UM–SJTU Joint Institute

Honors Mathematics II

(Vv186)

Recitation Class

Week three

TA: Yahoo

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1 The twelve properties for $\mathbb{Q}$

For natural number set $\mathbb{Q}$, it must satisfy P1-P12.

(P1)	(Associative law for addition)	$a+(b+c)=(a+b)+c$.
(P2)	(Existence of an additive identity)	$a+0=0+a=a$.
(P3)	(Existence of additive inverses)	$a+(-a)=(-a)+a=0$.
(P4)	(Commutative law for addition)	$a+b=b+a$.
(P5)	(Associative law for multiplication)	$a\cdot(b\cdot c)=(a\cdot b)\cdot c$.
(P6)	(Existence of a multiplicative identity)	$a\cdot 1=1\cdot a=a; \quad 1\neq 0$.
(P7)	(Existence of multiplicative inverses)	$a\cdot a^{-1}=a^{-1}\cdot a=1$, for $a\neq 0$.
(P8)	(Commutative law for multiplication)	$a\cdot b=b\cdot a$.
(P9)	(Distributive law)	$a\cdot(b+c)=a\cdot b+a\cdot c$.

(P10) (Trichotomy law) For every number a, one and only one of the following holds:

(i) $a=0$,
(ii) a is in the collection P,
(iii) $-a$ is in the collection P.

(P11) (Closure under addition) If a and b are in P, then $a+b$ is in P.

(P12) (Closure under multiplication) If a and b are in P, then $a\cdot b$ is in P.

2 Insufficiency in some cases

However, we may encounter some questions insufficient with only $\mathbb{Q}$.

The Square Root Problem

Reference Spivak, end of Chapter 2.

1.5.1. Theorem. There exists no number $a \in \mathbb{Q}$ such that $a^2 = 2$.

Proof.
Assume that $a = p/q \in \mathbb{Q}$ has the property that $(p/q)^2 = 2$, where $p, q \in \mathbb{N}$. We further assume that p and q have no common divisor greater than 1. Then

$$p^2 = 2q^2,$$

so p^2 is even. This implies that p is even, so $p = 2k$ for some $k \in \mathbb{N}$. But then

$$2q^2 = 4k^2 \qquad \Rightarrow \qquad q^2 = 2k^2 \text{ is even} \qquad \Rightarrow \qquad q \text{ is even.}$$

Thus p and q are both divisible by 2. $\lightning$ $\square$

Therefore, we need to introduce P13.

3 P13

Maxima and Minima of Sets

1.5.4. Definition. Let $U \subset \mathbb{Q}$ be a subset of the rational numbers. We say that a number $x_1 \in U$ is the ***minimum*** of U if

$$x_1 \leq x, \qquad \text{for all } x \in U$$

and we write $x_1 =: \min U$.

Similarly, we say that $x_2 \in U$ is the ***maximum*** of U if

$$x_2 \geq x, \qquad \text{for all } x \in U$$

and write $x_2 =: \max U$.

A set that is not bounded below has no minimum, and a set that is not bounded above has no maximum. However, even a bounded set does not need to have a maximum or a minimum.

Greatest Lower and Least Upper Bounds of Sets

1.5.5. Definition. We say that an upper bound $c_2 \in \mathbb{Q}$ of a set $U \subset \mathbb{Q}$ is the ***least upper bound*** of U if no $c \in \mathbb{Q}$ with $c < c_2$ is an upper bound. We then write $c_2 =: \sup U$.

We say that a lower bound $c_1 \in \mathbb{Q}$ is the ***greatest lower bound*** of U if no $c \in \mathbb{Q}$ with $c > c_1$ is a lower bound. We then write $c_1 =: \inf U$.

(P13) *If $A \subset \mathbb{R}$, $A \neq \emptyset$ is bounded above, then there exists a least upper bound for A in $\mathbb{R}$.*

Comment: P13 should be considered as the way that $\mathbb{R}$ is defined so far (Of course there are other definitions for $\mathbb{R}$).

Then in $\mathbb{R}$, there is a unique positive y such that $y^2 = x$ for a positive x $\in \mathbb{R}$.

4 Induction

The fifth Peano Axiom:

Induction axiom: *If a set $S \subset \mathbb{N}$ contains zero and also the successor of every number in S, then $S = \mathbb{N}$.*

5 Subsets of the Real Numbers

1.5.10. Definition. Let $A \subset \mathbb{R}$.

(i) We call $x \in \mathbb{R}$ an ***interior point of*** A if there exists some $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon) \subset A$. The set of interior points of A is denoted by $\operatorname{int} A$.

(ii) We call $x \in \mathbb{R}$ an ***exterior point of*** A if there exists some $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon) \cap A = \emptyset$.

(iii) We call $x \in \mathbb{R}$ a ***boundary point of*** A if for every $\varepsilon > 0$ $(x - \varepsilon, x + \varepsilon) \cap A \neq \emptyset$ and $(x - \varepsilon, x + \varepsilon) \cap A^c \neq \emptyset$. The set of boundary points of A is denoted by ∂A.

(iv) We call $x \in \mathbb{R}$ an ***accumulation point of*** A if for every $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon) \cap A \setminus \{x\} \neq \emptyset$.

6 Exercises

6.1 Properties for $\mathbb{R}$

1. If a $\in \mathbb{R}$, then a$\cdot$0=0.

2. If a$\cdot$c = b$\cdot$c, c $\neq$ 0, then a=b.

3. If a$\cdot$b = 0, then a=0 or b=0.

4. a-b = b-a if and only if a=b.

6.2 Max and Min

1. Prove max(x,y)=$\frac{x+y+|x-y|}{2}$ and min(x,y)=$\frac{x+y-|x-y|}{2}$

2. Derive max(x,(max(y,z)))

6.3 Binary system

Suppose we interpret number to mean either 0 or 1, and +,$\cdot$ to be operations so that

+	0	1
0	0	1
1	1	0

Table 1: Addition

$\cdot$	0	1
0	0	0
1	0	1

Table 2: Multiplication

Check P1-P9 hold for this system, even though 1+1=0.

6.4 Induction:Gang Shooting!

Shootout at the O.K. Corral: an odd number of lawless individuals, standing at mutually distinct distances to each other, fire pistols at each other in exactly the same instant. Every person fires at their nearest neighbor, hitting and killing this person. Then there is at least one survivor.

6.5 Infimum

For a set $\Omega \subset \mathbb{R}$, if Ω is bounded from below (it follows that a greatest lower bound exists). Prove y = inf $\Omega \Leftrightarrow$ ($\forall$x $\in \Omega$: x $\geq$ y) $\bigwedge$ ($\forall\varepsilon > 0$ $\exists$: x $\in \Omega$ x < y + ε) for some real number y.

References

[1] Hohberger, H., 2016, "vv186_main", https://sjtu-umich.instructure.com/courses/102/files. Retrieved 2016-09-25.

[2] Hohberger, H., 2016, "vv203_main", https://sjtu-umich.instructure.com/courses/102/files. Retrieved 2016-09-25.