An Efficient Semidefinite Relaxation Algorithm for Moving Source Localization Using TDOA and FDOA Measurements

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***Abstract*— In this letter, we address the moving source localization problem by using time-difference-of-arrival and frequency-difference-of-arrival measurements. The localization problem is first reformulated based on the robust least squares criterion and then perform semidefinite relaxation (SDR) to obtain a convex semidefinite programming problem, which can be solved efficiently via optimization toolbox. Unlike several existing SDR localization methods requiring the initial estimate, the proposed method does not require this priori knowledge. The simulation results also show the superior positioning performance of the proposed method at high noise level than other existing methods.**

***Index Terms*— Passive localization, semidefinite relaxation, robust least squares, time-difference-of-arrival (TDOA), frequency-difference-of- arrival (FDOA).**

I. Introduction

SOURCE localization is an important research topic since it has wide applications in the field of sensor, radar, communications and others. For a stationary emitter positioning scenario, time-difference-of-arrival (TDOA) based methods [1]–[3] can estimate the source location with high accuracy. When the source and sensors have relative motion, frequency-difference-of-arrival (FDOA) measurements should be utilized to determine both the source position and velocity accurately.

The positioning problem of using TDOA and FDOA measurements is challenging due to its high nonlinearity and nonconvexity, and the direct implementation of maximum likelihood (ML) by grid searching is very computationally expensive. An alternative is to linearize the nonlinear equations via Taylor series expansion [1]. However, this method requires a very good initial estimate to guarantee its convergence to the global solution or it may lead to wrong result. To tackle this problem, Chan and Ho proposed a well- known two-step weighted least squares (WLS) method [2], [4] transforming the TDOA and FDOA equations to a set of linear equations by introducing two nuisance parameters. Then linear weighted least-squares (LS) method was taken to estimate the source position. This closed form solution does not have convergence problem and can achieve Cramer-Rao lower bound (CRLB) at low noise level. The reference [5] is the extension of this method by considering the sensor location uncertainties. All these closed form solutions perform badly at sufficient high-level noise and suffer from the nonlinear threshold effect. More recently, the tool of convex optimization method for solving localization problem have attracted great interests due to its superior performance [6]–[12]. The method in [6] formulated the semidefinite programming (SDP) based on the ML criterion directly and then refined the SDP solution with local search algorithms. In [7], Wang et al. proposed to reformulate the localization problem based on WLS criterion and then solve it by semidefinite relaxation (SDR). It can avoid the local convergence problem if SDR is tight enough and thus the optimization step is possible to find the solution of the approximate WLS problem. Although these methods show a better performance than two-step WLS, an initial estimate is required to formulate the optimization problem and if the starting point is not accurate enough, the performance degradation occurs. Moreover, the approximation in [7] omitted the second-order noise terms, which may result in the local convergence problem at high-level noise. The work in [11] formulated an optimization problem to minimize the worst-case position estimation by using SDR which achieves the lower maximum position errors (MPEs). The non-line-of-sight (NLOS) condition was considered in [12] and NLOS errors were mitigated by SDR approximation. It is worth noting that only TDOA was utilized in both of [11] and [12].

In this letter, we propose an efficient semidefinite relaxation algorithm to determine the source location without any approximation and initial estimate. Different from the formulation method in [6] and [7], we start from the noise free TDOA and FDOA measurements and reformulate the problem based on robust LS criterion. Then it can be relaxed to a SDP problem and solved efficiently via optimization toolbox. Simulation results show that our method is more robust than other SDP methods and it can attain the CRLB at moderate and high-level noise.

II. Problem Formulation

Consider a network composed of N passive sensors whose locations and velocities are known, denoting by $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N$ and $\dot{\boldsymbol{s}}_1, \ldots, \dot{\boldsymbol{s}}_N$, respectively. A D-dimensional ($D = 2$ or 3) scenario is considered to determine the unknown source location $\boldsymbol{x}$ and its velocity $\dot{\boldsymbol{x}}$. The first sensor is chosen as the reference sensor. Firstly, the line-of-sight (LOS) TDOA measurements are given by

$$\tau_{i-1} = \frac{1}{c}(d_i - d_1) + n_{i-1}, \ i = 2, 3, \ldots, N \qquad (1)$$

where c is a constant of signal propagation speed. n_{i-1} denotes the TDOA measurements noise. The distance between source

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and i-th sensor is $d_k = \|\boldsymbol{x} - \boldsymbol{s}_k\|$ $(k = 1, 2, \ldots, N)$ and $\|\cdot\|$ is Euclid norm. Multiplying c to TDOA measurements in (1), the range difference of arrival (RDOA) is obtained by

$$z_{i-1} = d_i - d_1 + e_{i-1}, \ \ i = 2, 3, \ldots, N \tag{2}$$

where $e_{i-1} = cn_{i-1}$. The time derivation of (2) gives the range rate measurements as

$$\dot{z}_{i-1} = \dot{d}_i - \dot{d}_1 + \dot{e}_{i-1}, \quad i = 2, 3, \ldots, N \tag{3}$$

where

$$\dot{d}_k = \frac{(\dot{\boldsymbol{x}} - \dot{\boldsymbol{s}}_k)^{\mathrm{T}} (\boldsymbol{x} - \boldsymbol{s}_k)}{\|\boldsymbol{x} - \boldsymbol{s}_k\|}, \quad k = 1, 2, \ldots, N \tag{4}$$

where $\dot{e}_{i-1}$ is the FDOA measurement noise. Then we can write the following vector form by collecting all measurements in (2) and (3) as

$$\begin{aligned} \boldsymbol{z} &= \boldsymbol{G}\boldsymbol{d} + \boldsymbol{e} \\ \dot{\boldsymbol{z}} &= \boldsymbol{G}\dot{\boldsymbol{d}} + \dot{\boldsymbol{e}} \end{aligned} \tag{5}$$

where $\boldsymbol{G} = \left[-\boldsymbol{1}_{N-1}, \boldsymbol{I}_{N-1}\right]^{\mathrm{T}}$, $\boldsymbol{z} = \left[z_1, \ldots, z_{N-1}\right]^{\mathrm{T}}$, $\dot{\boldsymbol{z}} = \left[\dot{z}_1, \ldots, \dot{z}_{N-1}\right]^{\mathrm{T}}$, $\boldsymbol{e} = \left[e_1, \ldots, e_{N-1}\right]^{\mathrm{T}}$, $\dot{\boldsymbol{e}} = \left[\dot{e}_1, \ldots, \dot{e}_{N-1}\right]^{\mathrm{T}}$, $\boldsymbol{d} = \left[d_1, \ldots, d_{N-1}\right]^{\mathrm{T}}$ and $\dot{\boldsymbol{d}} = \left[\dot{d}_1, \ldots, \dot{d}_{N-1}\right]^{\mathrm{T}}$. $\boldsymbol{1}_{N-1}$ and $\boldsymbol{I}_{N-1}$ denote the $(N-1) \times 1$ all one column vector and $(N-1) \times (N-1)$ identity matrix. The measurement noise e_{i-1} and $\dot{e}_{i-1}$ are assumed to obey the stationary zero mean white Gaussian distribution with covariance $\boldsymbol{Q}_{\boldsymbol{e}} = \mathbb{E}\left[\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}\right]$ and $\dot{\boldsymbol{Q}}_{\boldsymbol{e}} = \mathbb{E}\left[\dot{\boldsymbol{e}}\dot{\boldsymbol{e}}^{\mathrm{T}}\right]$. The ML estimation of source location can be formulated by

$$\min_{\boldsymbol{x}, \dot{\boldsymbol{x}}} \boldsymbol{f}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \left(\tilde{z} - \tilde{\boldsymbol{G}}\tilde{\boldsymbol{d}}\right)^{\mathrm{T}} \boldsymbol{Q}^{-1} \left(\tilde{z} - \tilde{\boldsymbol{G}}\tilde{\boldsymbol{d}}\right) \tag{6}$$

where $\tilde{z} = \left[\boldsymbol{z}^{\mathrm{T}}, \dot{\boldsymbol{z}}^{\mathrm{T}}\right]^{\mathrm{T}}$, $\tilde{\boldsymbol{d}} = \left[\boldsymbol{d}^{\mathrm{T}}, \dot{\boldsymbol{d}}^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{G} = \mathrm{Diag}\{\boldsymbol{G}, \boldsymbol{G}\}$ and $\boldsymbol{Q} = \mathrm{Diag}\{\boldsymbol{Q}_{\boldsymbol{e}}, \dot{\boldsymbol{Q}}_{\dot{\boldsymbol{e}}}\}$. $\mathrm{Diag}\{\cdot\}$ is the diagonal operator.

A possible solution to solve the localization problem is based on WLS approximation proposed in [7]. However, an initial estimate of $\hat{\boldsymbol{\theta}}$ is required to estimate the distance information matrix between sensors and unknown source. This drawback may lead to local convergence at high-level noise if the initial estimate is not good enough.

A. Moving Source Localization Based on Robust LS

Firstly, we consider the noise free RDOA measurements

$$z_{i-1} = \|\boldsymbol{x} - \boldsymbol{s}_i\| - \|\boldsymbol{x} - \boldsymbol{s}_1\| \tag{7}$$

Squaring both side of (7) yields

$$z_{i-1}^2 = d_i^2 - d_1^2 + 2d_1^2 - 2d_i d_1 \tag{8}$$

and it is equivalent to

$$z_{i-1}^2 = \|\boldsymbol{s}_i\|^2 - \|\boldsymbol{s}_1\|^2 - 2(\boldsymbol{s}_i - \boldsymbol{s}_1)^{\mathrm{T}} \boldsymbol{x} - 2(d_i - d_1) d_1 \tag{9}$$

Taking the time derivation of (9), we have the following equations related FDOAs

$$\begin{aligned} \dot{z}_{i-1} z_{i-1} = \ & \boldsymbol{s}_i^{\mathrm{T}} \dot{\boldsymbol{s}}_i - \boldsymbol{s}_1^{\mathrm{T}} \dot{\boldsymbol{s}}_1 - (\dot{\boldsymbol{s}}_i - \dot{\boldsymbol{s}}_1)^{\mathrm{T}} \boldsymbol{x} - (\boldsymbol{s}_i - \boldsymbol{s}_1)^{T} \dot{\boldsymbol{x}} \\ & - \left(\dot{d}_i - \dot{d}_1\right) d_1 - (d_i - d_1) \dot{d}_1 \end{aligned} \tag{10}$$

Naturally, it is reasonable to solve these equations based on LS criterion

$$\begin{aligned} \min_{\boldsymbol{x}, \dot{\boldsymbol{x}}} \ & \sum_{i=2}^{N} z_{i-1}^2 - \|\boldsymbol{s}_i\|^2 + \|\boldsymbol{s}_1\|^2 + 2(\boldsymbol{s}_i - \boldsymbol{s}_1)^{\mathrm{T}} \boldsymbol{x} \\ & + 2(d_i - d_1) d_1 + \sum_{i=2}^{N} \dot{z}_{i-1} z_{i-1} - \boldsymbol{s}_i^{\mathrm{T}} \dot{\boldsymbol{s}}_i + \boldsymbol{s}_1^{\mathrm{T}} \dot{\boldsymbol{s}}_1 \\ & + (\dot{\boldsymbol{s}}_i - \dot{\boldsymbol{s}}_1)^{\mathrm{T}} \boldsymbol{x} + (\boldsymbol{s}_i - \boldsymbol{s}_1)^{\mathrm{T}} \dot{\boldsymbol{x}} + \left(\dot{d}_i - \dot{d}_1\right) d_1 \\ & + (d_i - d_1) \dot{d}_1 \end{aligned} \tag{11}$$

By replacing the $d_i - d_1$ and $\dot{d}_i - \dot{d}_1$ with $z_{i-1} - e_{i-1}$ and $\dot{z}_{i-1} - \dot{e}_{i-1}$ respectively, we obtain

$$\begin{aligned} \min_{\boldsymbol{x}, \dot{\boldsymbol{x}}} \ & \sum_{i=2}^{N} z_{i-1}^2 - \|\boldsymbol{s}_i\|^2 + \|\boldsymbol{s}_1\|^2 + 2(\boldsymbol{s}_i - \boldsymbol{s}_1)^{\mathrm{T}} \boldsymbol{x} \\ & + 2(z_{i-1} - e_{i-1}) d_1 \\ & + \sum_{i=2}^{N} \dot{z}_{i-1} z_{i-1} - \boldsymbol{s}_i^{\mathrm{T}} \dot{\boldsymbol{s}}_i + \boldsymbol{s}_1^{\mathrm{T}} \dot{\boldsymbol{s}}_1 + (\dot{\boldsymbol{s}}_i - \dot{\boldsymbol{s}}_1)^{\mathrm{T}} \boldsymbol{x} \\ & + (\boldsymbol{s}_i - \boldsymbol{s}_1)^{\mathrm{T}} \dot{\boldsymbol{x}} + (\dot{z}_{i-1} - \dot{e}_{i-1}) d_1 + (z_{i-1} - e_{i-1}) \dot{d}_1 \end{aligned} \tag{12}$$

The optimization problem in (12) cannot be solved because of the existence of unknown noise e_{i-1} and $\dot{e}_{i-1}$. However, we can reconsider (12) and solve an equivalent robust LS problem [9]. To this end, we introduce two variables $\dot{r} = \dot{d}_1 = [(\dot{\boldsymbol{x}} - \dot{\boldsymbol{s}}_1)^{\mathrm{T}} (\boldsymbol{x} - \boldsymbol{s}_1)] / \|(\boldsymbol{x} - \boldsymbol{s}_1)\|$ and $r = d_1 = \|(\boldsymbol{x} - \boldsymbol{s}_1)\|$, then (12) can be rewritten by

$$\begin{aligned} \min_{\boldsymbol{\theta} = [\boldsymbol{x}^{\mathrm{T}}, \dot{\boldsymbol{x}}^{\mathrm{T}}, r, \dot{r}]^{\mathrm{T}}} \ & \|(\boldsymbol{G} + \Delta \boldsymbol{G}) \boldsymbol{\theta} - \boldsymbol{g}\| \\ \text{s.t. } r = \ & \|\boldsymbol{x} - \boldsymbol{s}_1\| \\ \dot{r} = \ & \frac{(\dot{\boldsymbol{x}} - \dot{\boldsymbol{s}}_1)^{\mathrm{T}} (\boldsymbol{x} - \boldsymbol{s}_1)}{\|\boldsymbol{x} - \boldsymbol{s}_1\|} \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Delta \boldsymbol{G} &= \begin{bmatrix} \boldsymbol{O}_{(N-1) \times 2D} & -2\boldsymbol{e} & \boldsymbol{0}_{N-1} \\ \boldsymbol{O}_{(N-1) \times 2D} & \dot{\boldsymbol{e}} & \boldsymbol{e} \end{bmatrix} \\ \boldsymbol{G} &= \begin{bmatrix} 2(\boldsymbol{s}_2 - \boldsymbol{s}_1)^{\mathrm{T}} & \boldsymbol{0}_D^{\mathrm{T}} & 2z_1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 2(\boldsymbol{s}_N - \boldsymbol{s}_1)^{\mathrm{T}} & \boldsymbol{0}_D^{\mathrm{T}} & 2z_{N-1} & 0 \\ (\dot{\boldsymbol{s}}_2 - \dot{\boldsymbol{s}}_1) & (\boldsymbol{s}_2 - \boldsymbol{s}_1) & \dot{z}_1 & z_1 \\ \vdots & \vdots & \vdots & \vdots \\ (\dot{\boldsymbol{s}}_N - \dot{\boldsymbol{s}}_1) & (\boldsymbol{s}_N - \boldsymbol{s}_1) & \dot{z}_{N-1} & z_{N-1} \end{bmatrix} \\ \boldsymbol{g} &= \begin{bmatrix} \|\boldsymbol{s}_2\|^2 - \|\boldsymbol{s}_1\|^2 - z_1^2 \\ \vdots \\ \|\boldsymbol{s}_N\|^2 - \|\boldsymbol{s}_1\|^2 - z_{N-1}^2 \\ \boldsymbol{s}_2^{\mathrm{T}} \dot{\boldsymbol{s}}_2 - \boldsymbol{s}_1^{\mathrm{T}} \dot{\boldsymbol{s}}_1 - z_1^{T} \dot{z}_1 \\ \vdots \\ \boldsymbol{s}_N^{\mathrm{T}} \dot{\boldsymbol{s}}_N - \boldsymbol{s}_1^{\mathrm{T}} \dot{\boldsymbol{s}}_1 - z_{N-1}^{T} \dot{z}_{N-1} \end{bmatrix} \end{aligned} \tag{14}$$

Taking the expected value of objective function in (13), we can solve the stochastic robust LS (SRLS) problem

$$\begin{aligned} \min_{\boldsymbol{\theta} = [\boldsymbol{x}^{\mathrm{T}}, \dot{\boldsymbol{x}}^{\mathrm{T}}, r, \dot{r}]^{\mathrm{T}}} \ & \mathbb{E}_{p(\Delta \boldsymbol{G})} \left[\|(\boldsymbol{G} + \Delta \boldsymbol{G}) \boldsymbol{\theta} - \boldsymbol{g}\| | \boldsymbol{G}, \boldsymbol{g}\right] \\ \text{s.t. } r = \ & \|\boldsymbol{x} - \boldsymbol{s}_1\| \\ \dot{r} = \ & \frac{(\dot{\boldsymbol{x}} - \dot{\boldsymbol{s}}_1)^{\mathrm{T}} (\boldsymbol{x} - \boldsymbol{s}_1)}{\|\boldsymbol{x} - \boldsymbol{s}_1\|} \end{aligned} \tag{15}$$

where $p(\Delta\boldsymbol{G})$ is the pdf of $\Delta\boldsymbol{G}$ and '|' denotes 'under the condition'. The SRLS problem has the form of regularized LS (RLS)

$$\begin{aligned}&\mathbb{E}_{p(\Delta\boldsymbol{G})}\left[\| (\boldsymbol{G}+\Delta\boldsymbol{G})\boldsymbol{\theta}-\boldsymbol{g}\| | \boldsymbol{G},\boldsymbol{g}\right]\\&\quad=\mathbb{E}_{p(\Delta\boldsymbol{G})}\left[(\boldsymbol{G}\boldsymbol{\theta}-\boldsymbol{g}+\Delta\boldsymbol{G}\boldsymbol{\theta})^{\mathrm{T}}(\boldsymbol{G}\boldsymbol{\theta}-\boldsymbol{g}+\Delta\boldsymbol{G}\boldsymbol{\theta})\right]\\&\quad=\|\boldsymbol{G}\boldsymbol{\theta}-\boldsymbol{g}\|^2+\|\boldsymbol{P}^{1/2}\boldsymbol{\theta}\|^2 \qquad (16)\end{aligned}$$

where

$$\boldsymbol{P}=\mathbb{E}[\Delta\boldsymbol{G}^{\mathrm{T}}\Delta\boldsymbol{G}]=\begin{bmatrix}\boldsymbol{O}_{2D\times 2D} & \boldsymbol{0}_{2D} & \boldsymbol{0}_{2D}\\ \boldsymbol{0}_{2D}^{\mathrm{T}} & 4\mathrm{tr}(\boldsymbol{Q}_e)+\mathrm{tr}(\dot{\boldsymbol{Q}}_e) & 0\\ \boldsymbol{0}_{2D}^{\mathrm{T}} & 0 & \mathrm{tr}(\boldsymbol{Q}_e)\end{bmatrix} \qquad (17)$$

The regularized LS problem in (16) is also similar to the Tikhonov regularized problem [9] and it has standard solution

$$\hat{\boldsymbol{\theta}}_{\mathrm{RLS}}=\left(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}+\boldsymbol{P}\right)^{-1}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{g} \qquad (18)$$

The solution of (18) can be seen as a non-constraint solution of (15), but it is suboptimal in our localization problem since the constraints have not been considered.

B. Robust Moving Source Localization With SDR

Note that (15) has the same solution of the following problem

$$\begin{aligned}\min_{\boldsymbol{\theta}=[\boldsymbol{x}^{\mathrm{T}},\dot{\boldsymbol{x}}^{\mathrm{T}},r,\dot{r}]^{\mathrm{T}}}\ & \|\boldsymbol{H}\boldsymbol{\theta}-\boldsymbol{h}\|^2\\ \text{s.t. } & r=\|\boldsymbol{x}-\boldsymbol{s}_1\|\\ & \dot{r}=\frac{(\dot{\boldsymbol{x}}-\dot{\boldsymbol{s}}_1)^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{s}_1)}{\|\boldsymbol{x}-\boldsymbol{s}_1\|}z \qquad (19)\end{aligned}$$

where $\boldsymbol{H}$ is the matrix square root of $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}+\boldsymbol{P}$, i.e. $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}+\boldsymbol{P}=\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}$, and $\boldsymbol{h}=(\boldsymbol{H}^{\mathrm{T}})^{-1}\boldsymbol{G}\boldsymbol{f}$. We advocate rewriting the objective function in (19) as

$$(\boldsymbol{H}\boldsymbol{\theta}-\boldsymbol{h})^{\mathrm{T}}\boldsymbol{Q}'^{-1}(\boldsymbol{H}\boldsymbol{\theta}-\boldsymbol{h}) \qquad (20)$$

where $\boldsymbol{Q}'^{-1}=\boldsymbol{I}_{2D+2}$. The form of (20) is similar with the ML estimation, but its weighting matrix is independent of noise. The noise covariance is optimized in the symmetric matrix $\boldsymbol{H}$; therefore it will be robust to noise effect. Then we relax it with using SDP

$$\begin{aligned}\|\boldsymbol{H}\boldsymbol{\theta}-\boldsymbol{h}\|^2&=(\boldsymbol{H}\boldsymbol{\theta}-\boldsymbol{h})^{\mathrm{T}}\boldsymbol{Q}'^{-1}(\boldsymbol{H}\boldsymbol{\theta}-\boldsymbol{h})\\&=\mathrm{tr}\left\{\begin{bmatrix}\boldsymbol{Y} & \boldsymbol{\theta}\\ \boldsymbol{\theta}^T & 1\end{bmatrix}\boldsymbol{W}\right\} \qquad (21)\end{aligned}$$

where

$$\boldsymbol{Y}=\boldsymbol{\theta}\boldsymbol{\theta}^T,\ \boldsymbol{W}=\begin{bmatrix}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Q}'^{-1}\boldsymbol{H} & -\boldsymbol{H}^{\mathrm{T}}\boldsymbol{Q}'^{-1}\boldsymbol{h}\\ -\boldsymbol{h}^{\mathrm{T}}\boldsymbol{Q}'^{-1}\boldsymbol{H} & \boldsymbol{h}^{\mathrm{T}}\boldsymbol{Q}'^{-1}\boldsymbol{h}\end{bmatrix} \qquad (22)$$

and the constraints in (19) can be written as

$$\begin{aligned}r^2&=(\boldsymbol{x}-\boldsymbol{s}_1)^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{s}_1)\\ r\dot{r}&=(\boldsymbol{x}-\boldsymbol{s}_1)^{\mathrm{T}}(\dot{\boldsymbol{x}}-\dot{\boldsymbol{s}}_1) \qquad (23)\end{aligned}$$

TABLE I
POSITIONS AND VELOCITIES OF SENSORS

Sensor no.	x	y	z	$\dot{x}$	$\dot{y}$	$\dot{z}$
1	300	100	150	30	-20	20
2	400	150	100	-30	10	20
3	300	500	200	10	-20	10
4	350	200	100	10	20	30
5	-100	-100	-100	-20	10	10

That is

$$\begin{aligned}\boldsymbol{Y}(2D+1,2D+1)&=\mathrm{tr}\{\boldsymbol{Y}(1:D,1:D)\}-2\boldsymbol{s}_1^{\mathrm{T}}\boldsymbol{\theta}(1:D)\\&\quad+\|\boldsymbol{s}_1\|^2\\ \boldsymbol{Y}(2D+1,2D+2)&=\mathrm{tr}\{\boldsymbol{Y}(1:D,D+1:2D)\}-\dot{\boldsymbol{s}}_1^{\mathrm{T}}\boldsymbol{\theta}(1:D)\\&\quad-\boldsymbol{s}_1^{\mathrm{T}}\boldsymbol{\theta}(D+1:2D)+\dot{\boldsymbol{s}}_1^{\mathrm{T}}\boldsymbol{s}_1 \qquad (24)\end{aligned}$$

Furthermore, since $\dot{r}=\left[(\dot{\boldsymbol{x}}-\dot{\boldsymbol{s}}_1)^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{s}_1)\right]/\|\boldsymbol{x}-\boldsymbol{s}_1\|$, we can tighten the constraints by utilizing Cauchy-Schwartz inequality

$$\dot{r}^2\le\|(\dot{\boldsymbol{x}}-\dot{\boldsymbol{s}}_1)\|^2 \qquad (25)$$

which is equivalent to

$$\begin{aligned}\boldsymbol{Y}(2D+2,2D+2)&\le\mathrm{tr}\{\boldsymbol{Y}(D+1:2D,D+1,2D)\}\\&\quad-2\dot{\boldsymbol{s}}_1^T\boldsymbol{\theta}(D+1:2D)+\|\dot{\boldsymbol{s}}_1\|^2 \qquad (26)\end{aligned}$$

Then we obtain the optimization problem from (21), (24) and (26), which is given by

$$\begin{aligned}\min_{\boldsymbol{\theta}=[\boldsymbol{x}^{\mathrm{T}},\dot{\boldsymbol{x}}^{\mathrm{T}},r,\dot{r}]^{\mathrm{T}}}\ & \mathrm{tr}\left\{\begin{bmatrix}\boldsymbol{Y} & \boldsymbol{\theta}\\ \boldsymbol{\theta}^T & 1\end{bmatrix}\boldsymbol{W}\right\}\\ \text{s.t. } & (24),\ (26)\\ & \boldsymbol{Y}=\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} \qquad (27)\end{aligned}$$

Note that the constraints $\boldsymbol{Y}=\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}$ is equivalent to $\boldsymbol{Y}\succeq\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}$ and $\mathrm{rank}(\boldsymbol{Y})=1$. By dropping the rank-1 constraint which is nonconvex, we have the following SDP

$$\begin{aligned}\min_{\boldsymbol{\theta}=[\boldsymbol{x}^{\mathrm{T}},\dot{\boldsymbol{x}}^{\mathrm{T}},r,\dot{r}]^{\mathrm{T}}}\ & \mathrm{tr}\left\{\begin{bmatrix}\boldsymbol{Y} & \boldsymbol{\theta}\\ \boldsymbol{\theta}^T & 1\end{bmatrix}\boldsymbol{W}\right\}\\ \text{s.t. } & (24),\ (26)\\ & \left[\boldsymbol{Y}\ \boldsymbol{\theta};\ \boldsymbol{\theta}^{\mathrm{T}}\ 1\right]\succeq 0 \qquad (28)\end{aligned}$$

Solving the SDP in (28), we finally obtain the estimation of $\boldsymbol{\theta}$.

III. SIMULATIONS

In this section, simulations[1] are conducted to evaluate the performance of the proposed method, classic two-step WLS method [4], SDP method in [7] (labeled as SDP-Ini)[2] and RLS of (16). The CRLB [4] is also included for comparison.

The 3-D simulation scenarios follow the settings in [4], [5], and [7]. Five receiving sensors are considered and their positions and velocities are listed in Table I. The performance is evaluated by root mean square errors (RMSEs), defined by RMESs $=\left[(\sum_{i=1}^{N}\|\hat{\boldsymbol{x}}_i-\boldsymbol{x}\|^2)/L\right]^{1/2}$ and $\left[(\sum_{i=1}^{N}\|\hat{\dot{\boldsymbol{x}}}_i-\dot{\boldsymbol{x}}\|^2)/L\right]^{1/2}$, where $\hat{\boldsymbol{x}}$ and $\hat{\dot{\boldsymbol{x}}}$ are the estimate of true source position and velocity, respectively.

[1] All methods are implemented using MATLAB R2010b on a HP desktop equipped with a 2.80 GHz Intel Pentium CPU and 2 GB RAM.

[2] The initialization of SDP-Ini also follows the way proposed in [7].

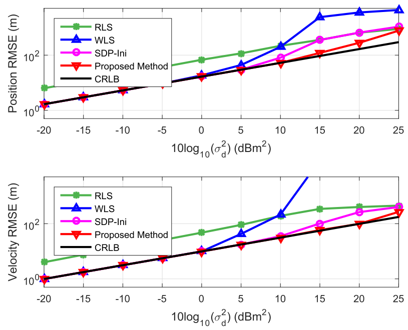

Fig. 1. Comparison of RMSEs using different methods in the near-field scenario.

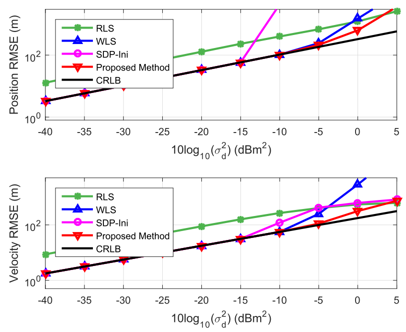

Fig. 2. Comparison of RMSEs using different methods in the far-field scenario.

The number of Monte Carlo runs in our simulation is chosen as $L = 1000$. The covariance of TDOA and FDOA noise are $\boldsymbol{Q_e} = \sigma_d^2 \boldsymbol{R}$ and $\dot{\boldsymbol{Q}}_{\boldsymbol{e}} = 0.1\sigma_d^2 \boldsymbol{R}$ where $\boldsymbol{R}$ is equal to unity in the diagonal matrix and 0.5 otherwise. The SDP based methods are solved using MATLAB toolbox CVX [10] and solver is chosen as SeduMi. We first consider the near-field localization scenario where the distances between sensor-to-sensor and sensor-to-source are comparable. The source is located at $(400, -500, 550)$ with velocity $(-20, 25, 30)$. From Fig. 1, we see clearly that the proposed method performs the best and it can attain CRLB even at high noise levels. Both of the WLS and SDP-Ini methods meet the threshold effect when the noise level is higher than 10dBm^2. RLS method performs the worst since it does not consider the constraints.

The far-field scenario where source is located far away from sensors is drawn in Fig. 2 and the source is assumed to be located at $(2500, -2000, 3000)$ with velocity $(-20, 25, 30)$. From Fig. 2, it can be seen that the proposed method still performs better than other methods. Note that SDP-Ini in far-field shows inferior to WLS; this is because the bad initial estimate leading to local convergence problem. In addition, we note that some inferior methods 'perform' comparable to others or even better at high noise level in both Fig. 1 and Fig. 2. This is mainly because of the nonlinear threshold and the estimation comes into an unpredictable region [7], [13].

Trajectory simulation with constant velocity (CV) model is shown in Fig. 3. The source moves starting from $(500, 150, 380)$ with constant velocity $(20, 15, 40)$ and 11 observation time intervals are intercepted. Noise effect is set as 0dBm^2. Fig. 3 shows that the proposed method almost holds the lowest estimation errors throughout the observation time.

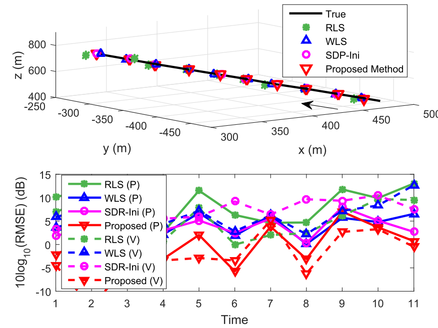

Fig. 3. Trajectory of a moving source and estimation RMSEs (The $10\log_{10}(\cdot)$ operator is utilized for explicit when both position and velocity errors are drawn. Notation 'P' and 'V' denote the position and velocity estimation, respectively).

IV. Conclusion

In this letter, we have presented a robust SDR method for moving source localization using TDOA and FDOA measurements. The proposed method formulates the localization problem based on robust LS criteria and solves the problem via SDR. Compared with the existing methods, it can achieve CRLB even at high-level noises and do not require any initial estimate.

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