Vv186 Mid2 Review

Yahoo

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1 Formula

1.
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2.
$$sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

3.
$$cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

4.
$$ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} (-1 < x \leq 1)$$

2 Exercise

1. Find the radius of convergence of the following series

   i)
   $$\sum_{k=0}^{\infty} \frac{k(x-4)^2}{k^3+1}$$

   ii)
   $$\sum_{k=0}^{\infty} \frac{k^2 x^k}{2k!!}$$

2. Discuss convergence of the following series
$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$$

3. Discuss convergence of the following series

$$\sum_{n=0}^{\infty}(1+\frac{1}{n})^{n^2}x^n$$

3 Solution

1. i) We apply the Hadamard formula of ratio form

$$\overline{\lim_{k\to\infty}}|\frac{\frac{k+1}{(k+1)^3+1}}{\frac{k}{k^3+1}}| = \lim_{k\to\infty}\frac{(k+1)(k^3+1)}{k((k+1)^3+1)} = 1$$

since the numerator and denominator both are polynomials of the type $k^4+c_3k^3+c_2k^2+c_1k+c_0$ for suitable constants $c_0; c_1; c_2; c_3$, so the radius of convergence is

$$\rho = \frac{1}{1} = 1$$

ii) We apply the Hadamard formula of ratio form(see assignment 7.3)

$$\overline{\lim_{k\to\infty}}|\frac{\frac{(k+1)^2}{(2k+2)!!}}{\frac{k^2}{2k!!}}| = \lim_{k\to\infty}\frac{k+1}{2k^2} = 0$$

so the radius of convergence is

$$\rho = +\infty$$

2. Let

$$a_n = \frac{(n!)^2}{(2n)!}$$

Then

$$\overline{\lim_{n\to\infty}}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} = 1/4$$

Therefore, the radius of convergence is

$$\rho = 4$$

When $\rho = 4$, the power series becomes

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}x^n = \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}4^n = \sum_{n=0}^{\infty}\frac{2n!!}{(2n-1)!!}$$

Let

$$b_n = \frac{2n!!}{(2n-1)!!}$$

Then we apply Raabe's Test

$$\lim_{n\to\infty} n(\frac{b_{n+1}}{b_n} - 1) = \lim_{n\to\infty} \frac{n}{2n+1} = 1/2 < 1$$

Then the power series is divergent when $\rho = 4$. When $\rho = -4$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} (-4)^n = \sum_{n=0}^{\infty} (-1)^n \frac{2n!!}{(2n-1)!!}$$

Let

$$b_n = (-1)^n \frac{2n!!}{(2n-1)!!}$$

Then we apply Raabe's Test

$$\lim_{n\to\infty} n(\frac{b_{n+1}}{b_n} - 1) = \lim_{n\to\infty} -\frac{(4n+3)n}{2n+1} = -\infty < 1$$

Then the power series is divergent when $\rho = 4$.

3. Let

$$a_n = (1 + \frac{1}{n})^{n^2}$$

Then

$$\overline{\lim_{n\to\infty}} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{(1+\frac{1}{n+1})^{(n+1)^2}}{(1+\frac{1}{n})^{n^2}} = \lim_{n\to\infty} (\frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^{n}})^n (1 + \frac{1}{n+1})^{n+1} = e$$

Therefore, the radius of convergence is

$$\rho = \pm 1/e$$

When $\rho = 1/e$, the power series becomes

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \pm \frac{(1+\frac{1}{n})^{n^2}}{e^n}$$

However, by Cauchy criteria

$$\lim_{n\to\infty} \pm \frac{(1+\frac{1}{n})^{n^2}}{e^n} = \pm 1 \neq 0$$

Then the power series is divergent when $\rho = \pm 1/e$.

References

[1] Horst, Hohberger, "vv186_main", sjtu-umich.instructure.com/courses/Vv 186 FA2106/files Retrieved 2016-10-25.