Vv186 Recitation

Week 6

By Yahoo

Outline

- Comments on RC last week
- Limit of a Function
- Continuous Function

Comments on RC last week

A wrong proof

Show that $\lim c_{\mathrm{n}} = \dfrac{\lim a_{\mathrm{n}}}{\lim b_{\mathrm{n}}}$.

- Step 1: Set $c_{\mathrm{n}} = \mathrm{a}_{\mathrm{n}}/\mathrm{b}_{\mathrm{n}}$, then $a_{\mathrm{n}} = \mathrm{b}_{\mathrm{n}}\mathrm{c}_{\mathrm{n}}$
- Step 2: $\lim a_{\mathrm{n}} = \lim \mathrm{b}_{\mathrm{n}}\mathrm{c}_{\mathrm{n}} = \lim \mathrm{b}_{\mathrm{n}} \lim c_{\mathrm{n}}$
- Step 3: $\lim c_{\mathrm{n}} = \dfrac{\lim a_{\mathrm{n}}}{\lim b_{\mathrm{n}}}$

Pitfall: $\lim \dfrac{a_{\mathrm{n}}}{b_{\mathrm{n}}}$ does not necessarily exist.

Practice

If a_{n} converges, then $S_n = \frac{1}{n}(a_1 + \cdots + a_{\mathrm{n}})$ converges to the same limit.

How to solve?

- Step 1: Suppose $\lim a_{\mathrm{n}}$= 0. Then for any epsilon, there is N so that $|a_{\mathrm{n}}|$ <epsilon for n>N. Then we need to prove there exist N', so that $|\frac{a_1+\cdots a_{\mathrm{n}}}{n}|$ <epsilon for n>N'.
- Step 2: Suppose $\lim a_{\mathrm{n}} \neq 0$. Then let $b_{\mathrm{n}} = a_{\mathrm{n}} - \lim a_{\mathrm{n}}$. Thus, $\lim b_{\mathrm{n}} = 0$. S(b_{n}) converges to 0.

Practice

Suppose x_n-x_{n-2} →0, as n→+∞, show that $(x_n$-$x_{n-1})/n$ →0, as n→+∞.

Limit of a Function

Limit of a Function as $x \to \infty$

Let f be a real- or complex-valued function defined on a subset of R that includes some interval (a,∞). Then f converges to L, as $x \to \infty$

$$\lim_{x\to\infty} f(x) = L \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{C>0}{\exists}\ \underset{x>C}{\forall}\ |f(x) - L| < \varepsilon.$$

Example

- Show that f(x) = $\frac{1}{x}$ $\rightarrow$0, as x $\rightarrow\infty$.

Limit of a Function as $x \to x_0$

Let f be a real- or complex-valued function defined on a subset of R called Ω and let x_0 in the closure of Ω. Then f converges to L, as $x \to x_0$

$$\lim_{x \to x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x \in \Omega \setminus \{x_0\}}{\forall}\ |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Example

- Show that f(x) = $\sqrt{x}$ $\rightarrow$1, as $x \rightarrow 1$.

- Show that f(x) = $\frac{x+1}{2x+1}$ $\rightarrow$1, as $x \rightarrow 1$.

Solution

- Show that f(x) = $\sqrt{x}$ →1, as *x* → *1*.

$$\delta = \varepsilon$$

- Show that f(x) = $\frac{x+1}{2x+1}$ →1, as *x* → *1*.

$$\delta = \min\{\frac{\varepsilon}{2}, \frac{1}{4}\}$$

Operation on limit

1. $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$,
2. $\lim_{x \to x_0} (f(x) \cdot g(x)) = (\lim_{x \to x_0} f(x))(\lim_{x \to x_0} g(x))$,
3. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$ if $\lim_{x \to x_0} g(x) \neq 0$.

These statements remain true if $x_0 = \pm\infty$.

The proof is similar to cases in sequences.

Practice

Calculate the limit of $\sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1}$ as x → +∞

One-sided Limits

Let f be a real- or complex-valued function defined on a subset of R called Ω and let x_0 in the closure of Ω. Then

$$\lim_{x \searrow x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{x \in \Omega \setminus \{x_0\}}{\forall} \ 0 < x - x_0 < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

$$\lim_{x \nearrow x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{x \in \Omega \setminus \{x_0\}}{\forall} \ 0 < x_0 - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Lemma

$f(x) \to L$ as $x \to x_0$ if and only if $f(x) \to L$ as $x \searrow x_0$ and $f(x) \to L$ as $x \nearrow x_0$.

Practice

- Show that the sign function has not limit as $x \to 0$.

$$Sig(x) = 1, \quad \text{if } x > 0$$

$$Sig(x) = 0, \quad \text{if } x = 0$$

$$Sig(x) = -1, \quad \text{if } x < 0$$

Limits of Functions using Sequences

Let f be a real- or complex-valued function defined on a subset of R called Ω and let x_0 in the closure of Ω. Then

$$\lim_{x \to x_0} f(x) = L \quad \Leftrightarrow \quad \underset{\substack{(a_n) \\ a_n \in \Omega \setminus \{x_0\}}}{\forall} \left(a_n \xrightarrow{n \to \infty} x_0 \Rightarrow f(a_n) \xrightarrow{n \to \infty} L \right)$$

Practice

- Show that f(x) = sin ($\frac{1}{x}$) does not have a limit as x $\rightarrow$ *0.*

The Big-O Landau Symbol

Let f be a real- or complex-valued function defined on a subset of R called Ω and let x_0 in the closure of Ω. Then

$$f(x) = O(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\exists} \; \underset{\varepsilon>0}{\exists} \; \underset{x\in\Omega}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| \leq C|\phi(x)|$$

The Big-O Landau Symbol

Let f be a real- or complex-valued function defined on a subset of R that includes some interval (L,∞). Then

$$f(x) = O(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\exists} \; \underset{M>L}{\exists} \quad x > M \quad \Rightarrow \quad |f(x)| \leq C|\phi(x)|$$

The Little-O Landau Symbol

Let f be a real- or complex-valued function defined on a subset of R called Ω and let x_0 in the closure of Ω. Then

$$f(x) = o(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\forall} \; \underset{\varepsilon>0}{\exists} \; \underset{x\in\Omega\setminus\{x_0\}}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| < C|\phi(x)|$$

The Little-O Landau Symbol

Let *f* be a real- or complex-valued function defined on a subset of R that includes some interval (L,∞). Then

$$f(x) = o(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\forall} \; \underset{M>L}{\exists} \quad x > M \quad \Rightarrow \quad |f(x)| < C|\phi(x)|$$

Example

(i) $\dfrac{1}{x^2} = O\left(\dfrac{1}{x}\right)$ as $x \to \infty$,

(ii) $\dfrac{1}{x} = O\left(\dfrac{1}{x^2}\right)$ as $x \to 0$,

Landau Symbols using Limits

Let f be a real- or complex-valued function defined on a subset of R called Ω and let x_0 in the closure of Ω. If there is C ≥ 0, and

$$\lim_{x \to x_0} \frac{|f(x)|}{|\phi(x)|} = C,$$

then we have

$$f(x) = O(\phi(x)) \qquad \text{as } x \to x_0$$

Landau Symbols using Limits

Let f be a real- or complex-valued function defined on a subset of Ω and let x_0 in the closure of Ω, then

$$\lim_{x \to x_0} \frac{|f(x)|}{|\phi(x)|} = 0 \qquad \Leftrightarrow \qquad f(x) = o(\phi(x)) \text{ as } x \to x_0.$$

Proof in assignment.

Continuous Functions

Continuous Function

Let f be a real- or complex-valued function defined on a subset of Ω and let x_0 in the closure of Ω, then f is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Condition for Continuity

- f needs to have a limit at x_0
- f needs to be defined at x_0
- The value of f must coincide with its limit

Continuous Function

Let f be a real- or complex-valued function defined on a subset of Ω and let x_0 in the closure of Ω. Then the following are equivalent:

1. f is continuous at x_0;
2. for any real sequence (a_n) with $a_n \to x_0$, $\lim_{n\to\infty} f(a_n) = f(x_0)$;
3. $\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\operatorname{dom} f}{\forall} : |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$

Practice

1) Suppose that $f(x) \leq g(x)$ for all x. Prove that $\lim f(x) \leq \lim g(x)$, provided that these limits exist.
2) If $f(x) < g(x)$ for all x, does it necessarily follow that $\lim f(x) < \lim g(x)$?

Continuous Function

Let f be a real- or complex-valued function defined on a subset of Ω and let x_0 in the closure of Ω. We say that f is continuous from the left or from below at x_0 if

$$\lim_{x \nearrow x_0} f(x) = f(x_0)$$

We say that f is continuous from the right or from above at x_0 if

$$\lim_{x \searrow x_0} f(x) = f(x_0)$$

Continuous Extension

2.5.6. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $\widetilde{\Omega} \supset \Omega$. Suppose that $f : \Omega \to \mathbb{R}$ and $\widetilde{f} : \widetilde{\Omega} \to \mathbb{R}$ are continuous functions. If $\widetilde{f}(x) = f(x)$ for all $x \in \Omega$, we say that $\widetilde{f}$ is a ***continuous extension*** of f to $\widetilde{\Omega}$.

Operation on Continuous Function

Let f and g be two real functions and x is in dom f and dom g. Assume that both f and g are continuous at x. Then

(i) $f + g$ is continuous at x and

(ii) $f \cdot g$ is continuous at x.

Practice

1. Show that $|f|$ is continuous if f is continuous
2. If f and g are continuous , then max{f, g} and min{f, g} are continuous.
3. Prove that every continuous f can be written $f = g - h$, where g and h re nonnegative and continuous.

Continuity for composition

If f and g are real functions with x in dom g, g(x) in dom f , g is continuous at x and f is continuous at g(x), then the composition $f \circ g$ is continuous at x.

Theorem (Proof)

Let f , g be real functions such that g(x) → L, as x → x_0, and f is continuous at L in dom f . Then we have

$$\lim_{x \to x_0} f(g(x)) = f(L)$$

Thank you for your attention!