Vv186 Recitation

Week 10

By Yahoo

Outline

- Vector space
- Sequences of Functions
- Series

Vector space

Vector space

1. V={(x,y)∈R^2: x=y}
2. V={(x_1, x_2,…, x_n)∈R^n: x_i∈R}
3. $C^\infty(\Omega, \mathbb{R}) = C^\infty(\Omega) = \{f : \Omega \to \mathbb{R} : f \in C^k(\Omega, \mathbb{R}) \text{ for all } k \in \mathbb{N}\}$
4. The set of all sequences converging to 0.
5. V={(x_1, x_2)∈R^2: x_1<0}

Definition

1. V is any set;
2. $+\colon V \times V \to V$ is a map (called addition) with the following properties:
   - $(u+v)+w = u+(v+w)$ for all $u, v, w \in V$ (***associativity***),
   - $u+v = v+u$ for all $u, v \in V$ (***commutativity***),
   - there exists an element $e \in V$ such that $v+e = v$ for all $v \in V$ (***existence of a unit element***),
   - for every $v \in V$ there exists an element $-v \in V$ such that $v+(-v) = e$;
3. $\cdot\colon \mathbb{R} \times V \to V$ is a map (called scalar multiplication) with the following properties:
   - $\lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v$ for all $\lambda \in \mathbb{R}$, $u, v \in V$,
   - $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$,
   - $(\lambda\mu) \cdot u = \lambda \cdot (\mu \cdot u)$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$.

Subspace

Suppose V={(x_1, x_2,…, x_n)∈R^n: x_i∈R}

1. V_1={(x_1, x_2,…, x_n)∈R^n: $x_1 = 0$}
2. V_2={(x_1, x_2,…, x_n)∈R^n: $x_1 = 1$}

Q: Are V_1 and V_2 subspaces of vector space V?

Definition

Let $(V, +, \cdot)$ be a real or complex vector space. If U is a subset of V and $(U, +, \cdot)$ is also a vector space, then we say that $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

How to prove a subspace

1. By definition: Check all the properties.
2. Let $(V, +, \cdot)$ be a real (complex) vector space and U is a subset of V. If $u1 + u2 \in U$ for $u1$, $u2 \in U$ and $\lambda u \in U$ for all $u \in U$ and $\lambda \in R$, then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

Normed Vector Spaces

Let V be a real (complex) vector space. Then a map $\|\cdot\|: V \to R$ is called a norm if for all $u, v \in V$ and all $\lambda \in R$:

1. $\|v\|$ 0 for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$
2. $\| \lambda \cdot v\| = | \lambda | \cdot \|v\|$,
3. $\|u+v\| \leq \|u\|+\|v\|$.

Example

1. $\mathbb{R}^n$ with $\|x\|_p = \left(\sum_{j=1}^{n} |x_i|^p\right)^{1/p}$ for any $p \in \mathbb{N} \setminus \{0\}$,
2. $\mathbb{R}^n$ with $\|x\|_\infty = \max_{1\le k\le n} |x_k|$,

Sequences of Functions

Pointwise V.S. uniform convergence

1. Pointwise convergence: Fix $\mathrm{x} \in \Omega, |f_n(x) -$

Example

1. $$f_n(x) = \sqrt{\frac{1}{(n+1)^2} + x^2}$$

2. $$f_n(x) = \begin{cases} 1 - nx, 0 \le x \le 1/n \\ 0, \quad otherwise \end{cases}$$

Continuity of uniform convergent functions

Let [a, b] ∈ R be a closed interval. Let (f_n) be a sequence of continuous functions defined on [a, b] such that f_n (x) converges to some f (x) ∈ R as n → 1 for every x ∈ [a, b]. If the sequence (f_n) converges uniformly to the thereby defined function f : [a, b] ∈ R, then f is continuous.

Remark

1. f_n must be continuous
2. f_n(x) must be pointwise continuous
3. f_n must be uniformly continuous to f

Complete vector space

A Complete vector space is a vector space in which every Cauchy sequence converges.

Example

The metric space $(C([a, b]), \varrho)$ is complete with $\varrho(f, g) = \|f - g\|_\infty = \sup_{x \in [a,b]} |f(x) - g(x)|$

Series

Summable series

Let (a_n) be a sequence in a normed vector space (V, ‖·‖). Then we say that (a_n) is summable with sum s ∈ V if

$$\lim_{n\to\infty} s_n = s, \qquad s_n := \sum_{k=0}^{n} a_k$$

Cauchy Criterion

Let $\sum a_k$ be a series in a complete vector space (V, ‖·‖). Then $\sum a_k$ converges is equivalent to $\|\sum_{k=n+1}^{m} a_k\| < \varepsilon$

Cauchy Criterion

1. If the series $\sum_{k=0}^{\infty} a_k$ converges, then the sequence $a_k \to 0$ as $k \to +\infty$.
2. If the series $\sum_{k=0}^{\infty} a_k$ converges, then the sequence $\sum_{k=n}^{\infty} a_k \to 0$ as $n \to +\infty$.

Caution

If the sequence $a_k \to 0$ as $k \to +\infty$, the series $\sum_{k=0}^{\infty} a_k$ does not necessarily converges.

Counterexample: a_k=1/k.

Absolute Convergence

A series $\sum_{k=0}^{\infty} a_k$ in a normed vector space $(V, \|\cdot\|)$ is called absolutely convergent if $\sum_{k=0}^{\infty} \|a_k\|$ converges.

An absolutely convergent series $\sum_{k=0}^{\infty} a_k$ in a complete vector space $(V, \|\cdot\|)$ is convergent.

Example

Show that $\sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^k}{k^2}$ converges.

Hint: $\sum_{\mathrm{k}=0}^{\infty} \left| \frac{(-1)^k}{k^2} \right|$ converges.

Infinite Triangle Inequality

Let $(V, \|\cdot\|)$ be a complete normed vector space and $\sum_{k=0}^{\infty} a_k$ an absolutely convergent series. Then

$$\left\| \sum_{k=0}^{\infty} a_k \right\| \leq \sum_{k=0}^{\infty} \|a_k\|$$

Comment: $b_n \to b$ implies $\|b_n\| \to \|b\|$.

Tests for convergent series

1. $0 \le a_k \le b_k, then \sum b_k \ converges \Rightarrow \sum a_k \ converges$
2. $\sqrt[k]{a_k} \le q < 1, then \sum a_k \ converges$
3. $\frac{a_{k+1}}{a_k} \le q < 1, then \sum a_k \ converges$
4. $\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k}, then \sum b_k \ converges \Rightarrow \sum a_k \ converges$

Example

1. $\sum \frac{n+3}{2n^3-n}$ converges : $\sum \frac{n+3}{2n^3-n} < \sum \frac{1}{n^2}$ for n>3
2. $\sum \frac{x^n}{n}$ absolutely converges when |x|<1 by Root test.

Weierstrass M-test

Let Ω *is a subset of* R and (f_k) be a sequence of functions defined on Ω, $f_k : \Omega \to$ C, satisfying $\sup_{x\in\Omega}|f_k(x)| \leq M_k$. Suppose that $\sum M_k$ converges, then the sequence (F_n) of partial sums converges uniformly to *f.*

$$f(x) := \sum_{k=0}^{\infty} f_k(x) \qquad F_n(x) = \sum_{k=0}^{n} f_k(x)$$

Conditionally Convergent Series

A series in a normed vector space $(V, \|\cdot\|)$ is called conditionally convergent if it is convergent, but not absolutely convergent.

Rearrangements of Terms in Series

Assume that $\sum a_k$ is an absolutely convergent series in a complete normed vector space. If the summands of the series are rearranged, the new series $\sum b_j$ converges absolutely with the same sum as $\sum a_k$.

Rearrangements of Terms in Series

Let $\sum a_k$ be a conditionally convergent series of real numbers. Then for any $a \in$ R there exists a rearrangement b_j of $\sum a_k$ such that $\sum b_i = a$.

The Leibniz Theorem

Let $\sum a_k$ be a complex series whose partial sums are bounded but need not converge. Let (α_k) be a decreasing convergent sequence with limit $\alpha_k = 0$. Then the series $\sum \alpha_k a_k$ converges.

Example: $\sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^k}{k}$ converges.

Cauchy Product

Let $\sum a_k$ and $\sum b_k$ be absolutely convergent series. Then the Cauchy product $\sum c_k$ given by $c_k=\sum_{i+j=k} a_i b_j$.

$$\begin{tikzcd}
a_1b_1 \arrow[r, bend left] & a_1b_2 \arrow[dl] & a_1b_3 & \cdots \\
a_2b_1 \arrow[d, bend right] & a_2b_2 \arrow[ur] & a_2b_3 & \cdots \\
a_3b_1 \arrow[ur] & a_3b_2 & a_3b_3 & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Remark

Given that $\sum a_k$ and $\sum b_k$ be absolutely convergent series, then does the Cauchy product $\sum c_k$ given by c_k=$\sum_{i+j=k} a_i b_j$ converge?

Example: (Hw)

$$\sum a_k = \sum b_k = \sum \frac{(-1)^n}{\sqrt{n}}, \sum c_k = (-1)^{n+1} \sum_{i+j=n+1} \frac{1}{\sqrt{ij}}$$

Thank you for your attention!