# Week 7 Recitation

## Convergence, Continuity, Functions and Derivatives

Yahoo

UM-SJTU Joint Institu(te

Summer 2017

# Selected Problems in Written Evaluation

Question.

Let $A \in Mat(n \times n; \mathbb{R})$ and let $I_n$ denote the $n \times n$ unit matrix. Suppose that $ran(A) \subset ker(A)$. Show that $I_n + A$ is invertible.(We know $A^2 = 0$, $\det(A)$=0).

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**Proof.**

$$(I_n + A)(I_n - A) = I_n + A - A + A^2 = I_n$$

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**Proof.**

$$(I_n + A)(I_n - A) = I_n + A - A + A^2 = I_n$$

□

**Comment.**

What's wrong with the following proof?

$$det(I_n + A) = \frac{det(A(I_n + A))}{det(A)} = \frac{det(A + A^2)}{det(A)} = \frac{det(A)}{det(A)} = 1$$

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$$AB \neq BA$$
$$A(B + C) = AB + BC$$
$$AA^{-1} = A^{-1}A = I_n$$
$$AI_n = I_nA = A$$

Why these equations are listed here?

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Why these equations are listed here?

**Comment.**

Are the following equations correct?

$$A \cdot (B+C) = AB + BC$$
$$A \cdot A^{-1} = A^{-1} \cdot A = I_n$$
$$A \cdot I_n = I_n \cdot A = A$$

# Equivalence of Norms

### Definition

Two norms are called equivalent if there exists two constants $C_1, C_2 > 0$ such that for all $v \in V$

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## Comment.

$$C_1||x||_1 \leq ||x||_2 \leq C_2||x||_1 \quad \Leftrightarrow \quad \frac{1}{C_2}||x||_2 \leq ||x||_1 \leq \frac{1}{C_1}||x||_2$$

# Equivalence of Norms

Example

In $\mathbb{R}^n$ we have the following two possible choices of norms:

$$\|x\|_2 := \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}, \qquad \|x\|_\infty := \max_{1\le i\le n} |x_i|.$$

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*Proof.*

$$\frac{1}{\sqrt{n}} \|x\|_2 \le \|x\|_\infty \le \|x\|_2$$

$\square$

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**Theorem**

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**Proof.**

The proof for equivalence of norms in a finite-dimensional vector space involves the theorem of Bolzano-Weierstraß in $\mathbb{R}^n$ and a basic norm inequality. □

# The Theorem of Bolzano-Weierstraß in $\mathbb{R}^n$

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**Proof.**

Apply the theorem of Bolzano-Weierstraß of real number to first item in the vector, and find a convergent subsequence. Then apply the theorem for the second item of the subsequence, and find a convergent sub-subsequence, which is of course a subsequence. $\square$

# A Basic Norm inequality

**Theorem**

*Let $(V, ||\cdot||)$ be a finite or infinite dimensional normed vector space and $v_1, ..., v_n$ an independent set in $V$. Then there exists a $C > 0$ such that for any $1, ..., n \in F$*

$$||\lambda_1 v_1 + ... + \lambda_n v_n|| \geq C_1(|\lambda_1| + ... + |\lambda_n|)$$

# Equivalence of Norms

Proof.

Let $v_1, ..., v_n$ be basis of $V$ and we have a linear combination of the bases

$$\|v\| = \|\lambda_1 v_1 + \cdots + \lambda_n v_n\| \leq \sum_{i=1}^{n} |\lambda_i| \|v_i\| \leq C \sum_{i=1}^{n} |\lambda_i|$$

where $C = \max_{1 \leq i \leq n} ||v_i||$. Therefore, we have from the basic norm inequality

$$C_1 \sum_{i=1}^{n} |\lambda_i| \leq \|v\| \leq C_2 \sum_{i=1}^{n} |\lambda_i|$$

Then an arbitrary norm is equivalent to norm $|| \cdot ||_1 = \sum_{i=1}^{n} |\lambda_i|$ □

# Equivalence of Norms

**Example**

However, consider the space of continuous functions on $[0,1]$ and you will show in the assignments that these two norms are not equivalent. Why?

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \qquad \|f\|_1 = \int_0^1 |f(x)|\, dx.$$

# Continuous Functions

**Theorem**

*Let $(U, ||\cdot||_1)$and $(V, ||\cdot||_2)$ be normed vector spaces and $f : U \to V$ a function. Then $f$ is continuous at $a \in U$ if and only if*

$$\forall_{\substack{(x_n)_{n\in\mathbb{N}}\\ x_n\in U}} \quad x_n \to a \quad \Rightarrow \quad f(x_n) \to f(a).$$

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**Comment.**

Just replace the modulus in previous definition with the new norms.

# Continuous functions

Question.

Continuous or not: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{xy}{x^2+y^2} & x^2 + y^2 \neq 0 \\ 0 & x^2 + y^2 = 0 \end{cases}$$

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**Comment.**

- $f(x,0) = f(0,y) = 0$, but $f(x,x) = \frac{1}{2} \neq 0$, as $x \to 0$
- $\lim_{y\to 0}(\lim_{x\to 0} f(x,y)) = \lim_{y\to 0}(0) = 0$
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# Continuous functions

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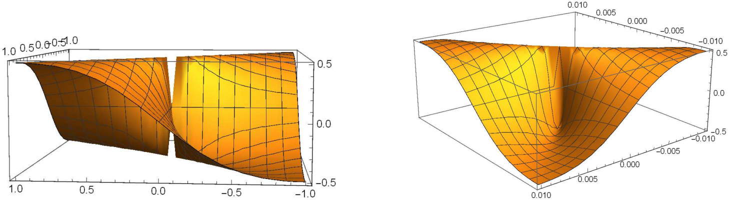


Figure: Plot of $f$ on $[-1,1] \times [-1,1]$ and $[-0.01, 0.01] \times [-0.01, 0.01]$

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**Question.**

Continuous or not: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

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**Comment.**

- $\lim\limits_{y \to 0} (\lim\limits_{x \to 0} f(x, y)) = \lim\limits_{y \to 0} (-1) = -1$
- $\lim\limits_{x \to 0} (\lim\limits_{y \to 0} f(x, y)) = \lim\limits_{x \to 0} (1) = 1$

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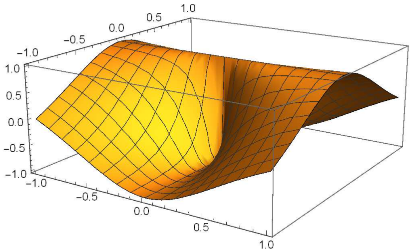


Figure: Plot of $f$ on $[-1,1]\times[-1,1]$

# Continuous Functions

### Theorem

*Let $(U, ||\cdot||_1)$ and $(V, ||\cdot||_2)$ be normed vector spaces and $f : U \to V$ a function. Then $f$ is continuous if and only if the pre-image $f^{-1}(\Omega)$ of every open set $\Omega \subset V$ is open.*

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**Example**

The determinant function is continuous.

# Compact Sets

Definition

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*Let $(V, ||\cdot||)$ be a (possibly infinite-dimensional) normed vector space and $K \subset V$ be compact. Then $K$ is closed and bounded.*

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**Theorem**

*Let $(V, ||\cdot||)$ be a finite-dimensional normed vector space and $K \subset V$ is closed and bounded, then $K$ is compact.*

# Compact Set and Continuous Functions

**Theorem**

*Let $(U, ||\cdot||_1)$ and $(V, ||\cdot||_2)$ be normed vector spaces and $K \subset V$ be compact. Let $f : K \to V$ be continuous. Then*

i) *ran $f = f(K)$ is compact in $V$.*

ii) *$f$ has a maximum in $K$.*

iii) *$f$ is uniformly continuous on $K$.*

# Differentiability

## Definition

There is a linear map $L_x \in \mathcal{L}(X, V)$ (called derivative) such that

$$f(x+h) = f(x) + L_x h + o(h) \qquad \text{as } h \to 0.$$

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### Comment.

We may thus regard the derivative as a linear map

$$D\colon C^1(\Omega, V) \to C(\Omega, \mathcal{L}(X, V)), \qquad f \mapsto Df.$$

# Derivative

Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = f(x_1, x_2) = x_1 + 2x_1x_2 + x_2^2$$

Take $h \to 0$, $L_x \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$

$$\begin{aligned} f(x+h) &= f(x_1 + h_1, x_2 + h_2) \\ &= (x_1 + h_1) + 2(x_1 + h_1)(x_2 + h_2) + (x_2 + h_2)^2 \\ &= f(x) + h_1 + 2(h_1x_2 + h_2x_1 + h_2x_2) + o(h) \\ &= f(x) + (1 + 2x_2, 2x_1 + 2x_2)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \end{aligned}$$

# Jacobian

If the function is differentiable, there is a quicker way to calculate the derivative.

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### Definition

The matrix is called the Jacobian of $f$.

$$J_f(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}\Bigg|_x$$

where the partial derivatives are defined as

$$\frac{\partial f}{\partial x_j}\bigg|_x := \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}$$

# Jacobian and continuity

**Theorem**

*Suppose all partial derivatives of $f$ exist*

i) *If all partial derivatives are bounded, then $f$ is continuous*
ii) *If all partial derivatives are continuous, then $f$ is continuously differentiable. The Jacobian is just the derivative.*

# Examples revisited

Example

i)

$$f(x) = \begin{cases} \frac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

The partial derivatives over $x$ or $y$ are the same (Calculate them correctly!)

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{y(y^2-x^2)}{(x^2+y^2)^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

However, the partial derivative is not bounded, so $f$ is not continuous.

# Examples revisited

Example

ii)

$$g(x) = \begin{cases} \frac{(x^2-y^2)}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

The partial derivatives over $x$ is

$$\frac{\partial g}{\partial x} = \begin{cases} \frac{4xy^2}{(x^2+y^2)^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

Similarly, the partial derivative is not bounded, so $g$ is not continuous.

# Chain Rule

Definition

$$D(f \circ g)|_x = Df|_{g(x)} \circ Dg|_x,$$

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## Example

Calculate the derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in spherical coordinate: $(r, \theta, \varphi) \in (0, +\infty) \times [0, \pi] \times [0, 2\pi)$

$$\Phi(r, \theta, \varphi) = \begin{pmatrix} r\cos\theta \\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \end{pmatrix}$$

# Chain Rule

Solution.

First, the derivative of $\Phi(r,\theta,\varphi)$

$$D\phi\Big|_{(r,\theta,\varphi)} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix}$$

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Then the derivative of $f$

$$Df\Big|_{\Phi(x,y,z)} = (2r\cos\theta, 2r\sin\theta\cos\varphi, 2r\sin\theta\sin\varphi)$$

# Chain Rule

Solution.

$$
\begin{aligned}
Df\Big|_{(r,\theta,\varphi)} =&(2r\cos\theta, 2r\sin\theta\cos\varphi, 2r\sin\theta\sin\varphi)\\
&\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix}\\
=&(2r, 0, 0)
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$$

# Chain Rule

**Solution.**

$$
\begin{aligned}
Df\Big|_{(r,\theta,\varphi)} =& (2r\cos\theta, 2r\sin\theta\cos\varphi, 2r\sin\theta\sin\varphi) \\
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=& (2r, 0, 0)
\end{aligned}
$$

**Comment.**

The derivative can be obtained in another way.

$$f(r,\theta,\varphi) = (r\cos\theta)^2 + (r\sin\theta\cos\varphi)^2 + (r\sin\theta\sin\varphi)^2 = r^2$$

$$Df\Big|_{(r,\theta,\varphi)} = \left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi}\right) = (2r, 0, 0)$$

# Chain Rule

**Question.**

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^3$

$$x = u^2 + v^2$$

$$y = u^2 - v^2$$

$$z = uv$$

Suppose $g : \mathbb{R}^2 \to \mathbb{R}^2$

$$u = r\cos\theta$$

$$v = r\sin\theta$$

What is the derivative of $f \circ g$?

# Chain Rule

Solution.

$$D(f \circ g) = \begin{pmatrix} 2r & 0 \\ 2r\cos 2\theta & -2r^2 \sin 2\theta \\ r \sin 2\theta & r^2 \cos 2\theta \end{pmatrix}$$

# Mean Value Theorem

**Theorem**

*Let $f \in C^1(\Omega, V)$. Let $0 \le t \le 1$, then*

$$f(x+y) - f(x) = \int_0^1 Df|_{x+ty} y \, dt = \Big(\int_0^1 Df|_{x+ty} \, dt\Big) y$$

**Theorem**

*Let $f : I \times \Omega \to V$ be a continuous function such that $Df(t, \cdot)$ exists and is continuous for every $t \in I$. Then $g(x)$ is differentiable and the derivative is*

$$g(x) = \int_a^b f(t,x) \, dt \qquad Dg(x) = \int_a^b Df(t, \cdot)|_x \, dt$$


Thank You!