Assignment 5

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CS21BTECH11003

Question: For a Poisson random variable X with parameter λ show that

$$(a)\Pr(0 < X < 2\lambda) > \frac{(\lambda-1)}{\lambda} \quad (1)$$
$$(b)E[X(X-1)] = \lambda^2 \quad (2)$$
$$(c)E[X(X-1)(X-2)] = \lambda^3 \quad (3)$$

Solution: Given X is a Poisson random variable so,

$$\Pr(X = x) = \frac{e^{-\lambda}.\lambda^x}{x!} \quad (4)$$
$$E(X) = \lambda \quad (5)$$
$$Var(X) = \sigma_X^2 = \lambda \quad (6)$$

(a) From Chebyshev's inequality we know that, for any $\epsilon > 0$

$$\Pr(|X - \eta| \geq \epsilon) \leq \frac{\sigma_X^2}{\epsilon^2} \quad (7)$$

Also by definition of probability,

$$\Pr(|X - \eta| \geq \epsilon) + \Pr(|X - \eta| < \epsilon) = 1 \quad (8)$$

Substituting 8 in 7 the inequality becomes,

$$\Pr(|X - \eta| < \epsilon) > 1 - \frac{\sigma_X^2}{\epsilon^2} \quad (9)$$

Since this is Poisson distribution,

$$\eta = \lambda \quad (10)$$
$$\sigma_X^2 = \lambda \quad (11)$$
$$\epsilon = \lambda \quad (12)$$

Substituting values in the equation.

$$\Pr(|X - \lambda| < \lambda) > 1 - \frac{\lambda}{\lambda^2} \quad (13)$$
$$\Rightarrow \Pr(|X - \lambda| < \lambda) > 1 - \frac{1}{\lambda} \quad (14)$$
$$\Rightarrow \Pr(0 < X < 2\lambda) > 1 - \frac{1}{\lambda} \quad (15)$$

Hence proved.

(b) To show,

$$E[X(X-1)] = \lambda^2 \quad (16)$$

Lets use the definitions,

$$E[X] = \sum_{x=0}^{x=\infty} x\frac{e^{-\lambda}.\lambda^x}{x!} \quad (17)$$

Also one the property of mean is that,

$$E[g(x)] = \sum_{x=0}^{x=\infty} g(x)\frac{e^{-\lambda}.\lambda^x}{x!} \quad (18)$$

Here $g(x) = x(x-1)$ substituting it,

$$E[X(X-1)] = \sum_{x=2}^{x=\infty} x(x-1)\frac{e^{-\lambda}.\lambda^x}{x!} \quad (19)$$
$$\Rightarrow E[X(X-1)] = e^{-\lambda}\lambda^2 \sum_{x=2}^{x=\infty} \frac{.\lambda^{x-2}}{(x-2)!} \quad (20)$$

We know that by Taylors expansion,

$$e^x = \sum_{k=0}^{k=\infty} \frac{x^k}{k!} \quad (21)$$

Which is same as

$$e^\lambda = \sum_{x=2}^{x=\infty} \frac{.\lambda^{x-2}}{(x-2)!} \quad (22)$$

Substituting it,

$$E[X(X-1)] = e^{-\lambda}\lambda^2 e^{-\lambda} \quad (23)$$
$$\Rightarrow E[X(X-1)] = \lambda^2 \quad (24)$$

Hence proved

(c) using the above results we get

$$E[X(X-1)(X-2)] = \sum_{x=3}^{x=\infty} x(x-1)(x-2)\frac{e^{-\lambda}.\lambda^x}{x!} \quad (25)$$

$$E[X(X-1)(X-2)] = e^{-\lambda}\lambda^3 \sum_{x=2}^{x=\infty} \frac{.\lambda^{x-3}}{(x-3)!} \quad (26)$$

$$E[X(X-1)(X-)2] = e^{-\lambda}\lambda^3 e^{-\lambda} \quad (27)$$
$$\Rightarrow E[X(X-1)(X-2)] = \lambda^3 \quad (28)$$