Written Evaluation II Review

Continuity,Derivatives and Curves

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Open and closed sets

2.1.2. Definition. Let $(V, \|\cdot\|)$ be a normed vector space. A set $U \subset V$ is called *open* if for every $a \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$.

2.1.18. Definition. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$. Then M is said to be *closed* if its complement $V \setminus M$ is open.

Continuous Functions

Theorem

Let $(U, ||\cdot||_1)$and $(V, ||\cdot||_2)$ be normed vector spaces and $f: U \to V$ a function. Then f is continuous at $a \in U$ if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in U}{\forall} \quad \|x-a\|_1 < \delta \qquad \Rightarrow \qquad \|f(x)-f(a)\|_2 < \varepsilon.$$

Continuous functions

Question.

Continuous or not: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} (1 - \cos \frac{x^2}{y})\sqrt{x^2 + y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

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Comment.

- $y = 0$, $f(x, y)$ is continuous.
- $y \neq 0$, we want to show $\lim\limits_{\sqrt{x^2+y^2} \to 0} f(x, y) = 0$. Let $\sqrt{x^2 + y^2} \to 0$, then

$$|f(x, y)| \leq |1 - \cos \frac{x^2}{y}|\sqrt{x^2 + y^2} \leq 2\sqrt{x^2 + y^2} \to 0$$

Continuous Functions

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Let $(U, ||\cdot||_1)$ and $(V, ||\cdot||_2)$ be normed vector spaces and $f : U \to V$ a function. Then f is continuous if and only if the pre-image $f^{-1}(\Omega)$ of every open set $\Omega \subset V$ is open.

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Example

The determinant function is continuous.

Compact Sets

Definition

K is said to be compact if every sequence in K has a convergent subsequence with limit contained in K.

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Let $(V, ||\cdot||)$ be a (possibly infinite-dimensional) normed vector space and $K \subset V$ be compact. Then K is closed and bounded.

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Let $(V, ||\cdot||)$ be a finite-dimensional normed vector space and $K \subset V$ is closed and bounded, then K is compact.

Compact Set and Continuous Functions

Theorem

Let $(U, ||\cdot||_1)$ and $(V, ||\cdot||_2)$ be normed vector spaces and $K \subset V$ be compact. Let $f : K \to V$ be continuous. Then

i) *ran $f = f(K)$ is compact in V.*

ii) *f has a maximum in K.*

iii) *f is uniformly continuous on K.*

Determinant as Continuous Function

Question.

Clarify the following

The set $\Omega = \{A \in \mathrm{Mat}(2 \times 2) \colon \det A = 1\}$ is

- ☐ bounded.
- ☐ open.
- ☒ closed.
- ☐ compact.

Determinant as Continuous Function

Solution.

- For example, $A = \begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \in \Omega$. Take the max norm for determinant $\|\cdot\| = \max_{i,j} |a_{ij}|$ and let $n \to \infty$, we can see that Ω is not bounded.

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- Since Ω is not bounded, then Ω is not compact.
- We argue that Ω is not open. Take $A = \begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$ $(n > 1) \in \Omega$ and the max norm, we find $B_\varepsilon(A)$ for $\varepsilon > 0$, $B = \begin{pmatrix} n + \frac{\varepsilon}{2} & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \in B_\varepsilon(A)$, but $B \notin \Omega$.

Determinant as Continuous Function

Solution.

- We claim Ω is closed. We know that for a continuous function, the pre-image of an open set is open. If the function is defined on the whole vector space, we can take the complement of the image set and pre-image set. Then we have the following conclusion

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Theorem

For a continuous function, the pre-image of a closed set is closed, if the function is defined on the whole vector space.

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Theorem

For a continuous function, the pre-image of a closed set is closed, if the function is defined on the whole vector space.

We know that the determinant is a continuous function, then the pre-image of a closed set will be closed. Since $\{1\} \in \mathbb{R}$ is closed, then Ω is closed.

Differentiability

Definition

There is a linear map $L_x \in \mathcal{L}(X, V)$ (called derivative) such that

$$f(x+h) = f(x) + L_x h + o(h) \qquad \text{as } h \to 0.$$

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Comment.

We may thus regard the derivative as a linear map

$$D\colon C^1(\Omega, V) \to C(\Omega, \mathcal{L}(X, V)), \qquad f \mapsto Df.$$

Derivative

Example

Exercise 3. Calculate the derivative of the map

$$\Psi\colon \mathrm{Mat}(n \times n, \mathbb{R}) \to \mathrm{Mat}(n \times n, \mathbb{R}), \qquad \Psi(A) = A \cdot A^T$$

where A^T denotes the transpose of A.

Derivative

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Solution.

Proof. We have

$$\Psi(A+H) = (A+H)(A+H)^T = AA^T + HA^T + AH^T + HH^T.$$

(1 Mark) We see that $HH^T = o(H)$, since using the operator norm,

$$\lim_{\|H\|\to 0} \frac{\|HH^T\|}{\|H\|} \le \lim_{\|H\|\to 0} \|H^T\| = 0.$$

(1 Mark) Hence,

$$D\Psi|_A H = HA^T + AH^T.$$

Jacobian

If the function is differentiable, there is a quicker way to calculate the derivative.

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Definition

The matrix is called the Jacobian of f.

$$J_f(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}\Bigg|_x$$

where the partial derivatives are defined as

$$\begin{aligned} \frac{\partial f}{\partial x_j}\bigg|_x &:= \lim_{h\to 0} \frac{f(x+he_j)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x_1,\dots,x_{j-1},x_j+h,x_{j+1},\dots,x_n)-f(x)}{h} \end{aligned}$$

Jacobian and continuity

Theorem

Suppose all partial derivatives of f exist

i) *If all partial derivatives are bounded, then f is continuous*
ii) *If all partial derivatives are continuous, then f is continuously differentiable. The Jacobian is just the derivative.*

Chain Rule

Definition

$$D(f \circ g)|_x = Df|_{g(x)} \circ Dg|_x$$

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Example

Calculate the derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in spherical coordinate: $(r, \theta, \varphi) \in (0, +\infty) \times [0, \pi] \times [0, 2\pi)$

$$\Phi(r, \theta, \varphi) = \begin{pmatrix} r \cos\theta \\ r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \end{pmatrix}$$

Chain Rule

Solution.

First, the derivative of $\Phi(r,\theta,\varphi)$

$$D\phi\Big|_{(r,\theta,\varphi)} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix}$$

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Then the derivative of f

$$Df\Big|_{\Phi(x,y,z)} = (2r\cos\theta, 2r\sin\theta\cos\varphi, 2r\sin\theta\sin\varphi)$$

Chain Rule

Solution.

$$
\begin{aligned}
Df\Big|_{(r,\theta,\varphi)} =& (2r\cos\theta, 2r\sin\theta\cos\varphi, 2r\sin\theta\sin\varphi) \\
& \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix} \\
=& (2r, 0, 0)
\end{aligned}
$$

Chain Rule

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$$
\begin{aligned}
Df\Big|_{(r,\theta,\varphi)} =& (2r\cos\theta, 2r\sin\theta\cos\varphi, 2r\sin\theta\sin\varphi) \\
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=& (2r, 0, 0)
\end{aligned}
$$

Comment.

The derivative can be obtained in another way.

$$f(r,\theta,\varphi) = (r\cos\theta)^2 + (r\sin\theta\cos\varphi)^2 + (r\sin\theta\sin\varphi)^2 = r^2$$

$$Df\Big|_{(r,\theta,\varphi)} = \left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi}\right) = (2r, 0, 0)$$

Mean Value Theorem

Theorem

Let $f \in C^1(\Omega, V)$. Let $0 \le t \le 1$, then

$$f(x+y) - f(x) = \int_0^1 Df|_{x+ty} y \, dt = \left(\int_0^1 Df|_{x+ty} \, dt \right) y.$$

Theorem

Let $f : I \times \Omega \to V$ be a continuous function such that $Df(t, \cdot)$ exists and is continuous for every $t \in I$. Then $g(x)$ is differentiable and the derivative is

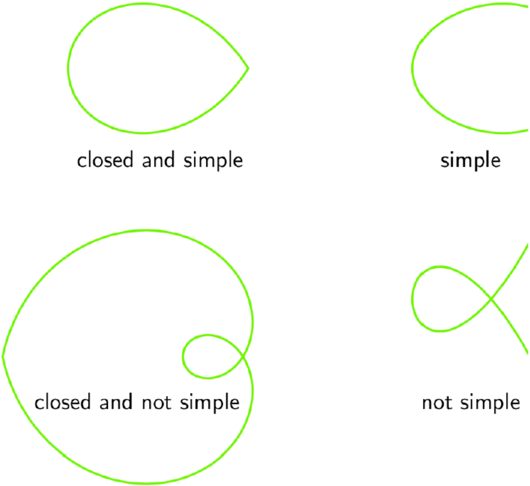
$$g(x) = \int_a^b f(t,x) \, dt \qquad Dg(x) = \int_a^b Df(t, \cdot)|_x \, dt$$

Curves and Parametrization

- A set $C \subset V$ for which there exists a continuous, surjective and locally injective map $\gamma : I \to C$ is called a curve.
- The map γ is called a parametrization of C.
- Locally injective means that in the neighborhood $B_\varepsilon(x) \cap I$ of any point $x \in I$ the parametrization is injective.

Simple, Open and Closed Curves

Example



Reparametrization

2.3.9. Definition. Let $\mathcal{C} \subset V$ be a curve with parametrization $\gamma\colon I \to \mathcal{C}$.

(i) Let $J \subset \mathbb{R}$ be an interval. A continuous, bijective map $r\colon J \to I$ is called a *reparametrization*.of the parametrized curve $(\mathcal{C}, \gamma)$.

Comment.

Given any two parametrizations $\gamma, \widetilde{\gamma}$ of an open curve, one can always find a reparametrization by setting $r = \gamma^{-1} \circ \widetilde{\gamma}$ (the continuity and local injectivity is enough for this definition to make sense).

Reparametrization

Question.

Can we find a reparametrization of curve S, $r : [0, 1] \to [0, 2\pi]$ such that

$$\gamma : [0, 2\pi] \to S, \qquad \gamma(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

$$\tilde{\gamma} : [0, 1] \to S, \qquad \tilde{\gamma}(t) = \begin{pmatrix} -\cos(2\pi t) \\ -\sin(2\pi t) \end{pmatrix}$$

$$\tilde{\gamma} = \gamma \circ r$$

Reparametrization

Solution.

$$\tilde{\gamma}(t) = \begin{pmatrix} -\cos(2\pi t) \\ -\sin(2\pi t) \end{pmatrix} = \begin{pmatrix} \cos(2\pi t \pm \pi) \\ \sin(2\pi t \pm \pi) \end{pmatrix}$$

$$r(t) = \begin{cases} 2\pi t + \pi & t \in [0, \frac{1}{2}] \\ 2\pi t - \pi & t \in (\frac{1}{2}, 1] \end{cases}$$

However, function r is not continuous at point $t = \frac{1}{2}$.

Tangent Vector, Normal Vector, Curve Length, Curvature

- Tangent Vector: $T \circ \gamma(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$
- Normal Vector: $N \circ \gamma(t) = \frac{(T\circ\gamma)'(t)}{\|(T\circ\gamma)'(t)\|}$
- Binormal vector: $B \circ \gamma(t) = T \circ \gamma(t) \times N \circ \gamma(t)$(Only in $\mathbb{R}^3$)
- Curve length: $l(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \, dt$
- Curve function: $l \circ \gamma(t) = \int_a^t \|\gamma'(t)\| \, dt$
- Curvature: $\kappa \circ l^{-1}(s) = \left\| \frac{d}{ds} T \circ l^{-1}(s) \right\| = \frac{\left\|(T\circ\gamma)'(t)\right\|}{\|\gamma'(t)\|}$
- Curvature in $\mathbb{R}^3$: $\kappa \circ l^{-1}(s) = \frac{\left\|\gamma'(t)\times\gamma''(t)\right\|}{\|\gamma'(t)\|}$
- Torsion: $\frac{d(B\circ l^{-\mathbf{1}}(s))}{ds} = -\tau(s)(N \circ l^{-1}(s))$, then $\tau(s) = -\frac{d(B\circ l^{-\mathbf{1}}(s))}{ds} \cdot (N \circ l^{-1}(s))$

Tangent Vector, Normal Vector, Curve Length, Curvature

Question.

Find the tangent vector, normal vector, curve length function, curvature of cycloid $\gamma(t),\ t \in (0, 2\pi)$

$$\gamma(t) = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$$

Tangent Vector, Normal Vector, Curve Length, Curvature

Solution.

- $$\gamma'(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}$$

  $$T \circ \gamma(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{1}{2\sin(t/2)} \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} \sin(t/2) \\ \cos(t/2) \end{pmatrix}$$

- $$(T \circ \gamma)'(t) = \frac{1}{2} \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix}$$

  $$N \circ \gamma(t) = \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|} = \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix}$$

Tangent Vector, Normal Vector, Curve Length, Curvature

Solution.

- $$\|\gamma'(t)\| = 2\sin(t/2)$$
  $$l \circ \gamma(t) = \int_0^t \|\gamma'(t)\|\,dt = 4(1 - \cos t/2), \quad t \in (0, 2\pi)$$
  $$l \circ \gamma(2\pi) = 8$$
- $$\kappa \circ l^{-1}(s) = \frac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|} = \frac{1}{4\sin(t/2)}$$


Thank You!