Scientific Computing - Session 3: Chebyshev Polynomials

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Introduction

In this session of Scientific Computing, we explore **Chebyshev Polynomials**, a sequence of orthogonal polynomials that play a crucial role in approximation theory and numerical analysis. They are particularly important for minimizing interpolation error.

Definition

The Chebyshev polynomial of degree n, denoted as $T_n(x)$, is defined for $x \in [-1, 1]$ by the formula:

$$T_n(x) = \cos(n \cos^{-1}(x)), \quad n \geq 0$$

Key Properties

Chebyshev polynomials exhibit several interesting and useful properties:

1. **Recurrence Relation**:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1$$

   with initial conditions $T_0(x) = 1$ and $T_1(x) = x$.

2. **Leading Coefficient**: The leading coefficient of $T_n(x)$ is 2^{n-1} for $n \geq 1$.

3. **Symmetry**:

   - If n is even, $T_n(x)$ is an even function.
   - If n is odd, $T_n(x)$ is an odd function.

4. **Roots**: The roots of $T_n(x)$ are real, distinct, and lie in $(-1, 1)$. They are given by:

$$x_j = \cos\left(\frac{2j-1}{2n}\pi\right), \quad j = 1, 2, \dots, n$$

5. **Extrema**: The absolute value of $T_n(x)$ is bounded by 1 on $[-1, 1]$:

$$|T_n(x)| \leq 1$$

   The extrema occur at $n+1$ points:

$$x_k = \cos\left(\frac{k\pi}{n}\right), \quad k = 0, 1, \dots, n$$

   At these points, $T_n(x_k) = (-1)^k$.

First Few Polynomials

Using the recurrence relation, we can derive the first few polynomials:

- $T_0(x) = 1$
- $T_1(x) = x$
- $T_2(x) = 2x^2 - 1$
- $T_3(x) = 4x^3 - 3x$

Minimax Property

One of the most significant properties of Chebyshev polynomials is related to polynomial approximation.

Theorem: Among all monic polynomials $P_n(x)$ of degree n (polynomials with leading coefficient 1), the one that has the smallest maximum absolute value on $[-1, 1]$ is:

$$\frac{1}{2^{n-1}} T_n(x)$$

This property is fundamental in **minimax approximation**.

Application in Interpolation

When interpolating a function $f(x)$ by a polynomial $P_n(x)$ of degree n at nodes $x_0, \dots, x_n$, the error is given by:

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

To minimize the error bound, we need to minimize the term $|\prod_{i=0}^{n}(x - x_i)|$. This is achieved by choosing the interpolation nodes x_i to be the roots of the Chebyshev polynomial $T_{n+1}(x)$.

Example

Consider approximating $f(x) = \sin(\frac{\pi}{2}x)$ on $[-1, 1]$. Using Chebyshev nodes significantly reduces the maximum error compared to using equidistant nodes.

Summary

Chebyshev polynomials provide an optimal way to choose interpolation points to minimize the worst-case error (Runge's phenomenon). They are a cornerstone of approximation theory.