RANDOM-NUMBERS

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CS21BTECH11003

CONTENTS

***Abstract*—This manual provides solutions to the Assignment of Random Numbers.**

I. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

I.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/yagnesh1708/Random
    –Numbers/blob/main/codes/1.1.c
wget https://github.com/yagnesh1708/Random
    –Numbers/blob/main/codes/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc 1.1.c
$ ./a.out
```

I.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1}$$

Solution: The following code plots Fig.I.2

```
wget https://github.com/yagnesh1708/Random
    –Numbers/blob/main/codes/1.2.py
```

Download the above files and execute the following commands to produce Fig.I.2

```
$ python3 1.2.py
```

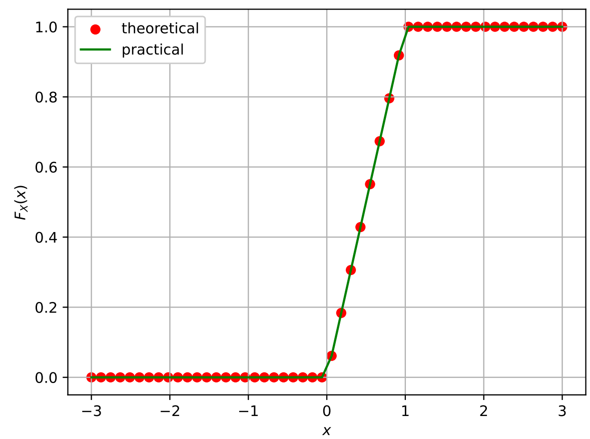

Fig. I.2. The CDF of U

I.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniform Random Variable.

$$p_U(x) = \frac{1}{10^6} \qquad 0 < x < 1 \tag{2}$$

$$p_U(x) = 0 \quad x \geq 1 \;\; and \;\; x \leq 0 \tag{3}$$

$$F_U(x) = \sum_{-\infty}^{x} p_U(x) \tag{4}$$

$$\boxed{\implies F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}} \tag{5}$$

I.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{6}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{7}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program.

```
wget https://github.com/yagnesh1708/Random
    –Numbers/blob/main/codes/1.4.c
wget https://github.com/yagnesh1708/Random
    –Numbers/blob/main/codes/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc 1.4.c
$ ./a.out
```

I.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{8}$$

Solution:

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2 \tag{9}$$

$$\implies \text{var}\left[U\right] = E\left[U^2\right] - E\left[U\right]^2 \tag{10}$$

$$E\left[U\right] = \int_{-\infty}^{\infty} x dF_U(x) \tag{11}$$

$$E\left[U\right] = \int_{0}^{1} x dx \tag{12}$$

$$\implies E\left[U\right] = \frac{1}{2} \tag{13}$$

Now,

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{14}$$

$$E\left[U^2\right] = \int_{0}^{1} x^2 dF_U(x) \tag{15}$$

$$E\left[U^2\right] = \int_{0}^{1} x^2 dx \tag{16}$$

$$\implies E\left[U^2\right] = \frac{1}{3} \tag{17}$$

Variance is,

$$\implies \text{var}\left[U\right] = \frac{1}{12} = 0.0833 \tag{18}$$

II. Central Limit Theorem

II.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{19}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
wget https://github.com/yagnesh1708/Random
    –Numbers/blob/main/codes/2.1.c
wget https://github.com/yagnesh1708/Random
    –Numbers/blob/main/codes/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc 2.1.c
$ ./a.out
```

II.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. II.2 using the code below

```
https://github.com/yagnesh1708/Random–
    Numbers/blob/main/codes/2.2.py
```

Download the above files and execute the following commands to produce Fig.II.2

```
$ python3 2.2.py
```

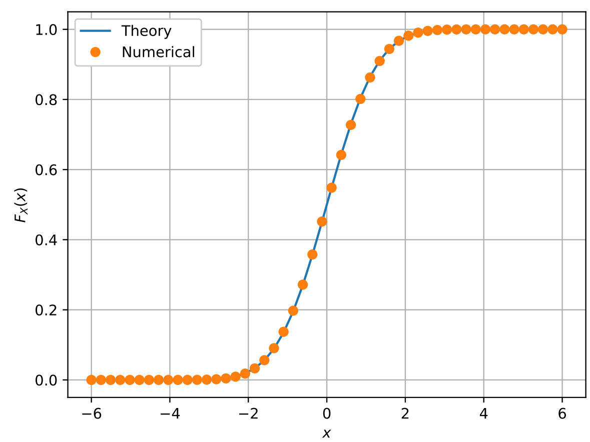

Fig. II.2. The CDF of X

Some of the properties of CDF

a) $\lim_{x\to\infty} F_X(x) = 1$
b) Symmetric about one point.

II.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{20}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. II.3 using the code below

```
https://github.com/yagnesh1708/Random-
    Numbers/blob/main/codes/2.3.py
```

Download the above files and execute the following commands to produce Fig.II.3

```
$ python3 2.3.py
```

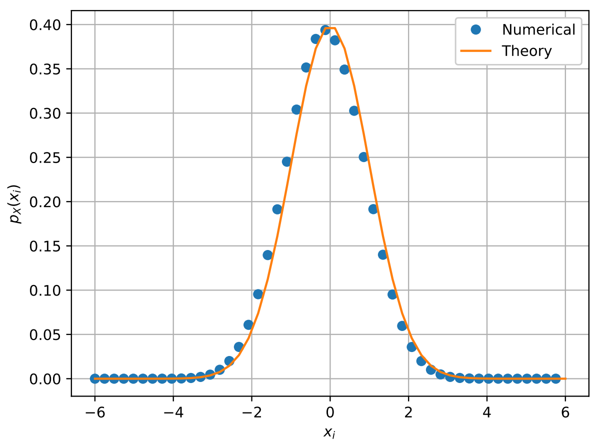

Fig. II.3. The PDF of X

Some of the properties of the PDF:

a) Symmetric about $x = \mu$ in this case
b) Decreasing function for $x > \mu$ and increasing for $x < \mu$ and attains maximum at $x = \mu$
c) Area under the curve is unity.

II.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program.

```
https://github.com/yagnesh1708/Random-
    Numbers/blob/main/codes/2.4.c
https://github.com/yagnesh1708/Random-
    Numbers/blob/main/codes/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc 2.4.c
$ ./a.out
```

II.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{21}$$

repeat the above exercise theoretically.

Solution:

1) CDF

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{22}$$

2) Mean

$$E(x) = \int_{-\infty}^{\infty} x p_X(x) dx \tag{23}$$

$$E(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} dx \tag{24}$$

$$\implies E(x) = 0 \tag{25}$$

3) Variance

$$var[X] = E[X^2] - (E[X])^2 \tag{26}$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \tag{27}$$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-x^2/2} dx \tag{28}$$

Let $x^2 = t$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} t e^{-t/2} \frac{dt}{2\sqrt{t}} \tag{29}$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{t} e^{-t/2} dt \tag{30}$$

If we apply Integration by parts we get

$$var[X] = \frac{1}{\sqrt{2\pi}} \left(\sqrt{t} \left(-2e^{-t/2}\right) \Big|0^{\infty} + \int_{0}^{\infty} \frac{e^{-t/2}}{\sqrt{t}} dt \right) \tag{31}$$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-t/2}}{\sqrt{t}} dt \tag{32}$$

Let $t = p^2$

$$\Rightarrow var[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} 2e^{-p^2/2} dp \tag{33}$$

$$\Rightarrow var[X] = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2} \Rightarrow var[X] \quad = 1 \tag{34}$$

III. From Uniform to Other

III.1 Generate samples of

$$V = -2 \ln (1 - U) \tag{35}$$

and plot its CDF.
Solution: Download the following files and execute the C program.

```
https://github.com/yagnesh1708/Random-Numbers/blob/main/codes/3.1.c
https://github.com/yagnesh1708/Random-Numbers/blob/main/codes/coeffs.h
```

Download the above files and execute the following commands

```
$ gcc 3.1.c -lm
$ ./a.out
```

The CDF of V is plotted in Fig. III.1 using the code below

```
https://github.com/yagnesh1708/Random-Numbers/blob/main/codes/3.1.py
```

Download the above files and execute the following commands to produce Fig.III.1

```
$ python3 3.1.py
```

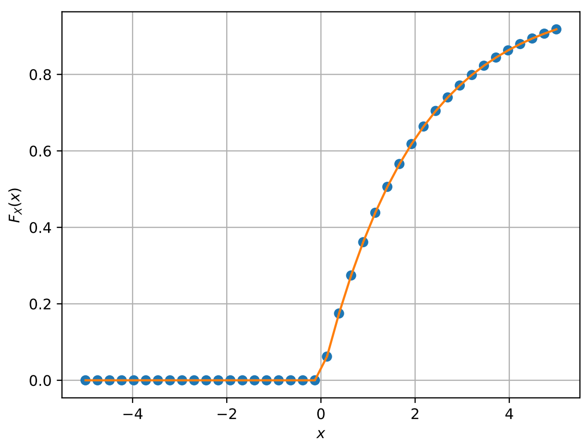

Fig. III.1. The CDF of V

III.2 Find a theoretical expression for $F_V(x)$.
Solution: If Y = g(X), we know that

$$F_Y(y) = F_X(g^{-1}(y)) \tag{36}$$

Now,

$$V = -2\ln(1-U) \tag{37}$$
$$1 - U = e^{\frac{-V}{2}} \tag{38}$$
$$U = 1 - e^{\frac{-V}{2}} \tag{39}$$
$$F_V(x) = F_U(1 - e^{\frac{-x}{2}}) \tag{40}$$

$$\implies F_U(1-e^{-x/2}) = \begin{cases} 0 & 1-e^{-x/2} < 0 \\ 1-e^{-x/2} & 0 \le 1-e^{-x/2} \le 1 \\ 1 & 1-e^{-x/2} > 1 \end{cases} \tag{41}$$

$$\implies F_V(x) = \begin{cases} 0 & x < 0 \\ 1-e^{\frac{-x}{2}} & x \ge 0 \end{cases} \tag{42}$$