UM–SJTU Joint Institute

Honors Mathematics II

(Vv186)

Recitation Class

Week three

Basic concepts in logic and sets

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1 Notation

Question: How to use **B** to define **A**?

$$A := B$$

Example: Exponential function.

$$exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbf{C}$$

Specifically, when $z = 0$

$$exp(z) = 1, \ z = 0$$

Therefore, we can define what is called Euler number:

$$e := exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$$

2 Statement

Definition: A statement (also called a proposition) is anything we can regard as being either true or false.

Question: How to define what exactly a statement is? What we mean by saying something is true or false? Are we able to define them? Example in real life: 'It is raining.'

Mathematical examples:

$$`1 > 0' \ is \ a \ true \ statement.$$

$$`x^3 > 0' \ is \ not \ a \ statement.$$

$$`\forall x \in \mathbf{R}, \ x^3 > 0' \ is \ a \ false \ statement.$$

In the final example, $\forall x \in \mathbf{R}$ is a quantifier, and $x^3 > 0$ is a statement frame or predicate.

Comment:

$$statement \ = \ predicate \ + \ quantifer(some \ value)$$

3 Working statements

3.1 Negation(A unary operation)

Let A be a statement. Then we define the negation of A, written as $\neg A$, to be the statement that is true if A is false and false if A is true.

Comment: We often use truth table to describe logic operations(Or proof!).

A	$\neg A$
T	F
F	T

Table 1: Truth table for negation of A

3.2 Conjuncion(A binary operation)

Let A and B be two statements. Then we define the conjunction of A and B, written A $\bigwedge$ B, which is spoken as A and B. A and B can be described by the following truth table:

A	B	A$\bigwedge$B
T	T	T
T	F	F
F	T	F
F	F	F

Table 2: Truth table for conjunction of A, B

3.3 Disjunction(A binary operation)

Let A and B be two statements. Then we define the disjunction of A and B, written A $\bigvee$ B, which is spoken as A or B. A and B can be described by the following truth table:

A	B	A$\bigvee$B
T	T	T
T	F	T
F	T	T
F	F	F

Table 3: Truth table for disjunction of A, B

4 Tautology and contradiction

A compound statement that is always true is called a tautology.

Example: A $\bigvee \neg$A is a tautology.(Easy proof.)

A compound statement that is always false is called a contradiction.

Example: A $\bigwedge \neg$A is a contradiction.(Easy proof.)

5 Implication

Let A and B be two statements. Then we define the implication of B and A, written A $\Rightarrow$ B, by the following truth table:

A	B	A$\bigvee$B
T	T	T
T	F	F
F	T	T
F	F	F

Table 4: Truth table for implication of B and A

Strange as it might seem at first sight, remember it is a definition!

Example: $\forall$ x$\in$ **R**, $x > 0 \;\Rightarrow\; x^3 \geq 0$. (Always a true statement, then a tautology)

Comment: Try illustrating implication of B and A by real life situation where A: It is raining, B: I bring an umbrella.

6 Equivalence

Let A and B be two statements. Then we define the equivalence of A and B, written A $\Leftrightarrow$ B, 'A is equivalent to B', or 'A if and only if B', by the following truth table:

A	B	A$\bigvee$B
T	T	T
T	F	F
F	T	F
F	F	T

Table 5: Truth table for equivalence of A and B

Two compound statements A and B are called logically equivalent if A $\Leftrightarrow$ B is a tautology, denoted by A $\equiv$ B.

Examples: De Morgan Rules are tautologies.

$$\neg(A \bigvee B) \equiv (\neg A) \bigwedge (\neg B)$$

$$\neg(A \bigwedge B) \equiv (\neg A) \bigvee (\neg B)$$

7 Contraposition

$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$$

Comment: Contraposition is useful in simplifying some proof.

8 Logical Quantifiers

8.1 Universal quantifier

Denoted by the symbol ∀, read as 'for all'.

Let M be a set and A(x) be a predicate. Then we define the quantifier ∀ by

$$\forall\ x \in M : A(x) \ \Leftrightarrow\ A(x)\ is\ true\ for\ all\ x \in M$$

8.2 Existential quantifier

Denoted by the symbol ∃, read as 'there exists'.

$$\exists\ x \in M : A(x) \ \Leftrightarrow\ A(x)\ is\ true\ for\ at\ least\ one\ x \in M$$

Comment:

$$\exists\ x \in M :\ A(x) \ \Leftrightarrow\ \neg\forall\ x \in M :\ (\neg A(x))$$

9 Vacuous Truth

If the domain of the universal quantifier ∀ is the empty set M = ∅, then the statement ∀x ∈ M: A(x) is defined to be true regardless of the predicate A(x). It is then said that A(x) is vacuously true.

Examples: All pink elephants can fly. All pink elephants can't fly.

10 Nesting Quantifiers

$$\forall x \exists y : x + y > 0$$
$$\exists x \forall y : x + y > 0$$

11 Naive Set Theorem

We indicate that an object (called an element) x is part of a collection (called a set) X by writing x ∈ X. We characterize the elements of a set X by some predicate P:

$$x \in X \ \Leftrightarrow\ \ P(x)$$

We write such a set X in the form X = {x : P(x)}.

We define the empty set ∅ := {x : x≠x }. The empty set has no elements, because the predicate x ≠ x is never true.

If every object x ∈ X is also an element of a set Y, then we say that X is a subset of Y.

We say that X = Y if and only if X is subset of Y and Y is subset of X.

We say that X is a proper subset of Y if X is subset of Y but X $\neq$ Y.

Example: A={a,b,c}, B={a,b,b,c,a,c}, a,b,c $\in$ **R** then A=B.

If a set X has a finite number of elements, we define the cardinality of X to be this number, denoted by #X, $|X|$ or card X.

We define the power set: P(M)={A:A is a subset of M}.

12 Operation on sets

If A = {x : P_1(x)}, B = {x : P_2(x)} we define the union, intersection and difference of A and B by

A $\cup$ B := {x : P_1(x) $\bigvee$ P_2(x)}, A $\cap$ B := {x : P_1(x) $\bigwedge$ P_2(x)}, A $\setminus$ B := {x : P_1(x) $\bigwedge$ $\neg P_2$(x)}

We then define the complement of A by A^c := M $\setminus$ A.

Comment: De morgan's rule for sets.

13 Ordered pair

An ordered pair denoted by (a,b) has the property

$$(a, b) = (c, d) \quad \Leftrightarrow (a = c) \vee (b = d)$$

There are (at least) two ways of defining an ordered pair as a set:

$$(a, b) := \{\{a\}, \{a, b\}\}$$

$$(a, b) := \{\{1, a\}, \{2, b\}\}$$

Cartesian product: A x B := $\{(a, b) : a \in A, b \in B\}$.

14 The Russel antinomy

The predicate P(x) : x $\neq$ x does not define a set A = $\{x : P(x)\}$.

Some other antinomy example: The sentence below is right.
The sentence above is wrong.

References

[1] Hohberger, H., 2016, ”vv186_main”, https://sjtu-umich.instructure.com/courses/102/files. Retrieved 2016-09-25.