TDOA Based Direct Positioning Maximum Likelihood Estimator and the Cramer-Rao Bound

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The maximum likelihood estimator (MLE) and its performance for the localization of a stationary emitter using a network of spatially separated passive stationary sensors is presented. The conventional approach for localization using multiple sensors is to first estimate the time differences of arrival (TDOAs) independently between pairs of sensors and then find the location of the emitter using the intersection point of the hyperbolas defined by these TDOAs. It has recently been shown that this two-step approach is suboptimal and an alternate direct position determination (DPD) approach has been proposed. In the work presented here we take the DPD approach to derive the MLE and show that the MLE outperforms the conventional two-step approach. We analyze the two commonly occurring cases of signal waveform unknown and signal waveform known with unknown transmission time. This paper covers a wide variety of transmitted signals such as narrowband or wideband, lowpass or bandpass, etc. Sampling of the received signals has a quantization-like effect on the location estimate and so a continuous time model is used instead. We derive the Fisher information matrix (FIM) and show that the proposed MLE attains the Cramer-Rao lower bound (CRLB) for high signal-to-noise ratios (SNRs).

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NOMENCLATURE

$(x_T,\ y_T)$ = Unknown location of the emitter
$(x_i,\ y_i)$ = Location of sensor i
2N – 1 = Number of non-zero Fourier coefficients
$\frac{N_0}{2}$ = Noise spectral density
τ_i = Unknown time of arrival of signal at sensor i
$\mathbf{A}$ = M × 1 vector of the unknown attenuation factors
$\mathbf{A}'$ = M – 1 × 1 vector of the unknown relative attenuation factors
$\mathbf{I}_N$ = An N × N identity matrix
A_i = Unknown attenuation factor at sensor i
a_i, b_i = Fourier coefficients of the signal
c = Propogation speed of signal
$d_{\max}$ = Distance between the farthest pair of sensors
F_0 = Fundamental frequency of the Fourier series
F_s = Sampling frequency
M = Number of sensors
R = Distance from the emitter to a sensor
$r_i(t)$ = Signal received at sensor i
$s(t)$ = Transmitted signal waveform
T = Length of observation interval
t_0 = Unknown transmission time
T_s = Non-zero length of the signal waveform
$w_i(t)$ = Additive Gaussian random process at sensor i
ε_{Si} = Signal energy at sensor i
$\mathcal{I}_\theta$ = Fisher information matrix of the unknown parameter vector $\boldsymbol{\theta}$
η = 3 × 1 vector of the unknown emitter location coordinates and the transmission time
ϕ = 2N — 1 × 1 vector of the unknown Fourier coefficients
τ = M × 1 vector of the unknown TOAs
τ' = M – 1 × 1 vector of the unknown TDOAs
$\boldsymbol{h}(t)$ = 2N – 1 × 1 vector as defined in Appendix III
ASNR = Average signal-to-noise ratio
CRLB = Cramer-Rao lower bound
FIM = Fisher information matrix
MLE = Maximum likelihood estimator
SNR = Signal-to-noise ratio
TDOA = Time difference of arrival
TOA = Signal to noise ratio

I. INTRODUCTION

Passive localization has been used for many years and has always been an important topic of research [1–4]. Localization can be performed using one or more of the emitter location dependent properties of the signal such as angle of arrival (AOA), time difference of arrival (TDOA), frequency difference of arrival (FDOA), or the energy of the received signal. Over the years the general approach to localization using the TDOAs, commonly referred to as the TDOA technique, has been to first estimate the difference in the times of arrival of the signal at a particular pair of sensors and then use these TDOAs to estimate the location of the emitter. Knapp and Carter [5, 6] proposed a generalized correlation method for the estimation of the TDOAs for stationary and relative motion cases. They modeled the signal as a stationary Gaussian random process. Stein [7] on the other hand modeled the signal as deterministic but unknown and derived the MLE for the differential delay and Doppler for a two-sensor case. Under similar assumptions for the signal, Yeredor and Angel [8] have derived the Cramer-Rao lower bound (CRLB) for the TDOAs. After the TDOAs are estimated they are used to estimate the location of the source [9–12]. Quite often, due to network capacity and computational constraints, not all sensor pair combinations are used. Fowler [13] addressed the problem of optimal selection of a subset of the sensor pairs. Torrieri [1] proposed a linear least squares estimator where the nonlinear relation between the TDOAs and the emitter location is linearized by expanding it in a Taylor series about a reference point and retaining the first two terms. This is an iterative method which requires some kind of a priori information in order to obtain an initial guess. Alternatively, Chan and Ho [2] use an intermediate variable, which is a function of the emitter location, in order to linearize the nonlinear equations. They use a two-step weighted least squares algorithm. Additionally, when the signal waveform is known, localization may be performed from the times of arrival (TOAs) instead of the TDOAs [14, 15]. In such TOA-based techniques the unknown transmission time occurs as a nuisance parameter which will have to be estimated. Sathyan et al. [16] have shown the theoretical equivalence of the TOA and the TDOA, taken against a common reference sensor, based position fixing techniques by comparing the Cramer-Rao bounds. This result is complemented by Kaune's [17] results which show the equivalence using Monte Carlo simulations. The above techniques may be called two-step techniques because the TOAs/TDOAs are first estimated at the local sensors and these TOA/TDOA estimates are used in a second step to compute the location of the emitter.

Weiss and Amar [18–21] have shown that the two-step approach is suboptimal and proposed a direct position determination (DPD) approach. Weiss had derived the maximum likelihood estimator (MLE) for the source location for the case of a stationary narrowband radio frequency transmitter using multiple stationary receivers in [18]. He uses a continuous time model and quickly considers the sampled version without discussing the effects of sampling on the emitter location estimate. He shows that the MLE of the emitter location is obtained by maximizing a quadratic form of the signal samples whose coefficients are functions of the emitter location. He considers the two cases of signal known and signal unknown but leaves out a more important case - signal known but transmission time unknown which is most likely to occur in real-world situations. He does not discuss the CRLB for this problem. There is an inherent ambiguity in the commonly used model and Weiss uses a constraint on the signal samples to resolve the ambiguity. No discussion is provided on the generality of the constraint as to why it is an appropriate constraint, how it resolves the ambiguity and whether it reduces the performance. In [19] Amar and Weiss extend the approach to a multiple emitters case. In [20] they address the problem of localization using only the Doppler frequency shifts and in [21] they consider the case of a single stationary emitter and moving receivers. In all these cases the results are similar, i.e., the MLE for the emitter location is obtained by maximizing a quadratic form of the signal whose coefficients are functions of the emitter location. The derivation of CRLB is attempted in the later papers but is not sufficiently simplified. The effect of sampling the signal is not discussed in any of the papers. Similar constraints are used to resolve the ambiguity in the papers [18–21], but no discussion is provided on the effects of the constraint.

In this paper we consider the case of a single stationary emitter and a network of stationary receive sensors. We address many of the shortcomings of [18–21]. We use a continuous time model and provide a straightforward derivation for the MLE of the emitter location for the two cases of signal waveform known with unknown transmission time and signal waveform unknown with unknown transmission time. Our model is valid for either narrowband or broadband signals, lowpass or highpass signals. We discuss the effect of sampling the signal on the emitter location estimate. Using simulations, we compare the MLE against a conventional TDOA technique. We show that the variance of the MLE is two to three orders of magnitude lower than the conventional TDOA technique.

A more difficult problem is deriving the CRLB. If the signal waveform is assumed unknown along with the TOA and the attenuation factor, then the commonly used model has an ambiguity. This ambiguity comes to light when deriving the CRLB. Because all the unknowns in the model cannot be uniquely resolved, the Fisher information matrix (FIM) becomes singular. We address this ambiguity in detail and derive the necessary steps to remove it. Then we derive the nonsingular FIM. The inverse of the FIM is the CRLB. CRLB gives the theoretical lower bound on the variance of any unbiased estimator. An important application of the CRLB is in deriving an optimal sensor configuration. The performance of a location estimator depends on the placement of sensors. A particular configuration of the sensors is called optimal if it optimizes a norm of the FIM. A quite common result [22, 23] is to place the sensors around the emitter in an equi-angular configuration. But when the sensors are geographically constrained the problem becomes much more difficult. We will investigate the problem of optimal sensor configuration in a future paper.

In Section II we provide a detailed description of the problem. In Section III we give the CRLBs and the MLEs.

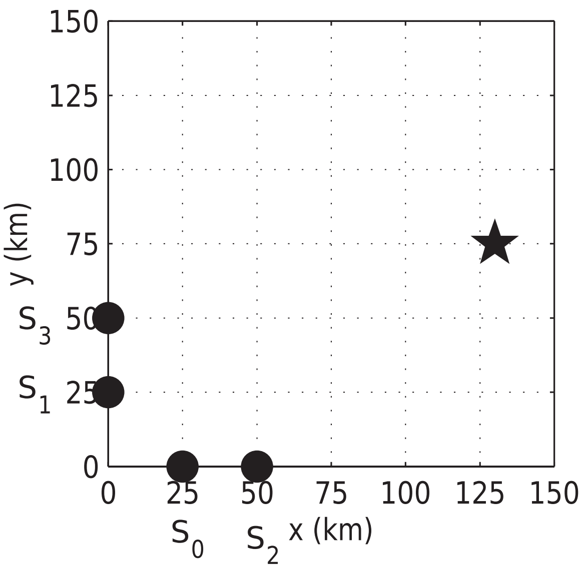

Fig. 1. Physical placement of sensors (for M = 4) and emitter position used for simulation.

Here we analyze the case of signal waveform unknown and the special case of signal waveform known, both cases with an unknown transmission time. In Section IV we use Monte Carlo simulations to compare the performance of the MLE against the conventionally used TDOA technique. We show that at higher signal-to-noise ratios (SNRs) the variance of the MLE approaches the CRLB. Conclusions are provided in Section V. Most of the mathematics are provided in the appendices. In Appendix I we derive a compact expression for the FIM. Appendix II has the derivation of the MLE. Appendix III presents the properties of a matrix we use in the model. In Appendix IV we discuss the transformation of the parameters and the constraints used in order to remove the ambiguity in the model.

II. PROBLEM STATEMENT

Suppose that a stationary emitter is located at an unknown location (x_T, y_T) and a network of M sensors are located at known locations $(x_i, y_i),\ i = 0, 1, \ldots, M-1$ as shown in Fig. 1. For simplicity we are assuming a two-dimensional case. Extension to the three-dimensional case is straightforward. The sensors are all synchronized in time and each of the sensors intercepts the signal within the time interval $(0, T)$. The emitter transmits a signal with unknown waveform $s(t)$ for an unknown duration $T_s < T$ starting at an unknown time $t_0 < T$. We assume that the transmitted signal waveform $s(t)$ is real. It can be narrowband or wideband, lowpass or bandpass. After interception, the signal received at sensor i in the presence of noise can be written as

$$r_i(t) = A_i s(t-\tau_i) + w_i(t),\ 0<t<T,\ i=0, 1, \ldots, M-1 \tag{1}$$

where $w_i(t)$ is a zero mean wide sense stationary additive white Gaussian random process with spectral density $\frac{N_0}{2}$, A_is are the unknown attenuations due to propagation loss, assumed real, and the τ_is are the unknown TOAs given by

$$\tau_i = \frac{\sqrt{(x_T-x_i)^2 + (y_T-y_i)^2}}{c} + t_0,\ \ i = 0, 1, \ldots, M-1 \tag{2}$$

where c is the propagation speed of the signal. In (2) the first term $\sqrt{(x_T - x_i)^2 + (y_T - y_i)^2}/c$ is the propagation delay and the second term t_0 is the unknown time of transmission. We assume that the noise at a sensor is independent of the noise at any other sensor, i.e., $w_i(t)$ and $w_j(t)$ are independent for $i \neq j$ and that the noise spectral density at all the sensors is equal to $\frac{N_0}{2}$. If the noise does not satisfy these conditions then the problem becomes more complex. For example, if the noise spectral density is different at each sensor but known, then the noise term does not factor out as in (6) but instead exists in each term. A more difficult problem is when the noise spectral density is different at each sensor and unknown, in which case, the noise spectral densities at each of the sensors need to be estimated as well. To keep the derivations simple we assumed the above conditions for the noise. Notice that here we do not assume as in [7], that $\tau_i \ll T$. Instead we just assume that $\max_{i,j}(\tau_i - \tau_j) < (T - T_s)$. That is, we are only assuming that the observation interval is large enough so that, within the observation interval, the signal reaches both the nearest and the farthest sensors from the emitter. Based on the sensor geometry it is possible to find a sufficient condition on the length of the observation interval. If $d_{\max}$ is the distance between the farthest pair of sensors, then the observation interval must be greater than $\frac{d_{\max}}{c}$.

Sampling the signal in time has a quantization-like effect on the estimate of the emitter location. This is because if the signal is sampled, then the TOA estimates are integer multiples of the sampling interval and hence quantized. For example, if the signal is sampled at a frequency of F_s samples/s, then the estimate of τ_i is quantized with a maximum quantization error of $\frac{1}{2F_s}$. This can introduce a maximum quantization error of $\frac{c}{2F_s}$ in the range estimate. Therefore, it is possible that the signal may have to be sampled at a rate much higher than the Nyquist rate in order to achieve a desired precision in the location estimate. We have used a continuous time model to avoid this problem in our analysis and to allow future studies of errors due to time synchronization effects.

Fig. 2 shows the signals received at the four sensors shown in Fig. 1 from an emitter located at (130, 75)km transmitting a Gaussian chirp. The propagation loss was modeled as a $\frac{1}{R}$ attenuation in the amplitude of the received signal, where R is the range. So, the farthest sensor has the largest TOA and smallest amplitude. We are assuming that the signal lies inside the observation interval $(0, T)$. So, we can assume that the unknown signal is periodic with period T and write it in terms of its

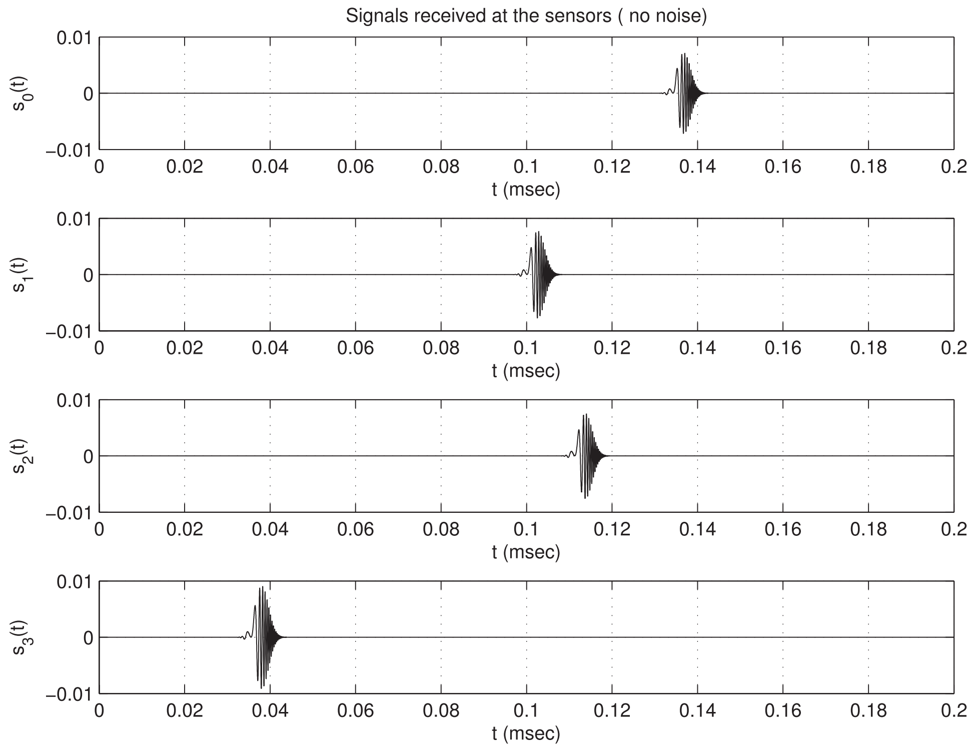

Fig. 2. Signals received at four sensors when Gaussian chirp is transmitted by emitter located at (130, 75) km.

Fourier series as

$$s(t) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty}(a_n \cos 2\pi n F_0 t + b_n \sin 2\pi n F_0 t) \quad (3)$$

where $F_0 = \frac{1}{T}$ and the Fourier coefficients are given by

$$a_0 = \tfrac{\sqrt{2}}{T}\int_0^T s(t)\,dt,\, a_n = \tfrac{2}{T}\int_0^T s(t)\cos 2\pi n F_0 t\,dt,$$
$$b_n = \tfrac{2}{T}\int_0^T s(t)\sin 2\pi n F_0 t\,dt. \quad (4)$$

We are using $\frac{a_0}{\sqrt{2}}$ for the DC component instead of the standard a_0 because it simplifies certain terms in the derivation of the CRLB. For a band-limited signal only a finite number of the Fourier coefficients are non-zero. If the signal is a lowpass signal, there exists an integer N such that the Fourier coefficients are all zero for $n \geq N$ and if the signal is a bandpass signal, then there exist integers N_1 and N_2, $N_1 < N_2$, such that the Fourier coefficients are zero for $n < N_1$ and for $n > N_2$. So we can approximate the lowpass signal $s(t)$ as (for a bandpass signal the summation is from N_1 to N_2)

$$s(t) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{N-1}(a_n \cos 2\pi n F_0 t + b_n \sin 2\pi n F_0 t).$$

This is an important step as it allows us to model any unknown signal and reduce it to a parameter estimation problem. Now, if we let $\boldsymbol{\phi} = [a_0\, a_1\, \cdots\, a_{N-1}\, b_1\, b_2\, \cdots\, b_{N-1}]^T$ be the $2N-1 \times 1$ vector of Fourier coefficients and $\boldsymbol{h}(t) = \big[\frac{1}{\sqrt{2}}\ \cos 2\pi F_0 t\ \cdots\ \cos 2\pi(N-1)F_0 t$ $\sin 2\pi F_0 t\ \cdots\ \sin 2\pi(N-1)F_0 t\big]^T$ then we have $s(t) = \boldsymbol{h}^T(t)\boldsymbol{\phi}$. This reduces the uncountable unknown parameter set $\{s(t) : t \in (0, T)\}$ to a finite countable number of unknown parameters $\boldsymbol{\phi}$. Therefore, we can rewrite the model in (1) as

$$r_i(t) = A_i \boldsymbol{h}^T(t - \tau_i)\boldsymbol{\phi} + w_i(t), 0 \leq t \leq T,$$
$$i = 0, 1, \ldots, M-1. \quad (5)$$

Let $\boldsymbol{\tau} = [\tau_0\, \tau_1\, \cdots\, \tau_{M-1}]^T$, $\mathbf{A} = [A_0\, A_1\, \cdots\, A_{M-1}]^T$, and $\boldsymbol{\phi} = [a_0\, a_1\, \cdots\, a_{N-1}\, b_1\, b_2\, \cdots\, b_{N-1}]^T$. Let $\boldsymbol{\theta} = [\boldsymbol{\tau}^T\, \mathbf{A}^T\, \boldsymbol{\phi}^T]^T$ be the $(2M + 2N - 1) \times 1$ vector of unknown parameters. If we let $\boldsymbol{\eta} = [x_T\ \ y_T\ \ t_0]^T$ and $\boldsymbol{\alpha} = [\boldsymbol{\eta}^T\, \mathbf{A}^T\, \boldsymbol{\phi}^T]^T$ then, using (2), we can write the TOA vector as a function of $\boldsymbol{\eta}$ as $\boldsymbol{\tau} = \mathbf{g}(\boldsymbol{\eta})$. So, the problem can be stated as, given the observations $r_i(t)$, $i = 0, 1, \ldots, M-1$ estimate the vector $\boldsymbol{\eta}$. We are only interested in the parameters (x_T, y_T) and the rest of the unknown parameters are nuisance parameters.

III. CRLB AND MLE OF THE EMITTER LOCATION

A. Signal Unknown with Unknown Transmission Time

For the continuous time model in (1) the log-likelihood function [24] for sensor i is given by

$$l = -\frac{1}{N_0}\int_0^T (r_i(t) - A_i s(t - \tau_i))^2\,dt. \quad (6)$$

Since the noise at different sensors is independent, by using (5) we can write the joint log-likelihood function as

$$l(\boldsymbol{\theta}) = -\frac{1}{N_0}\int_0^T \sum_{i=0}^{M-1}\left(r_i(t) - A_i \boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\right)^2 dt \quad (7)$$

The $(2M+2N-1 \times 2M+2N-1)$ FIM [25] for this model is given by (see Appendix I-A)

$$\mathcal{I}_\theta = \frac{(T/2)}{(N_0/2)}\begin{bmatrix} (2\pi F_0)^2 \boldsymbol{\phi}^T \mathbf{L}\mathbf{L}^T \boldsymbol{\phi}(\mathrm{diag}(\mathbf{A}))^2 & (2\pi F_0)(\boldsymbol{\phi}^T \mathbf{L}\boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) & (2\pi F_0)(\mathbf{A}\odot\mathbf{A})\boldsymbol{\phi}^T\mathbf{L} \\ (2\pi F_0)(\boldsymbol{\phi}^T \mathbf{L}^T \boldsymbol{\phi})(\mathrm{diag}(\mathbf{A}))^2 & (\boldsymbol{\phi}^T\boldsymbol{\phi})\mathbf{I}_M & \mathbf{A}\boldsymbol{\phi}^T \\ (2\pi F_0)\mathbf{L}^T\boldsymbol{\phi}(\mathbf{A}\odot\mathbf{A})^T & \boldsymbol{\phi}\mathbf{A}^T & (\mathbf{A}^T\mathbf{A})\mathbf{I}_M \end{bmatrix} \quad (8)$$

where $\odot$ represents the element by element product (Hadamard product), diag($\mathbf{A}$) is an $M \times M$ diagonal matrix with ith diagonal element as A_i, $\mathbf{I}_M$ is the $M \times M$ identity matrix, and the $(2N-1)\times(2N-1)$ matrix $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{bmatrix} \mathbf{0}_{(N,N)} & \begin{bmatrix} \mathbf{0}_{(1,N-1)} \\ \mathrm{diag}(1,2,\ldots,N-1) \end{bmatrix} \\ -\begin{bmatrix} \mathbf{0}_{(N-1,1)} & \mathrm{diag}(1,2,\ldots,N-1) \end{bmatrix} & \mathbf{0}_{(N-1,N-1)} \end{bmatrix}.$$

This FIM must be inverted in order to find the CRLB for the unknown parameter vector $\boldsymbol{\theta}$. By the form of the matrix in (8) it is easily shown (see Appendix IV) that the matrix is singular with rank equal to two less than full rank. Weiss [18] uses an ad hoc method to overcome this. We, however use the exact transformation of the parameters [26] that is required to eliminate the singularity of the information matrix. The singularity arises because there is an ambiguity in the model. It is not possible to uniquely determine all the unknown parameters in the model in (1). This is because of the relationship between the transmission time, attenuation factor, and the signal waveform. Suppose that in (1), $\bar{A}_i$ and $\bar{s}(t)$ are the true values of the gain and the signal waveform that generate $r_i(t)$. Then the pair of values $(\bar{A}_i/\gamma, \gamma\bar{s}(t))$ for any non-zero constant γ also generate the same $r_i(t)$. So, from the observation $r_i(t)$ it is impossible to determine the true values of A_i and $s(t)$. A similar relationship exists between the unknown transmission time and an unknown signal waveform. Suppose that the transmitted signal $s_T(t)$ is generated by an unknown signal waveform $\bar{s}(t)$ transmitted at an unknown transmission time $\bar{t}_0$ so that $s_T(t) = \bar{s}(t-\bar{t}_0)$. Now the same transmitted signal $s_T(t)$ can also be generated by the pair of values $((\bar{t}_0-\gamma), \bar{s}(t-\gamma))$ for any constant γ. This is more clearly demonstrated in Fig. 3. Notice that given a transmitted signal, it is not possible to determine whether the signal waveform is $s_1(t)$ with transmission time t_1 or $s_2(t)$ with transmission time t_2. This causes the information matrix to be at least rank two deficient, as shown in Appendix IV. The overparameterization can be resolved by applying an appropriate transformation that satisfies certain constraints [26]. As shown in Appendix IV, the appropriate transformation for this model is

$$\begin{aligned} \boldsymbol{\tau}' &= \begin{bmatrix} (\tau_1-\tau_0) & (\tau_2-\tau_0) & \cdots & (\tau_{M-1}-\tau_0) \end{bmatrix}^T \\ \mathbf{A}' &= (1/A_0)[A_1 \ \cdots \ A_{M-1}]^T \\ \boldsymbol{\phi}' &= A_0 \begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{I}_{N-1} \\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix} \end{bmatrix} \mathrm{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} \end{aligned} \quad (9)$$

and can be shown to be the least restrictive constraint for identifiability. Here we are using the $(*)'$ notation to represent the new parameters resulting from the transformation. Notice that the transformed parameter vectors $\boldsymbol{\tau}'$ and $\mathbf{A}'$ are TDOA and relative gain factor with respect to sensor 0 and are each reduced by one parameter from $\boldsymbol{\tau}$ and $\mathbf{A}$, respectively, while the $\boldsymbol{\phi}'$ is simply the Fourier coefficients of $r_0(t)$. Using these transformed parameters the model in (5) can be rewritten as

$$\begin{aligned} r_0(t) &= \boldsymbol{h}^T(t)\boldsymbol{\phi}' + w_0(t) \qquad 0 \le t \le T \\ r_i(t) &= A_i'\boldsymbol{h}^T(t-\tau_i')\boldsymbol{\phi}' + w_i(t), \quad 0 \le t \le T,\ i = 1,2,\ldots,M-1 \end{aligned} \quad (10)$$

where $A_i' = \frac{A_i}{A_0}$ and $\tau_i' = \tau_i - \tau_0$. So, the effect of the transformation is that the signal at sensor 0 is made the reference signal and the signals at all the other sensors are modeled relative to this reference signal. Although (10) seems intuitively obvious, by arriving at it from (5) using the transformation in (9), we have mathematically verified that (10) is indeed the correct model to use for the problem of localization under the unknown signal case. Weiss [18–21] uses (10) directly without this rigorous argument. This is a subtle but important result which is overlooked by Weiss. Now, let $\boldsymbol{\theta}' = [\boldsymbol{\tau}'^T \ \mathbf{A}'^{\mathbf{T}} \ \boldsymbol{\phi}'^T]^T$ be the $(2M+2N-3)\times 1$ vector of the unknown transformed parameters, $\boldsymbol{\eta}' = [x_T, y_T]^T$ and $\boldsymbol{\alpha}' = [\boldsymbol{\eta}'^T \ \mathbf{A}'^{\mathbf{T}} \boldsymbol{\phi}'^T]^T$. Notice that the TDOA vector $\boldsymbol{\tau}'$ is a function of only (x_T, y_T), i.e. $\boldsymbol{\tau}' = \mathbf{g}'(\boldsymbol{\eta}')$. The unknown transmission time t_0 does not appear and thus is not a nuisance parameter. The problem is now, given the observations $r_i(t),\ \ i = 0,1,\ldots,M-1$ estimate the vector $\boldsymbol{\eta}'$. The log-likelihood function for this model with the

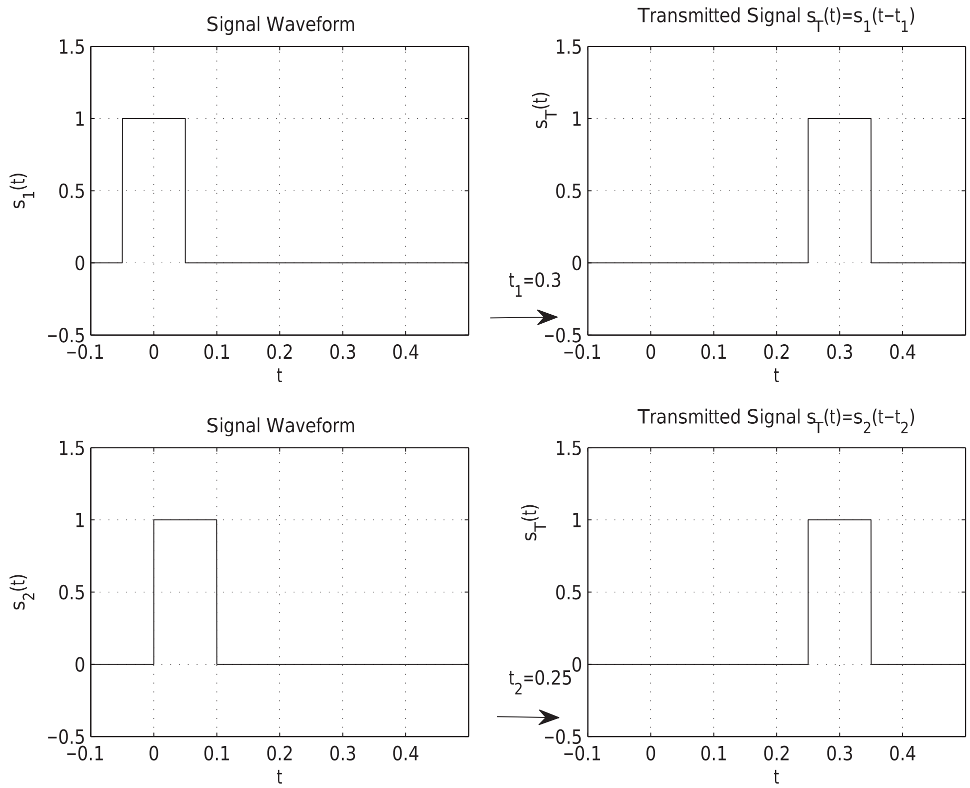

Fig. 3. Ambiguity when signal waveform and transmission time are both unknown.

transformed parameters is given by

$$l(\boldsymbol{\theta}') = -\frac{1}{N_0}\int_0^T \left(r_0(t) - \boldsymbol{h}^T(t)\phi'\right)^2 dt - \frac{1}{N_0}\int_0^T \sum_{i=1}^{M-1} \left(r_i(t) - A_i'\boldsymbol{h}^T(t-\tau_i')\phi'\right)^2 dt. \tag{11}$$

As shown in the Appendix I-A, the CRLB for this transformed parameter vector is

$$\mathcal{I}_{\theta'}^{-1} = \mathbf{H} I_\theta^\dagger \mathbf{H}^T \tag{12}$$

where $\mathbf{H}$ is given in (50), or equivalently the FIM is,

$$I_{\theta'} = \frac{(T/2)}{(N_0/2)}\begin{bmatrix} (2\pi F_0)^2\boldsymbol{\phi}'^T\mathbf{L}\mathbf{L}^T\boldsymbol{\phi}'(\mathrm{diag}(\mathbf{A}'))^2 & (2\pi F_0)(\boldsymbol{\phi}'^T\mathbf{L}\boldsymbol{\phi}')(\mathrm{diag}(\mathbf{A}')) & (2\pi F_0)(\mathbf{A}'\odot\mathbf{A}')\boldsymbol{\phi}'^T\mathbf{L} \\ (2\pi F_0)(\boldsymbol{\phi}'^T\mathbf{L}^T\boldsymbol{\phi}')(\mathrm{diag}(\mathbf{A}'))^2 & (\boldsymbol{\phi}'^T\boldsymbol{\phi}')\mathbf{I}_{(M-1)} & \mathbf{A}'\boldsymbol{\phi}'^T \\ (2\pi F_0)\mathbf{L}^T\boldsymbol{\phi}'(\mathbf{A}'\odot\mathbf{A}')^T & \boldsymbol{\phi}'\mathbf{A}'^T & (1+\mathbf{A}'^T\mathbf{A}')\mathbf{I}_{(2N-1)} \end{bmatrix}. \tag{13}$$

The FIM for the corresponding vector $\boldsymbol{\alpha}'$ is given by [25]

$$\mathcal{I}_{\alpha'} = \left(\frac{\partial\boldsymbol{\theta}'}{\partial\boldsymbol{\alpha}'^T}\right)^T I_{\theta'}\left(\frac{\partial\boldsymbol{\theta}'}{\partial\boldsymbol{\alpha}'^T}\right) = \left(\frac{\partial\boldsymbol{\theta}'}{\partial\boldsymbol{\alpha}'^T}\right)^T (\mathbf{H}\mathcal{I}_\theta^\dagger\mathbf{H}^T)^{-1}\left(\frac{\partial\boldsymbol{\theta}'}{\partial\boldsymbol{\alpha}'^T}\right) \tag{14}$$

where the Jacobian $\left(\frac{\partial\boldsymbol{\theta}'}{\partial\boldsymbol{\alpha}'^T}\right)$ is given in (39). In Appendix II-A we show that the MLE for the emitter location is obtained by maximizing over (x_T, y_T), the maximum eigenvalue of the $M \times M$ cross-correlation matrix $\mathbf{B}' = \mathbf{Y}'\mathbf{Y}'^T = \sum_{i=0}^{M-1}\mathbf{y}_i'\mathbf{y}_i'^{\mathbf{T}}$ where $\mathbf{Y}' = [\mathbf{y}_0' \quad \mathbf{y}_1' \quad \cdots \quad \mathbf{y}'_{M-1}]$ with

$$\mathbf{y}_0' = \int_0^T r_0(t)\boldsymbol{h}(t)\,dt \text{ and } \mathbf{y}_i' = \int_0^T r_i(t)\boldsymbol{h}(t-\tau_i')\,dt, \quad i = 1, 2, \cdots M-1. \tag{15}$$

That is,

$$\hat{\boldsymbol{\eta}}' = \arg\max_{\boldsymbol{\eta}'} \lambda_{\max}(\mathbf{B}') \tag{16}$$

or equivalently,

$$(\hat{x}_T, \hat{y}_T) = \arg\max_{(x_T, y_T)} \lambda_{\max}(\mathbf{B}') \tag{17}$$

where $\lambda_{\max}$ represents the maximum eigenvalue. The matrix $\mathbf{B}'$ is real symmetric and positive definite and so the maximum eigenvalue is real and positive.

B. Signal Known with Unknown Transmission Time

Quite often in practical situations it is possible that the signal waveform is known but the exact transmission time t_0 is unknown. In this case the number of unknowns is reduced to $2M$. Let $\boldsymbol{\zeta} = [\boldsymbol{\tau}^T\ \mathbf{A}^T]^T$ be the $2M \times 1$ unknown parameter vector. Similar to (7) the log-likelihood function is given by

$$l(\boldsymbol{\zeta}) = -\frac{1}{N_0}\int_0^T \sum_{i=0}^{M-1}\left(r_i(t) - A_i \boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\right)^2 dt \qquad (18)$$

where $\boldsymbol{\phi}$ is known. The FIM for this model is given by (see Appendix II-B)

$$\mathcal{I}_\zeta = \frac{(T/2)}{(N_0/2)} \begin{bmatrix} (2\pi F_0)^2 \boldsymbol{\phi}^T \mathbf{L}\mathbf{L}^T \boldsymbol{\phi}(\mathrm{diag}(\mathbf{A}))^2 & (2\pi F_0)(\boldsymbol{\phi}^T \mathbf{L}\boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) \\ (2\pi F_0)(\boldsymbol{\phi}^T \mathbf{L}^T \boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) & (\boldsymbol{\phi}^T\boldsymbol{\phi})\mathbf{I}_M \end{bmatrix}. \qquad (19)$$

This matrix is not singular because, for this case, the unknown parameters in the model can be uniquely determined. Therefore, there is no need to transform the parameters as in the case of the unknown signal. Although, the unknown transmission time t_0 is still retained here as the nuisance parameter. For this model it is shown in Appendix II-B that the MLE for emitter location and the unknown transmission time is given by

$$\hat{\boldsymbol{\eta}} = \arg\max_{\boldsymbol{\eta}} \boldsymbol{\phi}^T \mathbf{B}\boldsymbol{\phi} \qquad (20)$$

where $\mathbf{B} = \mathbf{Y}\mathbf{Y}^T$ and $\mathbf{Y} = [\mathbf{y}_0\ \mathbf{y}_1\ \cdots\ \mathbf{y}_{M-1}]$ with $\mathbf{y}_i = \int_0^T r_i(t)\boldsymbol{h}(t-\tau_i)\,dt, \quad i = 0, 1, \cdots M-1$. $\mathbf{B}$ is a function of (x_T, y_T, t_0). Using the fact that $\boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi} = s(t-\tau_i)$ and $\boldsymbol{\eta} = [x_T\ y_T\ t_0]^T$ we can rewrite (20) as

$$(\hat{x}_T, \hat{y}_T, \hat{t}_0) = \arg\max_{(x_T, y_T, t_0)} \sum_{i=0}^{M-1}\left(\int_0^T r_i(t)s(t-\tau_i)\,dt\right)^2. \qquad (21)$$

Equation (21) is simply the correlation values between the known signal waveform and the observed signal at each of the sensors, summed over all the sensors. The emitter location that yields the values of the TOAs that maximize the expression in (21) is the MLE of the emitter location.

IV. SIMULATION RESULTS

In order to evaluate the performance of the MLE we have run some simulations and compared the performance against the CRLB and against a typically used TDOA approach. The TDOA approach was implemented as a two-step algorithm where, in the first step, the TDOA estimates $\breve{\tau}_i'$ were obtained by cross-correlating the signal at each sensor with the signal at sensor 0. Then a 1 km $\times$ 1 km region around the true emitter location was split into 100 $\times$ 100 grid points and for each emitter location on the grid point the TDOAs were computed using the formula

$$\tau_i' = \frac{\sqrt{(x_T-x_i)^2+(y_T-y_i)^2}}{c} - \frac{\sqrt{(x_T-x_0)^2+(y_T-y_0)^2}}{c}, \quad i = 1, 2, \ldots, M-1.$$

Next the least squared error (LSE) between the estimated TDOAs and the computed TDOAs was calculated as

$$\mathrm{LSE} = \sum_{i=1}^{M-1}(\breve{\tau}_i' - \tau_i')^2.$$

This LSE is a function of the emitter location (x_T, y_T). The emitter location that minimized the LSE is the estimate of the emitter location. Since we know that the LSE is a 2-dimensional parabolic function of the emitter location, we improved the accuracy by fitting a parabola through the 10000 points at which the LSE was computed. Then by using the analytical formula for the minimum location of a 2-dimensional parabola, we computed the minimum. Fig. 4(a) shows a realization of the LSE function. For the simulation we have used 4 sensors placed at the coordinates shown in Fig. 1 and the emitter was placed at the coordinates (130, 75) km, also as shown in Fig. 1. The integrations in (4) and (15) are approximated using summations with $\delta t = 0.33$ ns which means the sampling frequency is $F_s = 300$ MHz. The reason for choosing such high sampling frequency is that at this frequency, the position quantization error due to sampling is on the order of $(c/F_s) = 10^{-3}$ km. A Gaussian chirp defined by

$$s(t) = \exp\left(-\frac{1}{2}\sigma_F^2\left(t - \frac{T_s}{2}\right)^2\right)\sin(2\pi m t^2)$$

was used as the unknown transmitted signal waveform. Fig. 5 shows the transmitted signal waveform. Notice that the signal is assumed to be approximately zero for $t < 0$ and for $t > T_s$. We set $T_s = 5\,\mu s$ and $\sigma_F = 0.2\pi$ MHz. The observation interval at each of the sensors was taken to be $T = 0.2$ ms. The unknown transmission time of the signal was set to $t_0 = 0.07$ ms. With this configuration the maximum TDOA is 0.0988 ms. The frequency spectrum of the Gaussian window is given by $|S(F)| = (\sqrt{2\pi}/\sigma_F)\exp(-2\pi^2F^2/\sigma_F^2)$ [25], and the bandwidth of the chirp is $BW = 1.5$ MHz. The rate of change of frequency for the linear chirp was chosen to be $m = \frac{BW}{T_s} = 3 \times 10^8$. A plot of the Fourier coefficients of the signal is shown in Fig. 6. A value of $N = 2 \times BW \times T = 600$ was used to have a total of $2N - 1 = 1199$ unknown Fourier coefficients. Notice that the Fourier coefficients are almost zero for $n > 600$. To measure the levels of the zero mean additive white Gaussian noise, we used a metric called the average signal-to-noise ratio (ASNR). The ASNR is the ratio of the average signal power to the noise power at each sensor averaged over all the sensors, i.e., if $\mathcal{P}_{si} = |A_i|^2\frac{1}{T}\int_0^T |s(t)|^2\,dt$ is the average power of the signal at the ith sensor and $\frac{N_0}{2}$ is the noise spectral density at the ith sensor, then the SNR averaged over M sensors is given by $10\log\left(\frac{1}{M}\sum_{i=0}^{M-1}\frac{\mathcal{P}_{si}}{(N_0/2)(F_s/2)}\right)$ dB. We set the

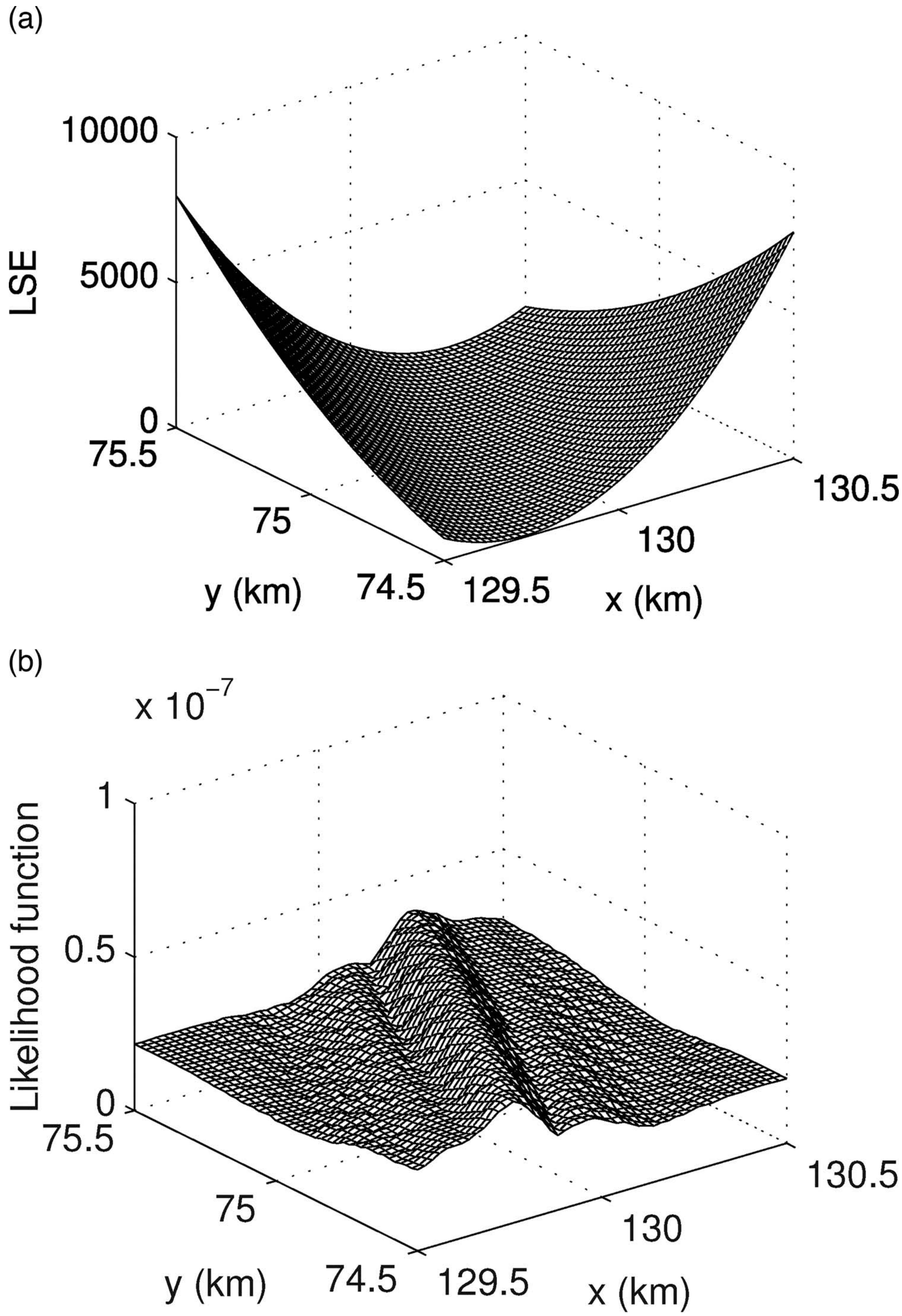

Fig. 4. (a) Realization of LSE at ASNR = –10 dB. (b) Realization of likelihood function at ASNR = –10 dB.

ASNR at –20 dB and ran a total of 300 Monte Carlo simulations to generate the scatter plot and the corresponding 95% error ellipse which are shown in Fig. 7. This is also called the 95% confidence ellipse. That is, if this estimator is used a large number of times for localization, then around 95% of those times the true location of the emitter will lie within this ellipse. To compute the MLE the maximization was performed using a grid search. We used a grid of size 1 km $\times$ 1 km with the grid points 0.01 km apart to have a total of 100 $\times$ 100 = 10000 points. The grid is shown with a dotted line in the figure. At –20 dB the variances of the MLEs of (x_T, y_T) were (0.0021, 0.0006) km^2 and the respective CRLBs were (0.0012,0.0003) km^2. The figure does not show 300 points because some points lie on top of the others due to position quantization induced by the finite number of grid points in the grid search for maximization. Due to the complex nature of the likelihood function (see Fig. 4(b)),

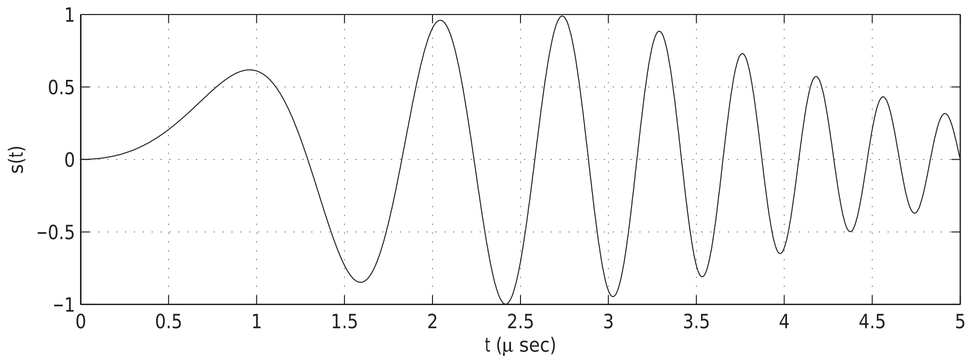

Fig. 5. Transmitted signal waveform.

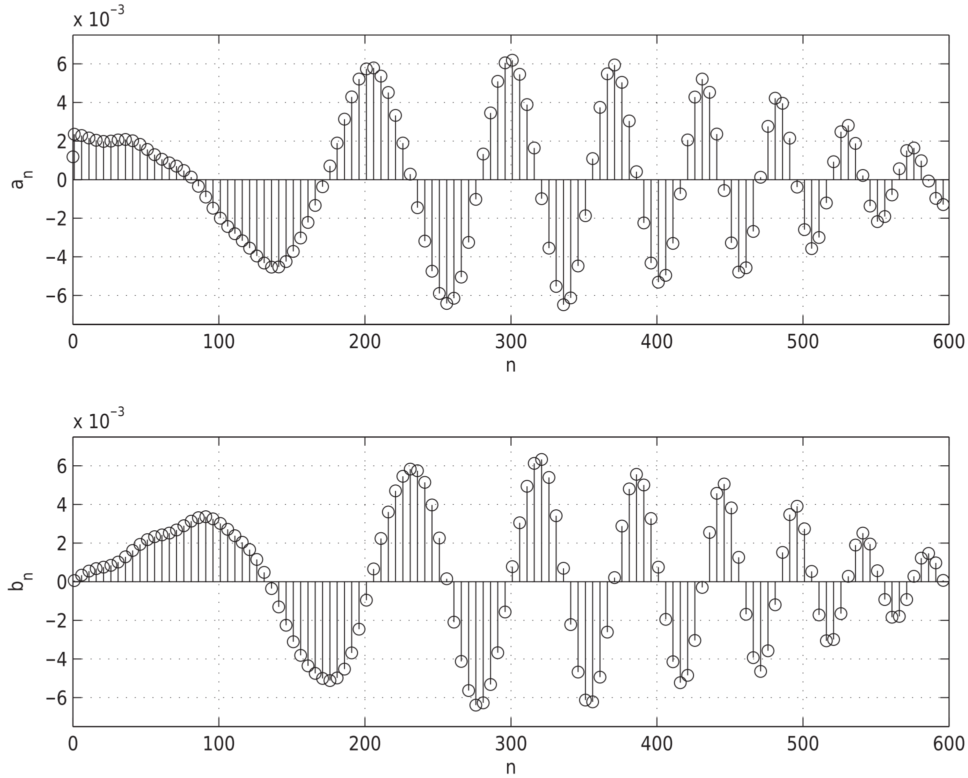

Fig. 6. Fourier coefficients plot.

it is not possible to use any curve fitting techniques to reduce this quantization effect as in the case of the conventional TDOA approach.

Fig. 8 shows the comparison of the variances of the MLE and the typical TDOA approach against the CRLB for different SNR values. Notice that for the ASNR values below –30 dB, the variance of the MLE remains flat. This is because of the restriction imposed by the finite grid size. As the ASNR increases above –30 dB, the variance of the MLE reduces rapidly to approach the CRLB at around –10 dB. Due to the nature of the conventional TDOA approach it breaks down for the ASNR values below –17 dB. So this figure has the variances of the TDOA approach only for the average SNR values above –17 dB. On the other hand, for this particular setup, the results for the MLE are reliable for the ASNR values as low as –30 dB. It is quite obvious from this figure that performance of the MLE is very much better than a typical TDOA approach. In this case the MLE performs as good as a typical TDOA approach for an ASNR value of about 10 dB less than that for the TDOA approach. Also notice that at around –10 dB, the variance of the MLE is almost two orders of magnitude less than that of the TDOA approach. We have noticed that for certain sensor-emitter configurations, particularly when the emitter was very close to the sensors, the variance of the MLE is up to three orders of magnitude less than that of the TDOA approach. Therefore, under low probability of intercept (LPI) scenarios where a conventional TDOA technique cannot be reliably used, the MLE can be used.

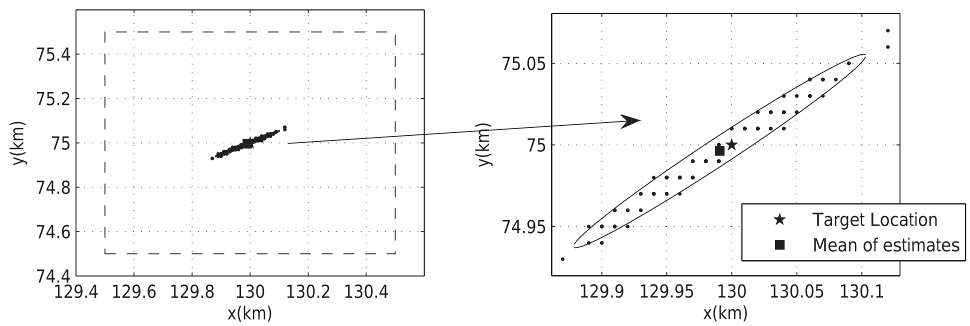

Fig. 7. Scatter plot and corresponding 95% error ellipse of MLE for ASNR = –20 dB.

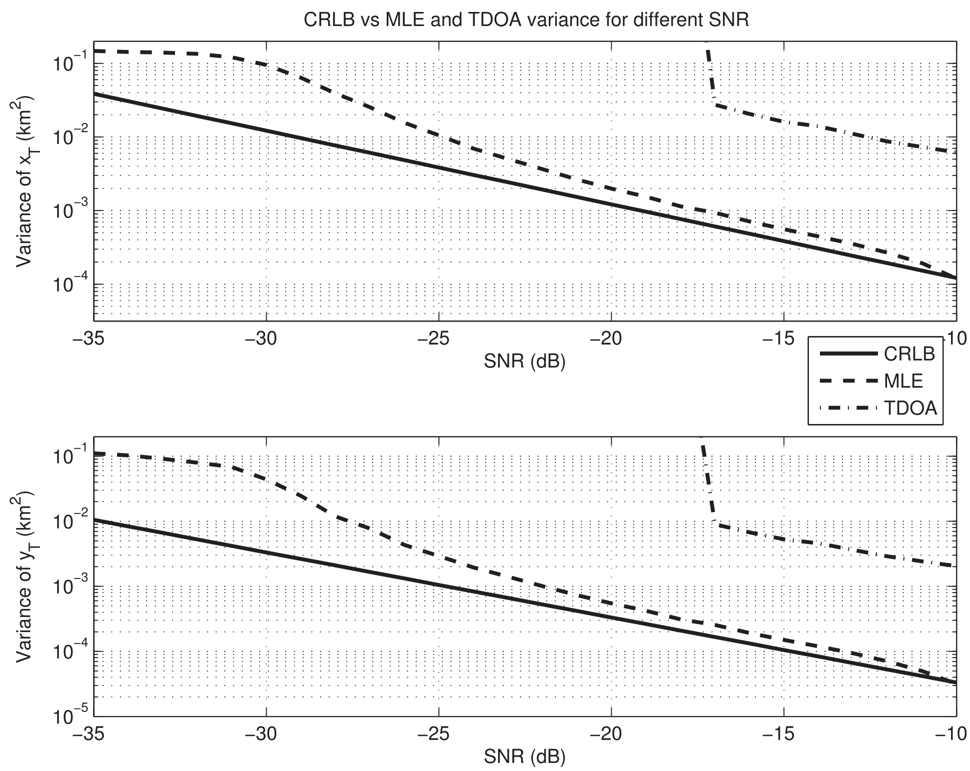

Fig. 8. Comparison of variances for emitter location estimate using MLE and typical TDOA approach against CRLB for different ASNR values.

V. CONCLUSIONS

We have derived a direct positioning estimator for an emitter location. This is the MLE. We have shown that for an unknown signal case, the model that is conventionally used has an inherent ambiguity and so all the unknown parameters cannot be uniquely determined. We derived an appropriate transformation of the parameters and reparameterized the model to remove the ambiguity. We have shown that for the special case of a known signal with unknown transmission time, there is no ambiguity in the model. We derived the MLE and the FIM for the model. The performance of the MLE was compared against a typical two-step TDOA-based localizer and against the CRLB. The performance of the MLE is significantly better than a typical two-step TDOA-based localizer.

APPENDIX I. CRLB

We derive the CRLB for the emitter location estimate. First, we show that the FIM for the model used for unknown signal with unknown transmission time case is singular. We then use a transformation of the parameters in the model and derive the CRLB. Let $\boldsymbol{\tau} = [\tau_0\ \tau_1\ \cdots\ \tau_{M-1}]^T$, $\mathbf{A} = [A_0\ A_1\ \cdots\ A_{M-1}]^T$, and $\boldsymbol{\phi} = [a_0\ a_1\ \cdots\ a_{N-1}\ b_1\ b_2\ \cdots\ b_{N-1}]^T$, where

$$a_0 = \frac{\sqrt{2}}{T}\int_0^T s(t)\,dt,\ a_n = \frac{2}{T}\int_0^T s(t)\cos 2\pi n F_0 t\ dt,$$

$$b_n = \frac{2}{T}\int_0^T s(t)\sin 2\pi n F_0 t\ dt.$$

Let $\boldsymbol{\theta} = [\tau^T\ \mathbf{A}^T\ \boldsymbol{\phi}^T]^T$. The TOAs τ_i are a function of the emitter location (x_T, y_T) and the signal transmission time t_0.

$$\tau_i = \frac{\sqrt{(x_T - x_i)^2 + (y_T - y_i)^2}}{c} + t_0$$

where c is the propagation speed of the signal. If $l(\boldsymbol{\theta})$ is the log-likelihood function, then the FIM is given by

$$\mathcal{I}_\theta = \begin{bmatrix} -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\tau}\partial \boldsymbol{\tau}^T}\right\} & -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\tau}\partial \mathbf{A}^T}\right\} & -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\tau}\partial \boldsymbol{\phi}^T}\right\} \\ -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mathbf{A}\partial \boldsymbol{\tau}^T}\right\} & -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mathbf{A}\partial \mathbf{A}^T}\right\} & -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mathbf{A}\partial \boldsymbol{\phi}^T}\right\} \\ -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\phi}\partial \boldsymbol{\tau}^T}\right\} & -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\phi}\partial \mathbf{A}^T}\right\} & -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\phi}\partial \boldsymbol{\phi}^T}\right\} \end{bmatrix}. \tag{22}$$

A. Signal Unknown with Unknown Transmission Time

From (7) we have the log-likelihood function as

$$l(\boldsymbol{\theta}) = -\frac{1}{N_0}\int_0^T \sum_{m=0}^{M-1} (x_m(t) - A_m \boldsymbol{h}^T(t-\tau_m)\boldsymbol{\phi})^2\, dt \tag{23}$$

where $\boldsymbol{h}(t)$ is as defined in Appendix III. Partial differentiation with respect to (w.r.t) τ_i gives

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \tau_i} = -\frac{1}{N_0}\int_0^T 2\left(r_i(t) - A_i \boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\right) \times \left(-A_i \frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\boldsymbol{\phi}\right) dt \tag{24}$$

and w.r.t A_i gives

$$\frac{\partial l(\boldsymbol{\theta})}{\partial A_i} = -\frac{1}{N_0}\int_0^T 2\left(r_i(t) - A_i \boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\right) \times \left(-\boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\right) dt \tag{25}$$

for $i = 0, 1, \ldots, M-1$. Partial differentiation w.r.t $\boldsymbol{\phi}$ gives

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \boldsymbol{\phi}} = -\frac{1}{N_0}\int_0^T \sum_{m=0}^{M-1} 2\left(x_m(t) - A_m \boldsymbol{h}^T(t-\tau_m)\boldsymbol{\phi}\right) \times \left(-A_m \boldsymbol{h}^T(t-\tau_m)\right) dt. \tag{26}$$

Next we evaluate the second derivatives. Partial differentiation of (24) w.r.t τ_i gives

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \tau_i^2} = -\frac{2}{N_0}\left[\int_0^T (r_i(t) - A_i \boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}) \times \left(-A_i \frac{\partial^2 \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i^2}\boldsymbol{\phi}\right) dt + \int_0^T \left(-A_i \frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\boldsymbol{\phi}\right)^2 dt\right].$$

Taking the negative of the expected value of both sides and using (48) gives

$$\begin{aligned} -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \tau_i^2}\right\} &= \frac{2}{N_0}\int_0^T \left(A_i \frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\boldsymbol{\phi}\right)^2 dt \\ &= \frac{A_i^2}{(N_0/2)}\boldsymbol{\phi}^T\left[\int_0^T \frac{\partial \boldsymbol{h}(t-\tau_i)}{\partial \tau_i}\frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\right]\boldsymbol{\phi} \\ &= \frac{(T/2)(2\pi F_0)^2}{(N_0/2)}(\boldsymbol{\phi}^T \mathbf{L}\mathbf{L}^T \boldsymbol{\phi})\, A_i^2 \end{aligned}$$

where the $2N-1 \times 2N-1$ matrix $\mathbf{L}$ is as defined in Appendix III. Since $\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \tau_i \partial \tau_j} = 0$ for $i \neq j$, we have

$$-\mathrm{E}\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\tau}\partial \boldsymbol{\tau}^T}\right\} = \frac{(T/2)(2\pi F_0)^2}{(N_0/2)}\boldsymbol{\phi}^T \mathbf{L}\mathbf{L}^T \boldsymbol{\phi}(\mathrm{diag}(\mathbf{A}))^2 \tag{27}$$

where $\mathrm{diag}(\mathbf{A})$ is an $M \times M$ diagonal matrix with ith diagonal element as A_i. Partial differentiation of (25) w.r.t τ_i gives

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \tau_i \partial A_i} = -\frac{2}{N_0}\left[\int_0^T \left(r_i(t) - A_i \boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\right) \times \left(-\frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\boldsymbol{\phi}\right) dt + \int_0^T \left(-A_i \frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\boldsymbol{\phi}\right)(-\boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi})\, dt\right].$$

Taking the negative of the expected value of both sides and using (49) gives

$$\begin{aligned} -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \tau_i \partial A_i}\right\} &= \frac{A_i}{(N_0/2)}\boldsymbol{\phi}^T\left[\int_0^T \frac{\partial \boldsymbol{h}(t-\tau_i)}{\partial \tau_i}\boldsymbol{h}^T(t-\tau_i)\, dt\right]\boldsymbol{\phi} \\ &= \frac{(T/2)(2\pi F_0)}{(N_0/2)}(\boldsymbol{\phi}^T \mathbf{L}\boldsymbol{\phi})\, A_i. \end{aligned}$$

Since $\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \tau_i \partial A_j} = 0$ for $i \neq j$, we have

$$-\mathrm{E}\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\tau}\partial \mathbf{A}^T}\right\} = \frac{(T/2)(2\pi F_0)}{(N_0/2)}(\boldsymbol{\phi}^T \mathbf{L}\boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})). \tag{28}$$

Partial differentiation of (26) w.r.t τ_i gives

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \tau_i \partial \boldsymbol{\phi}^T} = -\frac{2}{N_0}\left[\int_0^T \left(r_i(t) - A_i \boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\right) \times \left(-A_i \frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\right) dt + \int_0^T \left(-A_i \frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\boldsymbol{\phi}\right)(-A_i \boldsymbol{h}^T(t-\tau_i))\, dt\right].$$

Taking the negative of the expected value of both sides and using (49) gives

$$-E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\tau_i\partial\boldsymbol{\phi}^T}\right\}=\frac{A_i^2}{(N_0/2)}\boldsymbol{\phi}^T\left[\int_0^T\frac{\partial\boldsymbol{h}(t-\tau_i)}{\partial\tau_i}\boldsymbol{h}^T(t-\tau_i)\,dt\right]$$
$$=\frac{(T/2)(2\pi F_0)A_i^2}{(N_0/2)}\boldsymbol{\phi}^T\mathbf{L}.$$

So, we have

$$-\mathrm{E}\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\tau\partial\boldsymbol{\phi}^T}\right\}=\frac{(T/2)(2\pi F_0)}{(N_0/2)}(\mathbf{A}\odot\mathbf{A})\boldsymbol{\phi}^T\mathbf{L}. \tag{29}$$

Partial differentiation of (25) w.r.t A_i and using (47) gives

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial A_i^2}=-\frac{1}{(N_0/2)}\int_0^T(\boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi})^2$$
$$=-\frac{1}{(N_0/2)}\boldsymbol{\phi}^T\left[\int_0^T\boldsymbol{h}(t-\tau_i)\boldsymbol{h}^T(t-\tau_i)\right]\boldsymbol{\phi}$$
$$=-\frac{(T/2)}{(N_0/2)}\boldsymbol{\phi}^T\boldsymbol{\phi}.$$

and so

$$-\mathrm{E}\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\mathbf{A}\partial\boldsymbol{\phi}^T}\right\}=\frac{(T/2)}{(N_0/2)}\mathbf{A}\boldsymbol{\phi}^T. \tag{31}$$

Partial differentiation of (26) w.r.t $\boldsymbol{\phi}$ and using (47) gives

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}^T}=-\frac{2}{N_0}\int_0^T\sum_{i=0}^{M-1}\left(-A_i\boldsymbol{h}^T(t-\tau_i)\right)\left(-A_i\boldsymbol{h}^T(t-\tau_i)\right)dt$$
$$=-\sum_{i=0}^{M-1}\frac{A_i^2}{(N_0/2)}\int_0^T\boldsymbol{h}(t-\tau_i)\boldsymbol{h}^T(t-\tau_i)\,dt$$
$$=-\sum_{i=0}^{M-1}\frac{(T/2)}{(N_0/2)}\mathbf{I}_{(2N-1)}\,A_i^2$$

and so

$$-\mathrm{E}\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}^T}\right\}=\frac{(T/2)}{(N_0/2)}\mathbf{I}_{(2N-1)}\sum_{i=0}^{M-1}A_i^2=\frac{(T/2)\mathbf{A}^T\mathbf{A}}{(N_0/2)}\mathbf{I}_{(2N-1)}. \tag{32}$$

Putting (27), (28), (29), (30), (31), (32) back in (22), we have

$$\mathcal{I}_\theta=\frac{(T/2)}{(N_0/2)}\begin{bmatrix}(2\pi F_0)^2\boldsymbol{\phi}^T\mathbf{L}\mathbf{L}^T\boldsymbol{\phi}(\mathrm{diag}(\mathbf{A}))^2 & (2\pi F_0)(\boldsymbol{\phi}^T\mathbf{L}\boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) & (2\pi F_0)(\mathbf{A}\odot\mathbf{A})\boldsymbol{\phi}^T\mathbf{L}\\ (2\pi F_0)(\boldsymbol{\phi}^T\mathbf{L}^T\boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) & (\boldsymbol{\phi}^T\boldsymbol{\phi})\mathbf{I}_M & \mathbf{A}\boldsymbol{\phi}^T\\ (2\pi F_0)\mathbf{L}^T\boldsymbol{\phi}(\mathbf{A}\odot\mathbf{A})^T & \boldsymbol{\phi}\mathbf{A}^T & (\mathbf{A}^T\mathbf{A})\mathbf{I}_{(2N-1)}\end{bmatrix}. \tag{33}$$

Taking the negative of the expected value on both sides gives

$$-\mathrm{E}\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\mathbf{A}\partial\mathbf{A}^T}\right\}=\frac{(T/2)\boldsymbol{\phi}^T\boldsymbol{\phi}}{(N_0/2)}\mathbf{I}_M. \tag{30}$$

Partial differentiation of (26) w.r.t A_i gives

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial A_i\partial\boldsymbol{\phi}^T}=-\frac{2}{N_0}\left[\int_0^T\left(r_i(t)-A_i\boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\right)\left(-\boldsymbol{h}^T(t-\tau_i)\right)dt\right.$$
$$\left.+\int_0^T(-\boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi})(-A_i\boldsymbol{h}^T(t-\tau_i))\,dt\right].$$

Taking the negative of the expected value of both sides and using (47) gives

$$-E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial A_i\partial\boldsymbol{\phi}^T}\right\}=\frac{2A_i}{N_0}\int_0^T\boldsymbol{h}^T(t-\tau_i)\boldsymbol{\phi}\boldsymbol{h}^T(t-\tau_i)\,dt$$
$$=\frac{2A_i}{N_0}\boldsymbol{\phi}^T\int_0^T\boldsymbol{h}(t-\tau_i)\boldsymbol{h}^T(t-\tau_i)\,dt$$
$$=\frac{(T/2)}{(N_0/2)}A_i\boldsymbol{\phi}^T$$

The CRLB matrix for the unknown parameter vector $\boldsymbol{\theta}$ is the inverse of the matrix $\mathcal{I}_\theta$. But in Appendix IV it is shown that the null space of $\mathcal{I}_\theta$ is not empty and so it is not invertible. This is because the log-likelihood function is not uniquely defined by the model in (7). To eliminate the overparameterization we use the following transformations.

$$\boldsymbol{\tau}'=[(\tau_1-\tau_0)\quad(\tau_2-\tau_0)\quad\cdots\quad(\tau_{M-1}-\tau_0)]^T$$
$$\mathbf{A}'=(1/A_0)[A_1\ \cdots\ A_{M-1}]^T$$
$$\boldsymbol{\phi}'=A_0\begin{bmatrix}1 & \mathbf{0}_{(1,2N-2)}\\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix}\mathbf{I}_{N-1} & \mathbf{I}_{N-1}\\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1}\end{bmatrix}\end{bmatrix}\mathrm{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi}. \tag{34}$$

Let $\boldsymbol{\theta}'=[\tau'^T\ \mathbf{A}'^T\ \phi'^T]$. This is a function of $\boldsymbol{\theta}$. Let $\mathbf{H}=\left(\frac{\partial\boldsymbol{\theta}'}{\partial\boldsymbol{\theta}}\right)$ be the Jacobian. If $\mathbf{H}$ has row vectors that are linear combinations of those eigenvectors of $\mathcal{I}_\theta$ that have nonzero eigenvalues then the CRLB of $\boldsymbol{\theta}'$ is given by $\mathbf{H}\mathcal{I}_\theta^\dagger\mathbf{H}^T$ [26]. The † is used to represent the generalized inverse. This condition is verified in Appendix IV. Therefore,

$$\mathcal{I}_{\theta'}^{-1}=\mathbf{H}I_\theta^\dagger\mathbf{H}^T. \tag{35}$$

Alternately, the log-likelihood function for this model with the transformed parameters is given by

$$\mathcal{I}_{\theta'} = \frac{T/2}{(N_0/2)} \begin{bmatrix} (2\pi F_0)^2 \boldsymbol{\phi}'^T \mathbf{L}\mathbf{L}^T \boldsymbol{\phi}' (\mathrm{diag}(\mathbf{A}'))^2 & (2\pi F_0)(\boldsymbol{\phi}'^T \mathbf{L}\boldsymbol{\phi}')(\mathrm{diag}(\mathbf{A}')) & (2\pi F_0)(\mathbf{A}' \odot \mathbf{A}')\boldsymbol{\phi}'^T \mathbf{L} \\ (2\pi F_0)(\boldsymbol{\phi}'^T \mathbf{L}^T \boldsymbol{\phi}')(\mathrm{diag}(\mathbf{A}')) & (\boldsymbol{\phi}'^T \boldsymbol{\phi}')\mathbf{I}_{M-1} & \mathbf{A}'\boldsymbol{\phi}'^T \\ (2\pi F_0)\mathbf{L}^T \boldsymbol{\phi}' (\mathbf{A}' \odot \mathbf{A}')^T & \boldsymbol{\phi}' \mathbf{A}'^T & (1 + \mathbf{A}'^T \mathbf{A}')\mathbf{I}_{(2N-1)} \end{bmatrix}. \tag{37}$$

We have verified numerically that (35) is equivalent to (37). The elements of the TDOA vector $\boldsymbol{\tau}'$ are given by

$$\tau_i' = (\tau_i - \tau_0) = \frac{\sqrt{(x_T - x_i)^2 + (y_T - y_i)^2}}{c} - \frac{\sqrt{(x_T - x_0)^2 + (y_T - y_0)^2}}{c}$$

so that the new parameter vector $\boldsymbol{\tau}'$ is a function of only the emitter location (x_T, y_T). So, if we let $\boldsymbol{\eta}' = [x_T \; y_T]^T$ and $\boldsymbol{\alpha}' = [\boldsymbol{\eta}'^T \; \mathbf{A}'^T \; \boldsymbol{\phi}'^T]^T$ we have

$$\mathcal{I}_{\alpha'} = \left(\frac{\partial \boldsymbol{\theta}'}{\partial \boldsymbol{\alpha}'^T}\right)^T \mathcal{I}_{\theta'} \left(\frac{\partial \boldsymbol{\theta}'}{\partial \boldsymbol{\alpha}'^T}\right) = \left(\frac{\partial \boldsymbol{\theta}'}{\partial \boldsymbol{\alpha}'^T}\right)^T (\mathbf{H}\mathcal{I}_\theta^\dagger \mathbf{H}^T)^{-1} \left(\frac{\partial \boldsymbol{\theta}'}{\partial \boldsymbol{\alpha}'^T}\right). \tag{38}$$

The Jacobian is given by the $(2M + 2N - 3, \; M + 2N)$ matrix

$$\left(\frac{\partial \boldsymbol{\theta}'}{\partial \boldsymbol{\alpha}'^T}\right) = \begin{bmatrix} \left(\frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\eta}'^T}\right) & \left(\frac{\partial \boldsymbol{\tau}'}{\partial \mathbf{A}'^T}\right) & \left(\frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\phi}'^T}\right) \\ \left(\frac{\partial \mathbf{A}'}{\partial \boldsymbol{\eta}'^T}\right) & \left(\frac{\partial \mathbf{A}'}{\partial \mathbf{A}'^T}\right) & \left(\frac{\partial \mathbf{A}'}{\partial \boldsymbol{\phi}'^T}\right) \\ \left(\frac{\partial \boldsymbol{\phi}'}{\partial \boldsymbol{\eta}'^T}\right) & \left(\frac{\partial \boldsymbol{\phi}'}{\partial \mathbf{A}'^T}\right) & \left(\frac{\partial \boldsymbol{\phi}'}{\partial \boldsymbol{\phi}'^T}\right) \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\eta}'^T}\right) & \mathbf{0}_{(M,M)} & \mathbf{0}_{(M,2N-1)} \\ \mathbf{0}_{(M,2)} & \mathbf{I}_{M-1} & \mathbf{0}_{(M,2N-1)} \\ \mathbf{0}_{(2N-1,2)} & \mathbf{0}_{(2N-1,M)} & \mathbf{I}_{(2N-1)} \end{bmatrix} \tag{39}$$

where the $(M-1) \times 2$ matrix

$$\left(\frac{\partial \tau'}{\partial \boldsymbol{\eta}'^T}\right) = (1/c) \begin{bmatrix} \frac{(x_T - x_1)}{d_1} - \frac{(x_T - x_0)}{d_0} & \frac{(y_T - y_1)}{d_1} - \frac{(y_T - y_0)}{d_0} \\ \frac{(x_T - x_2)}{d_2} - \frac{(x_T - x_0)}{d_0} & \frac{(y_T - y_2)}{d_2} - \frac{(y_T - y_0)}{d_0} \\ \vdots & \vdots \\ \frac{(x_T - x_{M-1})}{d_{M-1}} - \frac{(x_T - x_0)}{d_0} & \frac{(x_T - x_{M-1})}{d_{M-1}} - \frac{(y_T - y_0)}{d_0} \end{bmatrix} \tag{40}$$

$$l(\boldsymbol{\theta}') = -\frac{1}{N_0}\int_0^T \left(r_0(t) - \boldsymbol{h}^T(t)\boldsymbol{\phi}'\right)^2 dt - \frac{1}{N_0}\int_0^T \sum_{i=1}^{M-1} \left(r_i(t) - A_i' \boldsymbol{h}^T(t - \tau_i')\boldsymbol{\phi}'\right)^2 dt. \tag{36}$$

So, computing the derivatives to find the FIM as done previously yields the FIM for the transformed parameter vector as

where $d_i, \; i = 0, \ldots, M-1$ is the distance between the sensor i and the emitter.

B. Signal Known with Unknown Transmission Time

From (18) we have the log-likelihood function as

$$l(\boldsymbol{\zeta}) = -\frac{1}{N_0}\int_0^T \sum_{i=0}^{M-1} \left(r_i(t) - A_i \boldsymbol{h}^T(t - \tau_i)\boldsymbol{\phi}\right)^2 dt. \tag{41}$$

Here the $2M \times 1$ unknown parameter vector is $\boldsymbol{\zeta} = [\boldsymbol{\tau}^T \ \mathbf{A}^T]^T$. So, the FIM is given by

$$\mathcal{I}_{\zeta} = \begin{bmatrix} -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\tau} \partial \boldsymbol{\tau}^T}\right\} & -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\tau} \partial \mathbf{A}^T}\right\} \\ -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mathbf{A} \partial \boldsymbol{\tau}^T}\right\} & -E\left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mathbf{A} \partial \mathbf{A}^T}\right\} \end{bmatrix}. \quad (42)$$

Using (27), (28), and (30), we have

$$\mathcal{I}_{\zeta} = \frac{(T/2)}{(N_0/2)} \begin{bmatrix} (2\pi F_0)^2 \boldsymbol{\phi}^T \mathbf{L}\mathbf{L}^T \boldsymbol{\phi} (\mathrm{diag}(\mathbf{A}))^2 & (2\pi F_0)(\boldsymbol{\phi}^T \mathbf{L} \boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) \\ (2\pi F_0)(\boldsymbol{\phi}^T \mathbf{L}^T \boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) & (\boldsymbol{\phi}^T \boldsymbol{\phi}) \mathbf{I}_M \end{bmatrix}. \quad (43)$$

APPENDIX II. MAXIMUM LIKELIHOOD ESTIMATOR

Here we derive the MLE for the two cases of signal unknown with unknown transmission time and signal known with unknown transmission time.

A. Signal Unknown with Unknown Transmission Time

The log-likelihood function with the transformed parameters is given by

$$l(\boldsymbol{\theta}') = -\frac{1}{N_0} \int_0^T \left(r_0(t) - \boldsymbol{h}^T(t)\phi'\right)^2 dt - \frac{1}{N_0} \int_0^T \sum_{i=1}^{M-1} \left(r_i(t) - A_i' \boldsymbol{h}^T(t-\tau_i')\phi'\right)^2 dt. \quad (44)$$

Partial differentiation w.r.t ϕ' gives

$$\frac{\partial l(\boldsymbol{\theta}')}{\partial \phi'} = -\frac{1}{N_0} \int_0^T 2\left(r_0(t) - \boldsymbol{h}^T(t)\phi'\right)\left(-\boldsymbol{h}^T(t)\right) dt - \frac{1}{N_0} \int_0^T \sum_{i=1}^{M-1} 2\big(r_i(t) - A_i' \boldsymbol{h}^T(t-\tau_i')\phi'\big)\left(-A_i' \boldsymbol{h}^T(t-\tau_i')\right) dt.$$

In order to find the maximum, we equate the above partial derivative to zero, which gives

$$\int_0^T r_0(t)\boldsymbol{h}^T(t)\, dt - \phi'^T \left(\int_0^T \boldsymbol{h}(t)\boldsymbol{h}^T(t)\, dt\right) + \sum_{i=1}^{M-1} A_i' \left(\int_0^T r_i(t)\boldsymbol{h}^T(t-\tau_i')\, dt\right) - A_i'^2 \phi'^T \left(\int_0^T \boldsymbol{h}(t-\tau_i')\boldsymbol{h}^T(t-\tau_i')\, dt\right) = \mathbf{0}.$$

Using the properties of the vector $\boldsymbol{h}(t)$ as shown in Appendix III, we have

$$\int_0^T r_0(t)\boldsymbol{h}^T(t)\, dt - (T/2)\phi'^T + \sum_{i=1}^{M-1} A_i' \left(\int_0^T r_i(t)\boldsymbol{h}^T(t-\tau_i')\, dt\right) - (T/2)\sum_{i=1}^{M-1} A_i'^2 \phi'^T = \mathbf{0}.$$

If we replace the integrals with

$$\mathbf{y}'_0 = \int_0^T r_0(t)\boldsymbol{h}(t)\, dt \quad \text{and} \quad \mathbf{y}_i' = \int_0^T r_i(t)\boldsymbol{h}(t-\tau_i')\, dt \ \text{ for } \ i = 1, 2, \ldots M-1$$

then we have

$$\mathbf{y}'^T_0 - (T/2)\phi'^T + \sum_{i=1}^{M-1} A_i' \mathbf{y}'^T_i - (T/2)\sum_{i=1}^{M-1} A_i'^2 \phi'^T = \mathbf{0}.$$

So, the MLE of ϕ' is

$$\hat{\phi}' = \frac{(2/T)\left(\mathbf{y}_0' + \sum_{i=1}^{M-1} \mathbf{A}'\mathbf{y}_i'\right)}{(1 + \boldsymbol{A}'^T \mathbf{A}')}.$$

Putting this back in (44), we have

$$l(\boldsymbol{\theta}') = -\frac{1}{N_0} \sum_{i=0}^{M-1} \int_0^T x_i^2(t)\, dt + \frac{1}{N_0} \frac{(2/T)\left(\mathbf{y}'^T_0 + \sum_{i=1}^{M-1} A_i' \mathbf{y}'^T_i\right)\left(\mathbf{y}_0' + \sum_{i=1}^{M-1} A_i' \mathbf{y}_i'\right)}{(1 + \mathbf{A}'^T \mathbf{A}')}.$$

Maximizing $l(\boldsymbol{\theta}')$ w.r.t $\mathbf{A}'$ and $\boldsymbol{\tau}'$ is equivalent to maximizing the second term. So, let

$$f(\mathbf{A}', \tau') = \frac{\left(\mathbf{y}'^T_0 + \sum_{i=1}^{M-1} A_i' \mathbf{y}'^T_i\right)\left(\mathbf{y}_0' + \sum_{i=1}^{M-1} A_i' \mathbf{y}_i'\right)}{(1 + \mathbf{A}'^T \mathbf{A}')}.$$

If we let $\mathbf{Y}' = [\mathbf{y}_0' \ \mathbf{y}_1' \cdots \mathbf{y}'_{M-1}]$ be the $2N - 1 \times M$ matrix then the maximum value of $f(\mathbf{A}', \ \boldsymbol{\tau}')$ w.r.t $\mathbf{A}'$ is $f_{\max}(\boldsymbol{\tau}')$ is equal to the maximum eigenvalue of $\mathbf{Y}'\mathbf{Y}^T$. Let $\mathbf{B}' = \mathbf{Y}\mathbf{Y}'^T$. The matrix $\mathbf{B}'$ is a function of $\boldsymbol{\tau}' = \mathbf{g}'(\boldsymbol{\eta}')$ which is a function of the emitter location (x_T, y_T). So, the MLE of the emitter location is found by maximizing the maximum eigenvalue of $\mathbf{B}'(x_T, y_T)$. i.e,

$$(\hat{x}_T, \hat{y}_T) = \underset{(x_T, y_T)}{\arg\max}\, \lambda_{\max}(\mathbf{B}'(x_T, y_T)). \quad (45)$$

B. Signal Known with Unknown Transmission Time

The log-likelihood function is given by

$$l(\boldsymbol{\zeta}) = -\frac{1}{N_0} \int_0^T \sum_{i=0}^{M-1} \left(r_i(t) - A_i \boldsymbol{h}^T(t-\tau_i)\phi\right)^2 dt. \quad (41)$$

Partial differentiation w.r.t A_k gives

$$\frac{\partial l(\boldsymbol{\theta})}{\partial A_k} = -\frac{1}{N_0} \int_0^T 2\big(r_k(t) - A_k \boldsymbol{h}^T(t-\tau_k)\phi\big)\big(-\boldsymbol{h}^T(t-\tau_k)\phi\big)\, dt = 0$$

for each of $k = 0, 1, \ldots, M-1$. Equating this to zero to find the maximum value gives

$$\phi^T\left(\int_0^T r_k(t)\boldsymbol{h}(t-\tau_k)\,dt\right)$$
$$-A_k\phi^T\left(\int_0^T \boldsymbol{h}(t-\tau_k)\boldsymbol{h}^T(t-\tau_k)\,dt\right)\phi=0.$$

If we replace the integral with $\mathbf{y}_k=\int_0^T r_k(t)\boldsymbol{h}(t-\tau_k)\,dt$ and use the properties of the vector $\boldsymbol{h}(t)$ as shown in Appendix III, we have

$$\phi^T\mathbf{y}_k-(T/2)A_k\phi^T\phi=0.$$

So, the MLE of A_k is

$$\hat{A}_k=\frac{\phi^T\mathbf{y}_k}{(T/2)\phi^T\phi},\ k=0,1,\ldots,M-1.$$

Putting this back in (41), we have

$$l(\boldsymbol{\zeta})=-\frac{1}{N_0}\sum_{i=0}^{M-1}\int_0^T r_1^2(t)\,dt-2\left(\frac{\phi^T\mathbf{y}_i}{(T/2)\phi^T\phi}\right)$$
$$\times\left(\int_0^T r_i(t)\boldsymbol{h}^T(t-\tau_i)\,dt\right)\phi$$
$$+\left(\frac{\phi^T\mathbf{y}_i}{(T/2)\phi^T\phi}\right)^2\phi^T\left(\int_0^T \boldsymbol{h}(t-\tau_i)\boldsymbol{h}^T(t-\tau_i)\,dt\right)\phi$$
$$=-\frac{1}{N_0}\sum_{i=0}^{M-1}\int_0^T r_i^2(t)\,dt+\frac{1}{N_0}\sum_{i=0}^{M-1}\left(\frac{\phi^T\mathbf{y}_i\mathbf{y}_i^T\phi}{(T/2)\phi^T\phi}\right).$$

Maximizing $l(\boldsymbol{\zeta})$ w.r.t τ is equivalent to maximizing the second term. So the MLE for $\boldsymbol{\eta}$ is given by

$$\hat{\boldsymbol{\eta}}=\arg\max_{\boldsymbol{\eta}}\sum_{i=0}^{M-1}\phi^T\mathbf{y}_i\mathbf{y}_i^T\phi=\arg\max_{\boldsymbol{\eta}}\phi^T\mathbf{B}\phi \qquad (46)$$

where $\mathbf{B}=\mathbf{Y}\mathbf{Y}^T$ and $\mathbf{Y}=[\mathbf{y}_0\,\mathbf{y}_1\ \cdots\ \mathbf{y}_{M-1}]$ with $\mathbf{y}_i=\int_0^T \tau_i(t)\boldsymbol{h}(t-\tau_i)\,dt,\ i=0,1,\ldots M-1$. $\mathbf{B}$ is a function or $(x_T,\ y_T,\ t_0)$.

APPENDIX III. PROPERTIES OF $\boldsymbol{h}(t)$

The time dependent vector $\boldsymbol{h}(t)$ that was used for modeling the problem in equation (5) has some interesting properties which simplify the derivation of the CRLB and the MLE. These properties are derived here. We have

$$\boldsymbol{h}(t-\tau_i)=\left[\frac{1}{\sqrt{2}}\ \cos 2\pi F_0(t-\tau_i)\ \cdots\ \cos 2\pi(N-1)F_0(t-\tau_i)\ \ \sin 2\pi F_0(t-\tau_i)\cdots\ \sin\ 2\pi(N-1)F_0(t-\tau_i)\right]^T.$$

Differentiating both sides w.r.t τ_i, we get

$$\frac{\partial\boldsymbol{h}(t-\tau_i)}{\partial\tau_i}=2\pi F_0[0\ \ \sin 2\pi F_0(t-\tau_i)\cdots 2\sin 2\pi 2F_0(t-\tau_i)\cdots(N-1)\sin 2\pi(N-1)F_0(t-\tau_i)$$
$$-\cos 2\pi F_0(t-\tau_i)-2\cos 2\pi 2F_0(t-\tau_i)\cdots-(N-1)\cos 2\pi(N-1)F_0(t-\tau_i)]^T.$$

Let

$$\mathbf{L}=\begin{bmatrix}\mathbf{0}_{(N,N)} & \begin{bmatrix}\mathbf{0}_{(1,N-1)}\\ \mathrm{diag}(1,2,\ldots,N-1)\\ \mathbf{0}_{(N-1,N-1)}\end{bmatrix}\\ -\begin{bmatrix}\mathbf{0}_{(N-1,1)} & \mathrm{diag}\,(1,2,\ldots,N-1)\end{bmatrix} & \end{bmatrix}.$$

So, we have the partial derivative of $\boldsymbol{h}(t-\tau_i)$ w.r.t to τ_i as

$$\frac{\partial\boldsymbol{h}(t-\tau_i)}{\partial\tau_i}=(2\pi F_0)\mathbf{L}\boldsymbol{h}(t-\tau_i).$$

Next we compute the integral $\int_0^T \boldsymbol{h}(t)\boldsymbol{h}^T(t)\,dt$. We have

$$\boldsymbol{h}(t)\boldsymbol{h}^T(t)=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \cos 2\pi F_0t\\ \vdots\\ \cos 2\pi(N-1)F_0t\\ \sin 2\pi F_0t\\ \vdots\\ \sin 2\pi(N-1)F_0t\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \cos 2\pi F_0t & \cdots & \cos 2\pi(N-1)F_0t & \sin 2\pi F_0t & \cdots & \sin 2\pi(N-1)F_0t\end{bmatrix}.$$

Let us compute each of the integrals in this $2N-1\times 2N-1$ product matrix separately. Integral of the first element is

$$\int_0^T \frac{1}{2}\,dt = [t]_0^T = \frac{T}{2}.$$

Integrals of the elements on the diagonal are given by

$$\begin{aligned}\int_0^T \cos^2 2\pi k F_0 t\,dt &= \frac{1}{2}\int_0^T (1+\cos 4\pi k F_0 t)\,dt\\ &= \frac{1}{2}\left[t + \frac{\sin 4\pi k F_0 t}{4\pi k F_0}\right]_0^T\\ &= \frac{T}{2}\end{aligned}$$

and

$$\begin{aligned}\int_0^T \sin^2 2\pi k F_0 t\,dt &= \frac{1}{2}\int_0^T (1-\cos 4\pi k F_0 t)\,dt\\ &= \frac{1}{2}\left[t - \frac{\sin 4\pi k F_0 t}{4\pi k F_0}\right]_0^T\\ &= \frac{T}{2}\end{aligned}$$

for $k = 1, 2, \ldots N-1$. Integrals of the rest of the elements are given by

$$\begin{aligned}&\int_0^T \cos 2\pi k F_0 t \sin 2\pi n F_0 t\,dt\\ &= \frac{1}{2}\int_0^T \sin 2\pi(n+k)F_0 t\,dt + \sin 2\pi(n-k)F_0 t\,dt\\ &= \frac{1}{2}\left[-\frac{\cos 2\pi(n+k)F_0 t}{4\pi(n+k)F_0} - \frac{\cos 2\pi(n-k)F_0 t}{4\pi(n-k)F_0}\right]_0^T\\ &= 0\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \cos 2\pi k F_0 t \cos 2\pi n F_0 t\,dt\\ &= \frac{1}{2}\int_0^T \cos 2\pi(n+k)F_0 t\,dt + \cos 2\pi(n-k)F_0 t\,dt\\ &= \frac{1}{2}\left[\frac{\sin 2\pi(n+k)F_0 t}{4\pi(n+k)F_0} + \frac{\sin 2\pi(n-k)F_0 t}{4\pi(n-k)F_0}\right]_0^T\\ &= 0\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \sin 2\pi k F_0 t \sin 2\pi n F_0 t\,dt\\ &= \frac{1}{2}\int_0^T \cos 2\pi(k-n)F_0 t\,dt - \cos 2\pi(n+k)F_0 t\,dt\\ &= \frac{1}{2}\left[\frac{\sin 2\pi(k-n)F_0 t}{4\pi(k-n)F_0} + \frac{\sin 2\pi(n+k)F_0 t}{4\pi(n+k)F_0}\right]_0^T\\ &= 0\end{aligned}$$

for $k, n = 1, 2, \ldots, N-1$. Therefore, we have the integral of $\boldsymbol{h}(t)\boldsymbol{h}^T$(t) as a scaled identity matrix given by

$$\int_0^T \boldsymbol{h}(t)\boldsymbol{h}^T(t)\,dt = \frac{T}{2}\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix} = (T/2)\mathbf{I}_{(2N-1)}.$$

Since $\boldsymbol{h}(t)$ is periodic with period T, for any τ,

$$\begin{aligned}\int_0^T \boldsymbol{h}(t-\tau)\boldsymbol{h}^T(t-\tau)\,dt &= \int_0^T \boldsymbol{h}(t)\boldsymbol{h}^T(t)\,dt\\ &= (T/2)\mathbf{I}_{(2N-1)}.\end{aligned} \tag{47}$$

Now, we compute the integral of the cross-product of the partial derivatives of $\boldsymbol{h}(t-\tau_i)$ w.r.t to τ_i

$$\begin{aligned}&\int_0^T \frac{\partial \boldsymbol{h}(t-\tau_i)}{\partial \tau_i}\frac{\partial \boldsymbol{h}^T(t-\tau_i)}{\partial \tau_i}\\ &= \int_0^T (2\pi F_0 \mathbf{L}\boldsymbol{h}(t-\tau_i))\,(2\pi F_0 \mathbf{L}\boldsymbol{h}(t-\tau_i))^T\;dt\\ &= (2\pi F_0)^2\mathbf{L}\left[\int_0^T \boldsymbol{h}(t-\tau_i)\boldsymbol{h}^T(t-\tau_i)\,dt\right]\mathbf{L}^T\\ &= (T/2)(2\pi F_0)^2\mathbf{L}\mathbf{L}^T\end{aligned} \tag{48}$$

and the integral of the cross-product of $\boldsymbol{h}(t-\tau_i)$ with its partial derivative w.r.t to τ_i is

$$\begin{aligned}&\int_0^T \frac{\partial \boldsymbol{h}(t-\tau_i)}{\partial \tau_i}\boldsymbol{h}^T(t-\tau_i)\\ &= \int_0^T 2\pi F_0 \mathbf{L}\boldsymbol{h}(t-\tau_i)\boldsymbol{h}^T(t-\tau_i)\,dt\\ &= (2\pi F_0)\mathbf{L}\left[\int_0^T \boldsymbol{h}(t-\tau_i)\boldsymbol{h}^T(t-\tau_i)\,dt\right]\\ &= (T/2)(2\pi F_0)\mathbf{L}.\end{aligned} \tag{49}$$

APPENDIX IV. TRANSFORMATION OF THE PARAMETERS

In Section III we discussed the relationship between the unknown attenuation factors and the unknown signal, and between the unknown TOAs and the unknown signal. Here we show that the FIM given in (8) is rank two deficient. Then we show that the transformation given in (9) satisfies the conditions given in [26]. From (8), we have

$$\mathcal{I}_\theta = \frac{(T/2)}{(N_0/2)}\begin{bmatrix}(2\pi F_0)^2\boldsymbol{\phi}^T\mathbf{L}\mathbf{L}^T\boldsymbol{\phi}(\mathrm{diag}(\mathbf{A}))^2 & (2\pi F_0)(\boldsymbol{\phi}^T\mathbf{L}\boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) & (2\pi F_0)(\mathbf{A}\odot\mathbf{A})\boldsymbol{\phi}^T\mathbf{L}\\ (2\pi F_0)(\boldsymbol{\phi}^T\mathbf{L}^T\boldsymbol{\phi})(\mathrm{diag}(\mathbf{A})) & (\boldsymbol{\phi}^T\boldsymbol{\phi})\mathbf{I}_M & \mathbf{A}\boldsymbol{\phi}^T\\ (2\pi F_0)\mathbf{L}^T\boldsymbol{\phi}(\mathbf{A}\odot\mathbf{A})^T & \boldsymbol{\phi}\mathbf{A}^T & (\mathbf{A}^T\mathbf{A})\mathbf{I}_{(2N-1)}\end{bmatrix}.$$

If $\boldsymbol{\nu}_1 = [\mathbf{1}_M^T\ \mathbf{A}^T\ -(\phi + (2\pi F_0)\mathbf{L}^T\phi)\ ]^T$ and $\boldsymbol{\nu}_2 = [\mathbf{1}_M^T - \mathbf{A}^T\ (\phi - (2\pi F_0)\mathbf{L}^T\phi)\ ]^T$ then it can be verified that $\mathcal{I}_\theta \boldsymbol{\nu}_1 = \mathbf{0}$ and $\mathcal{I}_\theta \boldsymbol{\nu}_2 = \mathbf{0}$. Therefore $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ are in the null space of $\mathcal{I}_\theta$. Also, $\mathcal{I}_\theta + (1/2)\boldsymbol{\nu}_1\boldsymbol{\nu}_1^T + (1/2)\boldsymbol{\nu}_2\ \boldsymbol{\nu}_2^T$ is nonsingular. This means that $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ are the basis vectors for the null space of $\mathcal{I}_\theta$ and so the matrix $\mathcal{I}_\theta$ is rank two deficient. The Jacobian of the transformation is given by

$$\mathbf{H} = \left(\frac{\partial \boldsymbol{\theta}'}{\partial \boldsymbol{\theta}}\right) = \begin{bmatrix} \frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\tau}} & \frac{\partial \boldsymbol{\tau}'}{\partial \mathbf{A}} & \frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\phi}} \\ \frac{\partial \mathbf{A}'}{\partial \boldsymbol{\tau}} & \frac{\partial \mathbf{A}'}{\partial \mathbf{A}} & \frac{\partial \mathbf{A}'}{\partial \boldsymbol{\phi}} \\ \frac{\partial \boldsymbol{\phi}'}{\partial \boldsymbol{\tau}} & \frac{\partial \boldsymbol{\phi}'}{\partial \mathbf{A}} & \frac{\partial \boldsymbol{\phi}'}{\partial \boldsymbol{\phi}} \end{bmatrix}. \tag{50}$$

Now, we compute each of the derivatives in the Jacobian matrix. The elements of the first subcolumn are given by

$$\begin{aligned} \frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\tau}} &= \frac{\partial}{\partial \boldsymbol{\tau}}([-\mathbf{1}_{M-1}\ \ \mathbf{I}_{M-1}]\boldsymbol{\tau}) \\ &= [-\mathbf{1}_{M-1}\ \ \mathbf{I}_{M-1}] \\ \frac{\partial \boldsymbol{\tau}'}{\partial \mathbf{A}} &= \mathbf{0}_{(M-1,M)} \\ \frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\phi}} &= \mathbf{0}_{(M-1,2N-1)}. \end{aligned}$$

The elements of the second subcolumn are given by

$$\begin{aligned} \frac{\partial \mathbf{A}'}{\partial \boldsymbol{\tau}} &= \mathbf{0}_{(M-1,M)} \\ \frac{\partial \mathbf{A}'}{\partial \mathbf{A}} &= \frac{\partial}{\partial \mathbf{A}}\left(\left([\mathbf{0}_{(M-1,1)}\ \ \mathbf{I}_{M-1}]\mathbf{A}\right)\left(\mathbf{e}_1^T\mathbf{A}\right)^{-1}\right) \\ &= (1/A_0)[-\mathbf{A}'\ \mathbf{I}_{M-1}] \\ \frac{\partial \mathbf{A}'}{\partial \boldsymbol{\phi}} &= \mathbf{0}_{(M-1,2N-1)} \end{aligned}$$

where $\mathbf{e}_1$ is the first column of an $M \times M$ identity matrix. The elements of the third subcolumn are given by

$$\begin{aligned} \frac{\partial \boldsymbol{\phi}'}{\partial \boldsymbol{\tau}} &= \frac{\partial}{\partial \boldsymbol{\tau}} A_0 \begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{I}_{N-1} \\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix} \end{bmatrix} \operatorname{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} \\ &= \left[A_0 \begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{I}_{N-1} \\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix} \end{bmatrix} \operatorname{diag}\left(\frac{\partial \boldsymbol{h}(-\tau_0)}{\partial \tau_0}\right)\boldsymbol{\phi} \quad \mathbf{0}_{(2N-1,1)} \quad \cdots \quad \mathbf{0}_{(2N-1,1)} \right] \\ &= \left[A_0 \begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{I}_{N-1} \\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix} \end{bmatrix} (2\pi F_0)\mathbf{L}\operatorname{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} \quad \mathbf{0}_{(2N-1,1)} \quad \cdots \quad \mathbf{0}_{(2N-1,1)} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial \boldsymbol{\phi}'}{\partial \mathbf{A}} &= \frac{\partial}{\partial \mathbf{A}} \left(\left(\mathbf{e}_1^T\mathbf{A}\right) \begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{I}_{N-1} \\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix} \end{bmatrix} \operatorname{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} \right) \\ &= \left[\begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{I}_{N-1} \\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix} \end{bmatrix} \operatorname{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} \quad \mathbf{0}_{(2N-1,1)} \quad \cdots \quad \mathbf{0}_{(2N-1,1)} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial \boldsymbol{\phi}'}{\partial \boldsymbol{\phi}} &= \frac{\partial}{\partial \boldsymbol{\phi}} \left(\left(\mathbf{e}_1^T\mathbf{A}\right) \begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{I}_{N-1} \\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix} \end{bmatrix} \operatorname{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} \right) \\ &= \left[A_0 \begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{I}_{N-1} \\ \mathbf{I}_{N-1} & -\mathbf{I}_{N-1} \end{bmatrix} \end{bmatrix} \operatorname{diag}(\boldsymbol{h}(-\tau_0)) \quad \mathbf{0}_{(2N-1,1)} \quad \cdots \quad \mathbf{0}_{(2N-1,1)} \right]. \end{aligned}$$

For the transformed parameters to have finite variance, the row vectors of $\mathbf{H}$ must be equal to the linear combinations of those eigenvectors of $\mathcal{I}_\theta$ that have nonzero eigenvalues [26]. In order to show that the row vectors of $\mathbf{H}$ are linear combinations of those eigenvectors of $\mathcal{I}_\theta$ that have nonzero eigenvalues, it is enough to show that the row vectors of $\mathbf{H}$ are orthogonal to the null space of $\mathcal{I}_\theta$. That is, it is enough to show that $\mathbf{H}\boldsymbol{\nu}_1 = \mathbf{0}$ and $\mathbf{H}\boldsymbol{\nu}_2 = \mathbf{0}$ Now,

$$\mathbf{H}\boldsymbol{\nu}_1 = \begin{bmatrix} \frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\tau}}\mathbf{1}_M + \frac{\partial \boldsymbol{\tau}'}{\partial \mathbf{A}}\mathbf{A} - \frac{\partial \boldsymbol{\tau}'}{\partial \boldsymbol{\phi}}(\boldsymbol{\phi} + (2\pi F_0)\mathbf{L}^T\boldsymbol{\phi}) \\ \frac{\partial \mathbf{A}'}{\partial \boldsymbol{\tau}}\mathbf{1}_M + \frac{\partial \mathbf{A}'}{\partial \mathbf{A}}\mathbf{A} - \frac{\partial \mathbf{A}'}{\partial \boldsymbol{\phi}}(\boldsymbol{\phi} + (2\pi F_0)\mathbf{L}^T\boldsymbol{\phi}) \\ \frac{\partial \boldsymbol{\phi}'}{\partial \boldsymbol{\tau}}\mathbf{1}_M + \frac{\partial \boldsymbol{\phi}'}{\partial \mathbf{A}}\mathbf{A} - \frac{\partial \boldsymbol{\phi}'}{\partial \boldsymbol{\phi}}(\boldsymbol{\phi} + (2\pi F_0)\mathbf{L}^T\boldsymbol{\phi}) \end{bmatrix}.$$

Substituting the partial derivatives that we computed previously and further simplifying gives

$$\mathbf{H}\boldsymbol{\nu}_1 = \begin{bmatrix} [-\mathbf{1}_{M-1}\mathbf{I}_{(M-1)}]\mathbf{1}_M + \mathbf{0}_{(M-1,1)} - \mathbf{0}_{(M-1,1)} \\ \mathbf{0}_{(M-1,1)} + (1/A_0)[-\mathbf{A}'\ \ \mathbf{I}_{(M-1)}]\mathbf{A} - \mathbf{0}_{(M-1,1)} \\ A_0\mathbf{P}(2\pi F_0)\mathbf{L}\mathrm{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} + A_0\mathbf{P}\mathrm{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} \\ -(A_0\mathbf{P}\mathrm{diag}(\boldsymbol{h}(-\tau_0))\boldsymbol{\phi} + A_0\mathbf{P}\mathrm{diag}(\boldsymbol{h}(-\tau_0))(2\pi F_0)\mathbf{L}^T\boldsymbol{\phi}) \end{bmatrix}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & \mathbf{0}_{(1,2N-2)} \\ \mathbf{0}_{(2N-2,1)} & \begin{bmatrix} \mathbf{I}_{(N-1)} & \mathbf{I}_{(N-1)} \\ \mathbf{I}_{(N-1)} & -\mathbf{I}_{(N-1)} \end{bmatrix} \end{bmatrix}.$$

Using the fact that $\mathbf{L}\mathrm{diag}\,(\boldsymbol{h}(-\tau_0)) = \mathrm{diag}\,(\boldsymbol{h}(-\tau_0))\mathbf{L}^T$, we have $\mathbf{H}\boldsymbol{\nu}_1 = \mathbf{0}$. Similarly it can be shown that $\mathbf{H}\boldsymbol{\nu}_2 = \mathbf{0}$.

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