Final Review

Pratical Integral

Yahoo

UM-SJTU Joint Institute

Summer 2017

Line Integral

Question.

Let $\mathcal{C} \subset \mathbb{R}^3$ be the intersection of the cylinder $\{x \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ and the plane $\{x \in \mathbb{R}^3 : x + y + z = 1\}$. Calculate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ given that $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$$F(x, y, z) = \begin{pmatrix} -y \\ x \\ -z \end{pmatrix}$$

Comment.

How is line integral defined?

Line Integral

Solution.

We parametrize the curve with $\gamma(t)$, $t \in [0, 2\pi)$. Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 1 - \cos t - \sin t \end{pmatrix}$$

Line Integral

Solution.

We parametrize the curve with $\gamma(t),\ t \in [0, 2\pi)$. Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 1 - \cos t - \sin t \end{pmatrix}$$

$$F(t) = \begin{pmatrix} -\cos t \\ \sin t \\ -1 + \cos t + \sin t \end{pmatrix}$$

Line Integral

Solution.

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$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 1 - \cos t - \sin t \end{pmatrix}$$

$$F(t) = \begin{pmatrix} -\cos t \\ \sin t \\ -1 + \cos t + \sin t \end{pmatrix}$$

$$F'(t) = \begin{pmatrix} \sin t \\ \cos t \\ \cos t - \sin t \end{pmatrix}$$

Line Integral

Solution.

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \langle \mathbf{F}, \mathbf{F}' \rangle dt \\
&= \int_0^{2\pi} (1 - \cos t - \sin t)(\cos t - \sin t) dt \\
&= \int_0^{2\pi} (\cos t - \sin t - \cos 2t) \ dt \\
&= 0
\end{aligned}$$

Integral over $\mathbb{R}^2$

Question.

Calculate the integral

$$I = \iint\limits_D \frac{3x}{y^2 + xy^3} dxdy$$

where D is the area bounded by $xy = 1, xy = 3, y^2 = x, y^2 = 3x$.

Comment.

Need to take a good substitution.

Integral over $\mathbb{R}^2$

Solution.

Let

$$u = xy$$

$$v = \frac{y^2}{x}$$

then D becomes $\{(u,x) : 1 \leq u, v \leq 3\}$. Take the inverse of the composition

$$x = u^{\frac{2}{3}} v^{-\frac{1}{3}}$$

$$y = (uv)^{\frac{1}{3}}$$

Now we need to calculate the modulus of the determinant of the Jacobian.

Integral over $\mathbb{R}^2$

Solution.

$$
\begin{aligned}
|\det J| &= |\det(\frac{\partial(x,y)}{\partial(u,v)})| \\
&= |\det \begin{pmatrix} \frac{2}{3}u^{-\frac{1}{3}}v^{-\frac{1}{3}} & -\frac{1}{3}u^{\frac{2}{3}}v^{-\frac{4}{3}} \\ \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{1}{3}} & \frac{1}{3}u^{\frac{1}{3}}v^{-\frac{2}{3}} \end{pmatrix} | \\
&= \frac{1}{3v}
\end{aligned}
$$

Integral over $\mathbb{R}^2$

Solution.

Therefore, by the substitution rule

$$\begin{aligned}\iint\limits_D \frac{3x}{y^2+xy^3}dxdy &= \iint\limits_D \frac{1}{\frac{y^2}{x}(1+xy)}dxdy\\ &= \iint \frac{1}{v(1+u)}\cdot\frac{1}{3v}dudv\\ &= \int\limits_1^3 \frac{du}{(1+u)}\int\limits_1^3\frac{dv}{v^2}\\ &= \frac{2}{3}\ln 2\end{aligned}$$

Integral over $\mathbb{R}^2$

Comment.

To calculate $\det(\frac{\partial(x,y)}{\partial(u,v)})$ more conveniently, you can first calculate $\det(\frac{\partial(u,v)}{\partial(x,y)})$, then take the reciprocal of it.

$$\det(\frac{\partial(u,v)}{\partial(x,y)}) = \frac{3y^2}{x} = 3v$$

$$\det(\frac{\partial(x,y)}{\partial(u,v)}) = \frac{1}{3v}$$

Surface Integral V.S. Volumn Intgral

Question.

Calculate $\int_\Omega (x, y, z)^T ds$ where

$$\Omega = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

Question.

Calculate $\int_\Omega xyzdxdydz$ where

$$\Omega = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$$

Integral over Polar Coordinate

Question.

Calculate the integral

$$I = \iint\limits_{D} \frac{1}{xy} dxdy$$

where

$$D = \{(x, y) : \frac{x}{x^2 + y^2} \in [2, 4], \frac{y}{x^2 + y^2} \in [2, 4]\}$$

Comment.

Use polar coordinate.

Integral over Polar Coordinate

Solution.

Let

$$x = r\cos t$$

$$y = r\sin t$$

D becomes

$$D = \{(r,t) : \frac{\cos t}{r} \in [2,4], \frac{\sin t}{r} \in [2,4]\}$$

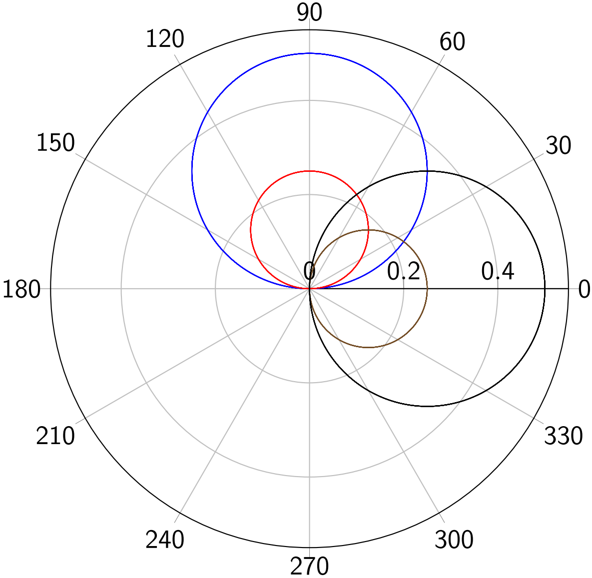
where

$$D = D_1 \bigcup D_2$$

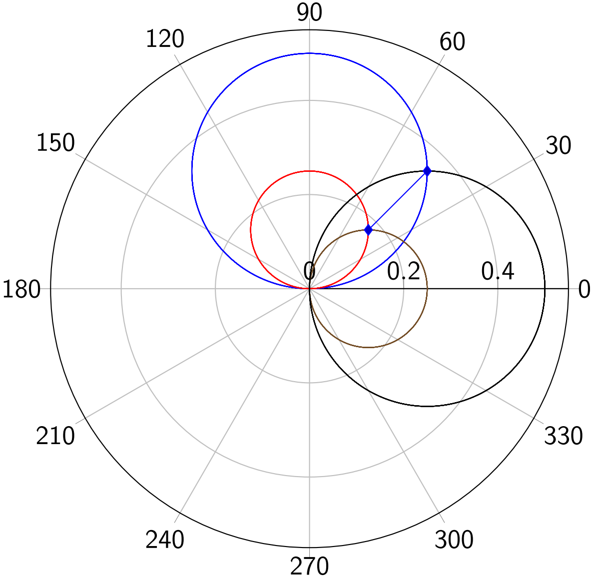
$$D_1 = \{(r,t) : r \in [\frac{1}{4}\cos t, \frac{1}{2}\sin t], t \in [arctan(\frac{1}{2}), \frac{\pi}{4}]\}$$

$$D_2 = \{(r,t) : r \in [\frac{1}{4}\sin t, \frac{1}{2}\cos t], t \in [\frac{\pi}{4}, arctan(2)]\}$$

Integral over Polar Coordinate



Integral over Polar Coordinate



Integral over Polar Coordinate

Solution.

$$
\begin{aligned}
\iint\limits_{D} \frac{1}{xy} dxdy &= 2 \iint\limits_{D_1} \frac{1}{xy} dxdy \\
&= 2 \int_{\tan^{-1} \frac{1}{2}}^{\frac{\pi}{4}} \int_{\frac{1}{4} \cos t}^{\frac{1}{2} \sin t} \frac{1}{r^2 \cos t \sin t} rdrdt \\
&= 2 \int_{\tan^{-1} \frac{1}{2}}^{\frac{\pi}{4}} \frac{dt}{\cos t \sin t} \ln(2 \tan t) \\
&= 2 \int_{\tan^{-1} \frac{1}{2}}^{\frac{\pi}{4}} \frac{\ln(2 \tan t)}{2 \tan t} d(2 \tan t) \\
&= 2 \int_{\tan^{-1} \frac{1}{2}}^{\frac{\pi}{4}} \ln(2 \tan t) d \ln(2 \tan t) \\
&= (\ln 2)^2
\end{aligned}
$$