Mid 1 Review

Matrix and Determinants

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UM-SJTU Joint Institute

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Matrices

- Matrix addition:

$$\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

Matrices

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$$\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

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- Product with scaler

$$\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$$

Matrices

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- Product with scaler

$$\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Matrices as Linear Maps

Theorem

Each matrix $A \in Mat(m \times n; \mathbb{R})$ $(m, n < \infty)$ *uniquely determines a linear map* $j(A) \in L(\mathbb{R}^n, \mathbb{R}^m)$ *such that the columns* $a_{\cdot k}$ *are the images of the standard basis vectors* $e_k \in \mathbb{R}^n$.

An important graphic illustration of matrix and Linear Maps

Theorem

Any linear map $L \in \mathcal{L}(U, V)$ *induces a matrix* $A = \Phi_A^B(L) = \varphi_B \circ L \circ \varphi_A^{-1}$ *through*

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_{\mathcal{A}}"'] & V \arrow[d, "\varphi_{\mathcal{B}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

Inverse of a Matrix

Theorem

Given an invertible matrix, we can find the inverse by following method:(Gauß-Jordan Algorithm)

$$(S \quad id) \sim (id \quad S^{-1})$$

Inverse Maps

Theorem

The inverse of any linear map $L \in \mathcal{L}(U, V)$ can be found by $L^{-1} = \varphi_A^{-1} \circ A^{-1} \circ \varphi_B$ through

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_{\mathcal{A}}"'] & V \arrow[d, "\varphi_{\mathcal{B}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

Active and Passive Points of View

$$x = \sum x_i e_i$$
$$x = \sum x_i' e_i'$$
$$e_i' = Te_i$$

- Active: the map acts on the vector

$$T^{-1}x = \sum x_i' T^{-1} e_i' = \sum x_i' e_i$$

- Active: the map acts on the basis

Reflection

i) Change to the basis $\mathcal{B}$

ii) Execute the reflection in this basis

iii) Change back to the standard basis

Fredholm Alternative

Theorem

Let A be an $n \times n$ matrix. Then

- $Ax = b$ *has a unique solution for any* $b \in \mathbb{R}^n$. *(Ker A={0})*
- $Ax = 0$ *has a non-trivial solution. (Ker A* $\neq$*{0})*

Matrix Rank

Definition

- the column rank of A

$$dim\ span\{a_{\cdot 1}, ..., a_{\cdot n}\}$$

- the row rank of A

$$dim\ span\{a_{1\cdot}, ..., a_{m\cdot}\}$$

- the rank of A

$$rank\ A\ = column\ rank\ A\ = row\ rank\ A$$

Determinants

- Determinant in $\mathbb{R}^2$:

$$|det(A(a,b))| = \text{area of parallelogram of the two vectors}$$

- Determinant in $\mathbb{R}^3$:

$$|det(A(a,b,c))| = \text{area of parallel epipeds of the three vectors}$$

$$V(a,b,c) = |\langle a \times b, c \rangle|$$

- Properties of determinant

$$\langle a \times b, c \rangle = \langle b \times c, a \rangle = \langle c \times a, b \rangle$$

Determinants in $\mathbb{R}^n$

- Permutation:

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

- Number of permutation

$$P_n = n!$$

Determinants in $\mathbb{R}^n$

- Product with scaler:

$$det(\lambda x, y) = \lambda det(x, y)$$

- Determinant addition ??

$$det(A + B) \neq det(A) + det(B)$$

p-Multilinear Maps

Definition

A function $f : \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is said to be ap-Multilinear map (or p-Multilinear form) if f is linear in each entry.

- The form is alternating if

$$f(a_1, ..., a_p) = 0 \quad if \quad a_j = a_k \ (j \neq k)$$

- The form is normed (p must be equal n) if

$$f(e_1, ..., e_n) = 1$$

p-Multilinear Maps

Theorem

Determinant is the only alternating, linear, normed n-multilinear form.

Formulas

Theorem

Leibnitz Formula

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn} \pi \; a_{1\pi(1)} \cdots a_{n\pi(n)}$$

Formulas

Theorem

Let $A \in Mat(n \times n)$ have upper triangular form

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$det(A) = \prod_{i=1}^{n} \lambda_i$$

Cramer's Rule

Theorem

The system $Ax = b$, $b \in \mathbb{R}^n$, has the solution

$$x_i = \frac{1}{\det A} \det(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n), \qquad i = 1, \dots, n.$$

Product Rule for Determinants

Theorem

Let $A, B \in Mat(n \times n)$. Then $det(AB) = det(A)det(B)$.

Theorem

Let $A \in Mat(n \times n)$ be invertible. Then

$$\det A^{-1} = \frac{1}{\det A}.$$

Exercise

Question.

Consider the complex numbers C as a real, two-dimensional vector space. Define linear map $T : C \to C$ by $Tz = e^{i\pi/2}(Re(z) - Im(z))/2$.

i. Find the representing matrix of T with respect to the basis $B = \{1, i\}$ in C.

ii. Decide whether T is bijective and prove your conclusion.

Exercise

Question.

Let $\varphi\colon a + bi \mapsto (a, b)$ be the basis isomorphism. Then $\varphi(Tz) = (0, (a - b)/2)$, so

$$A = \begin{pmatrix} 0 & 0 \\ 1/2 & -1/2 \end{pmatrix}$$

is the representing matrix.

The rank of A is less than 2 since the first row of A consists only of zeroes. therefore, A is not surjective and hence not bijective. It follows that T is also not bijective.

Ending

Thank you!