 2\mathbf{x_2} - 5 \qquad \{\because f(x) = 2x - 5\}$

    $\Rightarrow\ 2x_1 = 2x_2 - 5 + 5$

    $\Rightarrow\ 2x_1 = 2x_2$

    $\Rightarrow\ x_1 = x_2$

    Here, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

    Hence, $f(x) = 2x - 5$ is one–one function.

  - <u>onto</u>:

    Let $f(\mathbf{x}) = \mathbf{y}$ such that $y \in \mathbb{R}$ (codomain).

    Consider,

    $$f(\mathbf{x}) = \mathbf{y}$$

    $\Rightarrow\ 2\mathbf{x} - 5 = \mathbf{y} \qquad \{\because f(x) = 2x - 5\}$

    $\Rightarrow\ x = \dfrac{y + 5}{2}$

    Since, $y \in \mathbb{R}$

Then $x = \dfrac{y+5}{2} \in \mathbb{R}$ (domian).

Therefore, for every $y \in \mathbb{R}$ there is $x \in \mathbb{R}$ such that $f(x) = y$.

Hence, $f(x) = 2x - 5$ is onto function.

Here, function f is one–one and onto both.

Therefore, f is bijection.

So, f^{-1} exists.

$f^{-1}(x) = \dfrac{x+5}{2}$.

Examples of Method-11: Inverse Function

C	1	Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \dfrac{x+2}{4}$ for all $x \in \mathbb{R}$. State whether the function f is bijective or not. If yes, then find f^{-1}. **Answer: $f^{-1}(x) = 4x - 2$**
H	2	Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \dfrac{x+5}{7}$ for all $x \in \mathbb{R}$. State whether the function f is bijective or not. If yes, then find f^{-1}. **Answer: function f is bijective, $f^{-1}(x) = 7x - 5$.**
T	3	Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$ and let $f : X \to Y$ defined as follow: $f(a) = 2$, $f(b) = 1$, $f(c) = 3$. Find f^{-1} if exist. **Answer: $f^{-1}(2) = a$, $f^{-1}(1) = b$, $f^{-1}(3) = c$.**
C	4	Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3 - 4x$ be a bijective function. Show that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. **Hint: $f^{-1}(x) = \dfrac{3-x}{4}$**

T	5	Let $X = \{1, 2, 3\}$ then find inverse of following functions. **(1)** $f : X \to X,\ f = \{(1, 2), (2, 3), (3, 1)\}$ **(2)** $g : X \to X,\ g = \{(1, 1), (2, 2), (3, 3)\}$ **(3)** $h : X \to X,\ h = \{(1, 3), (2, 2), (3, 1)\}$ Also, verify that $(h \circ f)^{-1} = f^{-1} \circ h^{-1}$. **Answer:** $\mathbf{f^{-1}} = \{(\mathbf{2}, \mathbf{1}), (\mathbf{3}, \mathbf{2}), (\mathbf{1}, \mathbf{3})\}$, $\mathbf{g^{-1}} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3})\}$, $\mathbf{h^{-1}} = \{(\mathbf{3}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{1}, \mathbf{3})\}$, $(\mathbf{h} \circ \mathbf{f})^{-1} = \{(\mathbf{2}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{3}, \mathbf{3})\}$

* * * * * **End of the Unit** * * * * *