Week 2 Recitation

Systems of Linear Equations and Finite-Dimensional Vector Spaces

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UM-SJTU Joint Institute

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Definition

A linear system of m (algebraic) equations in n unknowns $x_1, ..., x_n \in$ V is a set of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

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Comment.

x_i can be whatever is in the set V.

Systems of Linear Equations

Comment.

1. Homogeneous($b_1 = ... = b_m = 0$) V.S. Inhomogeneous
2. Underdetermined($m < n$) V.S. Overdetermined(m>n)
3. Trivial solution: $x_1 = ... = x_m = 0$

Systems of Linear Equations

Example

This is an inhomogeneous system of equations in $\mathbb{R}^2$.

$$2x_1 + x_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad x_1 - x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Systems of Linear Equations

Theorem

An inhomogeneous system of equations may have either

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An inhomogeneous system of equations may have either

1. *a unique solution or*
2. *no solution or*
3. *an infinite number of solutions*

Systems of Linear Equations

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1. An inhomogeneous system of equations that has a unique solution

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2. An inhomogeneous system of equations that has no solution

$$x_1 + x_2 = 1, \quad x_1 + x_2 = 2$$

Systems of Linear Equations

Example

1. An inhomogeneous system of equations that has a unique solution

$$2x_1 + x_2 = \binom{2}{1}, \qquad x_1 - x_2 = \binom{0}{1}$$

2. An inhomogeneous system of equations that has no solution

$$x_1 + x_2 = 1, \quad x_1 + x_2 = 2$$

3. An inhomogeneous system of equations that has an infinite number of solution

$$x_1 + x_2 = 1, \quad x_1 + x_2 = 1$$

The Gauß – Jordan Algorithm

Forward Elimination

The Gauß – Jordan Algorithm

Forward Elimination

(i) Ensure that the top left hand element is equal to 1
(ii) Eliminate (transform to zero) all lower entries in the first column
(iii) Ensure that the entry in the second row and second column is equal to 1
(iv) Eliminate (transform to zero) all entries in the second column below the second row
(v) Ensure that the entry in the third row and third column is equal to 1

The Gauß – Jordan Algorithm

Backward Substitution

The Gauß – Jordan Algorithm

Backward Substitution

(i) Eliminate all entries in the third column above the third row
(ii) Eliminate all entries in the second column above the second row

Example

A detailed example can be found in the lecture slide 29-34.

Existence and Uniqueness of Solutions

A system of m equations with n unknowns will have a unique solution if and only if it is diagonizable, i.e., if it can be transformed into diagonal. Thus, $m \geq n$ is a necessary condition for the existence of a unique solution

$$
\left[\begin{array}{ccc|c}
1 & * & * & \diamond \\
0 & 1 & * & \diamond \\
0 & 0 & 1 & \diamond
\end{array}\right]
\qquad
\left[\begin{array}{ccc|c}
1 & * & * & \diamond \\
0 & 1 & * & \diamond \\
0 & 0 & 1 & \diamond \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

Figure: Upper triangular form

The Solution Set

Definition

The solution set S of a system of equations is the set of all n-tuples of numbers $x_1, \ldots, x_n$ that satisfy the system.

i. If a linear system has a unique solution, the set S contains a single point.

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ii. If there is no solution, $S = \emptyset$.
iii. If there is more than one solutions, S is an infinite set.

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Example

What is the solution set of the following system of equations?

$$2x_1 + x_2 = \binom{2}{1}, \qquad x_1 - x_2 = \binom{0}{1}$$

Fundamental Lemma for Homogeneous Equations

Theorem

The homogeneous system of m equations in n real or complex unknowns $x_1, \dots, x_n$ has a non-trivial solution if $n > m$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Linear Independence

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Let V be a real or complex vector space and $v_1, ..., v_n \in V$. Then the vectors $v_1, ..., v_n$ are said to be independent if for all $\lambda_1, ..., \lambda_n \in F$

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$$\sum_{k=1}^{n} \lambda_k v_k = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

Comment.

A finite set $M \subset V$ is called an independent set if the elements of M are independent.

Linear Independence

Example

Suppose a_1, a_2, a_3 are linear independent, show that $b_1 = a_1 + a_2$, $b_2 = a_2 + a_3$, $b_3 = a_3 + a_1$ are also linear independent.

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Proof.

$$\sum_{i=1}^{3} \lambda_i a_i = 0 \quad \Rightarrow \quad \lambda_i = 0 \ (i = 1, 2, 3)$$

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Proof.

$$\sum_{i=1}^{3} \lambda_i a_i = 0 \quad \Rightarrow \quad \lambda_i = 0 \ (i = 1, 2, 3)$$

$$\sum_{i=1}^{3} \lambda_i' b_i = 0 \quad \Rightarrow \quad \lambda_i' + \lambda_j' = 0 \ (i \neq j) \quad \Rightarrow \quad \lambda_i' = 0 \ (i = 1, 2, 3)$$

□

Linear Combinations and Span

Definition

Let $v_1, ..., v_n \in$ V and $\lambda_1, ..., \lambda_n \in$ F. Then the expression

$$\sum_{k=1}^{n} \lambda_k v_k = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

is called a linear combination of the vectors $v_1, ..., v_n$.

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Let $v_1, ..., v_n \in \mathrm{V}$ and $\lambda_1, ..., \lambda_n \in \mathrm{F}$. Then the expression

$$\sum_{k=1}^{n} \lambda_k v_k = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

is called a linear combination of the vectors $v_1, ..., v_n$.

$$\operatorname{span}\{v_1, \ldots, v_n\} = \left\{ y \in V : y = \sum_{k=1}^{n} \lambda_k v_k,\ \lambda_1, \ldots, \lambda_n \in \mathbb{F} \right\}$$

is called the (linear) span or the linear hull of the vectors $v_1, ..., v_n$.

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2. $\operatorname{span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}1\\3\\2\end{pmatrix} \right\} = \mathbb{R}^3$

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The vectors $v_1, \dots, v_n \in V$ are independent if and only if none of them is contained in the span of all the others.

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Proof.

We prove the contraposition of the statement:

$$
\begin{aligned}
& \underset{k\in\{1,\dots,n\}}{\exists} v_k \in \operatorname{span}\{v_1, \dots, v_{k-1}, v_{k+1}, v_n\} \\
\Leftrightarrow \quad & \underset{k\in\{1,\dots,n\}}{\exists} \underset{\substack{\lambda_i\in\mathbb{F} \\ i\in\{1,\dots,n\}\setminus\{k\} \\ \sum|\lambda_i|\neq 0}}{\exists} v_k = \sum_i \lambda_i v_i \\
\Leftrightarrow \quad & \underset{\substack{\lambda_i\in\mathbb{F} \\ i\in\{1,\dots,n\} \\ \sum|\lambda_i|\neq 0}}{\exists} \sum_i \lambda_i v_i = 0
\end{aligned}
$$

$\square$

Basis

Definition

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Example

The standard basis or canonical basis of $\mathcal{R}^n$ is the tuple of vectors $(e_1, ..., e_n)$, $e_i \in \mathbb{R}^n$

$$e_i = (0, \ldots 0, \underset{\substack{\uparrow \\ i\text{th} \\ \text{entry}}}{1}, 0, \ldots, 0), \qquad i = 1, \ldots, n,$$

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ii. *$V = \text{span } \mathcal{B}$.*

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i. *the vectors $b_1, ..., b_n$ are linearly independent, i.e., B is an independent set, and*
ii. *$V = span\ \mathcal{B}$.*

Proof.

The proof can be found in slide 59-61. $\square$

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Theorem

Let V be a real or complex finite-dimensional vector space, $V \neq \{0\}$. Then any basis of V has the same length (number of elements) n.

Dimension

Definition

Let V be a finite-dimensional real or complex vector space. We define the dimension of V, denoted $\dim\mathcal{V}$, as follows

1. If $V = \{0\}$, $\dim\mathcal{V} = 0$.
2. If $V \neq \{0\}$, $\dim\mathcal{V} = n$, where n is the length of any basis of V.
3. If V is an infinite-dimensional vector space, $\dim\mathcal{V} = \infty$.

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Comment.

The infinite-dimensional vector spaces are always tricky to handle. For example, we can ask ourselves, do infinite-dimensional vector spaces have basis? What is the length of basis of infinite-dimensional vector spaces?. The answers may be ambiguous. Fortunately, we usually have the situation under finite-dimensional vector spaces, but we should also be alert when dealing with infinite-dimensional vector spaces.

Maximal Subsets and Basis Extension Theorem

Theorem

Let V be a vector space, $A \subset V$ a finite set. Then every independent subset $A' \subset A$ lies in some maximal subset $F \subset A$.

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1. We define the sum of U and W by

$$U + W := \left\{ v \in V : \underset{u \in U}{\exists} \; \underset{w \in W}{\exists} : v = u + w \right\}.$$

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$$U + W := \left\{ v \in V : \underset{u \in U}{\exists} \underset{w \in W}{\exists} : v = u + w \right\}.$$

2. If U and W are subspaces of V with $U \cap W = \{0\}$, the sum U+W is called direct, and we denote it by $U \oplus W$.

Sums of Vector Spaces

Theorem

The sum $U+W$ of vector spaces U,W is direct if and only if all $x \in U+W$, $x \neq 0$, have a unique representation $x = u + w$, $u \in U$, $w \in W$.

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Proof.

$\Rightarrow$ We show the contraposition: if the representation is not unique for all $x \in U+W$, then the sum is not direct. Let $x = u + w = u' + w'$ with $u, u' \in U$, $w, w' \in W$. Then $u - u' = w' - w$, so $u - u' \in U$ and $u - u' \in W$. Thus $U \cap W \neq \{0\}$.

□

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Proof.

$\Leftarrow$ We again show the contraposition: if the sum is not direct, then there exists some x $\in$ U +W with a non-unique representation. This is obvious, because if 0 $\neq \in$ U$\cap$W, then we may write

$$x = \underbrace{x}_{\in U} + \underbrace{0}_{\in W} = \underbrace{\frac{1}{2}x}_{\in U} + \underbrace{\frac{1}{2}x}_{\in W},$$

so this x has more than one representation. $\square$

Sums of Vector Spaces

Theorem

Let V be a vector space and U,W⊂V be finite-dimensional subspaces of V. Then

$$dim(\mathcal{U}+\mathcal{W}) + dim(\mathcal{U}\cap\mathcal{W}) = dim\mathcal{U} + dim\mathcal{W}$$

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Proof.

Suppose the basis that only belong to $\mathcal{U}$ and $\mathcal{W}$ are $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$. Suppose the basis that belong to both $\mathcal{U}$ and $\mathcal{W}$ is $b_1, \ldots, b_t$, then we know

$$dim(\mathcal{U}+\mathcal{W}) = n+m+t, \quad dim(\mathcal{U}\cap\mathcal{W}) = t, \quad dim(\mathcal{U}) = n+t, \quad dim(\mathcal{W}) = m+t$$

$\square$

Thank you for your attention!