Problem Set 1

Name: SID:

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1. Wason's Experiment

Suppose we have four cards on a table:

- 1st about Alice, 2nd about Bob, 3rd about Charlie, and 4th about Donna.
- For each person, one side of the card indicates their dessert, the other what they did after dinner.
- Theory: "If a person has ice cream for dessert, he/she has to do the dishes after dinner."
- Cards: Alice: fruit, Bob: watched TV, Charlie: ice cream, Donna: did dishes

Whose cards do you have to flip to test the theory?

Solution

2. Prove or Disprove

Prove or disprove the following:

a. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

Solution

b. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

Solution

c. $A \implies (B \wedge C) \equiv (A \implies B) \wedge (A \implies C)$

Solution

d. $A \implies (B \vee C) \equiv (A \implies B) \vee (A \implies C)$

Solution

3. Equivalences

Determine whether the following equivalences hold, and give brief justifications for your answers. Clearly state whether or not each pair is equivalent.

a. $\forall x\, \exists y\, (P(x) \Rightarrow Q(x,y)) \equiv \forall x\, (P(x) \Rightarrow (\exists y\, Q(x,y)))$

Solution

b. $\neg \exists x\, \forall y\, (P(x) \Rightarrow \neg Q(x,y)) \equiv \forall x\, \exists y\, (P(x) \wedge Q(x,y))$

Solution

c. $\forall x\, \exists y\, (Q(x,y) \Rightarrow P(x)) \equiv \forall x\, ((\exists y\, Q(x,y)) \Rightarrow P(x))$

Solution

4. Propositions

Decide whether each of the following propositions is true, when the domain for x and y is the real numbers $\mathbb{R}$. Prove your answers.

a. $\forall x \, \exists y \, (xy > 0 \Rightarrow y > 0)$

Solution

b. $\neg \forall x \, \exists y \, (xy \geq x^2)$

Solution

c. $\exists y \, \forall x \, (xy \geq x^2)$

Solution

5. Magical World

Here are statements about a magical world:

I Duck Dynasty viewers don't read the candidates' positions.

II No one, who votes, ever fails to do their homework (on the issues).

III No one is well-informed, if he or she is confused.

IV Everyone who has done their homework (on the issues) is well-informed.

V A person is always confused if he or she doesn't read the candidates positions.

VI No one wears a party hat, unless he or she votes.

a. Write each of the above six sentences as a quantified proposition over the universe of all people. You should use the following symbols for the various elementary propositions: $V(x)$ for "x votes", $H(x)$ for "x has done their homework", $W(x)$ for "x is well-informed", $C(x)$ for "x is confused", $D(x)$ for "x is a Duck Dynasty viewer", $I(x)$ for "x doesn't read the candidates' positions", and $P(x)$ for "x wears a party hat".

Solution

b. Now rewrite each proposition equivalently using the contrapositive.

Solution

c. You now have twelve propositions in total. What can you conclude from them about a person who wears a party hat? Explain clearly the implications you used to arrive at your conclusion.

Solution

6. Karnaugh Maps

Below is the truth table where F is encoded as 0 and T is encoded as 1 for the boolean function

$$Y \equiv (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge C) \vee (A \wedge B \wedge C)$$

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

In this question, we will explore a different way of representing a truth table, the *Karnaugh map.* A Karnaugh map is just a grid-like representation of a truth table, but as we will see, the mode of presentation can give more insight. The values inside the squares are copied from the output column of the truth table, so there is one square in the map for every row in the truth table.

Around the edge of the Karnaugh map are the values of the input variables, where again F is encoded as 0 and T is encoded is 1. Note that the sequence of numbers across the top of the map is not in binary sequence, which would be $00, 01, 10, 11$. It is instead $00, 01, 11, 10$, which is called *Gray code* sequence. Gray code sequence only changes one binary bit as we go from one number to the next in the sequence. That means that adjacent cells will only vary by one bit, or Boolean variable. In other words, *cells sharing common Boolean variable values are adjacent.*

For example, here is the Karnaugh map for Y:

A \ BC	00	01	11	10
0	0	1	0	1
1	0	1	1	0

The Karnaugh map provides a simple and straight-forward method of minimizing boolean expressions by visual inspection. The technique is to examine the Karnaugh map for any groups of adjacent ones that occur, which can be combined to simplify the expression. Note that "adjacent" here means in the modular sense, so adjacency wraps around the top/bottom and left/right of the Karnaugh map; for example, the top-most cell of a column is adjacent to the bottom-most cell of the column.

For example, the ones in the second column in the Karnaugh map above can be combined because $(\neg A \wedge \neg B \wedge C) \vee (A \wedge C) \vee (\neg A \wedge B \wedge \neg C)$ simplifies to $(\neg B \wedge C)$. Applying this technique to the Karnaugh map (illustrated below), we obtain the following simplified expression for Y:

$$Y = (\neg B \wedge C) \vee (A \wedge C) \vee (\neg A \wedge B \wedge \neg C)$$

A \ BC	00	01	11	10
0	0	1	0	1
1	0	1	1	0

a. Write the truth table for the boolean function

$$Z = (\neg A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (\neg A \wedge \neg B \wedge C \wedge \neg D) \vee (A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (A \wedge \neg B \wedge C \wedge \neg D)$$

Solution

b. Using your truth table from Part 1, fill in the Karnaugh map for Z below

AB \ CD	00	01	11	10
00				
01				
11				
10				

Solution

AB \ CD	00	01	11	10
00				
01				
11				
10				

c. Using your Karnaugh map from Part 2, write down a simplified expression for Z.

Solution

d. Show that this simplification could also be found algebraically by factoring the expression for Z in (1). (Recall the distributive property of multiplication over addition; think about the distributive property of $\wedge$ over $\vee$.)

Solution

7. Proof by?

a. Prove that if $x, y, a \in \mathbb{Z}$, if a does not divide xy, then a does not divide x and a does not divide y. In notation: $(\forall x, y \in \mathbb{Z}) a \nmid xy \implies (a \nmid x \wedge a \nmid y)$. What proof technique did you use?

Solution

b. Prove or disprove the contrapositive.

Solution

c. Prove or disprove the converse.

Solution

8. Prove or Disprove

a. For all natural numbers n, if n is odd then $n^2 + 3n$ is even.

Solution

b. For all natural numbers $n, n^2 + 7n$ is even.

Solution

c. For all real numbers a, b, if $a + b \geq 10$ then $a \geq 7$ or $b \geq 3$.

Solution

d. For all real numbers r, if r is irrational then $r + 1$ is irrational.

Solution

e. For all natural numbers $n, 10n^2 > n!$.

Solution

f. For all natural numbers a where a^5 is odd, then a is odd.

Solution