The original signal which is a row of an image is shown in Fig. 3(a) ($L = 256$). Fig. 3(b) shows the signal degraded by impulses. This signal was filtered by SM and RM filters with sizes 3 and 5, and the LOR filter with size 3, and then the above threshold algorithm with $T = 25$ was applied. The results are shown in Fig. 3(c)–(l). The LOR filter with size 3 performed like the SM and RM filters with size 5.

VI. Conclusions

The LOR filter which is useful for impulsive noise suppression was introduced. Impulse suppression via LOR filtering seems more effective, and is often simpler to implement than that via SM or RM filtering. This is so because the LOR filter often requires less computation than the SM or RM filter with the same window, and moreover, a smaller window can be used in LOR filtering.

Since RM filters have the threshold decomposition property [9], from Observation 2, RM filters can be implemented by using the LOR filter structure as follows. The input is decomposed into a set of binary signals, and each binary signal is LOR filtered with window $N + 1$. Recombining the filtered binary data results in the output of the RM filter of span $2N + 1$. This seems to be another promising application of LOR filtering, and requires further research.

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Frequency Difference of Arrival Accuracy

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***Abstract*—The problem considered occurs in the geolocation of a fixed emitter using observations from two moving collectors. Estimates for the time difference and frequency difference of signal arrivals are employed. Frequency difference of arrival (FDOA) is estimated through the use of a mixing product. Standard regression analysis procedures are then applied to estimate the slope of the unwrapped phase angle. The derivation of an expression for the standard error of FDOA is given, and this result is related to several other well-known results.**

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Introduction

The June 1981 issue of the IEEE Transactions on Acoustics, Speech, and Signal Processing was a special issue devoted largely to time delay estimation. The papers by Piersol [2] and Stein [3] were of particular interest to this author.

Piersol developed a method for computing the relative time delay of a signal as observed by two different collectors through the technique of fitting a straight line to the phase of the mixing product, and then finding the delay as the slope of that line. An advantage of this method is that well-known regression analysis techniques enable one to find the sample variability of the resulting delay. This sample variability is very useful when evaluating results.

This correspondence addresses the problem of computing relative Doppler, or *frequency difference of arrival* (FDOA), using the regression technique Piersol used for delay, and then gives a derivation of the standard error formula of Stein based on the expression for standard error found in regression analysis. This derivation is elementary and requires few assumptions as to the nature of the signal and the noise functions received. It is not general in that it assumes narrow-band filtering of the mixing product.

Definition of the Problem

Assume that a signal of interest is being collected by two spatially separated receivers, so that $r_1(t)$ is seen by receiver one and $r_2(t)$ is the signal observed by receiver two. Next suppose that the two collectors see slightly different versions of the signal of interest, with different noise being added into each signal. Denote the noise by $n_1(t)$ and $n_2(t)$. The noise is assumed to be stationary with means and variances given by

$$\mu = E[n_1(t)] = E[n_2(t)] = 0$$
$$\sigma_k^2 = E\left[|n_k(t)|^2\right], \; k = 1, 2. \tag{1}$$

Also, the assumption is made that the noise is uncorrelated:

$$E[n_1(t) n_2^*(t + \tau)] = 0, \quad \text{all } \tau. \tag{2}$$

A somewhat simplified model will be employed, one that ignores time delay. Effectively, the assumption is that all delay has been removed from the data. The received signals then are

$$r_k(t) = A_k m(t)\, e^{j(\omega_k t + \phi_k)} + n_k(t), \qquad k = 1, 2. \tag{3}$$

The definitions of these terms are as follows:

$$\begin{aligned} m(t) &= \text{the modulation} \\ A_k &= \text{the received amplitude of signal } k \\ \omega_k &= \text{the received carrier frequency of signal } k \\ \phi_k &= \text{the received random phase of signal } k. \end{aligned} \tag{4}$$

Next, define the *mixing product* function $p(t)$ by

$$\begin{aligned} p(t) &= r_1(t)\, r_2^*(t) \\ &= A_1 A_2^* |m(t)|^2\, e^{j[(\omega_1 - \omega_2)t + (\phi_1 - \phi_2)]} \\ &\quad + A_1 m(t)\, e^{j(\omega_1 t + \phi_1)}\, n_2^*(t) + A_2^* m(t)\, e^{-j(\omega_2 t + \phi_2)} n_1(t) \\ &\quad + n_1(t)\, n_2^*(t). \end{aligned} \tag{5}$$

Taking the expectation of the mixing product, we have

$$\begin{aligned} E[p(t)] &= E[r_1(t) r_2^*(t)] = A_1 A_2^* |m(t)|^2 e^{j[(\omega_1 - \omega_2)t + (\phi_1 - \phi_2)]} \\ &= A_1 A_2^* |m(t)|^2\, e^{j(\omega_0 t + \phi_0)} \end{aligned} \tag{6}$$

where we define $\omega_0 = \omega_1 - \omega_2$ and $\phi_0 = \phi_1 - \phi_2$. The ω_0 term is the FDOA in radians, and is the object of the measurements.

Sine Wave Only Case

Now let us suppose the modulation is unity $[m(t) = 1]$. That is, the signal is reduced to a sine wave only. This is reasonable for a number of signals, and for *fine* analysis: this work is often done in stages, with a coarse estimate being calculated first using ambiguity plane procedures; this coarse estimate is removed before computing the mixing product; in the fine estimation portion, the bandwidths of the time histories are narrowed through the use of bandpass filters for the purpose of reducing the noise.

In this reduced case, the mixing product becomes

$$p(t) = [A_1 A_2^* e^{j\phi_0}] e^{j\omega_0 t} + n_p(t) = A e^{j\omega_0 t} + n_p(t) \tag{7}$$

where

$$A = [A_1 A_2^* e^{j\phi_0}]$$

$$n_p(t) = A_1 e^{j(\omega_1 t + \phi_1)} n_2^*(t) + A_2^* e^{-j(\omega_2 t + \phi_2)} n_1(t) + n_1(t)\, n_2^*(t). \tag{8}$$

Note that while $n_p(t)$ is a random variable, it is also dependent on the signal power. Also note that it has dimensionality different from $n_1(t)$ and $n_2(t)$. The expected value of $n_p(t)$ is zero, and its variance is taken to be $\sigma_p^2 = |A_1|^2 \sigma_2^2 + |A_2|^2 \sigma_1^2 + \sigma_1^2 \sigma_2^2$.

There are a number of methods that could be applied other than the phase fitting to be employed here: Marple [1] discusses a number of high-resolution power spectral density techniques; many of these could be applied to this problem in one form or another. On the other hand, the phase method is simple to apply, has relatively few assumptions, and leads to a straightforward error analysis. Good error analysis is often lacking in the modern methods.

Fig. 1 shows the real and imaginary part of $p(t)$ plotted in polar/vector form, with the noise acting as a small vector on the end of the rotating sinusoid vector. This figure and the following analysis assume a relatively high signal-to-noise ratio. This is reasonable for fine FDOA processing: at this point, there should be substantial gain in the processing due to having the coarse estimates available and to the filtering that normally would have been done. Note that one standard deviation in the noise *plus* the amplitude $|A|$ of the signal causes an uncertainty in the angle ω given by

$$\sigma_\omega \approx \arctan\left(\frac{\sigma_p}{|A|}\right). \tag{9}$$

It is important to note that the size of σ_ω is dependent on both σ_p and $|A|$, and that σ_p must be small relative to $|A|$ in order for this analysis to work. Also note that σ_ω is dimensionless.

The time history of the phase is a new function which will be denoted as $y(t)$. It is the arctangent of the ratio of the imaginary to the real parts of $p(t)$:

$$y(t) = S\left[\arctan \frac{\mathfrak{I}[p(t)]}{\mathfrak{R}[p(t)]}\right] \tag{10}$$

where the S operation indicates that the phase has been straightened. If we assume for the moment that there is no noise, then we would have

$$y(t) = S\left[\arctan \frac{|A| \sin(\omega_0 t + \phi_0)}{|A| \cos(\omega_0 t + \phi_0)}\right] = \omega_0 t + \phi_0. \tag{11}$$

The algorithms that might be employed to do the unwinding are not examined here. Rather, we simply assume that it can be done given a high SNR.

The final assumption is that the noise at this stage is small enough so that the actual measurement after processing can be modeled as

$$y(t) = \omega_0 t + \phi_0 + n(t). \tag{12}$$

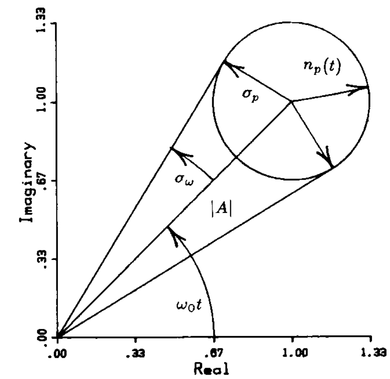

Fig. 1. Sketch of $p(t)$ signal and noise.

The noise term $n(t)$ is the result of two nonlinear operations involving $n_p(t)$. This would at first appear to lead to a function whose characteristics might be difficult to analyze. From the above pictorial analysis, this turns out not to be the case. For high signal-to-noise ratio, $n(t)$ has zero mean and variance σ_ω^2. It will tend to be Gaussian in most cases.

Minimum Mean Square Fit to a Line

The method of estimating the fine FDOA frequency amounts to estimating the *slope* of the unwound phase. We will model this as being of the form $\hat{y}(t) = at + b$ where the a parameter is the one desired; it should be equal to the radian frequency. In particular, suppose that $t = iT$ where T is the sampling interval in seconds. That is,

$$y(i) = (aT)i + b + n(i). \tag{13}$$

We wish to find parameters a and b such that

$$\epsilon^2 = \sum_{i=1}^{N} [y(i) - (aT)i - b]^2 \tag{14}$$

is minimized. The solution to this is worked out in a number of places. In particular, Williams [4] derives the solution that follows. First, define $\bar{x}$ as the average of the $x(i)$ values. Then the minimum mean-square error solution for a, written $\bar{a}$, and $s_{\bar{a}}^2$, the sample variance of $\bar{a}$, are given by

$$\bar{a} = \frac{\sum_{i=1}^{N} y(i)[x(i) - \bar{x}]}{\sum_{i=1}^{N} [x(i) - \bar{x}]^2},$$

$$s_{\bar{a}}^2 = \frac{\sum_{i=1}^{N} y(i)[x(i) - \bar{x}]^2 - \frac{\left[\sum_{i=1}^{N} y(i)[x(i) - \bar{x}]\right]^2}{\sum_{i=1}^{N} [x(i) - \bar{x}]^2}}{(N-2)\sum_{i=1}^{N} [x(i) - \bar{x}]^2}. \tag{15}$$

These are very useful results. The former allows us to estimate a, while the latter allows us to analyze the variability of the estimate. We next assume that the y term is a linear function of time:

$$y(i) = \omega_0 x(i) + \phi_0 + n(i) = \omega_0(iT + i_0 T) + \phi_0 + n(i). \tag{16}$$

Inserting this into the formula for $\bar{a}$, we have that

$$\bar{a} = \frac{\sum_{i=1}^{N} [\omega_0(iT + i_0 T) + \phi_0 + n(i)] \left(i - \frac{N+1}{2}\right)}{TQ(N)} \tag{17}$$

where $Q(N) = (N - 1)(N + 1)(2N)/24$. Expanding the numerator of $\bar{a}$, we have that

$$\sum_{i=1}^{N} [\omega_0(iT + i_0 T) + \phi_0 + n(i)] \left(i - \frac{N+1}{2}\right) = \omega_0 TQ(N) + \sum_{i=1}^{N} n(i) \left(i - \frac{N+1}{2}\right) \tag{18}$$

so that

$$\bar{a} = \frac{\omega_0 TQ(N) + \sum_{i=1}^{N} n(i) \left(i - \frac{N+1}{2}\right)}{TQ(N)}. \tag{19}$$

Thus, $E|\bar{a}| = \omega_0$, as would be expected. The next step is to evaluate $E[s_{\bar{a}}^2]$. In order to do this, it is necessary to have an expression for $\bar{y}$:

$$\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y(i) = \omega_0 \bar{x} + \phi_0 + \bar{n}, \quad \bar{n} = \frac{1}{N} \sum_{i=1}^{N} n(i). \tag{20}$$

We now evaluate the term

$$E\left[\sum_{i=1}^{N} [y(i) - \bar{y}]^2\right] = E\left[\omega_0^2 T^2 Q(N) + 2\omega_0 \sum_{i=1}^{N} [x(i) - \bar{x}] \cdot [n(i) - \bar{n}] + \sum_{i=1}^{N} [n(i) - \bar{n}]^2\right] = \omega_0^2 T^2 Q(N) + 0 + (N - 1)\sigma_\omega^2. \tag{21}$$

The last term that needs to be evaluated is

$$E\left[\left(\sum_{i=1}^{N} y(i)(x(i) - \bar{x})\right)^2\right] = E\left[\omega_0^2 T^4 Q^2(N) + 2\omega_0 T^3 Q(N) \sum_{i=1}^{N} n(i) \left(i - \frac{N+1}{2}\right) + T^2 \sum_{i=1}^{N} \sum_{l=1}^{N} n(i) n(l) \left(i - \frac{N+1}{2}\right)\left(l - \frac{N+1}{2}\right)\right] = \omega_0^2 T^4 Q(N) + T^2 \sigma_\omega^2 Q(N). \tag{22}$$

Putting this altogether, we have

$$E[s_{\bar{a}}^2] = \frac{\omega_0^2 T^2 Q(N) + (N-1)\sigma_\omega^2 - \frac{\omega_0^2 T^4 Q^2(N) + T^2 \sigma_\omega^2 Q(N)}{T^2 Q(N)}}{(N-2) T^2 Q(N)} = \frac{\sigma_\omega^2}{T^2 Q(N)} = \frac{24\sigma_\omega^2}{T^2 (N+1)(N-1)2N} \approx \frac{12\sigma_\omega^2}{T^2 N^3} \quad \text{for } N \text{ large.} \tag{23}$$

It is important to note that this sample variance is expected to decrease in proportion to the cube of N. Assuming that there are enough data available and that the unwrapping can be done, this method provides a tremendous lever for increasing the accuracy of the estimate.

Comparison to the Stein Results

Now let us apply this to the analysis by Stein. Using (10) and assuming that the estimator is unbiased, we have that

$$E[s_{\bar{a}}^2] = \sigma_{\bar{a}}^2 = \frac{12}{(NT)^2 N\left(\frac{|A|^2}{\sigma_\omega^2}\right)} = \frac{12}{(NT)^2 N\gamma}. \tag{24}$$

The term $\gamma = |A|^2/\sigma_\omega^2$ is the signal-to-noise ratio. Taking the square root, we have finally

$$\sigma_{\bar{a}} = \frac{3.46}{P\sqrt{N\gamma}} \tag{25}$$

where P is the length of record in seconds ($= NT$) and N is the number of data points. This is almost exactly in the form shown by Stein in his (12b). In particular, dividing by 2π and noting that the sampling rate $S = 1/T$, we can write

$$\sigma_{\bar{a}}^{f} = \frac{0.55}{P\sqrt{N\gamma}} = \frac{0.55}{P\sqrt{NTS\gamma}} = \frac{0.55}{P\sqrt{PS\gamma}}. \tag{26}$$

Conclusions

It has been shown that fitting a line to the unwound phase can yield an error variance on the order of N^3, a tremendous improvement for large N over other potential methods. Additionally, the methods of computing the sample variance of the estimate are very simple. Finally, these results can be reduced to that of Stein.

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