Week 5 Recitation

Theory of Systems of Linear Equations and Determinants

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UM-SJTU Joint Institute

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Theorem

Let A be an $n \times n$ matrix. Then

- $Ax = b$ *has a unique solution for any* $b \in \mathbb{R}^n$. *(Ker A={0})*
- $Ax = 0$ *has a non-trivial solution. (Ker A ≠{0})*

Theorem

A matrix $A \in Mat(n \times n)$ *is invertible if and only if* $det(A) \neq 0$.

Matrix Rank

Definition

- the column rank of A

$$dim\ span\{a_{\cdot 1}, ..., a_{\cdot n}\}$$

- the row rank of A

$$dim\ span\{a_{1\cdot}, ..., a_{m\cdot}\}$$

- the rank of A

$$rank\ A\ = column\ rank\ A\ = row\ rank\ A$$

Matrix Rank

Theorem

There exists a solution x for $Ax = b$ if and only if rank A = rank $(A \mid b)$, where

$$(A \mid b) = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix} \in \mathrm{Mat}((n+1) \times m).$$

Determinants in $\mathbb{R}^2$

- Determinant in $\mathbb{R}^2$:

$$|det(A(a,b))| = \text{area of parallelogram of the two vectors}$$

- Properties of determinant
  - det is normed
  - det is bilinear
  - det is alternating

Determinants in $\mathbb{R}^3$

- Determinant in $\mathbb{R}^3$:

$$|det(A(a,b,c))| = \text{area of parallel epipeds of the three vectors}$$

$$V(a,b,c) = |\langle a \times b, c\rangle|$$

- Properties of determinant

$$\langle a \times b, c\rangle = \langle b \times c, a\rangle = \langle c \times a, b\rangle$$

Determinants in $\mathbb{R}^n$

- Permutation:

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

- Number of permutation

$$P_n = n!$$

p-Multilinear Maps

Definition

A function $f : \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is said to be ap-Multilinear map (or p-Multilinear form) if f is linear in each entry.

- The form is alternating if

$$f(a_1, ..., a_p) = 0 \quad if \quad a_j = a_k \ (j \neq k)$$

- The form is normed (p must be equal n) if

$$f(e_1, ..., e_n) = 1$$

p-Multilinear Maps

Theorem

Let $f : \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ be p-multilinear map. Then the following are equivalent:

(i) f is alternating

(ii) $$f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_p) = -f(a_1, \dots, a_{j-1}, a_k, a_{j+1}, \dots, a_{k-1}, a_j, a_{k+1}, \dots, a_p)$$

(iii) $f(a_1, \dots, a_p) = 0$ if $a_1, \dots, a_p$ are linearly dependent.

p-Multilinear Maps

Theorem

Determinant is the only alternating, linear, normed n-multilinear form.

Theorem

For $n \times n$ matrix

$$det(A) = det(A^T)$$

Comment.

The properties of n-multilinear form also work for determinant.

Question.

Calculate the determinant:

$$D_{(2n\times 2n)} = \begin{vmatrix} a & & & & & & & b \\ & a & & & & & b & \\ & & \ddots & & & \iddots & & \\ & & & a & b & & & \\ & & & b & a & & & \\ & & \iddots & & & \ddots & & \\ & b & & & & & a & \\ b & & & & & & & a \end{vmatrix}$$

Formulas

Theorem

Leibnitz Formula

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn} \pi \ a_{1\pi(1)} \cdots a_{n\pi(n)}$$

Theorem

Let $A \in Mat(n \times n)$ have upper triangular form

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$det(A) = \prod_{i=1}^{n} \lambda_i$$

Formulas

Question.

Calculate the following determinant:

$$A = \begin{pmatrix} 2 & 10 & -5 & 4 & -1 & 7 \\ 8 & 9 & -3 & 9 & 11 & 4 \\ -4 & 6 & -8 & 7 & 1 & 13 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 7 & 8 & 9 \end{pmatrix}$$

Formulas

Question.

Calculate the following determinant:

$$A = \begin{pmatrix} 2 & 10 & -5 & 4 & -1 & 7 \\ 8 & 9 & -3 & 9 & 11 & 4 \\ -4 & 6 & -8 & 7 & 1 & 13 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 7 & 8 & 9 \end{pmatrix}$$

Comment.

Block matrix method.

Cramer's Rule

Theorem

The system $Ax = b$, $b \in \mathbb{R}^n$, has the solution

$$x_i = \frac{1}{\det A} \det(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n), \qquad i = 1, \dots, n.$$

Proof.

$$\begin{aligned}
&\det(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \\
&= \det(a_1, \dots, a_{i-1}, Ax, a_{i+1}, \dots, a_n) \\
&= \det\left(a_1, \dots, a_{i-1}, \sum_{k=1}^{n} x_k a_k, a_{i+1}, \dots, a_n\right) \\
&= \sum_{k=1}^{n} x_k \det\left(a_1, \dots, a_{i-1}, a_k, a_{i+1}, \dots, a_n\right) \\
&= x_i \det\left(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n\right) + 0 \\
&= x_i \det A.
\end{aligned}$$

Product Rule for Determinants

Theorem

Let $A, B \in Mat(n \times n)$. Then $det(AB) = det(A)det(B)$.

Theorem

Let $A \in Mat(n \times n)$ be invertible. Then

$$\det A^{-1} = \frac{1}{\det A}.$$

Product Rule for Determinants

Question.

Suppose A is invertible($n \times n$), u, v be vectors of length n. Show that

$$det(A + uv^T) = (1 + v^T A^{-1} u) det(A)$$

Jin, Liu, 'Mid-term exam 1, VV214, Linear algebra', SU 2017. Retrieved Jun.12, 2017.

Selected Exercises in HW3

- Ex3.2.2: Discuss the case $j = 1$.
- Ex3.3.4: $(\text{ran}A)^{\perp}$ and $(\text{ker}A)^{\perp}$ are sets. To prove $(\text{ran}A)^{\perp}$=kerA^*, please prove $(\text{ran}A)^{\perp} \subset$ker$A^*$ and ker$A^* \subset(\text{ran}A)^{\perp}$
- Ex3.3.5: Orthonormal Bases
$$A^T = A^{-1}$$
- Ex3.3.6: It is possible for $B \neq C$ even if the direct sum satisfies
$$V = A + B = A + C$$

Selected Exercises in HW3

Thank you!