Week 3 Recitation

Inner Product Spaces and Linear Map

Yahoo

UM-SJTU Joint Institute

Summer 2017

1 Matrices

2 Selected Problems in HW2

How to look at a matrix

Question.

How do you look at matrix multiplication?

$$Ax = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 a_{11} + \cdots + x_n a_{1n} \\ \vdots \\ x_1 a_{m1} + \cdots + x_n a_{mn} \end{pmatrix}$$

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Comment.

How to look at matrix multiplication AB?

Matrices as Linear Maps

Theorem

Each matrix $A \in Mat(m \times n; \mathbb{R})$ $(m, n < \infty)$ *uniquely determines a linear map* $j(A) \in L(\mathbb{R}^n, \mathbb{R}^m)$ *such that the columns* $a_{\cdot k}$ *are the images of the standard basis vectors* $e_k \in \mathbb{R}^n$*.*

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Theorem

Each matrix $A \in Mat(m \times n; \mathbb{R})$ $(m, n < \infty)$ uniquely determines a linear map $j(A) \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that the columns $a_{\cdot k}$ are the images of the standard basis vectors $e_k \in \mathbb{R}^n$.

Comment.

$$j : Mat(m \times n; \mathbb{R}) \to L(\mathbb{R}^n, \mathbb{R}^m)$$

is an isomorphism.(Linear map view: what are the basis? Not necessarily standard normal.)

An important graphic illustration of matrix and Linear Maps

Theorem

Any linear map $L \in \mathcal{L}(U, V)$ induces a matrix $A = \Phi_A^B(L) = \varphi_B \circ L \circ \varphi_A^{-1}$ through

$$\begin{array}{ccc} U & \xrightarrow{L} & V \\ \Big\downarrow \varphi_{\mathcal{A}} & & \Big\downarrow \varphi_{\mathcal{B}} \\ \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m \end{array}$$

Matrix of Complex Conjugation

Example

$A = \Phi_B^B(L) = \varphi_B \circ L \circ \varphi_B^{-1}$

$$
\begin{tikzcd}
U \arrow[r, "L"] \arrow[d, "\varphi_{\mathcal{A}}"'] & V \arrow[d, "\varphi_{\mathcal{B}}"] \\
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m
\end{tikzcd}
$$

Base $\mathcal{B} = (1, i)$ for vector space $\mathbb{C}$. Conjugate map $L : z \mapsto \bar{z}$.
$\varphi_B : 1 \mapsto (1,0)^T, i \mapsto (0,1)^T$, $\varphi(a + bi) = (a, b)^T$,
$\varphi(L(a + bi)) = \varphi(a - bi) = (a, -b)^T$, $A(a, b)^T = (a, -b)^T$, so $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Matrix of Complex Conjugation

Example

$A = \Phi_A^A(L) = \varphi_A \circ L \circ \varphi_A^{-1}$

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_{\mathcal{A}}"'] & V \arrow[d, "\varphi_{\mathcal{B}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

Base $\mathcal{A} = (1+i, 1-i)$ for vector space $\mathbb{C}$. Conjugate map $L : z \mapsto \bar{z}$.
$\varphi_A : 1+i \mapsto (1,0)^T, 1-i \mapsto (0,1)^T$, $\varphi(a+bi) = (a+b, a-b)^T/2$,
$\varphi(L(a+bi)) = \varphi(a-bi) = (a-b, a+b)^T/2$,
$A(a+b, a-b)^T = (a-b, a+b)^T$, so $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Inverse of a Matrix

Theorem

Given an invertible matrix, we can find the inverse by following method:(Gauß-Jordan Algorithm)

$$(S \quad id) \sim (id \quad S^{-1})$$

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Theorem

Matrix A is invertible if and only if $det(A) \neq 0$.

Inverse Maps

Theorem

The inverse of any linear map $L \in \mathcal{L}(U, V)$ can be found by $L^{-1} = \varphi_A^{-1} \circ A^{-1} \circ \varphi_B$ through

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_{\mathcal{A}}"'] & V \arrow[d, "\varphi_{\mathcal{B}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

Inverse Maps

Example

Let $\mathcal{P}_2$ be the space of polynomials of degree not more than 2. Consider the linear map

$$L\colon \mathcal{P}_2 \to \mathcal{P}_2, \qquad ax^2 + bx + c \mapsto \frac{a+b+c}{3}x^2 + \frac{a+b}{2}x + \frac{a-c}{2}$$

We choose Base $\mathcal{B} = (x^2, x, 1)$, then

$$\varphi_B(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad , \quad \varphi_B(L(ax^2 + bx + c)) = \begin{pmatrix} (a+b+c)/3 \\ (a+b)/2 \\ (a-c)/2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \end{pmatrix} \quad , \quad A^{-1} = \begin{pmatrix} 3 & -2 & 2 \\ -3 & 4 & -2 \\ 3 & -2 & 0 \end{pmatrix}$$

Inverse Maps

Now we are able to calculate the inverse of L:

$$\begin{aligned}
L^{-1}(ax^2+bx+c) &= \varphi_{\mathcal{B}}^{-1}\circ A^{-1}\circ\varphi_{\mathcal{B}}(ax^2+bx+c)\\
&= \varphi_{\mathcal{B}}^{-1}\circ A^{-1}\begin{pmatrix} a\\ b\\ c\end{pmatrix}\\
&= \varphi_{\mathcal{B}}^{-1}\begin{pmatrix} 3 & -2 & 2\\ -3 & 4 & -2\\ 3 & -2 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\end{pmatrix}\\
&= \varphi_{\mathcal{B}}^{-1}\begin{pmatrix} 3a-2b+2c\\ -3a+4b-2c\\ 3a-2b\end{pmatrix}\\
&= (3a-2b+2c)x^2+(-3a+4b-2c)x+3a-2b
\end{aligned}$$

Active and Passive Points of View

$$x = \sum x_i e_i$$

$$x = \sum x_i' e_i'$$

$$e_i' = T e_i$$

- Active: the map acts on the vector

$$T^{-1}x = \sum x_i' T^{-1} e_i' = \sum x_i' e_i$$

- Active: the map acts on the basis

Reflection

i) Change to the basis $\mathcal{B}$

ii) Execute the reflection in this basis

iii) Change back to the standard basis

Example

Reflection in $\mathbb{R}^2$ along line $y = 2x$. Base $\mathcal{A} = (b_1, b_2)$, $b_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $b_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Then $Lb_1 = b_1, Lb_2 = -b_2$. Since $b_1 = Te_1, b_2 = Te_2$, then $T = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$, $T^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, so $L = TAT^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$.

Matrices as Linear Maps

Question.

What if the vector space is infinite? $AB = id \Rightarrow BA = id$?

Matrices as Linear Maps

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Example

Suppose $l^\infty = \{(a_n) : sup|a_n| < \infty\}$, $R : (a_0, a_1, a_2, ...) \mapsto (0, a_0, a_1, a_2, ...)$, $L : (a_0, a_1, a_2, ...) \mapsto (a_1, a_2, ...)$, then we have

$$LR = id$$

$$RL \neq id$$

1 Matrices

2 Selected Problems in HW2

Selected Problems in HW2

- Ex2.4
- Ex2.5
- Ex2.6

Thank you!