Vv186 Recitation

Week 5

By Yahoo

Outline

- Comments on homework 1
- Complex numbers
- Functions and map
- Sequence
- Real functions

Homework 1

- Prove set A=B by proving separately A is a subset of B and B is a subset of A. (Or truth table for Ex1.2.2)
- Cite sources you find elsewhere

Complex numbers

- Defined as $\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2$
- Addition: $(a_1, b_1) + (a_2, b_2) := (a_1 + a_2, b_1 + b_2)$
- Multiplication:

$$(a_1, b_1) \cdot (a_2, b_2) := (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$$

$$\lambda \cdot (a, b) = (\lambda a, \lambda b), \qquad \lambda \in \mathbb{R}, \ (a, b) \in \mathbb{C}.$$

- Imaginary number i := (0 , 1)

$$(a, b) = (a, 0) + (0, b) = a \cdot (1, 0) + b \cdot (0, 1).$$

Practice

- Prove: The square of (0 , 1) is a real number, which happens to be -1.

- Comment: Can we *define* i in another way, for example, the square of i to be -1? Or can we define complex number by Euler equation?

Modulus of complex numbers

- Modulus: $|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}}.$
- Complex conjugate of (a + b i) is (a – b i)
- Open ball : $B_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$

Function

- Let *X* and *Y* be sets and let *P* be a predicate with domain *X* x *Y* . Let $f := \{(x,y) \in X \times Y : P(x,y)\}$

  and assume that *P* has the property that

$$\forall_{(x_1,y_1)\in f} \; \forall_{(x_2,y_2)\in f} \quad x_1 = x_2 \;\Rightarrow\; y_1 = y_2.$$

- Domain of f = $\left\{x \in X : \underset{y\in Y}{\exists} : (x,y) \in f\right\}$
- Range of f = $\left\{y \in Y : \underset{x\in X}{\exists} : (x,y) \in f\right\}$

Different ways to write functions

- $f = \{(x, y) \in X \times Y : P(x, y)\}$
- $x \mapsto f(x).$
- $y = f(x)$

- Functions are also called maps

Sequence

- A sequence is simply a map from the natural to the real or complex numbers (Infinite sequences).
- We write $(a_n)_{n\in\Omega} = (a_n) = a_0, a_1, a_2, \ldots .$

Convergence sequences

- If $\forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n>N} \; |a_n - a| < \varepsilon$, then we say $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ as $n \to \infty$ (Drawing)
- Also convergence is equivalent to say

  $\forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n>N} \; a_n \in B_\varepsilon(a)$. where $B_\varepsilon(a) = \{z \in X : |a - z| < \varepsilon\}$
- A sequence that does not converge (to any limit) is called divergent.

Practice

- Show that the alternating sequence $a_n = (-1)^n$ is divergent.

- Hint: How to describe divergence in mathematics language?

Divergence to Infinity

- A real sequence (a_n) is called divergent to infinity if we have (drawing)

$$\underset{C>0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad a_n > C.$$

General Results

- A sequence is bounded if its range is a bounded set.
- A sequence is bounded if and only if $\sup|a_n| < +\infty$

General Results

- A convergent sequence is bounded.*

- Step 1: Choose epsilon = 1 (Remember the definition)
- Step 2: a_n are bounded when n > N
- Step 3: a_n are bounded for all n

General Results

- A convergent sequence has precisely one limit. (*Uniqueness**)

- Step 1: Suppose there are *two** different limits
- Step 2: Choice epsilon = |b-a|/2
- Step 3: Triangle inequaity

Operations on limits

- $\lim(a_n + b_n) = a + b$
- $\lim(a_n \cdot b_n) = ab$
- Have you solved the proof of the second statement?

Practice

Use epsilon-delta language to prove $\lim (a_n/b_n) = \lim a_n / \lim b_n$ if $\lim b_n \neq 0$.

How to solve ?

- Any volunteer? Not a difficult problem.

- Step 1: Definition of limits for $\mathrm{a_n}$ and b_{n}
- Step 2: Write out the final expression we want to prove
- Step 3: Try using inequality skills to obtain epsilon on the other side

How to solve ?

- Or we can use a Vv156 method…

- Step 1: Set $c_{\mathrm{n}} = \mathrm{a_n}/\mathrm{b_n}$, then $a_{\mathrm{n}} = \mathrm{b_n c_n}$
- Step 2: lim $a_{\mathrm{n}} = \lim \mathrm{b_n c_n} = \lim \mathrm{b_n} \lim c_{\mathrm{n}}$
- Step 3: $\lim c_{\mathrm{n}} = \dfrac{\lim a_{\mathrm{n}}}{\lim b_{\mathrm{n}}}$

- Comments: Try using bricks we have made! (Anyfood)

Squeeze Theorem

- Let (a_{n}), (b_{n}) and (c_{n}) be real sequences with $a_{\mathrm{n}} < b_{\mathrm{n}} < c_{\mathrm{n}}$ for sufficiently large *n*. Suppose that lim a_{n} = lim b_{n} = *a*. Then (c_{n}) converges and lim c_{n} = *a*.

- The proof is just about inequality.

Practice

Use Squeeze Theorem to calculate

$$\lim \frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2}.$$

Practice

Find the limit of the sequence $a_{\mathrm{n}} = \mathrm{n}(\frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n})$.

Limits of power function

- $\lim q^n = 0$ if $|q| < 1$ for complex q
- What if $|q| > 1$?
- What if $|q| = 1$?

Practice

Prove $|q|^n$ diverges if $|q| > 1$ for complex q.

- Method 1: Bernoulli’s inequality
- Method 2: Triangle form of complex numbers

Real sequence

- Increasing $a_n \leq a_{n+1}$
- Decreasing $a_n \geq a_{n+1}$
- Strictly increasing $a_n < a_{n+1}$
- Strictly decreasing $a_n > a_{n+1}$
- Monotonic

Theorem

Every monotonic and bounded (real) sequence (a_{n}) is convergent.

Subsequence

- Let (a_{n}) be a sequence and let (n_{k}) be a strictly increasing sequence of natural numbers. Then the composition

$$(a_{n_k})_{k\in\mathbb{N}} := (a_n)_{n\in\mathbb{N}} \circ (n_k)_{k\in\mathbb{N}} = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$$

  is called a subsequence of (a_{n}).

- A functional perspective

Same-limit Lemma

Let (a_{n}) be a convergent sequence with limit a. Then any subsequence of (a_{n}) is convergent with the same limit.

Monotonic Subsequence

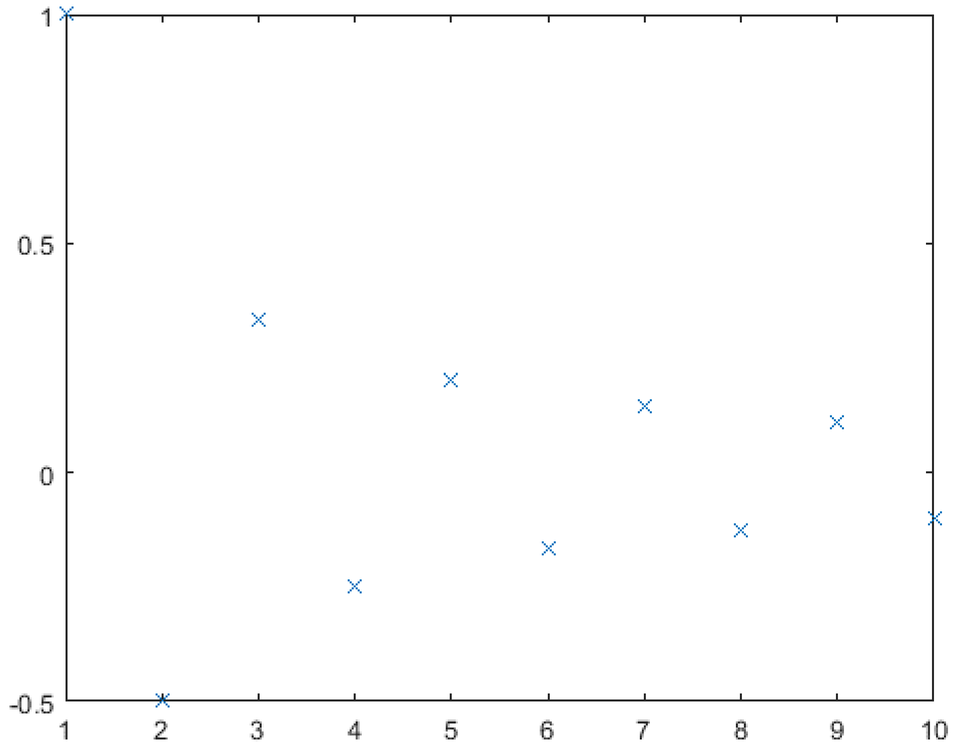
Every real sequence has a monotonic subsequence.*

Proof: Suppose $A := \{n \in \mathbb{N} : a_n \geq a_k \quad \text{for all } k > n\}$.

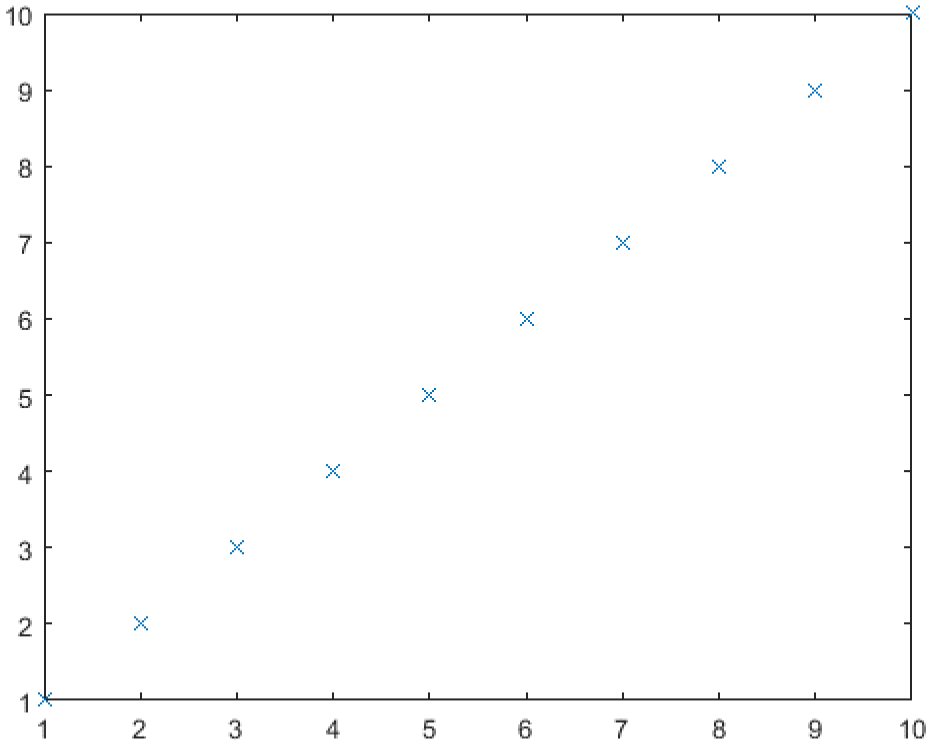
Case 1: A is not finite , $a_{\mathrm{n}} = (-1)^n \frac{1}{n}$

Case 2: A is finite. Then $\underset{n>N}{\forall} \; \underset{l>n}{\exists} \quad a_l > a_n$. Fix $n_0, n = n_0, find\ l, l = n_1$ … , (a_{n} = n)

$$a_{\mathrm{n}} = (-1)^n \frac{1}{n}$$



$$a_n = n$$



Accumulation Points

Let (a_n) be a sequence. Then a number a is called an accumulation point of (a_n) if

$$\underset{\varepsilon>0}{\forall} \ \underset{N\in\mathbb{N}}{\forall} \ \underset{n>N}{\exists} \ |a_n - a| < \varepsilon.$$

- Remark: If a sequence converges, the limit is one accumulation point and the only one.

How to solve

1. The limit is one accumulation point.
2. The accumulation point is the limit.

1) Lemma. 2.2.34. A number *a* is an accumulation point of a sequence if and only if there exists a subsequence of that converges to *a.*
2) The limit of a convergent sequence is equal to the limit of its subsequence.

3. A convergent sequence has a unique limit.

Bolzano-Weierstraß Theorem

Every bounded real sequence has an accumulation point.

Proof

- A bounded sequence of real numbers has a monotonic subsequence.
- The subsequence is bounded.
- The bounded and monotonic subsequence converges.
- The limit of the subsequence is an accumulation point.

Practice

Show that every bounded complex sequence has an accumulation point, which is also complex of course, and the real and imaginary part of the accumulation point is the accumulation point of the sequence of the real and imaginary part of the complex sequence.

How to solve

- Step 1. The sequence of the real and imaginary part of the complex sequence are bounded.
- Step 2. The sequence of the real part of the complex sequence has an accumulation point, which is the limit of a subsequence of the real part sequence.

How to solve

- Step 3. For the index of the convergent subsequence of real part sequence, find the corresponding subsequence of imaginary part sequence, which is bounded.
- Step 4. The subsequence of imaginary part sequence has an accumulation point, whichi s the limit of a subsubsequence of the subsequence of imaginary part sequence.

Comment

- We have considered separately the real part and the imagery part of complex number in the previous proof.
- Is such a method efficient ? Do we have to repeat the procedure more times if we have higher dimensions sets (vector spaces actually), such as (x i + y j + z k)?

Metric space

A map ρ: M x M → R is called a metric if

- ρ(x, y) > 0 for all x, y in M and ρ(x, y) = 0 if and only if x = y.
- ρ(x, y) = ρ(y, x)
- ρ(x, z) <= ρ(x, y) + ρ(y, z)

Then the pair (*M*, ρ) is called a ***metric space***.

Example

- $M = \mathbb{R}$, $\varrho(x,y) = | x - y |$.
- $M = \mathbb{C}$, $\varrho(z,w) = |\mathrm{Re}\, z - \mathrm{Re}\, w| + |\mathrm{Im}\, z - \mathrm{Im}\, w|$.
- $M = \mathbb{C}$, $\varrho(z,w) = \sqrt{|\mathrm{Re}\, z - \mathrm{Re}\, w|^2 + |\mathrm{Im}\, z - \mathrm{Im}\, w|^2}$.

Definitions under metric space

- (a_n) is called bounded if $\underset{R>0}{\exists} \ \underset{n\in\mathbb{N}}{\forall} \ a_n \in B_R(x)$ where $B_\varepsilon(a) = \{y \in M : \varrho(y, a) < \varepsilon\}$.
- Note that this definition does not depend on the point x, so boundedness is well-defined. (Drawing)

Cauchy Sequence

A sequence (a_n) in a metric space (M, ϱ) is called a Cauchy sequence if

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \varrho(a_m, a_n) < \varepsilon$$

Clearly, every convergent sequence is a Cauchy sequence, since

$$\varrho(a_n, a_m) \leq \varrho(a_n, a) + \varrho(a_m, a)$$

Boundedness of Cauchy

Every Cauchy sequence in a metric space (M, ϱ) is bounded.

Are Cauchy sequences convergent?

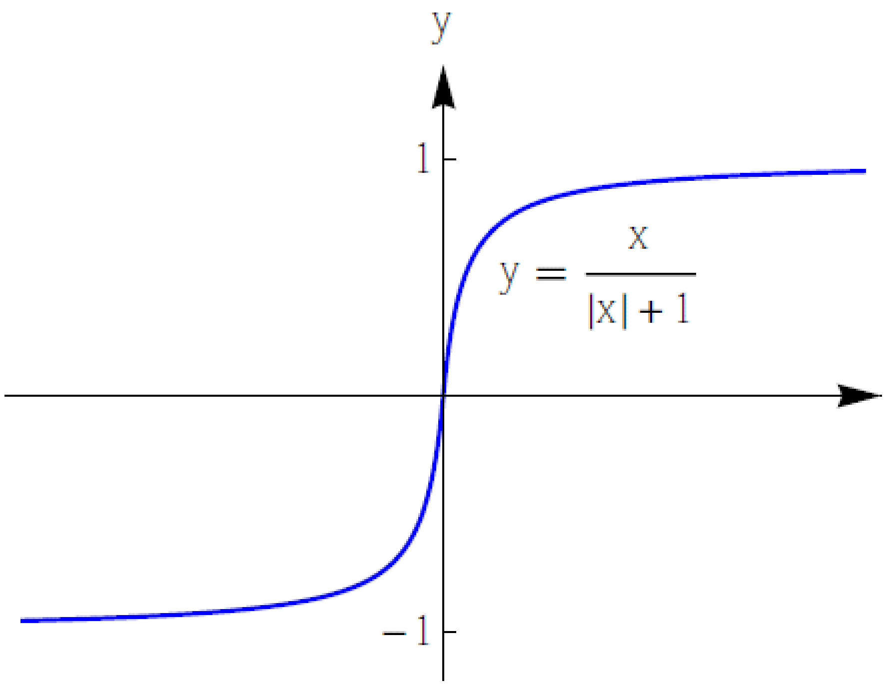
Every Cauchy sequence in R with the metric $\varrho(x,y) = | x - y |$ is convergent.

However, Cauchy sequence in metric

$$\varrho(x,y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|$$

may be not convergent.

$a_n = n$



Practice

Show that the sequence $a_\mathrm{n} = 1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}$ converges. (Cauchy sequence is convergent in $\varrho(x,y) = | x - y |$.)

Complete metric space

A metric space (M, ϱ) is called ***complete*** if every Cauchy sequence converges in M.

The Real Numbers

- An equivalence relation is defined as if

$$(a_n) \sim (b_n) \qquad :\Leftrightarrow \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

- The set of all sequences with the same limit as a sequence (*an*) is denoted by [(*an*)], called a (equivalence) class.
- The set of all classes is denoted Conv(Q)/~ (Q).
- R := Cauchy(Q)/~

Practice

If a_{n} converges, then $S_n = \frac{1}{n}(a_1 + \cdots + a_{\mathrm{n}})$ converges to the same limit.

Thank you for your attention!