Vv186 Recitation

Week 9

By Yahoo

Outline

- Differentiability
- Property of differentiability

Differentiability

Definition

Let Ω (a subset of R) be some set, x is an interior point of Ω and f : Ω → R a real function. Then f is differentiable at x if there exists a linear map L_x such that for all sufficiently small h in R

$$f(x+h) = f(x) + L_x(h) + o(h) \qquad \text{as } h \to 0.$$

$$L_x(h) = L_x \cdot h$$

And L_x is called the derivative of f at x.

Uniqueness of derivative

For given f and x, the derivative L_x is unique.

Derivative when mapped to R

Let Ω be a set, x ∈ Ω is an interior point and f : Ω → R a function that is differentiable at x with derivative L_x = f ′(x). Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example

1. $f: R \to R$, $f(x) = x^n$, $n \in N$. Then by binomial formula, we have $f'(x) = n \cdot x^{n-1}$.
2. $f: R \to R$, $f(x) = x^n$, $n \in R$. Then we also have $f'(x) = n \cdot x^{n-1}$ in domain of f.

Derivative of common functions

- (c)’=0
- (a^x)’=a^xln(a)
- (e^x)’=e^x
- ln’(x)=1/x
- sin’(x)=cos(x)
- cos’(x)=-sin(x)

Practice

Suppose f(x) = $\begin{cases} x^2, x\ is\ rational \\ 0, is\ irrational \end{cases}$. Find the derivative of f(x) at point 0.

Continuity

Let f be a function that is differentiable at some $x \in \operatorname{dom} f$. Then f is continuous at x.

Practice

Suppose F(x) = $\begin{cases} x^2, x \leq x_0 \\ ax + b, x > x_0 \end{cases}$ has a derivative at point x_0, find a & b.

Operation on derivative

Let f and g be functions on R, x ∈ dom f and x ∈ dom g. Assume that f and g are both differentiable at x, then

$$(f+g)'(x) = f'(x) + g'(x).$$

$$(\lambda f)'(x) = \lambda f'(x)$$

Product rule

Let f and g be functions on R, x ∈ dom f and x ∈ dom g. Assume that f and g are both differentiable at x, then

$$(f \cdot g)'(x) = f'(x)g(x) + g'(x)f(x)$$

Quotient Rule

Let f and g be functions on R, $x \in \text{dom } f$ and $x \in \text{dom } g$. Assume that f and g are both differentiable at x, then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

Chain Rule

Let f and g be functions on R, g(x) ∈ dom f and x ∈ dom g. Assume that f is differentiable at g(x) and g is differentiable at x, then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Practice

1. Find f’(x) of f(x) = $\frac{2x}{1-x^2}$ by Quotient Rule.
2. Find f’(x) of f(x) = $x\sqrt{1+x^2}$ by Product Rule and Chain Rule.

Practice

Suppose that f(x) is differentiable at a and f(a)≠0. Show that |f|(x) is also differentiable at a. (Think of the counterexample when f(a)=0. Why under such circumstance |f|(x) is not differentiable?)

Inverse Function Theorem

Let I be an open interval and let f : I → R be differentiable and strictly monotonic. Then the inverse map f^{-1} = g : f (I) → I exists and is differentiable at all points y ∈ f (I) for which f′(g(y)) ≠ 0. Furthermore we have

$$g'(y) = \frac{1}{f'(g(y))}$$

Property of differentiability

Extrema

Let f be a function and Ω is a subset of dom f . Then x ∈ Ω is called a (global) maximum point for f on Ω if

$$f(x) \geq f(y) \qquad \text{for all } y \in \Omega$$

Extrema

Let f be a function and (a, b) in dom f an open interval. If $x \in (a, b)$ is a maximum (or minimum) point for f on (a, b) and if f is differentiable at x, then $f'(x) = 0$.

Critical point

A function f is said to have a critical point at $x \in \operatorname{dom} f$ if $f'(x) = 0$.

Remark

The critical point at x ∈ dom f where f ′(x) = 0 does not necessarily to be an extrema. However, there are still chances that the critical point is an extrema.

Example: f(x)=x^3 has a critical point at 0 but 0 is not extrema point. f(x)=x^2 has a critical point at 0 and 0 is an extrema point.

Practice

Find the extrema of f(x)=$x\sqrt[3]{x-1}$.

Remark: What is f'(x)? Does f(x) reach extrema only when f'(x)=0?

Rolle's Theorem

Let f be a real function and $a < b \in R$ such that $[a, b] \in \mathrm{dom}\, f$. Assume that f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a)=f(b)$. Then there exists a number $x \in (a, b)$ such that $f'(x) = 0$.

Practice

Suppose $f''(x)$ exists on $[0,1]$ for some function $f(x)$ and $f(0)=f(1)=0$. Let $F(x)=xf(x)$, show that there exists a $t \in R$ such that $F''(t)=0$.

Solution

1. We can easily check $F(0)=F(1)=0$. Then by Rolle's Theorem, there exists a $p \in [0,1]$ such that $F'(p)=0$.
2. Also, we can easily check $F'(0)=F'(p)=0$. Then by Rolle's Theorem there exists a $t \in [0,p]$ such that $F'(t)=0$.

Mean Value Theorem

Let f be a real function and $a < b \in R$ such that $[a, b] \in \text{dom } f$. Assume that f is continuous on [a, b] and differentiable on (a, b). Then there exists a number $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Corollary

1. Let f be a real function and I in dom f . Assume that $f'(x) = 0$ on I . Then f is constant on I .
2. Let f and g be real functions and I in dom f and I in dom g. Assume that $f'(x)=g'(x)$ on I . Then there exists some $c \in R$ such that $f(x)=g(x)+c$.

Corollary

3. Let f be a real function and I in dom f. Assume that $f'(x) > 0$ on I. Then f is strictly increasing on I. If $f'(x) < 0$ on I, f is strictly decreasing on I.

Practice

Suppose $f'(x)$ exists for some real continuous function $f(x)$, and the limit of $f(x)$ and $f'(x)$ both exist as $x \to +\infty$. Show that the limit of $f'(x)$ as $x \to +\infty$ is actually 0.

Solution

1. Suppose the limit of f(x) as x $\rightarrow +\infty$ is L. Then the limit of f(x+1) as x $\rightarrow +\infty$ is also L.
2. By Mean Value Theorem, there exists a number t ∈ (x,x+1) such that f(x+1)-f(x)=f'(t).
3. Since the limit of f'(x) as x $\rightarrow +\infty$ exists, take the limit of both sides of eqn f(x+1)-f(x)=f'(t). Then we have the limit of f'(x) as x $\rightarrow +\infty$ is actually 0.

Maxima and Minima

Let f be a real function and $x \in \text{dom } f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x. If $f''(x) < 0$, then f has a local maximum at x.

Maxima and Minima

Let f be a real function and $x \in \text{dom } f$ such that $f'(x) = 0$. If f has a local minimum at x, then $f''(x) \geq 0$. If f has a local maximum at x, then $f''(x) \leq 0$.

Example

1. f(x)=x^2 has a minimum at 0 because f'(0)=0 and f''(0)=2>0.
2. f(x)=x^3 does not have a minimum or maximum at 0 because f''(0)=0 even if f'(0)=0.

Convexity and Concavity

Let Ω be a subset of R and I ∈ Ω an interval. A function f : Ω → R is called strictly convex on I if for all a, x, b ∈ I with a < x < b

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}.$$

Convexity and Concavity

Let $f : I \rightarrow R$ be be strictly convex on I and differentiable at $a, b \in I$. Then

1. For any $h > 0$ such that $a + h \in I$, the graph of f over the interval $(a, a + h)$ lies below the secant line through the points $(a, f(a))$, $(a + h, f(a + h))$.
2. The graph of f over all of I lies above the tangent line through the point $(a, f(a))$.
3. If $a < b$, then $f'(a) < f'(b)$.

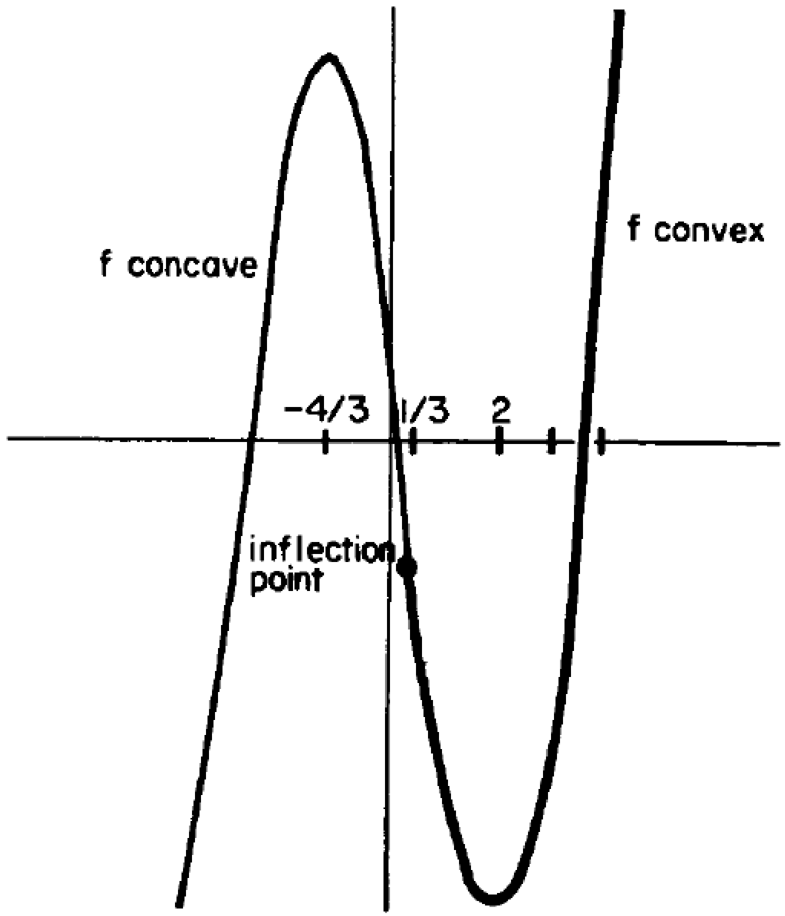
Convexity and Concavity

Let I be an open interval, $f : I \to R$ differentiable and $f'(x)$ strictly increasing. If $a, b \in I$, $a < b$, and $f(a) = f(b)$, then $f(x) < f(a) = f(b)$ for all $x \in (a, b)$.

Example

f(x) = $x^3 - x^2 - 8x + 1$ on [-2,2] is concave on [-2,1/3) and convex on (1/3,2], where point x=1/3 is the inflection point.

Graph of f(x) = $x^3 - x^2 - 8x + 1$



Curve Sketching

1. Ruler & pencil
2. Don’t draw small graphs. At least 8 cm wide.
3. Label the function
4. Title the graph

Curve Sketching

1. Coordinate system: origin, the domain and range
2. Continuity and behavior near points of discontinuity
3. The behavior as $x \to \pm\infty$; in particular asymptotes
4. Local and global extrema
5. Intervals where the function is increasing, decreasing or constant
6. Inflection points, where the second derivative changes sign

Example

Draw the graph of f(x) = $x^3 - x^2 - 8x + 1$, where the domain is R.

Cauchy Mean Value Theorem

Let f and g be real functions and [a, b] in dom f ∩ dom g. If f and g are continuous on [a, b] and differentiable on (a, b), then there exists an x ∈ (a, b) such that

$$\big(f(b) - f(a)\big)g'(x) = \big(g(b) - g(a)\big)f'(x).$$

Practice

Suppose f'(x) exists for some continuous f(x) on [0,1] and f(0)≠f(1). Show that there exists p,q ∈ (0,1) such that $f'(p) = \frac{f'(q)}{2q}$.

Solution

1. Let g(x)=x, then by Cauchy Mean Value Theorem, there exists p ∈ (0,1) such that f(1)-f(0)=(b-a)f’(p)
2. Let h(x)=x^2, then by Cauchy Mean Value Theorem, there exists q ∈ (0,1) such that [f(1)-f(0)]2q=(b-a)f’(q)
3. To sum, f’(q)/f’(p)=2q

L'Hôpital's Rule

Let f and g be real functions and $b \in \operatorname{dom} f \cap \operatorname{dom} g$ with

$$\lim_{x \searrow b} f(x) = 0 \qquad \text{and} \qquad \lim_{x \searrow b} g(x) = 0.$$

Suppose that there exists a $\delta > 0$ such that f and g are defined and differentiable on the interval $(b, b+\delta)$ and $g'(x) \neq 0$ for all $x \in (b, b+\delta)$. Suppose the limit $\lim_{x \searrow b} f'(x)/g'(x) =: L$ exists. Then

$$\lim_{x \searrow b} \frac{f(x)}{g(x)} = L.$$

Example

1. $\lim_{x\to 0}\frac{\ln(x+1)}{x} = \lim_{x\to 0}\frac{\frac{1}{x+1}}{1} = 1$
2. $\lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{1}{2x} = \frac{1}{2}$
3. $\lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{x-1}{(x+1)(x-1)} = \lim_{x\to 1}\frac{1}{x+1} = \frac{1}{2}$
4. $\lim_{x\to 1} x = \lim_{x\to 1}\frac{x^2}{x} = \lim_{x\to 1}\frac{2x}{1} = 2.$???

Thank you for your attention!