Week 11 Recitation

Riemann Integral and Measurable Sets, Integration in Practice

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Jordan Measurable

Definition

3.3.3. Definition. Let $\Omega \subset \mathbb{R}^n$ be a bounded non-empty set. We define the *outer* and *inner volume* of Ω by

$$\overline{V}(\Omega) := \inf\Big\{\sum_{k=0}^{r}|Q_k| : r \in \mathbb{N},\ Q_0, \dots, Q_r \in \mathcal{Q}_n,\ \Omega \subset \bigcup_{k=1}^{r} Q_k\Big\},$$

$$\underline{V}(\Omega) := \sup\Big\{\sum_{k=0}^{r}|Q_k| : r \in \mathbb{N},\ Q_0, \dots, Q_r \in \mathcal{Q}_n,\ \Omega \supset \bigcup_{k=1}^{r} Q_k,\ \bigcap_{k=1}^{r} Q_k = \emptyset\Big\}$$

It is easy to see that $0 \le \underline{V}(\Omega) \le \overline{V}(\Omega)$.

Jordan Measurable

Definition

3.3.4. Definition. Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then Ω is said to be *(Jordan) measurable* if either

(i) $\overline{V}(\Omega) = 0$ or

(ii) $\overline{V}(\Omega) = \underline{V}(\Omega)$.

In the first case, we say that Ω has (Jordan) *measure zero*, in the second case we say that

$$|\Omega| := \overline{V}(\Omega) = \underline{V}(\Omega)$$

is the Jordan measure of Ω.

Jordan Measurable

Example

3.3.5. Examples.

(i) A set $\{x\}$ consisting of a single point $x \in \mathbb{R}^n$ is a set of measure zero.

(ii) A subset of $\mathbb{R}^n$ consisting of a finite number of single points is a set of measure zero.

(iii) A curve of finite length $\mathcal{C} \subset \mathbb{R}^n$, $n \geq 2$, is a set of measure zero.

(iv) A bounded section of a plane in $\mathbb{R}^3$ is a set of measure zero.

(v) The set of rational numbers in the interval $[0, 1]$ has measure zero.

(vi) The set of irrational numbers in the interval $[0, 1]$ is not (Jordan) measurable.

Integration in Practice

Example

3.4.1. Fubini's Theorem. Let Q_1 be and n_1-cuboid and Q_2 an n_2-cuboid so that $Q := Q_1 \times Q_2 \subset \mathbb{R}^{n_1+n_2}$ is an (n_1+n_2)-cuboid. Assume that $f\colon Q \to \mathbb{R}$ is integrable on Q and that for every $x \in Q_1$ the integral

$$g(x) = \int_{Q_2} f(x, \cdot)$$

exists. Then

$$\int_Q f = \int_{Q_1 \times Q_2} f = \int_{Q_1} g = \int_{Q_1} \left(\int_{Q_2} f \right).$$

Practical Integration over $\mathbb{R}^2$

Question.

Calculate the integral $\iint\limits_D xydxdy$ over domain D, where D is the area bounded by $y^2 = x$ and $y = x - 2$

Practical Integration over $\mathbb{R}^2$

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Solution.

$$\iint\limits_D xydxdy = \int_{-1}^{2} dy \int_{y^2}^{y+2} xydx$$
$$= \frac{1}{2}\int_{-1}^{2} y[(y+2)^2 - y^4]dy$$
$$= \frac{45}{8}$$

Substitution Rule

Theorem

3.4.12. Substitution Rule. Let $\Omega \subset \mathbb{R}^n$ be open and $g: \Omega \to \mathbb{R}^n$ injective and continuously differentiable. Suppose that $\det J_g(y) \neq 0$ for all $y \in \Omega$. Let K be a compact measurable subset of Ω. The $g(K)$ is compact and measurable and if $f: g(K) \to \mathbb{R}$ is integrable, then

$$\int_{g(K)} f(x)\,dx = \int_K f(g(y)) \cdot |\det J_g(y)|\,dy.$$

Substitution Rule

Example

- Polar coordinates in $\mathbb{R}^2$:

$$|\det J_\varphi| = r$$

- Cylindrical coordinates in $\mathbb{R}^3$:

$$|\det J_\varphi| = r$$

- Spherical coordinates in $\mathbb{R}^3$:

$$|\det J_\varphi| = r^2 \sin\theta$$

- *Spherical coordinates in $\mathbb{R}^n$:

$$|\det J_\varphi| = r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2}$$

Substitution Rule

Question.

Calculate the volume of an ellipsoid in $\mathbb{R}^3$, $\Omega = \{(x, y, z) \in \mathbb{R}^3 \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$

Substitution Rule

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Solution.

Define new variable of $r \in [0,1], \varphi \in [0,\pi], \theta \in [0,2\pi]$

$$x = ar \sin\varphi \cos\theta$$

$$y = br \sin\varphi \sin\theta$$

$$z = cr \cos\varphi$$

Substitution Rule

Solution.

$$J_\varphi = \begin{pmatrix} a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta \\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta \\ c\cos\varphi & -cr\sin\varphi & 0 \end{pmatrix}$$

$$|\det J_\varphi| = abcr^2\sin\varphi$$

Therefore, we have

$$\begin{aligned}\iiint\limits_{\Omega} dxdydz &= \iiint\limits_{\Omega'} abcr^2\sin\varphi dr d\theta d\varphi \\ &= abc\int_0^1 r^2 dr \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \\ &= \frac{4\pi}{3}abc\end{aligned}$$

Substitution Rule

Question.

Calculate the volume of the n-dimensional ball $\mathcal{B}_n = \{x \in \mathbb{R}^n \big| x_1^2 + ... + x_n^2 \leq 1\}$

Solution.

Under spherical coordinates, domain $E_n = \{(r, \theta_1, ..., \theta_n) \Big| 0 \leq r \leq 1, 0 \leq \theta_1 \leq \pi,$
$0 \leq \theta_2 \leq \pi, ..., 0 \leq \theta_{n-1} \leq 2\pi\}$

$$\begin{aligned}
V_n &= \int\limits_{\mathcal{B}_n} dx_1 dx_2 \cdots dx_n \\
&= \int\limits_{\mathcal{E}_n} r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} dr d\theta_1 \cdots d\theta_{n-1} \\
&= \int\limits_0^1 r^{n-1} dr \int\limits_0^\pi \sin^{n-2}\theta_1 d\theta_1 \int\limits_0^\pi \sin^{n-3}\theta_2 d\theta_2 \cdots \int\limits_0^\pi \sin\theta_{n-2} d\theta_{n-2} \int\limits_0^{2\pi} d\theta_{n-1}
\end{aligned}$$

Substitution Rule

Solution.

when $k \in \mathbb{N}$, we have $k - 1 \leq 0$, $\sin\theta = \sin(\pi - \theta)$

$$\int_0^{\pi} \sin^{k-1}\theta d\theta = 2\int_0^{\frac{\pi}{2}} \sin^{k-1}\theta d\theta$$

we now let

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta$$

Substitution Rule

Solution.

Use integral by parts, we can find

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = \int_0^{\frac{\pi}{2}} -\sin^{n-1}\theta d\cos\theta \\
&= (-\sin^{n-1}\theta\cos\theta)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos\theta d(-\sin^{n-1}\theta) \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\cos^2\theta d\theta \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta - (n-1)\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta
\end{aligned}
$$

Substitution Rule

Solution.

The formula simply means

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

We can find the base case

$$I_0 = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin\theta d\theta = 1$$

Substitution Rule

Solution.

Combine the recursive formula, we can find out

$$I_n = \begin{cases} \frac{(2m-1)(2m-3)\cdots 3}{(2m)(2m-2)\cdots 2}\frac{\pi}{2}, & n = 2m \\ \frac{(2m)(2m-2)\cdots 2}{(2m+1)(2m-1)\cdots 3}, & n = 2m+1 \end{cases}$$

or more elegantly

$$I_n = \begin{cases} \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2}, & n = 2m \\ \frac{(2m)!!}{(2m+!)!!}, & n = 2m+1 \end{cases}$$

Then the volume is (product of I_ns !!)

$$V_n = \begin{cases} \frac{\pi^m}{m!}, & n = 2m \\ \frac{2^{m+\mathbf{1}}\pi^m}{(2m+1)!!}, & n = 2m+1 \end{cases}$$

Green's Theorem

Theorem

3.4.18. Green's Theorem. Let $R \subset \mathbb{R}^2$ be a bounded, simple region and $\Omega \supset R$ an open set containing R. Let $F\colon \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field. Then

$$\int_{\partial R^*} F\, d\vec{s} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dx \tag{3.4.1}$$

where ∂R^* denotes the boundary curve of R with positive (counter-clockwise) orientation.

Green's Theorem

Comment.

Green's Theorem in $\mathbb{R}^3$, Stokes' Theorem in $\mathbb{R}^3$ (Important!)

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

General form:

$$\int_S \mathrm{rot}\mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

Green's Theorem

Question.

Calculate the line integral

$$\oint_L 2xydx + x^2dy$$

Green's Theorem

Question.

Calculate the line integral

$$\oint_L 2xydx + x^2dy$$

Solution.

Let

$$P = 2xy$$

$$Q = x^2$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x} = 2x - 2x = 0$$

Therefore, we get

$$\oint_L 2xydx + x^2dy = 0$$


Thank You!