Week 13 Recitation – Integrals

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December 5, 2016

The Fundamental Theorem of Calculus

Theorem.

Let f: [a,b] $\to$ C be continuous and set

$$F(x) := \int_a^x f(x)$$

Then F is differentiable on (a, b) and

$$F'(x) = f(x), \quad \textit{for } x \in (a, b)$$

Practical Integrals

Integral table.

1) $\int 0\,dx = C$

2) $\int x^{\alpha} dx = \frac{1}{\alpha+1} x^{\alpha+1} + C$

3) $\int \frac{1}{x} dx = \ln|x| + C$

4) $\int a^{x} dx = \frac{a^{x}}{\ln a} + C (a > 0, a \neq 1)$

5) $\int e^{x} dx = e^{x} + C$

6) $\int \sin x\,dx = -\cos x + C$

7) $\int \cos x\,dx = \sin x + C$

8) $\int \sec^{2} x\,dx = \tan x + C$

9) $\int \csc^{2} x\,dx = -\cot x + C$

10) $\int \sec x \tan x\,dx = \sec x + C$

11) $\int \csc x \cot x\,dx = -\csc x + C$

12) $\int \frac{dx}{\sqrt{1-x^{2}}} = \arcsin x + C$

13) $\int \frac{dx}{1+x^{2}} = \arctan x + C$

Practical Integrals

Practice.

Calculate the following integrals

1. $$\int cos^2(x)dx$$

2. $$\int \frac{1+x+x^2}{x(1+x^2)}dx$$

Substitution Rule

Substitution Rule.

Let $f \in Reg([\alpha, \beta])$ and g: $[a, b] \to [\alpha, \beta]$ continuously differentiable. Then

$$\int_a^b (f \circ g)(x) g'(x) dx = \int_{g(a)}^{g(b)} f(y) dy$$

Example.

$$\int xe^{-x^2} dx = \int -\frac{1}{2} e^{-x^2} d(-x^2) = -\frac{1}{2} \int d(e^{-x^2}) = -\frac{1}{2} e^{-x^2}$$

Practice.

Calculate the following integrals

1. $$\int x\sqrt{1-x^2}dx \; (Two \;\; method)$$

2. $$\int tan(x)dx$$

3. $$\int \frac{1}{sin(x)}dx$$

Integration by Parts

Theorem.

Let f,g: [a,b] $\rightarrow$ C be continuously differentiable. Then

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx$$

Example.

$$\int lnx\ dx = x\ lnx - \int x\ dlnx = x\ lnx - \int dx = x\ lnx - x$$

Integration by Parts

Practice.

Calculate the following integrals

1. $$\int x sin(x) dx$$
2. $$\int e^x sin(x) dx$$

Improper Integrals

Improper Integrals.

Assume that b $\leq \infty$ and that f: [a, b) $\to$ C is regulated on any closed subinterval [a, x], x $<$ b. Then

$$\int_a^b f(t)dt$$

is called an improper integral and is said to converge or exist if

$$\lim_{x \nearrow b} \int_a^x f(t)dt = L$$

exists. The number L$\in$ C is then called the value of the improper integral and we write

$$L = \int_a^b f(t)dt$$

Improper Integrals

Example.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx &= arctan(x)\Big|_{-\infty}^{\infty} \\
&= \lim_{x\to+\infty} arctan(x) - \lim_{x\to-\infty} arctan(x) \\
&= \frac{\pi}{2} - \frac{\pi}{2} \\
&= \pi
\end{aligned}$$

Improper Integrals

Practice.

Calculate the following abnormal integral

$$\int_{-1}^{1} \frac{dx}{x^2}$$

Improper Integrals

Solution

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1}$$
$$= -1 - 1$$
$$= -2$$

Comment.

Is the answer correct? If not, where is wrong?

Cauchy Property of Functions

Theorem.

The limit $\lim\limits_{x \to a} F(x)$ exists is equivalent to saying the function F(x) satisfies the Cauchy property

$$\underset{\varepsilon > 0}{\forall} \; \underset{\delta > 0}{\exists} \; \underset{x, y \in I}{\forall} \; |x - a| < \delta \wedge |y - a| < \delta \; \Rightarrow |f(x) - f(y)| < \varepsilon$$

Cauchy Criterion

Theorem.

Let a $\in$ R and f: [a,∞) $\rightarrow$ R be integrable on every interval [a, x], x $\in$ R. The improper integral

$$\int_a^\infty f(x)dx$$

converges if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{R>0}{\exists}\ \underset{x,y\in I}{\forall}\ x > R \wedge y > R \ \Rightarrow \left| \int_x^y f(t)dt \right| < \varepsilon$$

Comparison Test

Theorem.

Let I $\subset$ R and f: I $\to$ C, g: I $\to$ $[0,+\infty)$. If $|f(t) < g(t)$, then

$$\int_I g(x)dx \ converges \Rightarrow \int_I f(x)dx \ converges$$

Comparison Test

Corollary.

Let I $\subset$ R and f, g: I $\rightarrow$ [0,$+\infty$), and

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = l$$

Then if $0 \leq l < +\infty$,

$$\int_I g(x)dx \ converges \Rightarrow \int_I f(x)dx \ converges$$

Then if $0 < l \leq +\infty$,

$$\int_I g(x)dx \ diverges \Rightarrow \int_I f(x)dx \ diverges$$

Finer Versions of Comparison Test

Theorem.

Let a $\in R^+$ and f: [a,∞) $\to$ R be integrable on every interval [a, x], x $\in$ R, and $f(x) \geq 0$. Moreover

$$\lim_{x\to+\infty} x^p f(x) = l$$

Then, if $0 \leq l < +\infty$, $p > 1$, we have $\int_a^\infty f(x)dx$ converges.
If $0 < l \leq +\infty$, $p \leq 1$, we have $\int_a^\infty f(x)dx$ diverges.

Finer Versions of Comparison Test

Practice.

Determine whether the following abnormal integral is convergent or not

$$\int_1^{+\infty} \frac{dx}{x\sqrt{1+x^2}}$$

Euler Gamma Function

Definition.

$$\Gamma(t) = \int_0^{+\infty} z^{t-1} e^{-z} dz$$

Corollary.

$$\Gamma(t+1) = t\Gamma(t), \qquad t > 0$$

Corollary.

$$\Gamma(t+1) = t! \quad \textit{for} \quad t \in N, \quad \textit{since} \quad \Gamma(1) = 1$$

Exercises

Calculate the following integrals

1. $$\int_0^1 \frac{ln(1+x)}{1+x^2}dx \quad (x = tan(t))$$

2. $$\int_0^{\frac{\pi}{2}} \frac{sin^2(x)}{sin(x)+cos(x)}dx \quad (t = \frac{\pi}{2} - x)$$

Integral Test

Theorem.

Let m $\in$ N and f: [m,∞) $\to$ [0,$+\infty$) be a decreasing function, such that $\int_m^{+\infty} f(t)dt$ exists. Then

$$\int_m^{\infty} f(t)dt < \infty \quad \Leftrightarrow \quad \sum_{n=m}^{\infty} f(n) < \infty$$

Example.

$$\sum_{n=1}^{\infty} \frac{ln(n)}{n^2} < \infty \quad \Leftrightarrow \quad \int_1^{\infty} \frac{ln(x)}{x^2} dx = -\frac{1+ln(x)}{x}\Big|_1^{\infty} = 1 < \infty$$

Series of functions

Theorem.

If f is continuous on $(-\rho, \rho)$ and f is differentiable, and we have

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then within the radius of convergence we can "interchange" differentiation and summation as well as integration and summation of a power series.

Series of functions

Example.

Differentiation:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} \frac{d}{dx} (a_n x^n) \\
&= \sum_{n=0}^{\infty} n a_n x^{n-1}
\end{aligned}
$$

Series of functions

Example.

Integral:

$$\begin{aligned}\int_0^x f(y)dy &= \int_0^x (\sum_{n=0}^{\infty} a_n y^n) dy \\ &= \sum_{n=0}^{\infty} \int_0^x (a_n y^n) dy \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}\end{aligned}$$

Taylor series

Definition.

Let I $\subset$R be an open interval and f$\in C^\infty(I)$. We say that f is real-analytic or just analytic at $x_0 \in$ I if there exists a neighborhood $B_\epsilon(x_0) \subset$ I such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

for all $x \in B_\epsilon(x_0)$.

Taylor series

Some common Taylor Series:

1. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. $$sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

3. $$cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

4. $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

5. $$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Taylor series

Example.

$$
\begin{aligned}
arctan(x) &= \int_0^x \frac{1}{1+y^2}\,dy \\
&= \int_0^x \sum_{n=0}^{\infty} (-1)^n y^{2n}\,dy \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^x y^{2n}\,dy \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}
\end{aligned}
$$

Taylor series

Practice.

Find the Taylor series of the following function at point x=0

$$f(x) = \frac{x}{(1-x)^2}$$

Solution

We can calculate the $n-th$ derivative of function f(x), which turns out to be

$$f(0) = \left.\frac{x}{(1-x)^2}\right|_{x=0} = 0$$

$$f'(0) = \left.\frac{1+2x}{(1-x)^2}\right|_{x=0} = 1$$

$$f''(0) = \left.\frac{4+2x}{(1-x)^4}\right|_{x=0} = 4$$

*Then plug the derivatives into the formula (**Don't forget n!**.), or we can (see next page)*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{n=0}^{\infty} nx^n$$

Solution

We can observe the function more carefully

$$
\begin{aligned}
f(x) &= x \cdot \frac{1}{(1-x)^2} \\
&= x\frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\
&= x\sum_{n=0}^{\infty} nx^{n-1} \\
&= \sum_{n=0}^{\infty} nx^n
\end{aligned}
$$

Ending

Thank you for your attention!

Integral is all about practice!