Week 8 Recitation

Coordinates and curves

Yahoo

UM-SJTU Joint Institu(te

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Three kinds of Coordinates

- Cartesian Coordinates
- Cylindrical coordinates: $(r, \theta, z) \in (0, +\infty) \times [0, 2\pi] \times (-\infty, +\infty)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Bigg|_{(r,\theta,z)} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

- Spherical coordinate: $(r, \theta, \varphi) \in (0, +\infty) \times [0, \pi] \times [0, 2\pi)$

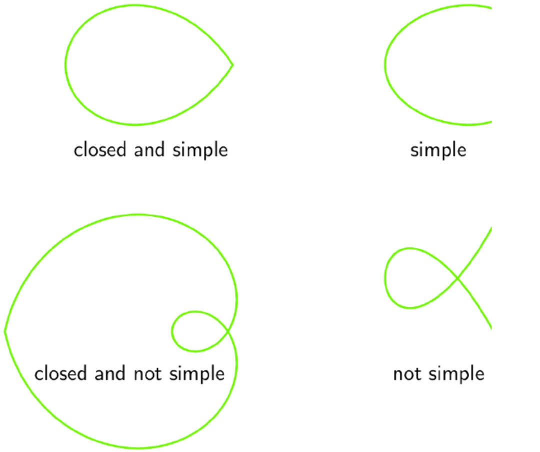
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Bigg|_{(r,\theta,\varphi)} = \begin{pmatrix} r\cos\theta \\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \end{pmatrix}$$

Curves and Parametrization

- A set $C \subset V$ for which there exists a continuous, surjective and locally injective map $\gamma : I \to C$ is called a curve.
- The map γ is called a parametrization of C.
- Locally injective means that in the neighborhood $B_\varepsilon(x) \cap I$ of any point $x \in I$ the parametrization is injective.

Simple, Open and Closed Curves

Example



Curves in Polar Coordinates

Definition

$$r = f(t)$$

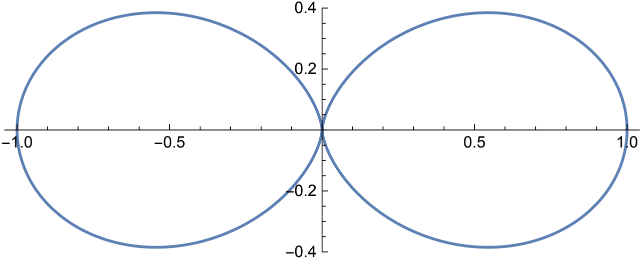
$$\gamma(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$$

Curves in Polar Coordinates

Example

Lemniscate of Bernoulli:

$$r^2 = \cos^2 t$$

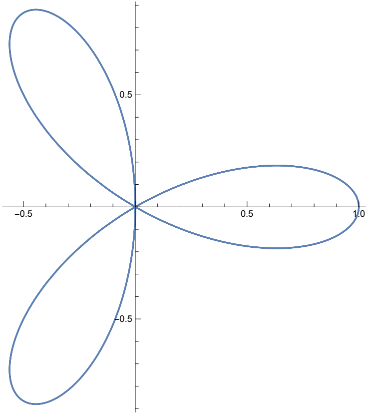


Curves in Polar Coordinates

Example

Four Leaf Clover:

$$r = \cos 3t$$

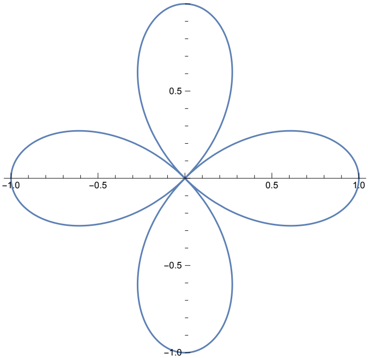


Curves in Polar Coordinates

Example

Kepler oval:

$$r = \cos 2t$$



Smooth Curves

- γ is continuously differentiable on int I.
- $D_\gamma\big|_t \neq 0$ for all $t \in$ int I.

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Example

The curve $y = 2x$ in $\mathbb{R}^2$ is smooth. We can consider a parametrization of this curve

$$\gamma(t) = \begin{pmatrix} t \\ 2t \end{pmatrix}$$

or for the curve in x^+ plane

$$\gamma(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$$

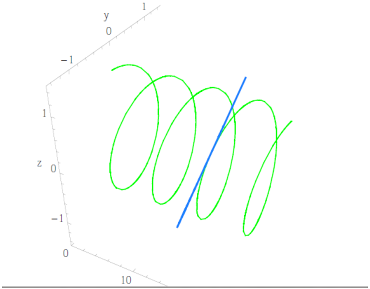
Tangent Lines of Curves

The linear approximation to the parametrization gives the tangent line

$$x = \gamma(t_0) + \gamma'(t_0)t$$

Example

$$\gamma(t_0 + t) = \begin{pmatrix} t_0 \\ \cos t_0 \\ \sin t_0 \end{pmatrix} + \begin{pmatrix} 1 \\ -\sin t_0 \\ \cos t_0 \end{pmatrix} t + o(t)$$



Tangent Vector, Normal Vector, Curve Length, Curvature

- Tangent Vector: $T \circ \gamma(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$
- Normal Vector: $N \circ \gamma(t) = \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}$
- Curve length: $l(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \, dt$
- Curve function: $l \circ \gamma(t) = \int_a^t \|\gamma'(t)\| \, dt$
- Curvature: $\kappa \circ l^{-1}(s) = \left\| \frac{d}{ds} T \circ l^{-1}(s) \right\| = \frac{\left\|(T \circ \gamma)'(t)\right\|}{\|\gamma'(t)\|}$
- Curvature in $\mathbb{R}^3$: $\kappa \circ l^{-1}(s) = \frac{\left\|\gamma'(t) \times \gamma''(t)\right\|}{\|\gamma'(t)\|}$

Tangent Vector, Normal Vector, Curve Length, Curvature

Question.

Find the tangent vector, normal vector, curve length function, curvature of cycloid $\gamma(t),\ t \in (0, 2\pi)$

$$\gamma(t) = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$$

Tangent Vector, Normal Vector, Curve Length, Curvature

Solution.

- $$\gamma'(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}$$
  $$T \circ \gamma(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{1}{2\sin(t/2)} \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} \sin(t/2) \\ \cos(t/2) \end{pmatrix}$$
- $$(T \circ \gamma)'(t) = \frac{1}{2} \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix}$$
  $$N \circ \gamma(t) = \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|} = \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix}$$

Tangent Vector, Normal Vector, Curve Length, Curvature

Solution.

- $$\|\gamma'(t)\| = 2\sin(t/2)$$
  $$l \circ \gamma(t) = \int_0^t \|\gamma'(t)\|\, dt = 4(1 - \cos t/2), \quad t \in (0, 2\pi)$$
  $$l \circ \gamma(2\pi) = 8$$
- $$\kappa \circ l^{-1}(s) = \frac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|} = \frac{1}{4\sin(t/2)}$$


Thank You!