# Vv186 Mid1 Review

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## 1 Properties of continuous function at a certain point

1. Let $\Omega \subset \mathbb{R}$ be some set and f $:\to \mathbb{R}$ a function that is continuous at some point $x_0 \in \Omega$ and f($x_0$)>0. Then there exists $\delta > 0$ such that f(x)>0 for any $x \in$($x_0 - \delta$, $x_0 + \delta$) $\subset \Omega$.

2. Let $\Omega \subset \mathbb{R}$ be some set and f $:\to \mathbb{R}$ a function that is continuous at some point $x_0 \in \Omega$. Then there exists $\delta > 0$ such that f is bounded on ($x_0 - \delta$, $x_0 + \delta$) $\subset \Omega$.

## 2 Properties of continuous function on a closed interval

1. Let $a < b$ and f: $[a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded on $[a, b]$.

2. Let $a < b$ and f: $[a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Then there exists some x$\in$ $[a, b]$ such that f(x)=0.

3. Let $a < b$ and f: $[a, b] \to \mathbb{R}$ be a continuous function. Then for y$\in$[min{f(a), f(b)}, max{f(a), f(b)}] there exists some x$\in$ $[a, b]$ such that y=f(x).

4. Let $a < b$ and f:$[a, b] \to \mathbb{R}$ be a continuous function $\to \mathbb{R}$ with ran f $\subset [a, b]$. Then f has a fixed point, i.e., there exists some x$\in$ $[a, b]$ such that f(x)=x.

5. Let $a < b$ and f:$[a, b] \to \mathbb{R}$ be a continuous function $\to \mathbb{R}$. Then there exists a y$\in$ $[a, b]$ such that $f(x) \leq f(y)$ for all x$\in$ $[a, b]$.

## 3 Inverse Function

1. Basic concepts:

   - Injectivity: $f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$

- Surjectivity: For all $y$ there exists a $x$ such that $f(x) = y$. (ran f = Codomain)
- Bijectivity: Injectivity and surjectivity.

2. A bijective function has an inverse function.

3. A strictly increasing and continuous function has an inverse function that is also strictly increasing and continuous.

4. A continuous and bijective function must be strictly monotonic.

# 4 Uniformly continuous function

1. Definition:

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in I}{\forall} |x-y| < \delta \ \Rightarrow |f(x) - f(y)| < \varepsilon$$

2. A uniformly continuous function is continuous.

3. A continuous function on a closed interval is uniformly continuous.

4. The sum of two uniformly continuous function is still uniformly continuous.

5. The product of two uniformly continuous function is *not* necessarily uniformly continuous. (Counterexample: f(x)=g(x)=x on $[0, +\infty)$)

6. The product of two uniformly continuous function on *a compact interval* is uniformly continuous. (Example: f(x)=g(x)=x on $[0, 1]$)

# 5 Outlook

1. The image of a continuous function on a closed interval is also a closed interval.

2. A close and bounded set is called compact.

3. A continuous function on a compact set is uniformly continuous, bounded and has maximum and minimum value. Also, the image is also compact.

## 6 Operation on uniformly continuous function

1. The sum of two uniformly continuous function is still uniformly continuous.
2. The product of two uniformly continuous function is *not* necessarily uniformly continuous. (Counterexample: f(x)=g(x)=x on $[0, +\infty)$)
3. The product of two uniformly continuous function on *a compact interval* is uniformly continuous. (Example: f(x)=g(x)=x on $[0, 1]$)

## 7 Exercise

1. Find the root of the function

$$p(x) = 2x^3 - 3x^2 - 3x + 2 = 0$$

2. Suppose f(x) is continuous on $[0, 2]$ and f(0)=f(2). Show that there exists $x, y \in [0, 2]$ such that y-x=1 and f(x)=f(y).
3. Show that f(x) is uniformly continuous on domain $X \Leftrightarrow$ For any two sequences $x_n,\ y_n$ in $X$, if $\lim\limits_{x\to\infty}(x_n - y_n)$=0, then $\lim\limits_{x\to\infty}(f(x_n) - f(y_n))$=0.
4. Show that if f is uniformly continuous on (0, 1), then $\lim\limits_{x \searrow 0} f(x)$ exists.

## 8 Solution

1. By calculation, we can find

| | | | | |
|---|---|---|---|---|
| x | -2 | 0 | 1 | 3 |
| p(x) | -20 | 2 | -2 | 20 |

   Therefore, by Intermediate Value Theorem we know that p(x)=0 has three roots.

2. Let F(x)=f(x+1)-f(x), then F(x) is continuous on $[0, 1]$. Moreover,

$$F(0) = f(1) - f(0) = f(1) - f(2) = -(f(2) - f(1)) = -F(1)$$

   (a) If F(1)=F(0)=0, then we simply have x=0 and y=1 such that

$$f(0) = f(1)$$

(b) If F(1)≠0, then we apply the Intermediate Value Theorem to $[0,1]$: there must have a $x_0 \in [0,1]$ such that

$$F(x_0) = f(x_0+1) - f(x_0) = 0$$

Then we have $x = x_0$ and $y = x_0 + 1$.

3. $\Rightarrow$:

Since f(x) is uniformly continuous o $X$, then we have

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x,y\in I} |x-y| < \delta \ \Rightarrow |f(x) - f(y)| < \varepsilon$$

For this $\delta$, since $\lim\limits_{x\to\infty} (x_n - y_n)$=0, then

$$\exists N, \forall n > N : |x_n - y_n| < \delta \ \Rightarrow |f(x_n) - f(y_n)| < \varepsilon$$

Therefore, $\lim\limits_{x\to\infty} (f(x_n) - f(y_n))$=0.

$\Leftarrow$:

Suppose f(x) is not uniformly continuous o $X$, then we have

$$\exists_{\varepsilon>0} \forall_{\delta>0} \exists_{x,y\in I} |x-y| < \delta \ \Rightarrow |f(x) - f(y)| \geq \varepsilon$$

Take $\delta = \frac{1}{n}$, then there exists $x_n,\ y_n$ such that

$$|x_n - y_n| < \frac{1}{n} \quad |f(x_n) - f(y_n)| \geq \varepsilon$$

Therefore, $\lim\limits_{x\to\infty} (x_n - y_n)$=0 but $f(x_n) - f(y_n)$ does not converge to 0, which leads to a contradiction.

4. Since f(x) is uniformly continuous on $(0,1)$, then we simply have

$$\forall_{\varepsilon>0} \exists_{\delta_0>0} \forall_{x,y\in I} |x-y| < \delta \ \Rightarrow |f(x) - f(y)| < \varepsilon$$

Suppose $(x_n)$ is a sequence in $(0,1)$ and converges to 0. We simply want the sequence $(f(x_n))$ also converges, so we check whether $(f(x_n))$ is Cauchy in the normal modulus metric space. Since $(x_n)$ is a convergent sequence, then $(x_n)$ is Cauchy

$$\forall \delta \exists M : \forall m, n > M : |x_n - x_m| < \delta$$

We simply let $\delta$=$\delta_0$, then we have

$$|f(x_n) - f(y_n)| < \varepsilon$$

which means $(f(x_n))$ is Cauchy and then convergent in the normal modulus metric space.

However, we still need to prove that no matter what $(x_n)$ we choose, we simply have one unique limit of $(f(x_n))$. The proof can be simplified with the result in (3).

Suppose there are two sequences $(x_n)$ and $(y_n)$ in (0,1) that both converge to 0. Then we simply know that

$$\lim_{x\to\infty}(x_n - y_n) = \lim_{x\to\infty} x_n - \lim_{x\to\infty} y_n = 0 - 0 = 0$$

However, f(x) is uniformly continuous on (0,1). Then by (3) we simply know that

$$\lim_{x\to\infty}(f(x_n) - f(y_n)) = 0$$

Since both $\lim\limits_{x\to\infty} f(x_n)$ and $\lim\limits_{x\to\infty} f(y_n)$ exist (We have already proved it), then

$$\lim_{x\to\infty} f(x_n) = \lim_{x\to\infty} f(y_n)$$

To sum, for any $(x_n)$ in (0,1) that converges to 0, $\lim\limits_{x\to\infty} f(x_n)$ exists and is unique. Therefore, $\lim\limits_{x\searrow 0} f(x)$ exists.

# References

[1] Horst, Hohberger, "vv186_main", sjtu-umich.instructure.com/courses/Vv 186 FA2106/files Retrieved 2016-10-25.