Vv186 Recitation

Week 7

By Yahoo

Outline

- Properties of Continuous Function
- Inverse function
- Uniform continuity
- Comment on Hw3

Properties of Continuous Function

1. Unchanged sign
2. Boundedness of Continuous Functions
3. Existence of max and min on closed interval
4. Closed interval mapped to closed interval

Unchanged sign

Unchanged sign in a small region

Let Ω (a subset of R) be some set, $f : \Omega \to R$ a function that is continuous at some point a in Ω and assume that $f(a) > 0$. Then there exists a $\varepsilon > 0$ such that $f(x) > 0$ for all x in $(a - \varepsilon, a + \varepsilon)$ in Ω.

Bolzano Intermediate Value Theorem

Let $a < b$ and $f : [a, b] \to R$ be a continuous function with $f(a) < 0 < f(b)$. Then there exists some x in $[a, b]$ such that $f(x) = 0$.

Proof

Consider $A := \{x \in [a, b] \colon f(y) < 0 \text{ for } y \in [a, x]\}$

1. Sup A exsits (α)： A is bounded (for all x in A, a ≤ x ≤ b) and A is non-empty (a in A)
2. α < b. (*b* - *δ* is an upper bound for *A* where *f* (*x*) > 0 for all *x in* (*b* - *δ* , *b*])
3. *f* (α) = 0 :*f* (α) > 0 and *f* (α) < 0 lead to contradictions (Sup A might not be in A!)

Bolzano Intermediate Value Theorem

Let $a < b$ and $f : [a, b] \to R$ be a continuous function. Then for y in $[\min\{ f (a), f (b)\}, \max\{ f (a), f (b)\}]$ there exists some x in $[a, b]$ such that $y = f (x)$.

Practice

If a continuous function is not equal to zero at any point in R, then f(x) is always positive or always negative.

A Fixed Point Theorem

Let $f : [a, b] \to R$ be a continuous function with ran f is a subset of $[a, b]$. Then f has a fixed point, i.e., there exists some x in $[a, b]$ such that $f(x) = x$.

Boundedness of Continuous Functions

Boundedness in a small region

Let Ω (a subset of R) be some set and $f : \Omega \to R$ a function that is continuous at some point a in Ω. Then there exists a $\varepsilon > 0$ such that f is bounded above on $(a - \varepsilon, a + \varepsilon)$ in Ω.

Boundedness on a closed interval

Let $a < b$ and $f : [a, b] \to R$ be a continuous function. Then f is bounded above.

Caution

This is not always true for an open interval!

Example:

1. $F(x) = \frac{1}{x}$ for $0 < x < \infty$
2. $F(x) = \frac{1}{x}$ for $1 < x < \infty$
3. $F(x) = \frac{1}{x}$ for $1 < x < 100$

Practice

F(x) is continuous on [a, +∞), and lim f(x) = c < ∞, as x → ∞. Then f(x) is bounded on [a, +∞).

Existence of max and min on closed interval

Existence of max on closed interval

Let $a < b$ and $f : [a, b] \to R$ be a continuous function. Then there exists a y in $[a, b]$ such that $f(x) \leq f(y)$ for all x in $[a, b]$. Hence $\max\{ f(x) : x \text{ in } [a, b] \}$ exists.

Closed interval mapped to closed interval

Continuity on closed interval

Let Ω (a subset of R) and f : Ω → R continuous. Suppose that I (a subset of Ω) is a closed interval. Then

$$f(I) = \left\{ y \in \mathbb{R} : \underset{x \in I}{\exists} f(x) = y \right\}$$

is also a closed interval.

Compact sets

A closed and bounded set K (a subset of R) is called compact.

Continuity on compact sets

Let K (a subset of R) be a compact set and f : K → R be continuous. Then f is uniformly continuous on K, f is bounded and has a maximum and minimum. Furthermore, the image f (K) is also compact.

Inverse function

Inverse function

- Injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in \Omega$
- Surjective: range of f = domain
- Bijective: injective and surjective

Inverse function

A bijective function f has an inverse function, denoted by f^{-1}.

$$f^{-1}: \widetilde{\Omega} \to \Omega, \qquad f(x) \mapsto x.$$

Inverse function

A strictly increasing and continuous function must have an inverse function.

Inverse function

A continuous function that has an inverse function is strictly monotonic.

Proof

Suppose f(a_0) < f(b_0) for a continuous function on I and a_0, b_0 are in I. For $0 \le t \le 1$

$$x_t := (1-t)a_0 + ta_1, \qquad y_t := (1-t)b_0 + tb_1.$$

where $a_1 < b_1$. Then

$$x_t < y_t \qquad \text{for } t \in [0.1]$$

Define the function g : [0, 1] *in* R, $g(t) = f\,(y_t\,)$- $f\,(x_t)$. Then g(t) ≠ 0, g(t) > 0 since g(0) > 0.

Uniform continuity

Uniform continuity

Let I (a subset of R) be an interval and f : Ω → R a function with I (a subset of Ω). Then f is called uniformly continuous on I if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in I}{\forall}\ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon.$$

Continuity V.S. Uniform continuity

If f is uniformly continuous on I , then f is also continuous on I .

Continuity V.S. Uniform continuity

Let $f : \Omega \to R$ a function with $I = [a, b]$ (a subset of Ω). If f is continuous on [a, b], then f is also uniformly continuous on [a, b].

Example

1. F(x)= $\frac{1}{x}$ is continuous on (0,1), but not uniformly continuous on (0,1). ($\frac{1}{n}$, $\frac{1}{n+1}$)
2. Given uniformly continuous function f(x) and g(x) on (a,b) , then f(x)+g(x) and f(x)g(x) are also uniformly continuous on (a,b).

Comments on Hw

Ex 3.5.4 upper limit

Let (x_n) be a bounded real sequence and A be the set of all accumulation points of (x_n). Prove that $\min A$, $\max A$ exist and

$$\varliminf_{n\to\infty} x_n = \min A \qquad \text{and} \qquad \varlimsup_{n\to\infty} x_n = \max A.$$

Proof

- Step 1: Sup A exsit.
- Step 2: Sup A = max A. (Sup A is also an accumulation point of x_n !)
- Step 3: Upper limit ≥ max A
- Step 4: Upper limit ≤ max A (Upper limit is also an accumulation point of x_n !)

Thank you for your attention!