Geolocation of a Known Altitude Object From TDOA and FDOA Measurements

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Most satellite systems for locating an object on Earth use only time difference of arrival (TDOA) measurements. When there are relative motions between an emitter and receivers, frequency difference of arrival (FDOA) measurements can be used as well. Often, the altitude of an object is known (it is zero, for example) or can be measured with an altimeter. Two sets of geolocation solutions are proposed which exploit the altitude constraint to improve the localization accuracy. One is for TDOAs alone and the other for the combination of TDOA and FDOA measurements. The additional complexity by imposing the constraint is a one-dimensional Newton's search and the rooting of a polynomial. The covariance matrices of the new estimators are derived under a small measurement noise assumption and shown to attain the constrained Cramèr-Rao lower bound (CRLB). When there is a bias error in the assumed altitude, using the altitude constraint will introduce a bias to the solution. Since applying the constraint decreases the variance, there is a tradeoff between variance and bias in the mean square error (MSE). The maximum allowable altitude error such that the constraint solution will remain superior to the unconstraint is given. Simulation results are included to corroborate the theoretical development.

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I. INTRODUCTION

Determining the location of an object on Earth has many applications, such as in navigation and remote sensing. The location system generally consists of a number of spatially well separated receivers that capture the radiated or reflected signal from the object. Due to their large coverage, localization of a close to Earth object by satellites has become popular in recent years. The geolocation systems currently in operation are VOR/DME, OMEGA, LORAN C, GPS, GLONASS, and GEOSTAR, with each of them having different coverage and accuracy [1]. These systems are originally developed for search and rescue, and the military. However, some of them such as GPS are available for civilian applications after being intentionally degraded in accuracy.

Geolocation is performed by first measuring the time difference of arrival (TDOA) and/or the frequency difference of arrival (FDOA) of a transmitted signal to or from a number of satellites in which their positions are known [2–3]. Then solving a set of nonlinear equations relating the measurements and unknown will give the object position estimate. Conventional solution methods are either multidimensional searches or iteration by linearization [4–5]. The former is computationally intensive while the latter cannot guarantee convergence to the correct solution unless the initial guess is close to it. As a consequence, various efforts have been devoted to finding a computationally simple solution; and [6–12] have derived closed-form hyperbolic fix solutions with 3 or more TDOA measurements while [13–15] have provided the GPS solutions with pseudo range measurements. A technique using two-stage least-squares computation [16] has also been proposed recently. All these computationally efficient algorithms work with TDOAs only. The incorporation of FDOAs for a better estimate has not yet been considered. Another factor not taken into account is a priori knowledge of the object. In some circumstances the altitude of the target is known. For example, an object on the Earth's surface has zero altitude at sea level. It is advantageous to include this constraint to increase location accuracy.

Currently, very little work in literature has considered the constrained localization problem. Moreover, no algebraic closed-form solution is available. It is the intention of this work to develop two sets of altitude constrained solutions with either TDOA or combined TDOA and FDOA measurements, and to study the effect on location accuracy when the altitude information is wrong.

Section II develops two sets of geolocation solutions which incorporate the prior altitude information to improve estimation accuracy. The first set is for TDOA only while the other is for combining TDOA and FDOA measurements. The

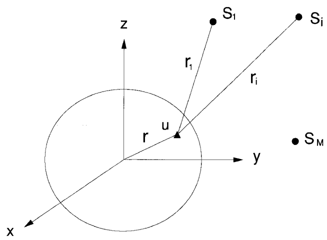

Fig. 1. Geolocation of object close to Earth.

proposed solutions are more computationally efficient than that by exhaustive search and do not suffer from convergence problems inherent in the iterative linearization method. The computations require a one-dimensional Newton's search and finding the positive root of a polynomial. Section III gives two special cases that can reduce the computations. Section IV compares the location variance with the constrained Cramèr-Rao lower bound (CRLB). The proposed solutions assume that the altitude of the object is known exactly which may not be valid in practice. An error in the assumed altitude will lead to a location bias that cannot be eliminated even when there is no measurement noise. Section V proposes the use of an oblate spheroid model to improve altitude accuracy. It also studies the maximum tolerable error on the altitude below which the altitude constraint remains useful. Section VI gives a performance study by simulations and conclusions are drawn in Section VII.

The following notations are used. Bold face upper case letter denotes matrix and bold face lower case letter denotes vector. $\{\cdot\}^o$ is the noise-free quantity of $\{\cdot\}$ and $\Delta\{\cdot\}$ is its random component so that $\{\cdot\} = \{\cdot\}^o + \Delta\{\cdot\}$. $\{\dot{\cdot}\}$ represents the time derivative of $\{\cdot\}$. Finally, $\mathbf{A} \equiv \mathbf{B} - \mathbf{C}$ means that $\mathbf{A}$ is defined as $\mathbf{B} - \mathbf{C}$.

II. ALGEBRAIC SOLUTION

Fig. 1 is the localization configuration in geocentric coordinate system [7]. We wish to find the location of an emitter on Earth, denoted by $\mathbf{u} \equiv [x, y, z]^T$, by measurements from M satellites, whose positions $\mathbf{s}_i \equiv [x_i, y_i, z_i]^T$ and speeds $\dot{\mathbf{s}}_i \equiv [\dot{x}_i, \dot{y}_i, \dot{z}_i]^T$, $i = 1, 2, \ldots, M$, are known. This requires an Earth station (not shown in the figure) to determine a set of $M - 1$ TDOAs, $d_{i,1}$, $i = 2, 3, \ldots, M$ between receiver (satellite) pair i and 1. If the receivers are in geosynchronous or lower circulating orbits, a set of FDOAs $f_{i,1} = f_c \dot{d}_{i,1}$, $i = 2, 3, \ldots, M$ can also be measured, where f_c is the carrier frequency. The problem is to locate the emitter from the TDOA only, or from combined TDOA and FDOA measurements, given the altitude of the emitter.

The first step in the localization problem is to relate the measurements to the unknown. Let r_i be the Euclidean distance between the emitter and receiver i:

$$r_i \equiv |\mathbf{s}_i - \mathbf{u}| \equiv \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2}, \quad i = 1, 2, \ldots, M. \tag{1}$$

Let c be the signal propagation speed. Then $cd_{i,1}$ is the range difference so that the TDOA measurement equations are

$$r_{i,1} \equiv cd_{i,1} = r_i - r_1, \qquad i = 2, 3, \ldots, M. \tag{2}$$

Substituting (1) into (2) results in a set of hyperbolae with the receivers as the foci. When the receivers are moving with speeds $\dot{\mathbf{s}}_i \equiv [\dot{x}_i, \dot{y}_i, \dot{z}_i]^T$, taking time derivative of (2) yields a set of FDOA measurement equations:

$$\dot{r}_{i,1} = c\dot{d}_{i,1} = \dot{r}_i - \dot{r}_1, \qquad i = 2, 3, \ldots, M \tag{3}$$

where $\dot{r}_i$ is the rate of change of r_i. From the time derivative of (1), $\dot{r}_i$ is related to the unknown location $\mathbf{u}$ by

$$\dot{r}_i = \frac{(\mathbf{s}_i - \mathbf{u})^T \dot{\mathbf{s}}_i}{r_i}. \tag{4}$$

Different from TDOA, the FDOA surface is not hyperbolic and is quite complicated. Let the sum of the known emitter altitude and the known local Earth radius (Earth radius at emitter location) be r. It is clear that the emitter location satisfies

$$\mathbf{u}^T \mathbf{u} = r^2. \tag{5}$$

Imposition of the altitude constraint (5) allows one to solve for a solution with only two measurements, both TDOAs, both FDOAs, or a combination of them. In the TDOA only case, the unknown is solved by (2) and (5). In the combined case, the unknown is the solution of (2), (3), and (5).

The solution is not simple because of the nonlinearity between measurements and unknown. Previous methods rely on a two-dimensional search in the parameter space or iterative solution by linearization. The former is computationally intensive and the latter suffers from convergence problems. Currently, there are no closed-form solutions, although many are available for the unconstrained case [6–16]. We provide computationally attractive solutions to the emitter location problem.

In practice, the measurements are corrupted by noise so that a variance exists in the location estimate. In order to study the localization accuracy, the following measurement models are assumed

$$\mathbf{d} = \mathbf{d}^0 + \Delta\mathbf{d}; \qquad E[\Delta\mathbf{d}] = \mathbf{0}, \qquad E[\Delta\mathbf{d}\,\Delta\mathbf{d}^T] = \mathbf{Q}_t \tag{6}$$

and

$$\dot{\mathbf{d}} = \dot{\mathbf{d}}^0 + \Delta\dot{\mathbf{d}}, \qquad E[\Delta\dot{\mathbf{d}}] = \mathbf{0}, \qquad E[\Delta\dot{\mathbf{d}}\,\Delta\dot{\mathbf{d}}^T] = \mathbf{Q}_f \tag{7}$$

where $\mathbf{d} = [d_{2,1}, d_{3,1}, \ldots, d_{M,1}]^T$ is a vector of TDOAs, $\dot{\mathbf{d}} = [\dot{d}_{2,1}, \dot{d}_{3,1}, \ldots, \dot{d}_{M,1}]^T$ is a vector of FDOAs and $\mathbf{0}$ is a zero vector of appropriate size. Furthermore, $\Delta\mathbf{d}$ and $\Delta\dot{\mathbf{d}}$ are assumed to be uncorrelated. This uncorrelated condition is valid when a maximum likelihood (ML) estimator for TDOA and FDOA is employed.

The presentation of the algebraic, closed-form solutions is as follows. Subsection A is the estimation methods for TDOA case while Subsection B the methods for the combined case. Each subsection discusses separately the critically determined and overdetermined situations, and each contains the solution and the associated location accuracy. Matrix notation is used throughout the derivation for simplicity.

A. TDOA

The method starts by transforming the TDOA equations (2) to another set of equations [16]. Rewriting (2) as $r_{i,1} + r_1 = r_i$, squaring both sides and substituting in (1), we obtain

$$r_{i,1}^2 + 2r_{i,1}r_1 + r_1^2 = r^2 + \mathbf{s}_i^T\mathbf{s}_i - 2\mathbf{s}_i^T\mathbf{u}, \quad i = 2,3,\ldots,M \tag{8a}$$

where r_1 is from (1)

$$r_1^2 = r^2 + \mathbf{s}_1^T\mathbf{s}_1 - 2\mathbf{s}_1^T\mathbf{u}. \tag{8b}$$

Note that (5) has been used in both (8a) and (8b). Subtraction of (8b) from (8a) yields

$$r_{i,1}^2 + 2r_{i,1}r_1 = \mathbf{s}_i^T\mathbf{s}_i - \mathbf{s}_1^T\mathbf{s}_1 - 2(\mathbf{s}_i - \mathbf{s}_1)^T\mathbf{u}, \quad i = 2,3,\ldots,M. \tag{9}$$

Solving {(2), (5)} is equivalent to solving {(8b), (9), (5)}. The second set of equations is algebraically simpler than the original one as the square root in (2) has been removed by introducing a temporary variable r_1. To find the unknown location $\mathbf{u}$, we first use (8b) and (9) to express it in terms of the variable r_1. The use of (5) will then give a polynomial in r_1 whose roots can be evaluated by numerical methods. Substitution of the computed r_1 value into (8a) and (9) and solving gives $\mathbf{u}$.

1) $M = 3$: From (8b) and (9), the solution of $\mathbf{u}$ in terms of r_1 is

$$\mathbf{u} = \mathbf{G}_1^{-1}\mathbf{h};$$
$$\mathbf{G}_1 \equiv -2\begin{bmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T - \mathbf{s}_1^T \\ \mathbf{s}_3^T - \mathbf{s}_1^T \end{bmatrix}, \tag{10}$$
$$\mathbf{h} \equiv \begin{bmatrix} -r^2 - \mathbf{s}_1^T\mathbf{s}_1 & 0 & 1 \\ r_{2,1}^2 - \mathbf{s}_2^T\mathbf{s}_2 + \mathbf{s}_1^T\mathbf{s}_1 & 2r_{2,1} & 0 \\ r_{3,1}^2 - \mathbf{s}_3^T\mathbf{s}_3 + \mathbf{s}_1^T\mathbf{s}_1 & 2r_{3,1} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ r_1 \\ r_1^2 \end{bmatrix}.$$

Substitution of (10) into (5) generates a fourth order polynomial in r_1. Putting the positive root(s) back into (10) yields the location estimate. On some occasions, the polynomial may contain more than one positive root, giving more than one solution. The ambiguity can be resolved by choosing the one that satisfies the original measurement equations (2). On some rare occasions, some prior knowledge about the emitter region may be needed.

The solution requires the matrix $\mathbf{G}_1$ to be invertible. This is equivalent to the observability condition that not any three of $\{\mathbf{0}, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ lie on a straight line, where $\mathbf{0}$ is the Earth center. In addition, the three receivers should have sufficient spatial separation to eliminate the ill-conditioned problem.

Since the solution is the intersection of (2) and (5), taking differentials of these equations with respect to $\mathbf{u}$ and evaluating them at the noise free values give

$$\mathbf{H}\Delta\mathbf{u} = c\begin{bmatrix} \Delta\mathbf{d} \\ 0 \end{bmatrix}$$
$$\mathbf{H} \equiv -\begin{bmatrix} (\mathbf{s}_2 - \mathbf{u}^0)^T/r_2^0 - (\mathbf{s}_1 - \mathbf{u}^0)^T/r_1^0 \\ (\mathbf{s}_3 - \mathbf{u}^0)^T/r_3^0 - (\mathbf{s}_1 - \mathbf{u}^0)^T/r_1^0 \\ \mathbf{u}^{0T} \end{bmatrix}. \tag{11}$$

Hence the solution is unbiased with small zero mean measurement noise and the location covariance matrix is

$$\Psi = \mathrm{cov}(\mathbf{u}) = c^2\mathbf{H}^{-1}\begin{bmatrix} \mathbf{Q}_t & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}\mathbf{H}^{-T}. \tag{12}$$

This result is identical to that in [6 and 16].

2) $M \geq 4$: This is an overdetermined situation in which the number of equations is larger than the number of unknowns. The equations are often inconsistent because of measurement noise. In this case, we minimize the equation error in (9) with two constraints: one is the nonlinear relationship (8b) while the other is the quadratic constraint (5). Introducing Lagrange multipliers λ_1 and λ_2, the augmented cost function is

$$\xi \equiv (\mathbf{h} - \mathbf{G}_1\mathbf{u} - \mathbf{g}_2 r_1)^T\mathbf{W}(\mathbf{h} - \mathbf{G}_1\mathbf{u} - \mathbf{g}_2 r_1) + \lambda_1(2\mathbf{s}_1^T\mathbf{u} - \mathbf{s}_1^T\mathbf{s}_1 - r^2 + r_1^2) + \lambda_2(\mathbf{u}^T\mathbf{u} - r^2);$$
$$\mathbf{h} \equiv \begin{bmatrix} r_{2,1}^2 - \mathbf{s}_2^T\mathbf{s}_2 + \mathbf{s}_1^T\mathbf{s}_1 \\ r_{3,1}^2 - \mathbf{s}_3^T\mathbf{s}_3 + \mathbf{s}_1^T\mathbf{s}_1 \\ \vdots \\ r_{M,1}^2 - \mathbf{s}_M^T\mathbf{s}_M + s_1^T\mathbf{s}_1 \end{bmatrix}, \tag{13}$$
$$\mathbf{G}_1 \equiv -2\begin{bmatrix} \mathbf{s}_2^T - \mathbf{s}_1^T \\ \mathbf{s}_3^T - \mathbf{s}_1^T \\ \vdots \\ \mathbf{s}_M^T - \mathbf{s}_1^T \end{bmatrix}, \qquad \mathbf{g}_2 \equiv -2\begin{bmatrix} r_{2,1} \\ r_{3,1} \\ \vdots \\ r_{M,1} \end{bmatrix}$$

where $\mathbf{W}$ is a weighting matrix. Setting derivatives of ξ with respect to $\mathbf{u}$ and r_1 to zero produces

$$\begin{aligned}\mathbf{u} &= \mathbf{G}_4(\mathbf{G}_1^T\mathbf{W}\mathbf{G}_5\mathbf{r}_1 - \lambda_1\mathbf{s}_1)\\ \mathbf{G}_4 &\equiv (\mathbf{G}_1^T\mathbf{W}\mathbf{G}_1 + \lambda_2\mathbf{I})^{-1},\\ \mathbf{r}_1 &\equiv [1, r_1, r_1^2]^T,\\ \mathbf{G}_5 &\equiv [\mathbf{h}, -\mathbf{g}_2, \mathbf{0}]\end{aligned} \tag{14}$$

and

$$-\mathbf{g}_2^T\mathbf{W}(\mathbf{G}_5\mathbf{r}_1 - \mathbf{G}_1\mathbf{u}) + \lambda_1 r_1 = 0 \tag{15}$$

where $\mathbf{I}$ is an identity matrix. Let $\mathbf{g}_3 \equiv [\mathbf{s}_1^T\mathbf{s}_1 + r^2, 0, -1]^T$. Then (8b) becomes $2\mathbf{s}_1^T\mathbf{u} = \mathbf{g}_3^T\mathbf{r}_1$. Inserting $\mathbf{u}$ in (14) into (8b) gives λ_1:

$$\lambda_1 = \mathbf{g}_6^T\mathbf{r}_1; \qquad \mathbf{g}_6^T \equiv \frac{2\mathbf{s}_1^T\mathbf{G}_4\mathbf{G}_1^T\mathbf{W}\mathbf{G}_5 - \mathbf{g}_3^T}{2\mathbf{s}_1^T\mathbf{G}_4\mathbf{s}_1}. \tag{16}$$

Putting (16) into (14) yields

$$\mathbf{u} = \mathbf{G}_7\mathbf{r}_1; \qquad \mathbf{G}_7 \equiv \mathbf{G}_4(\mathbf{G}_1^T\mathbf{W}\mathbf{G}_5 - \mathbf{s}_1\mathbf{g}_6^T). \tag{17}$$

Substitution of (16) and (17) into (15) forms a 3rd-order polynomial in r_1,

$$\begin{aligned} r_1\mathbf{g}_6^T\mathbf{r}_1 - \mathbf{g}_8^T\mathbf{r}_1 &= 0\\ \mathbf{g}_8^T &\equiv \mathbf{g}_2^T\mathbf{W}(\mathbf{G}_5 - \mathbf{G}_1\mathbf{G}_7).\end{aligned} \tag{18}$$

The equation can be solved for a particular λ_2. There is only one positive root in most cases since this is an overdetermined situation. Inserting the positive r_1 into (17) produces an emitter location estimate in terms of λ_2. The appropriate λ_2 is the one that makes the solution satisfy (5). If (18) has more than one positive root, some prior knowledge about emitter region is necessary to choose the proper root.

Newton's method [23] is effective for finding the zeros of $\rho(\lambda_2) = \mathbf{u}^T\mathbf{u} - r^2$. Several zeros may occur because of the nonlinear nature of the problem. The solution with $\lambda_2 = 0$ is the one without exploiting the altitude information. It follows that the correct λ_2 must be closest to zero and Newton's method should start with $\lambda_2 = 0$. An inspection of $\mathbf{G}_4$ in (14) reveals that λ_2 modifies its eigenvalues only. Hence an eigenvalue decomposition of $\mathbf{G}_1^T\mathbf{W}\mathbf{G}_1$ will avoid matrix inversion and reduce computations. Our technique is more computationally efficient than two-dimensional exhaustive search and does not require a proper initial position guess as in iterative linearization methods.

The weighting matrix is given by $\mathbf{W} \equiv E[\phi_t\phi_t^T]^{-1}$, where ϕ_t is the equation error vector in (9) due to TDOA measurement noise:

$$\phi_t \equiv \mathbf{h} - \mathbf{G}_1\mathbf{u}^0 - \mathbf{g}_2 r_1^0 \tag{19a}$$

where the matrix $\mathbf{G}_1$, vectors $\mathbf{h}$ and $\mathbf{g}_2$ are defined in (13). Expressing $\mathbf{h}$ and $\mathbf{g}_2$ as the sum of their true and random components, another form of (19a) is

$$\phi_t = \Delta\mathbf{h} - \Delta\mathbf{g}_2 r_1^0. \tag{19b}$$

Using measurement model (6) in $\mathbf{h}$ and $\mathbf{g}_2$, produces [16]

$$\begin{aligned}\phi_t &= c\mathbf{B}^0\Delta\mathbf{d} + c^2\Delta\mathbf{d}\odot\Delta\mathbf{d}\\ \mathbf{B} &\equiv \mathrm{diag}\{2r_2, 2r_3, \ldots, 2r_M\}\end{aligned} \tag{19c}$$

where $\odot$ is the Schur product. With small measurement noise, the weighting matrix simplifies to (ignoring the scaling factor c)

$$\mathbf{W} \approx (\mathbf{B}^0\mathbf{Q}_t\mathbf{B}^0)^{-1}. \tag{20a}$$

The weighting matrix is not known because $\mathbf{B}^0$ depends on the true emitter location as seen in (1). In practice satellites are at high altitudes so that the r_i^0, $i = 2, 3, \ldots, M$ are roughly the same. Since a scaling of $\mathbf{W}$ does not affect the location estimate, an approximation of it is

$$\mathbf{W} \approx \mathbf{Q}_t^{-1}. \tag{20b}$$

For a more precise result, the coarse solution from (20b) can provide an estimate of $\mathbf{B}^0$ for (20a).

Perturbation analysis is employed to compute the location covariance matrix. Representing $\mathbf{h}$ as $\mathbf{h}^0 + \Delta\mathbf{h}$, $\mathbf{g}_2$ as $\mathbf{g}_2^0 + \Delta\mathbf{g}_2$, r_1 as $r_1^0 + \Delta r_1$, and $\mathbf{u}$ as $\mathbf{u}^0 + \Delta\mathbf{u}$, the cost function ξ in (13) becomes

$$\begin{aligned}\xi &= (\phi_t - \mathbf{G}_1\Delta\mathbf{u} - \mathbf{g}_2^0\Delta r_1)^T\mathbf{W}(\phi_t - \mathbf{G}_1\Delta\mathbf{u} - \mathbf{g}_2^0\Delta r_1)\\ &\quad + \lambda_1\{2(\mathbf{s}_1 - \mathbf{u}^0)^T\Delta\mathbf{u} + 2r_1^0\Delta r_1\} + \lambda_2(2\mathbf{u}^{0T}\Delta\mathbf{u})\end{aligned} \tag{21}$$

where second-order perturbation terms have been ignored and (19b) has been used. Note that r^2 has been replaced by $\mathbf{u}^T\mathbf{u}$ in the first constraint. Since the location solution occurs at the minimum of ξ, this implies $\Delta\mathbf{u}$ minimizes ξ as well. Setting derivatives of ξ with respect to $\Delta\mathbf{u}$, Δr_1, λ_1 and λ_2 to zero yields

$$-\mathbf{G}_1^T\mathbf{W}(\phi_t - \mathbf{G}_1\Delta\mathbf{u} - \mathbf{g}_2^0\Delta r_1) + \lambda_1(\mathbf{s}_1 - \mathbf{u}^0) + \lambda_2\mathbf{u}^0 = 0 \tag{22}$$

$$-\mathbf{g}_2^{0T}\mathbf{W}(\phi_t - \mathbf{G}_1\Delta\mathbf{u} - \mathbf{g}_2^0\Delta r_1) + \lambda_1 r_1^0 = 0 \tag{23}$$

$$\Delta r_1 = \frac{-1}{r_1^0}(\mathbf{s}_1 - \mathbf{u}^0)^T\Delta\mathbf{u} \tag{24}$$

$$\mathbf{u}^{0T}\Delta\mathbf{u} = 0. \tag{25}$$

Substituting (24) into (23) gives

$$\begin{aligned}\lambda_1 &= \frac{1}{r_1^0}\mathbf{g}_2^{0T}\mathbf{W}(\phi_t - \mathbf{G}_6^0\Delta\mathbf{u})\\ \mathbf{G}_6 &\equiv \mathbf{G}_1 - \frac{1}{r_1}\mathbf{g}_2(\mathbf{s}_1 - \mathbf{u})^T.\end{aligned} \tag{26}$$

Putting (24) and (26) into (22) yields

$$\Delta\mathbf{u} = (\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\{\mathbf{G}_6^{0T}\mathbf{W}\phi_t - \lambda_2\mathbf{u}^0\}. \tag{27}$$

If we premultiply (27) by $\mathbf{u}^{0T}$ and use (25),

$$\lambda_2 = \frac{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{G}_6^{0T}\mathbf{W}\phi_t}{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0} \tag{28}$$

and (27) now becomes

$$\Delta\mathbf{u} = \left\{\mathbf{I} - \frac{(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0\mathbf{u}^{0T}}{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}\right\}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{G}_6^{0T}\mathbf{W}\phi_t. \tag{29}$$

When the noise is small, (19c) implies the location estimate is unbiased. In addition,

$$\Psi \equiv \mathrm{cov}(\mathbf{u}) = \left\{\mathbf{I} - \frac{(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0\mathbf{u}^{0T}}{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}\right\}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1} \tag{30}$$

in which $\mathbf{W} = E[\phi_t\phi_t^T]^{-1}$ has been used for simplification. It is easy to verify that $\mathrm{cov}(\mathbf{u})\mathbf{u}^0 = 0$. That means that the variation in location estimate is always orthogonal to its true value. It also implies that $\mathrm{cov}(\mathbf{u})$ is rank deficient with the null space $\mathbf{u}^0$ as a result of the constraint (5).

B. TDOA and FDOA

For inclined geosynchronous or lower orbit satellites, relative motions between emitter and receivers induce frequency shifts in the received signals. FDOAs can be estimated and used as additional information to improve a position estimate. With both kinds of measurements, a minimum of 2 satellites is sufficient to produce a solution. This is because two spatially separated receivers with at least one moving can generate one TDOA and one FDOA. Together with the altitude constraint (5), they provide three equations to fix the emitter location.

The FDOA surface is very complicated. However, the availability of TDOA permits one to form a simple equation relating TDOA, FDOA, and emitter location. It is obtained by taking time derivative of (9), giving

$$\begin{aligned} 2r_{i,1}\dot{r}_{i,1} + 2r_{i,1}\dot{r}_1 + 2\dot{r}_{i,1}r_1 - 2\mathbf{s}_i^T\dot{\mathbf{s}}_i + 2\mathbf{s}_1^T\dot{\mathbf{s}}_1 \\ = -2(\dot{\mathbf{s}}_i - \dot{\mathbf{s}}_1)^T\mathbf{u}, \qquad i = 2,3,\ldots,M. \end{aligned} \tag{31}$$

Apart from r_1, the combined case has another intermediate variable $\dot{r}_1$. It is related to r_1 and $\mathbf{u}$ as in (4). The solution is found by the TDOA equations: {(8b), (9)}, TDOA and FDOA equations: {(31), (4)}and the altitude constraint {(5)}. The solution approach is to first express $\mathbf{u}$ in terms of the temporary variables r_1 and $\dot{r}_1$. Application of (4) then reduces $\mathbf{u}$ to a function of r_1 only. Substitution of the intermediate result into (5) gives a polynomial in r_1. Putting the positive root back forms the final solution.

1) $M = 2$: With one TDOA $d_{2,1}$ and one FDOA $\dot{d}_{2,1}$, applying (8b), (9), and (31) yields $\mathbf{u}$ in terms of r_1 and $\dot{r}_1$:

$$\begin{aligned}
\mathbf{u} &= \mathbf{G}_1^{-1}\mathbf{h} = \mathbf{G}_4\mathbf{r}_1 + \mathbf{g}_5\dot{r}_1 \\
\mathbf{G}_1 &\equiv -2\begin{bmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T - \mathbf{s}_1^T \\ \dot{\mathbf{s}}_2^T - \dot{\mathbf{s}}_1^T \end{bmatrix}, \\
\mathbf{h} &\equiv \mathbf{G}_2\mathbf{r}_1 + \mathbf{g}_3\dot{r}_1 \\
&\equiv \begin{bmatrix} -r^2 - \mathbf{s}_1^T\mathbf{s}_1 & 0 & 1 \\ r_{2,1}^2 - \mathbf{s}_2^T\mathbf{s}_2 + \mathbf{s}_1^T\mathbf{s}_1 & 2r_{2,1} & 0 \\ 2r_{2,1}\dot{r}_{2,1} - 2\mathbf{s}_2^T\dot{\mathbf{s}}_2 + 2\mathbf{s}_1^T\dot{\mathbf{s}}_1 & 2\dot{r}_{2,1} & 0 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 \\ r_1 \\ r_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2r_{2,1} \end{bmatrix}\dot{r}_1 \\
\mathbf{G}_4 &\equiv \mathbf{G}_1^{-1}\mathbf{G}_2, \qquad \mathbf{g}_5 \equiv \mathbf{G}_1^{-1}\mathbf{g}_3.
\end{aligned} \tag{32}$$

The variable $\dot{r}_1$ can be expressed in terms of r_1 when substituting (32) into (4), as

$$\begin{aligned}
\dot{r}_1 &= \frac{1}{r_1 + p}\mathbf{g}_6^T\mathbf{r}_1 \\
p &\equiv \dot{\mathbf{s}}_1^T\mathbf{g}_5 \\
\mathbf{g}_6 &\equiv \begin{bmatrix} \mathbf{s}_1^T\dot{\mathbf{s}}_1 \\ 0 \\ 0 \end{bmatrix} - \mathbf{G}_4^T\dot{\mathbf{s}}_1
\end{aligned} \tag{33}$$

so that (32) now becomes

$$\begin{aligned}
\mathbf{u} &= \frac{\mathbf{G}_7\mathbf{r}_1}{r_1 + p} \\
\mathbf{G}_7 &\equiv \begin{bmatrix} p\mathbf{G}_4 + \mathbf{g}_5\mathbf{g}_6^T & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{G}_4 \end{bmatrix} \\
\mathbf{r}_1 &\equiv \begin{bmatrix} 1 \\ r_1 \\ r_1^2 \\ r_1^3 \end{bmatrix}.
\end{aligned} \tag{34}$$

Putting (34) into (7) gives a 6th-order polynomial in r_1,

$$\begin{aligned}
\mathbf{r}_1^T\mathbf{G}_8\mathbf{r}_1 - \mathbf{g}_9^T\mathbf{r}_1 &= 0 \\
\mathbf{G}_8 &\equiv \mathbf{G}_7^T\mathbf{G}_7 \\
\mathbf{g}_9 &\equiv r^2[p^2, 2p, 1, 0]^T.
\end{aligned} \tag{35}$$

The positive roots define possible solutions according to (34). The solution ambiguity is resolved by choosing the one that fulfils the measurement equations (2) and (3). In some rare cases, a priori knowledge about the expected region of the emitter may be needed.

The solution requires the matrix $\mathbf{G}_1$ to be invertible. The conditions are: 1) the Earth center and the two receivers do not lie on a straight line, and 2) the non-zero relative motion between the two receivers is neither in the direction of $\mathbf{s}_1$ nor that of $\mathbf{s}_1 - \mathbf{s}_2$.

Schmidt [24] has also derived a solution for this case by using a different formulation. The y and z coordinates of the emitter are expressed in terms of the x coordinate. The location is found by selecting a trial x, computing y and z, and checking whether these values satisfy (5). This procedure is mathematically equivalent to finding the positive root of a 12th-order polynomial. Our method is expected to be less computationally intensive.

To compute location variance, we take differentials of (2), (3), and (5) to form

$$
\begin{aligned}
&\mathbf{H}\Delta\mathbf{u} = c\begin{bmatrix}\Delta d_{2,1}\\ \Delta\dot{d}_{2,1}\\ 0\end{bmatrix}\\
&\mathbf{H} \equiv \begin{bmatrix} -(\mathbf{s}_2-\mathbf{u}^0)^T/r_2^0 + (\mathbf{s}_1-\mathbf{u}^0)^T/r_1^0\\ (\mathbf{s}_2-\mathbf{u}^0)^T\dot{r}_2^0/r_2^{02} - (\mathbf{s}_1-\mathbf{u}^0)^T\dot{r}_1^0/r_1^{02} - \dot{\mathbf{s}}_2^T/r_2^0 + \dot{\mathbf{s}}_1^T/r_1^0\\ \mathbf{u}^{0T}\end{bmatrix}
\end{aligned}
\tag{36}
$$

which implies at small measurement noise, the solution is unbiased and the covariance matrix is

$$
\Psi = c^2\mathbf{H}^{-1}\begin{bmatrix}\mathbf{Q}_t & 0 & 0\\ 0 & \mathbf{Q}_f & 0\\ 0 & 0 & 0\end{bmatrix}\mathbf{H}^{-T}. \tag{37}
$$

2) $M \geq 2$: In the overdetermined situation, we minimize the equation error of (9) and (31), with the constraints (4) at $i = 1$, (5), and (8b). The cost function is

$$
\begin{aligned}
\xi \equiv{}& (\mathbf{h}-\mathbf{G}_1\mathbf{u}-\mathbf{g}_2 r_1-\mathbf{g}_3\dot{r}_1)^T\mathbf{W}(\mathbf{h}-\mathbf{G}_1\mathbf{u}-\mathbf{g}_2 r_1-\mathbf{g}_3\dot{r}_1) + \lambda_1(2\mathbf{s}_1^T\mathbf{u}-\mathbf{s}_1^T\mathbf{s}_1-r^2+r_1^2)\\
&+\lambda_2(2\dot{\mathbf{s}}_1^T\mathbf{u}-2\mathbf{s}_1^T\dot{\mathbf{s}}_1+2r_1\dot{r}_1)+\lambda_3(\mathbf{u}^T\mathbf{u}-r^2)\\
\mathbf{h} \equiv{}& \begin{bmatrix} r_{2,1}^2-\mathbf{s}_2^T\mathbf{s}_2+\mathbf{s}_1^T\mathbf{s}_1\\ r_{3,1}^2-\mathbf{s}_3^T\mathbf{s}_3+\mathbf{s}_1^T\mathbf{s}_1\\ \vdots\\ r_{M,1}^2-\mathbf{s}_M^T\mathbf{s}_M+\mathbf{s}_1^T\mathbf{s}_1\\ 2r_{2,1}\dot{r}_{2,1}-2\mathbf{s}_2^T\dot{\mathbf{s}}_2+2\mathbf{s}_1^T\dot{\mathbf{s}}_1\\ 2r_{3,1}\dot{r}_{3,1}-2\mathbf{s}_3^T\dot{\mathbf{s}}_3+2\mathbf{s}_1^T\dot{\mathbf{s}}_1\\ \vdots\\ 2r_{M,1}\dot{r}_{M,1}-2\mathbf{s}_M^T\dot{\mathbf{s}}_M+2\mathbf{s}_1^T\dot{\mathbf{s}}_1\end{bmatrix},\qquad
\mathbf{G}_1 \equiv -2\begin{bmatrix}\mathbf{s}_2^T-\mathbf{s}_1^T\\ \mathbf{s}_3^T-\mathbf{s}_1^T\\ \vdots\\ \mathbf{s}_M^T-\mathbf{s}_1^T\\ \dot{\mathbf{s}}_2^T-\dot{\mathbf{s}}_1^T\\ \dot{\mathbf{s}}_3^T-\dot{\mathbf{s}}_1^T\\ \vdots\\ \dot{\mathbf{s}}_M^T-\dot{\mathbf{s}}_1^T\end{bmatrix},\\
\mathbf{g}_2 \equiv{}& -2\begin{bmatrix} r_{2,1}\\ r_{3,1}\\ \vdots\\ r_{M,1}\\ \dot{r}_{2,1}\\ \dot{r}_{3,1}\\ \vdots\\ \dot{r}_{M,1}\end{bmatrix},\qquad
\mathbf{g}_3 \equiv -2\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ r_{2,1}\\ r_{3,1}\\ \vdots\\ r_{M,1}\end{bmatrix}
\end{aligned}
\tag{38}
$$

where $\mathbf{W}$ is a weighing matrix and λ_1, λ_2, and λ_3 are the Lagrange multipliers. Setting the partial derivative of ξ with respect to $\mathbf{u}$ to zero gives

$$
\begin{aligned}
&\mathbf{u} = \mathbf{G}_6\mathbf{r}_1 - \lambda_1\mathbf{G}_4\mathbf{s}_1 - \lambda_2\mathbf{G}_4\dot{\mathbf{s}}_1\\
&\tilde{\mathbf{r}}_1 \equiv [1, r_1, r_1^2, \dot{r}_1, r_1\dot{r}_1]^T\\
&\mathbf{G}_4 \equiv (\mathbf{G}_1^T\mathbf{W}\mathbf{G}_1 + \lambda_3\mathbf{I})^{-1}\\
&\mathbf{G}_5 \equiv [\mathbf{h}, -\mathbf{g}_2, \mathbf{0}, -\mathbf{g}_3, \mathbf{0}],\\
&\mathbf{G}_6 \equiv \mathbf{G}_4\mathbf{G}_1^T\mathbf{W}\mathbf{G}_5.
\end{aligned}
\tag{39}
$$

If we premultiply $\mathbf{u}$ by $2\mathbf{s}_1^T$ and $2\dot{\mathbf{s}}^T$ and use the constraints (8a) and (4), λ_1 and λ_2 are found to be

$$
\begin{aligned}
\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} &= \begin{bmatrix}\mathbf{g}_9^T\\ \mathbf{g}_{10}^T\end{bmatrix}\tilde{\mathbf{r}}_1\\
&\equiv \frac{1}{2}\begin{bmatrix}\mathbf{s}_1^T\mathbf{G}_4\mathbf{s}_1 & \mathbf{s}_1^T\mathbf{G}_4\dot{\mathbf{s}}_1\\ \dot{\mathbf{s}}_1^T\mathbf{G}_4\mathbf{s}_1 & \dot{\mathbf{s}}_1^T\mathbf{G}_4\dot{\mathbf{s}}_1\end{bmatrix}^{-1}\begin{bmatrix}2\mathbf{s}_1^T\mathbf{G}_6-\mathbf{g}_7^T\\ 2\dot{\mathbf{s}}_1^T\mathbf{G}_6-\mathbf{g}_8^T\end{bmatrix}\tilde{\mathbf{r}}_1 \qquad (40)\\
\mathbf{g}_7 &\equiv [\mathbf{s}_1^T\mathbf{s}_1 + r^2, 0, -1, 0, 0]^T,\\
\mathbf{g}_8 &\equiv [2\dot{\mathbf{s}}_1^T\mathbf{s}_1, 0, 0, 0, -2]^T.
\end{aligned}
$$

Inserting (40) into (39) allows one to express $\mathbf{u}$ in terms of r_1 and $\dot{r}_1$ as

$$
\mathbf{u} = \mathbf{G}_{11}\tilde{\mathbf{r}}_1; \qquad \mathbf{G}_{11} \equiv \mathbf{G}_6 - \mathbf{G}_4(\mathbf{s}_1\mathbf{g}_9^T + \dot{\mathbf{s}}_1\mathbf{g}_{10}^T). \tag{41}
$$

Equating the derivative of ξ with respect to $\dot{r}_1$ to zero and using (41) yields

$$\begin{aligned} & r_1 \mathbf{g}_{10}^T \tilde{\mathbf{r}}_1 - \mathbf{g}_3^T \mathbf{W} \mathbf{G}_{12} \tilde{\mathbf{r}}_1 = 0 \\ & \mathbf{G}_{12} \equiv \mathbf{G}_5 - \mathbf{G}_1 \mathbf{G}_{11}. \end{aligned} \tag{42}$$

Since the last two elements of $\tilde{\mathbf{r}}_1$ contain $\dot{r}_1$, $\dot{r}_1$ can be written in terms of r_1 as

$$\begin{aligned} \dot{r}_1 &= \frac{\mathbf{g}_{15}^T \mathbf{r}_1}{\mathbf{g}_{14}^T \mathbf{r}_1} \\ \mathbf{r}_1 &\equiv [1, r_1, r_1^2, r_1^3, r_1^4]^T \\ \mathbf{g}_{13} &\equiv \mathbf{G}_{12}^T \mathbf{W}^T \mathbf{g}_3 \\ \mathbf{g}_{14}^T &\equiv [-g_{13}(4), g_{10}(4) - g_{13}(5), g_{10}(5), 0, 0]^T \\ \mathbf{g}_{15} &\equiv [g_{13}(1), g_{13}(2) - g_{10}(1), g_{13}(3) - g_{10}(2), -g_{10}(3), 0]^T \end{aligned} \tag{43}$$

where $g(i)$ denotes the ith element of a vector $\mathbf{g}$. Using the definition of $\tilde{\mathbf{r}}_1$ given in (39), one can deduce from (43) that

$$\begin{aligned} (\mathbf{g}_{14}^T \mathbf{r}_1) \tilde{\mathbf{r}}_1 &= \mathbf{G}_{16} \mathbf{r}_1 \\ \mathbf{G}_{16} &\equiv [\mathbf{g}_{14}, \mathbf{C}\mathbf{g}_{14}, \mathbf{C}^2 \mathbf{g}_{14}, \mathbf{g}_{15}, \mathbf{C}\mathbf{g}_{15}]^T \\ \mathbf{C} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned} \tag{44}$$

and r_1 can be found by first setting the derivative of ξ with respect to r_1 to zero. Upon multiplying the resulting equation by $(\mathbf{g}_{14}^T \mathbf{r}_1)^2$ and then substituting in (40), (43), and (44), we have

$$\begin{aligned} & r_1 \mathbf{r}_1^T \mathbf{G}_{17} \mathbf{r}_1 + \mathbf{r}_1^T \mathbf{G}_{18} \mathbf{r}_1 = 0 \\ & \mathbf{G}_{17} = \mathbf{g}_{14} \mathbf{g}_9^T \mathbf{G}_{16} \\ & \mathbf{G}_{18} = (\mathbf{g}_{15} \mathbf{g}_{10}^T - \mathbf{g}_{14} \mathbf{g}_2^T \mathbf{W} \mathbf{G}_{12}) \mathbf{G}_{16}. \end{aligned} \tag{45}$$

Because the last two elements of $\mathbf{g}_{14}$ and the last element of $\mathbf{g}_{15}$ are zero, (45) is a 7th-order polynomial in r_1 which can be solved with a chosen λ_3. There is, in most cases, only one positive root as the system of equations are overdetermined. Substituting the positive r_1 into (43) gives $\dot{r}_1$ and putting these values into (41) then forms a position estimate. The proper λ_3 is the value that makes the location estimate satisfy (5). Searching for λ_3 is achieved by the Newton's method, similar to the TDOA case.

The weighting matrix $\mathbf{W}$ is $E[[\phi_t^T, \phi_f^T]^T [\phi_t^T, \phi_f^T]]$, where ϕ_t is defined in (19a) and ϕ_f is the equation error vector associated with (31), i.e.,

$$\begin{bmatrix} \phi_t \\ \phi_f \end{bmatrix} \equiv \mathbf{h} - \mathbf{G}_1 \mathbf{u}^0 - \mathbf{g}_2 r_1^0 - \mathbf{g}_3 \dot{r}_1^0 \tag{46a}$$

where $\mathbf{G}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ are defined in (38). Using the measurement model (6) and (7) gives

$$\begin{aligned} \phi_f &= c\dot{\mathbf{B}}^0 \Delta\mathbf{d} + c\mathbf{B}^0 \Delta\dot{\mathbf{d}} + 2c^2 \Delta\mathbf{d} \odot \Delta\dot{\mathbf{d}} \\ &\approx c\dot{\mathbf{B}}^0 \Delta\mathbf{d} + c\mathbf{B}^0 \Delta\dot{\mathbf{d}}, \\ \dot{\mathbf{B}} &\equiv 2\,\mathrm{diag}\{\dot{r}_2, \dot{r}_3, \ldots, \dot{r}_M\} \end{aligned} \tag{46b}$$

in which $\mathbf{B}$ is defined in (19c) and the assumption that $r_i^0 \gg c\,\Delta d_{i,1}$ has been used. Hence (ignoring the scaling factor c)

$$\mathbf{W} \approx \begin{bmatrix} \mathbf{B}^0 & \dot{\mathbf{B}}^0 \\ \mathbf{O} & \mathbf{B}^0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Q}_t^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{Q}_f^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B}^0 & \mathbf{O} \\ \dot{\mathbf{B}}^0 & \mathbf{B}^0 \end{bmatrix}^{-1}. \tag{47a}$$

The weighting matrix is not available in practice because $\mathbf{B}^0$ and $\dot{\mathbf{B}}^0$ contain the unknown true emitter location. If satellites are far from Earth, $\dot{\mathbf{B}}^0$ will be small compared with $\mathbf{B}^0$ and the diagonal elements of $\mathbf{B}^0$ are almost equal. In this case $\mathbf{W}$ can be approximated by

$$\mathbf{W} \approx \begin{bmatrix} \mathbf{Q}_t^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{Q}_f^{-1} \end{bmatrix}. \tag{47b}$$

If, on the other hand, satellites are close to Earth or moving at a relatively high speed, (47b) is first used to compute an initial solution to estimate $\mathbf{B}^0$ and $\dot{\mathbf{B}}^0$. Equation (47a) then can generate a more accurate solution.

Using perturbation analysis and following similar steps as in the TDOA case, one can show that when measurement noise is small the solution is unbiased and has a location covariance matrix in the same form as (30), with $\mathbf{W}$ given by (47a) and

$$\mathbf{G}_6 = \mathbf{G}_1 - \frac{1}{r_1} \mathbf{g}_2 (\mathbf{s}_1 - \mathbf{u})^T + \frac{\dot{r}_1}{r_1} \mathbf{g}_3 (\mathbf{s}_1 - \mathbf{u})^T - \frac{1}{r_1} \mathbf{g}_3 \dot{\mathbf{s}}_1^T. \tag{48}$$

where the matrix $\mathbf{G}_1$ and vectors $\mathbf{g}_2$ and $\mathbf{g}_3$ are defined in (38).

In contrast with the TDOA only case, the combined case needs more computation since it requires solving λ_1 and λ_2 from (40) and finding the positive root of a 7th-order polynomial instead of a 3rd. However, the combined case can give a better position estimate as it utilizes more information.

III. SPECIAL CASES

The solutions in Section II apply to general situations. For some special cases, a modification of the method can decrease the computations. We examine two such cases.

The first special case is when the receivers are geostationary. This implies the z-coordinates of receivers are zero (in the geocentric coordinate system) and only TDOAs are available. The position estimate with two TDOA measurements can be found in [7]. For more than three TDOAs, the one given in Subsection IIA2 is not applicable as the matrix $\mathbf{G}_1$ in (13) is singular. When $z_i = 0$, (8b) and (9) can fix only x and y. Since z can always be chosen to satisfy the constraint (5) given x and y, the Newton's search in Subsection IIA2 is not needed. In particular, x and y are found by minimizing the cost function

$$\begin{aligned}\xi &\equiv (\mathbf{h} - \mathbf{G}_1\mathbf{u} - \mathbf{g}_2 r_1)^T \mathbf{W}(\mathbf{h} - \mathbf{G}_1\mathbf{u} - \mathbf{g}_2 r_1)\\ &\quad + \lambda_1(2\mathbf{s}_1^T\mathbf{u} - \mathbf{s}_1^T\mathbf{s}_1 - r^2 + r_1^2)\\ \mathbf{G}_1 &\equiv -2\begin{bmatrix} x_2 - x_1 & y_2 - y_1\\ x_3 - x_1 & y_3 - y_1\\ \vdots & \\ x_M - x_1 & y_M - y_1\end{bmatrix} \qquad (49)\\ \mathbf{u} &\equiv \begin{bmatrix} x\\ y\end{bmatrix}, \qquad \mathbf{s}_1 \equiv \begin{bmatrix} x_1\\ y_1\end{bmatrix}\end{aligned}$$

where $\mathbf{h}$ and $\mathbf{g}_2$ are defined in (13). After some algebraic manipulations, λ_1 and $\mathbf{u}$ are, respectively, given by (16) and (17), with λ_2 in $\mathbf{G}_4$ set to zero. Equating the derivative of ξ with respect to r_1 to zero gives (15). Substitutions of (16) and (17) into (15) produce a 3rd-order polynomial in r_1. Hence $\mathbf{u}$ is readily determined and z is found from (5) to be

$$z = \pm\sqrt{r^2 - \mathbf{u}^T\mathbf{u}}. \qquad (50)$$

The sign ambiguity is resolved by some extra knowledge such as the pointing directions of the receiving antennae.

The second case involves the combined TDOA and FDOA scenario. The solution method in Section IIB requires two intermediate variables r_1 and $\dot{r}_1$. If $\mathbf{s}_1$ is geostationary, $\dot{r}_1$ as well as $\dot{\mathbf{s}}$ will be zero. Upon using (5), solving a 4th-order polynomial instead of a 6th is sufficient for a solution in Subsection IIB1. For $M \geq 3$, ξ in (38) reduces to (13), with λ_2 replaced by λ_3 and the relevant vectors and matrices as defined in (38). As a consequence, only the positive root of a 3rd instead of 7th-order polynomial is needed.

IV. PERFORMANCE EVALUATION

The CRLB [25] is known to be the best accuracy that an estimator can achieve. It is originally developed for unconstrained estimation. Recently, it has extended (see for example [26, 27]) the bound to incorporate constraints. The derivation of the bound has been simplified in [28] for unbiased constrained estimators. The constrained CRLB of an unbiased estimator for a parameter vector $\mathbf{u}$ is given by [28]

$$\mathrm{cov}(\mathbf{u})_{\min} = \mathbf{J}^{-1} - \mathbf{J}^{-1}\mathbf{F}(\mathbf{F}^T\mathbf{J}^{-1}\mathbf{F})^{-1}\mathbf{F}^T\mathbf{J}^{-1}|_{\mathbf{u}=\mathbf{u}^0} \qquad (51)$$

where $\mathbf{J}$ is the Fisher information matrix [25] and $\mathbf{F}$ is the gradient matrix of the set of constraints with respect to the unknown parameters. The unconstrained CRLB is simply equal to $\mathbf{J}^{-1}$. Equation (51) shows that incorporating constraints, that is, prior information, can always reduce the bound. This section compares the accuracy of the proposed solutions with the constrained CRLB, under the assumption that the measurement noises $\Delta\mathbf{d}$ and $\Delta\dot{\mathbf{d}}$ in (6)–(7) are small and Gaussian.

When the measurement noise is Gaussian, the Fisher information matrix in the TDOA case is

$$\begin{aligned}\mathbf{J}_{\mathrm{TDOA}} &= \left(\frac{\partial\mathbf{d}^{0T}}{\partial\mathbf{u}}\mathbf{Q}_t^{-1}\frac{\partial\mathbf{d}^0}{\partial\mathbf{u}^T}\right)_{\mathbf{u}=\mathbf{u}^0}\\ \frac{\partial\mathbf{d}^0}{\partial\mathbf{u}^T} &= -\begin{bmatrix}(\mathbf{s}_2 - \mathbf{u})^T/r_2 - (\mathbf{s}_1 - \mathbf{u})^T/r_1\\ (\mathbf{s}_3 - \mathbf{u})^T/r_3 - (\mathbf{s}_1 - \mathbf{u})^T/r_1\\ \vdots\\ (\mathbf{s}_M - \mathbf{u})^T/r_M - (\mathbf{s}_1 - \mathbf{u})^T/r_1\end{bmatrix}.\end{aligned} \qquad (52)$$

For the combined case with Gaussian noise, it is

$$\begin{aligned}&\mathbf{J}_{\mathrm{TDOA,FDOA}}\\ &= \left(\begin{bmatrix}\frac{\partial\mathbf{d}^{0T}}{\partial\mathbf{u}} & \frac{\partial\dot{\mathbf{d}}^{0T}}{\partial\mathbf{u}}\end{bmatrix}\begin{bmatrix}\mathbf{Q}_t & 0\\ 0 & \mathbf{Q}_f\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial\mathbf{d}^0}{\partial\mathbf{u}^T}\\ \frac{\partial\dot{\mathbf{d}}^0}{\partial\mathbf{u}^T}\end{bmatrix}\right)_{\mathbf{u}=\mathbf{u}^0}\\ &\frac{\partial\dot{d}^0}{\partial\mathbf{u}^T} = \begin{bmatrix}(\mathbf{s}_2 - \mathbf{u})^T\dot{r}_2/r_2^2 - (\mathbf{s}_1 - \mathbf{u})^T\dot{r}_1/r_1^2 - \dot{\mathbf{s}}_2^T/r_2 + \dot{\mathbf{s}}_1^T/r_1\\ (\mathbf{s}_3 - \mathbf{u})^T\dot{r}_3/r_3^2 - (\mathbf{s}_1 - \mathbf{u})^T\dot{r}_1/r_1^2 - \dot{\mathbf{s}}_3^T/r_3 + \dot{\mathbf{s}}_1^T/r_1\\ \vdots\\ (\mathbf{s}_M - \mathbf{u})^T\dot{r}_M/r_M^2 - (\mathbf{s}_1 - \mathbf{u})^T\dot{r}_1/r_1^2 - \dot{\mathbf{s}}_M^T/r_M + \dot{\mathbf{s}}_1^T/r_1\end{bmatrix}.\end{aligned} \qquad (53)$$

In the altitude constrained geolocation problem, $\mathbf{F}$ is found from (5) to be $\mathbf{u}$. Substitution of (52) into (51) gives the constrained CRLB for the TDOA case and putting (53) into (51) yields the bound for the combined case.

Comparison of the bound (51) with the covariance matrix (30) reveals that they are of the same form. To show that our methods achieve the constrained CRLB, it is sufficient to prove that $(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}$ is equal to $\mathbf{J}$.

Let us first consider the TDOA only case. The matrices $\mathbf{G}_6$ and $\mathbf{W}$ are given by (26) and (20a), respectively, where $\mathbf{B}$ in (20a) is defined in (19c). It can be shown after some algebraic manipulations that

$$\mathbf{B}^{0-1}\mathbf{G}_6^0 = \left(\frac{\partial\mathbf{d}^0}{\partial\mathbf{u}^T}\bigg|\right)_{\mathbf{u}=\mathbf{u}^0}. \qquad (54)$$

Substitution of (20a) and (54) into $(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}$ gives the Fisher information matrix in (52). Hence the proposed solution in TDOA case attains the constrained CRLB.

For the combined case, $\mathbf{G}_6$, $\mathbf{W}$, $\mathbf{B}$, and $\dot{\mathbf{B}}$ are defined in (48), (47a), (19c), and (46b), respectively.

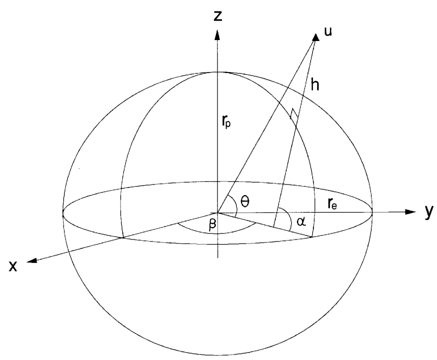

Fig. 2. Oblate spheroid Earth model.

It can be proved that

$$\begin{bmatrix} \mathbf{B}^0 & \mathbf{0} \\ \dot{\mathbf{B}}^0 & \mathbf{B}^0 \end{bmatrix}^{-1} \mathbf{G}_6^0 = \left(\left. \begin{bmatrix} \dfrac{\partial \mathbf{d}^0}{\partial \mathbf{u}^T} \\ \dfrac{\partial \dot{\mathbf{d}}^0}{\partial \mathbf{u}^T} \end{bmatrix} \right|_{\mathbf{u}=\mathbf{u}^0} \right). \tag{55}$$

Substituting (47a) and (55) into $(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}$ yields $\mathbf{J}_{\mathrm{TDOA,FDOA}}$ in (53). It follows that the proposed solution for the combined case also achieves the constrained CRLB. In the derivations, the noise is assumed to be small since (30) ignores the second or higher order perturbation terms. It is expected that when the noise exceeds certain threshold, the accuracy of the proposed methods will deviate from the bound.

V. INACCURATE KNOWLEDGE OF r

The solutions in Sections II, III, and IV assume that the distance r between the emitter and the Earth center is known, where r is the local Earth radius plus the altitude of the emitter. Uncertainty in r always exists because first of all, the non-spherical Earth makes local Earth radius depend on emitter location. Using a constant such as the equatorial radius for local Earth radius will certainly introduce error. Secondly, the altitude of the emitter may not be known or measured precisely. Unlike the random errors in TDOA and FDOA estimates, the error in r is generally deterministic, that is, it is a bias error. That means that an error in r will lead to a bias in the location estimate. This section considers the two kinds of error in r separately. It is demonstrated that using a better Earth model can eliminate the first kind of error. Whereas for the second kind, we compute the solution bias and then determine the tolerable limit on altitude error so that the constrained solution is still preferred to the unconstrained approach.

A. Non-spherical Earth Model

A spherical shape is a common model for the Earth because of its simple form. A spherical model is accurate when an emitter is close to the equator. The modeling error for a spherical Earth increases as the latitude of the emitter increases. It requires a more accurate Earth model to obtain better estimates. An ellipsoid can provide an improved description of the shape of the Earth. Indeed, an oblate spheroid model shown in Fig. 2 provides a simplification in geometry without introducing a significant modeling error [29]. An oblate spheroid is characterized by three parameters, equatorial radius r_e, eccentricity e, and a polar radius r_p where

$$\begin{aligned} r_e &= 6378.137 \text{ km} \\ e &= 0.0818191908426214957 \\ r_p &= r_e\sqrt{1-e^2}. \end{aligned} \tag{56}$$

Referring to the geocentric coordinate system, a point on the oblate sphere is

$$x = (\gamma + h)\cos(\alpha)\cos(\beta) \tag{57a}$$

$$y = (\gamma + h)\cos(\alpha)\sin(\beta) \tag{57b}$$

$$z = \{(1-e^2)\gamma + h\}\sin(\alpha) \tag{57c}$$

where β is the longitude, h is the altitude above the Earth, α is the geodetic latitude which relates to the geocentric latitude θ by

$$\tan(\theta) = (1-e^2)\tan(\alpha) \tag{58}$$

and γ is defined as

$$\gamma \equiv \frac{r_e}{\sqrt{1-e^2\sin^2(\alpha)}}. \tag{59}$$

After eliminating α and β, an oblate spheroid can be represented by

$$\frac{x^2}{(\gamma+h)^2} + \frac{y^2}{(\gamma+h)^2} + \frac{z^2}{\{(1-e^2)\gamma+h\}^2} = 1. \tag{60}$$

In vector form,

$$\begin{aligned} \mathbf{u}^T\mathbf{P}\mathbf{u} &= (\gamma+h)^2, \\ \mathbf{P} &\equiv \mathrm{diag}\left\{1, 1, \frac{(\gamma+h)^2}{\{(1-e^2)\gamma+h\}^2}\right\}. \end{aligned} \tag{61}$$

The problem now becomes one of solving the measurement equations with constraint (61) instead of (5). Unless $h = 0$, (61) is not quadratic because γ is dependent on the emitter location through the geodetic latitude α. Since $r^2 = \mathbf{u}^T\mathbf{u}$ is unknown, (8b) is a quadratic instead of a linear constraint. Finding the location directly becomes a difficult task. The following gives an iterative approach to solving the problem.

Consider the TDOA case first. The algorithm begins by setting r in (8b) to the equatorial radius r_e plus h and $\alpha = 0$. The solution is found in the same way as in Section IIA, with $\mathbf{G}_4$ changed to

$$\mathbf{G}_4 = (\mathbf{G}_1^T\mathbf{W}\mathbf{G}_1 + \lambda_2\mathbf{P})^{-1}. \tag{62}$$

The correct λ_2 is the one that makes the computed solution satisfy (61). This answer provides better estimates of r:

$$r^2 = \mathbf{u}^T\mathbf{u} \tag{63}$$

and the geodetic latitude:

$$\alpha = \tan^{-1}\left\{\frac{u(3)}{\sqrt{u(1)^2 + u(2)^2}(1-e^2)}\right\}. \tag{64}$$

These updated quantities are then used in (8b) and (61) again to generate an improved estimate. The process is repeated until r reaches a steady-state value and the corresponding $\mathbf{u}$ will be the final answer. In our simulations only 3 to 4 repetitions are enough to reach a solution.

The same idea applies to the combined case after replacing $\mathbf{G}_4$ in (40) by

$$\mathbf{G}_4 = (\mathbf{G}_1^T\mathbf{W}\mathbf{G}_1 + \lambda_3\mathbf{P})^{-1}. \tag{65}$$

B. Altitude Error

While choosing an appropriate Earth model can eliminate the unknown local Earth radius problem, there appears no possible way to reduce the solution bias caused by an altitude error. Although (51) indicates that using constraint can reduce uncertainty, an error in altitude will generate a bias when used in conjunction with the constraint (5). Thus whether to use the constraint will depend on the amount of location bias due to altitude error. Intuitively, when signal-to-noise ratio (SNR) is low, it would be better to incorporate the constraint as location variance will dominate mean square error (MSE). On the contrary, the constraint should not be used at high SNR as the bias from altitude error will then dominate the MSE. It is useful to determine the tolerable error in r so that imposing the constraint will still give an improvement in MSE over the nonconstraint solution.

Perturbing (5) around the true emitter location gives

$$r^0\,\Delta r = \mathbf{u}^0\,\Delta\mathbf{u} \tag{66}$$

where second-order perturbation terms have been ignored and Δr is the altitude error. The auxiliary function (21) for TDOA localization becomes

$$\begin{aligned}\xi &= (\phi_t - \mathbf{G}_1\,\Delta\mathbf{u} - \mathbf{g}_2^0\,\Delta r_1)^T\mathbf{W}(\phi_t - \mathbf{G}_1\,\Delta\mathbf{u} - \mathbf{g}_2^0\,\Delta r_1)\\ &\quad + \lambda_1\{2(\mathbf{s}_1 - \mathbf{u}^0)^T\,\Delta\mathbf{u} + 2r_1^0\,\Delta r_1\}\\ &\quad + \lambda_2(2\mathbf{u}^{0T}\,\Delta\mathbf{u} - 2r^0\,\Delta r).\end{aligned} \tag{67}$$

Taking derivatives of ξ with respect to $\Delta\mathbf{u}$, Δr_1, λ_1, and λ_2 and following the same mathematical steps as before, it can be shown that

$$\begin{aligned}\Delta\mathbf{u} &= \left\{\mathbf{I} - \frac{(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0\mathbf{u}^{0T}}{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}\right\}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{G}_6^{0T}\mathbf{W}\phi_t\\ &\quad + \frac{(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}r^0\,\Delta r.\end{aligned} \tag{68}$$

Taking expectation of (68) yields the bias

$$E[\Delta\mathbf{u}] = \frac{(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}r^0\,\Delta r. \tag{69}$$

Equation (69) is directly proportional to the altitude error and depends on the relative geometry between emitter and receivers, but is independent of TDOA measurement noise power. Postmultiplying (68) by its transpose and then taking the expectation, we have

$$\begin{aligned}\mathrm{MSE} = E[\Delta\mathbf{u}\,\Delta\mathbf{u}^T] &= (\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\\ &\quad - \frac{(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}}{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}\\ &\quad \times\left\{1 - \frac{(r^0\,\Delta r)^2}{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}\right\}.\end{aligned} \tag{70}$$

A comparison with (30) reveals that the altitude error induces a positive term to counteract the negative term due to constraint. Note that $\Delta r = 0$ in (70) is the covariance without constraint. In order to make the constraint useful, the second term in (70) must be positive, i.e., the tolerable limit on errors in r is

$$\Delta r \le \frac{1}{r^0}\sqrt{\mathbf{u}^{0T}(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}\mathbf{u}^0}. \tag{71}$$

An intuitive explanation of (71) is that the constraint is very effective when the unconstrained covariance matrix $(\mathbf{G}_6^{0T}\mathbf{W}\mathbf{G}_6^0)^{-1}$ is large. On the other hand, if an unconstrained solution is already quite accurate, the altitude error must be small in order to enhance the position estimate. Otherwise, the solution will have an increased MSE due to the bias given by (69).

Following the same approach, it is straightforward to show that the combined case yields the same conclusions as above, with the definition of $\mathbf{G}_6$ changed to the one in (48).

VI. SIMULATION RESULTS

In our simulation study, the emitter was in Ottawa which has a longitude and latitude of 75.9°W and 45.35°N, respectively. The local Earth radius at this point is 6367.287 km. The emitter altitude was set to zero. The receivers were geosynchronous satellites having a distance 42164 km from the Earth center. They were at $s_1 = (50.0°\mathrm{W}, 2.0°\mathrm{N})$, $s_2 = (47.0°\mathrm{W}, 0.0°\mathrm{N})$, $s_3 = (53.0°\mathrm{W}, 0.0°\mathrm{N})$ and $s_4 = (51.5°\mathrm{W}, 3.0°\mathrm{N})$. Their relative speeds to Earth were $\dot{\mathbf{s}}_1 = (-15.48, -13.0, -772.04)$ km/h, $\dot{\mathbf{s}}_2 = (-30.78, -28.70, 972.72)$ km/h, $\dot{\mathbf{s}}_3 = (-0.054, -0.041, -38.60)$ km/h, and $\dot{\mathbf{s}}_4 = (-119.62, -95.15, 1920.34)$ km/h. The measured TDOAs and FDOAs were generated by adding to the true values Gaussian noises with correlation matrices $\mathbf{Q}_t$ and $\mathbf{Q}_f$, respectively. For simplicity, $\mathbf{Q}_f$ is set identical to $\mathbf{Q}_t$, whose elements are $c^2\sigma_d^2$ in the diagonal and $0.5c^2\sigma_d^2$ otherwise, where σ_d^2 is the

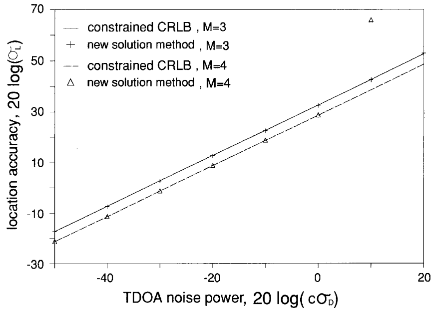

Fig. 3. Localization accuracy of geolocation solution from TDOA measurements versus TDOA noise power.

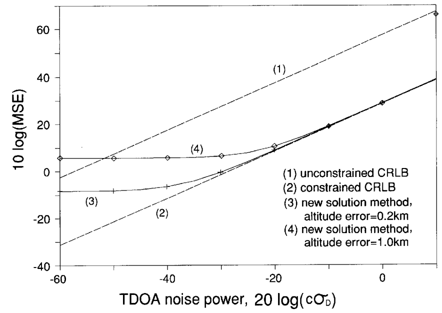

Fig. 4. Effect of altitude error on position estimate from TDOA measurements, $M = 4$. Theory: solid lines. Simulation: symbol.

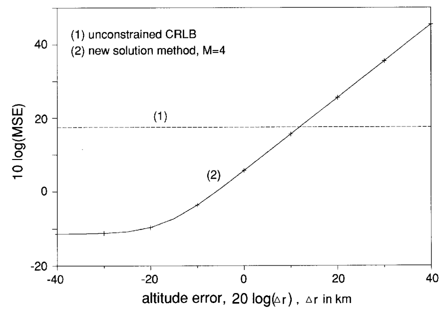

Fig. 5. Comparison between constrained and unconstrained solution in presence of altitude error, TDOA measurements. Theory: solid lines. Simulation: symbol.

TDOA variance. All computations were performed in MATLAB with double precision arithmetic and the results reported were the average of 2000 independent runs.

A. TDOA

Fig. 3 shows the location accuracy of the proposed method as TDOA noise power varies. The weighting matrix was approximated by (20b) throughout the simulations and r was assumed to be known exactly to be 6367.287 km. Two curves are provided, one with three receivers $\mathbf{s}_1$ to $\mathbf{s}_3$ and the other four receivers $\mathbf{s}_1$ to $\mathbf{s}_4$. Also shown are the constrained CRLBs computed from (51)–(52) for the two cases. It is clear that our solution variance is close to the optimum bound. Note that at high noise levels, the bound is not attained. The threshold effect for $M = 4$ occurs earlier than the $M = 3$ case at $c^2\sigma_d^2 > 1$, counter to expectation. The peculiarity may probably be due to insufficient precision in the computations.

The local Earth radius is not known in actual situation as it depends on the emitter location. To handle this difficulty, the algorithm described in Section VA was tested with r initialized to r_e. After 3 iterations, the algorithm reached a steady-state solution and the MSE curves were matched exactly to those in Fig. 3. The feasibility of the algorithm is thus corroborated.

We next study the effects of altitude error in the solution estimate. Two altitude errors $\Delta r = 1.0$ km and $\Delta r = 0.2$ km were tried. The number of receivers was set to 4 and Fig. 4 shows the simulation results (represented by symbols). At low SNR, i.e., when the TDOA noise is large, the curves are close to the constrained CRLBs. As TDOA noise decreases, bias caused by altitude error comes into effect and the MSEs become constant. Obviously, a larger altitude error produces a higher MSE. The theoretical MSEs computed from (65) are shown in solid line. Note that simulations agree closely with the theoretical results.

Keeping $c^2\sigma_4^2 = 0.0001$ and $M = 4$, Fig. 5 depicts location MSE (symbols) as the altitude error increases. Also shown are the theoretical MSE (70) (solid lines) and the CRLB for the unconstrained solution. Location MSE increases with Δr in the constrained case. The unconstrained solution does not use altitude information and the corresponding MSE curve is independent of Δr. Clearly, the constrained solution is preferred when altitude error is small. For the simulated location geometry, (71) indicates that Δr must be less than 3.49 km in order to make the constrained solution useful. This is exactly the point where the two curves meet in the figure. The validity of (71) is confirmed.

B. TDOA and FDOA

Fig. 6 shows the results computed from the solution method in Section IIB. The three curves are, respectively, for two receivers $\mathbf{s}_1$ and $\mathbf{s}_2$, three receivers $\mathbf{s}_1$ to $\mathbf{s}_3$ and four receivers $\mathbf{s}_1$ to $\mathbf{s}_4$. The location accuracy is in close agreement with the constrained CRLB in all three cases. Due to the nonlinear nature of the problem, threshold effect occurs when measurement noise is large. Compared with Fig. 3, incorporating FDOAs improves the accuracy.

The iterative procedure in Section VA was tested when the local Earth radius is not known. As in the TDOA case, the algorithm converged to a steady solution at 3 iterations and they matched those shown in Fig. 6.

Fig. 7 gives location MSE (symbols) in the presence of altitude error for $M = 3$. Again, two

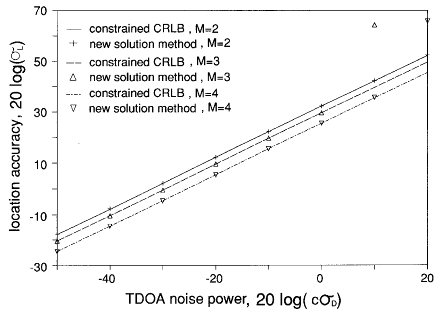

Fig. 6. Accuracy of geolocation solution using both TDOA and FDOA measurements.

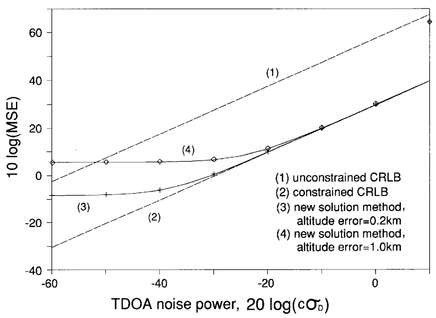

Fig. 7. Effect of altitude error on position estimate from TDOA and FDOA measurements, $M = 3$. Theory: solid lines. Simulation: symbol.

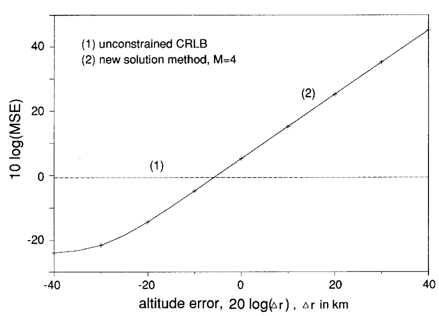

Fig. 8. Comparison between constrained and unconstrained solution in presence of altitude error, TDOA and FDOA measurements, and $M = 4$. Theory: solid lines. Simulation: symbol.

altitude errors $\Delta r = 1.0$ km and $\Delta r = 0.2$ km were tested. Simulation and theoretical results are in close agreement. The bias due to altitude error is obvious when measurement noise is small.

Fig. 8 shows the dependency of MSEs for the constrained and the unconstrained solutions on the altitude error when $c^2\sigma_d^2 = 0.00001$ and $M = 4$. The MSE for the unconstrained solution is also given. As predicted by (71), the two curves intersect at $\Delta r = 0.508$ km.

Niezgoda, et al. [30] has recently extended the Taylor-series approach to solve the localization problem with altitude constraint. Apart from initialization problems, another shortcoming is that the method relies on a small linearization error. A direct consequence is that its performance degrades in cases where the geometric dilution of precision is large (measurement curves intersect at small angles) [13]. In contrast, our proposed technique does not require any linear approximation and hence will perform better than the Taylor-series approach. To be more specific, we have compared the Taylor-series and the proposed methods by simulations. The Taylor-series method was initialized using the true emitter location. For the simulated location geometry, both achieved the constrained CRLB when the noise power was small. Differences occurred when the noise power was large. In the TDOA case with $M = 4$, Taylor-series can not converge when $(c\sigma_D)^2 > 0.316$ whereas our new method can extend the thresholding point to $(c\sigma_D)^2 = 1$ as shown in Fig. 3. In the combined case with $M = 4$, Taylor-series has a thresholding point at $(c\sigma_D)^2 = 3.0$ while our new method deviates from the constrained CRLB at $(c\sigma_D)^2 = 10$ (Fig. 6). These simulations confirmed that the proposed technique has a larger noise threshold than the Taylor-series method.

VII. CONCLUSIONS

Two sets of solutions (one from TDOA, the other from TDOA and FDOA measurements) for localizing an emitter with known altitude were proposed. The solution method transforms the measurement equations, introduces nuisance variable(s) and solves the emitter location by a constrained least-squares minimization. The complexity is a one-dimensional Newton's search and the root finding of a polynomial (3rd-order for TDOA, 7th-order for the combined case). Emitter position variances have been derived and shown to achieve the constrained CRLB when the measurement noise is small. An iterative procedure to handle the unknown local Earth radius by using an oblate spheroid Earth model was given. Location bias due to inexact knowledge of emitter altitude is directly proportional to altitude error. The bias is dependent on localization geometry but unaffected by measurement noise. A tolerable limit on altitude measurement error in which the constrained approach is better than the unconstrained was derived. It is proportional to the unconstrained CRLB so that the more accurate the unconstrained solution, the smaller will be the altitude error needed to improve accuracy. Simulations have confirmed the theoretical developments and the effectiveness of the proposed method.

This study assumes the emitter is stationary on earth. Extension of the solution method to a moving source is a subject for further investigation.

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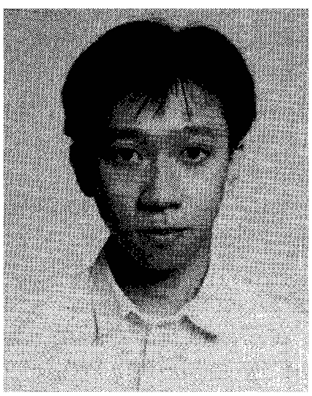

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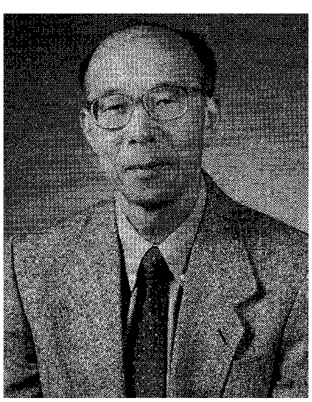

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