Week 10 Recitation

Vector Fields and Line Integrals, Circulation and Flux

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UM-SJTU Joint Institu(te

Summer 2017

Line Integral of a Potential Function

Definition

3.1.1. Definition. Let $\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ be a continuous potential function and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve with parametrization $\gamma : I \to \mathcal{C}$. We then define the *line integral of the potential f along* $\mathcal{C}^*$ by

$$\int_{\mathcal{C}^*} f\, ds := \int_I (f \circ \gamma)(t) \cdot |\gamma'(t)|\, dt$$

Line Integral of a Potential Function

Example

Evaluate $\int_{\mathcal{C}^*}(2+x^2y)\ ds$, where $\mathcal{C}^*$ is $x^2+y^2=1(y>0)$.

Line Integral of a Potential Function

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Evaluate $\int_{\mathcal{C}^*}(2+x^2y)\ ds$, where $\mathcal{C}^*$ is $x^2+y^2=1(y>0)$.

Solution.

We take parametrization $\gamma(t),\ \ t\in[0,\pi]$ to be

$$\gamma(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$$

Therefore, we have

$$\begin{aligned}\int_{\mathcal{C}^*}(2+x^2y)\ ds&=\int_0^{\pi}(2+\cos^2 t\sin t)|\gamma'(t)|\ dt\\&=\int_0^{\pi}(2+\cos^2 t\sin t)\sqrt{\sin^2 t+\cos^2 t}\ dt\\&=2\pi+\frac{2}{3}\end{aligned}$$

Vector Fields

Definition

3.1.4. Definition. Let $\Omega \subset \mathbb{R}^n$. Then a function $F\colon \Omega \to \mathbb{R}^n$,

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{pmatrix}.$$

is called a *vector field* on Ω.

Line Integral of a Vector Field

Definition

3.1.8. Definition. Let $\Omega \subset \mathbb{R}^n$, $F\colon \Omega \to \mathbb{R}$ be a continuous vector field and $\mathcal{C}^* \subset \Omega$ an oriented open, smooth curve in $\mathbb{R}^n$. We then define the *line integral of the vector field F along $\mathcal{C}^*$* by

$$\int_{\mathcal{C}^*} F\, d\vec{s} := \int_{\mathcal{C}^*} \langle F, T \rangle\, ds \tag{3.1.5}$$

Line Integral of a Vector Field

Comment.

If we calculate the line integral using a concrete parametrization $\gamma\colon I \to \mathcal{C}$, we obtain

$$\begin{aligned}\int_{\mathcal{C}^*} F\, d\vec{s} = \int_{\mathcal{C}^*} \langle F, T\rangle\, ds &= \int_I \langle F \circ \gamma(t), T \circ \gamma(t)\rangle \|\gamma'(t)\|\, dt \\ &= \int_I \left\langle F \circ \gamma(t), \frac{\gamma'(t)}{\|\gamma'(t)\|} \right\rangle \|\gamma'(t)\|\, dt \\ &= \int_I \langle F \circ \gamma(t), \gamma'(t)\rangle\, dt \qquad (3.1.6)\end{aligned}$$

Line Integral of a Vector Field

Example

Evaluate $\int_{\mathcal{C}^*} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \begin{pmatrix} xy \\ yz \\ zx \end{pmatrix}$ and $\mathcal{C}^*$ is parametrized by

$\gamma(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$, $t \in [0, 1]$.

Line Integral of a Vector Field

Example

Evaluate $\int_{\mathcal{C}^*} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \begin{pmatrix} xy \\ yz \\ zx \end{pmatrix}$ and $\mathcal{C}^*$ is parametrized by

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Solution.

$$\int_{\mathcal{C}^*} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle \mathbf{F} \circ \gamma(t), \gamma'(t) \rangle \ dt$$

$$= \int_0^1 \left\langle \begin{pmatrix} t^3 \\ t^5 \\ t^4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \right\rangle dt = \frac{27}{28}$$

Potential Fields

Definition

3.1.11. Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set. A vector field $F\colon \Omega \to \mathbb{R}^n$ is said to be a *potential field* if there exists a differentiable potential function $U\colon \Omega \to \mathbb{R}$ such that

$$F(x) = \nabla U(x).$$

Potential Fields

Comment.

Potential fields are very useful, as the integral along an oriented open curve $\mathcal{C}^*$ depends only on the initial and the final point of the curve. This can be seen from

$$\int_I \langle F \circ \gamma(t), \gamma'(t) \rangle \, dt = \int_I \langle \nabla U \circ \gamma(t), \gamma'(t) \rangle \, dt = \int_I DU|_{\gamma(t)}(\gamma'(t)) \, dt = \int_I (U \circ \gamma)'(t) \, dt.$$

where we have used the chain rule.

Supposing that the initial point of the curve is p_{initial} and the final point is p_{final}, we have from the fundamental theorem of calculus

$$\int_{\mathcal{C}^*} F \, d\vec{s} = \int_I (U \circ \gamma)'(t) \, dt = U(p_{\text{final}}) - U(p_{\text{initial}}).$$

Conservative Fields

Definition

3.1.14. Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F\colon \Omega \to \mathbb{R}^n$ a vector field. If the integral along any open curve $\mathcal{C}^*$ depends only on the initial and final points or, equivalently,

$$\oint_{\mathcal{C}} F\,d\vec{s} = 0 \qquad \text{for any closed curve } \mathcal{C},$$

then F is called *conservative*.

Potential Fields and Conservative Fields

Potential Fields are Conservative.

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Definition

3.1.16. Definition. Let $\Omega \subset \mathbb{R}^n$. Then Ω is said to be *(pathwise) connected* if for any two points in Ω there exists an open curve within Ω joining the two points.

Potential Fields and Conservative Fields

Potential Fields are Conservative.

Definition

3.1.16. Definition. Let $\Omega \subset \mathbb{R}^n$. Then Ω is said to be *(pathwise) connected* if for any two points in Ω there exists an open curve within Ω joining the two points.

Theorem

3.1.17. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F: \Omega \to \mathbb{R}^n$ is a continuous, conservative field. Then F is a potential field.

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Theorem

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Comment.

Continuous conservative field on connected open set is potential field.

Criteria for Potential Fields

Theorem

3.1.18. Lemma. Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F\colon \Omega \to \mathbb{R}^n$ is continuously differentiable. Then F is a potential field only if for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}. \tag{3.1.9}$$

Criteria for Potential Fields

Theorem

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$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}. \tag{3.1.9}$$

Proof.

If F is a potential field, then $F = \nabla U$ for some potential function U, i.e., $F_i = \frac{\partial U}{\partial x_i}$. Since the second derivative of U is continuous, we have

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial^2 U}{\partial x_j \partial x_i} = \frac{\partial^2 U}{\partial x_i \partial x_j} = \frac{\partial F_j}{\partial x_i}$$

□

Criteria for Potential Fields

Theorem

3.1.21. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a *simply connected* open set and suppose that $F\colon \Omega \to \mathbb{R}^n$ is continuously differentiable. If for all $i, j = 1, \ldots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

then F is a potential field.

Criteria for Potential Fields

Theorem

3.1.21. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a *simply connected* open set and suppose that $F\colon \Omega \to \mathbb{R}^n$ is continuously differentiable. If for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

then F is a potential field.

Loosely speaking, a set $\Omega \subset \mathbb{R}^n$ is said to be simply connected if

(i) Ω is pathwise connected and

(ii) every closed curve in Ω can be contracted to a single point within Ω.

Simply Connected set

Definition

3.1.23. Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set.

(i) A closed curve $\mathcal{C} \subset \Omega$ given as the image of a map $g\colon S^1 \to \mathcal{C}$ is said to be *contractible to a point* if there exists a continuous function $G\colon D \to \Omega$ such that $G|_{S^1} = g$.

(ii) The set Ω is said to be *simply connected* if it is connected and every closed curve in Ω is contractible to a point.

Determining Potentials

Example

Determine whether or not the vector field

$$\mathbf{F}(x, y) = \begin{pmatrix} x - y \\ x - 2 \end{pmatrix}$$

is conservative.

Determining Potentials

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Determine whether or not the vector field

$$\mathbf{F}(x, y) = \begin{pmatrix} x - y \\ x - 2 \end{pmatrix}$$

is conservative.

Solution.

$$\frac{\partial F_1}{\partial y} = -1$$

$$\frac{\partial F_2}{\partial x} = 1$$

so $\mathbf{F}$ is not conservative.

Determining Potentials

Example

Determine whether or not the vector field on $\mathbb{R}^2$

$$\mathbf{G}(x, y) = \begin{pmatrix} 3 + 2xy \\ x^2 - 3y^2 \end{pmatrix}$$

is conservative.

Determining Potentials

Example

Determine whether or not the vector field on $\mathbb{R}^2$

$$\mathbf{G}(x, y) = \begin{pmatrix} 3 + 2xy \\ x^2 - 3y^2 \end{pmatrix}$$

is conservative.

Solution.

$$\frac{\partial G_1}{\partial y} = 2x$$

$$\frac{\partial G_2}{\partial x} = 2x$$

Since $\mathbb{R}^2$ is simply connected open set, and $\mathbf{G}$ is continuously differentiable, then $\mathbf{G}$ is conservative.

Determining Potentials

Example

Determine the potential of the vector field on $\mathbb{R}^2$

$$\mathbf{G}(x,y) = \begin{pmatrix} 3+2xy \\ x^2-3y^2 \end{pmatrix}$$

Determining Potentials

Example

Determine the potential of the vector field on $\mathbb{R}^2$

$$\mathbf{G}(x,y) = \begin{pmatrix} 3+2xy \\ x^2-3y^2 \end{pmatrix}$$

Solution.

Suppose $\mathbf{G}(x,y) = \nabla g(x,y)$, then $\frac{\partial g}{\partial x} = 3+2xy, \quad \frac{\partial g}{\partial y} = x^2-3y^2$

$$g(x,y) = \int (3+2xy)\ dx + C_1(y) = 3x + x^2y + C_1(y)$$

$$\frac{\partial g}{\partial y} = x^2 + \frac{dC_1(y)}{dy} = x^2 - 3y^2, \quad C_1(y) = -y^3 + C$$

$$g(x,y) = 3x + x^2y - y^3 + C$$

Differential Forms

Definition

3.1.25. Definition. Let $F_1, \dots, F_n \colon \mathbb{R}^n \to \mathbb{R}$ be scalar functions. Then

$$\alpha = F_1\,dx_1 + \cdots + F_n\,dx_n$$

is said to be a *differential one-form*.

Differential Forms

Definition

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$$\alpha = F_1\,dx_1 + \cdots + F_n\,dx_n$$

is said to be a *differential one-form*.

Example

$$\int_{\mathcal{C}^*} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}^*} F_1 \; dx_1 + F_2 \; dx_2 + \dots + F_n \; dx_n$$

Circulation and Flux

Definition

3.2.2. Definition. Let $\Omega \subset \mathbb{R}^2$ be an open set, $F\colon \Omega \to \mathbb{R}^2$ a continuously differentiable vector field and $\mathcal{C}^*$ a positively oriented closed curve in $\mathbb{R}^2$. Then

$$\int_{\mathcal{C}^*} \langle F, T\rangle \, ds \tag{3.2.1}$$

is called the (total) *circulation* of F along $\mathcal{C}$ and

$$\int_{\mathcal{C}^*} \langle F, N\rangle \, ds \tag{3.2.2}$$

is called the (total) *flux* of F through $\mathcal{C}$.

Flux Density and Divergence

Definition

3.2.5. Definition. Let $\Omega \subset \mathbb{R}^n$ and $F\colon \Omega \to \mathbb{R}^n$ be a continuously differentiable vector field. Then

$$\operatorname{div} F := \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

is called the *divergence* of F.

Circulation Density - Rotation / Curl

Definition

3.2.6. Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F : \Omega \to \mathbb{R}^n$ a continuously differentiable vector field. Then the anti-symmetric, bilinear form

$$\operatorname{rot} F|_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad \operatorname{rot} F|_x(u, v) := \langle DF|_x u, v\rangle - \langle DF|_x v, u\rangle \tag{3.2.3}$$

is called the *rotation* (in mainland Europe) or *curl* (in anglo-saxon countries) of the vector field F at $x \in \mathbb{R}^n$.

Rotation / Curl

Theorem

3.2.7. Theorem. Let $\Omega \subset \mathbb{R}^2$ be open and $F\colon \Omega \to \mathbb{R}^2$ a continuously differentiable vector field. Then here exists a uniquely defined continuous potential function $\operatorname{rot} F\colon \Omega \to \mathbb{R}$ such that

$$\mathscr{rot}\, F|_x(u, v) = \operatorname{rot} F(x) \cdot \det(u, v). \tag{3.2.5}$$

Comment.

This scalar function is given by

$$\operatorname{rot} F = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}. \tag{3.2.6}$$

Rotation / Curl

Theorem

3.2.9. Theorem. Let $\Omega \subset \mathbb{R}^3$ be open and $F\colon \Omega \to \mathbb{R}^3$ a continuously differentiable vector field. Then here exists a uniquely defined continuous vector field $\operatorname{rot} F\colon \Omega \to \mathbb{R}^3$ such that

$$\mathscr{rot}\, F|_x(u, v) = \det(\operatorname{rot} F(x), u, v) = \langle \operatorname{rot} F(x), u \times v \rangle. \tag{3.2.7}$$

Comment.

Similarly, we calculate the other components of $\operatorname{rot} F$ to obtain

$$\operatorname{rot} F(x) = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}.$$

Rotation in Higher Dimensions

Theorem

In general, the space of bilinear maps in $\mathbb{R}^n$ is isomorphic to the space of $n \times n$ matrices (see (2.5.6) and the discussion there). In particular, there must exist a matrix $A(x)$ such that

$$\mathcal{rot}\, F|_x(u, v) = \langle u, A(x)v\rangle \tag{3.2.9}$$

Since the rotation is anti-symmetric, $\mathcal{rot}\, F|_x(v, u) = -\mathcal{rot}\, F|_x(u, v)$, i.e.,

$$\langle u, A(x)v\rangle = -\langle v, A(x)u\rangle = \langle -A(x)^T v, u\rangle = \langle u, -A(x)^T v\rangle.$$

Hence, A must satisfy

$$A(x)^T = -A(x).$$

for all x. In fact, we can see directly from the definition (3.2.3) that

$$A(x) = (DF|_x)^T - DF|_x.$$

Triangle Calculus

It is convenient to introduce the formal "vector"

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

Then the gradient of a potential function f is just

$$\nabla f = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} .$$

The divergence of a vector field F can be expressed as

$$\operatorname{div} F = \langle \nabla, F \rangle = \left\langle \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}, \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \right\rangle = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Triangle Calculus

The rotation of a vector field F can be formally written as

$$\operatorname{rot} F = \nabla \times F(x) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{pmatrix},$$

where e_1, e_2, e_3 are the standard basis vectors in $\mathbb{R}^3$. Finally, we can formally write

$$\begin{aligned} \langle \nabla, \nabla \rangle &= \left(\frac{\partial}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial}{\partial x_n} \right)^2 \\ &= \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \\ &= \Delta \end{aligned}$$

so it is common for physicists to write ∇^2 instead of Δ.

Curl

Theorem

If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is twice continuous differentiable, then

$$\nabla \times (\nabla f) = 0$$

Curl

Theorem

If $f: \mathbb{R}^3 \to \mathbb{R}$ is twice continuous differentiable, then

$$\nabla \times (\nabla f) = 0$$

Comment.

The rotation of a potential field is 0.

Curl

Example

Show that **F** is conservative given

$$\mathbf{F} = \begin{pmatrix} y^2z^3 \\ 2xyz^3 \\ 3xy^2z^2 \end{pmatrix}$$

Curl

Example

Show that $\mathbf{F}$ is conservative given

$$\mathbf{F} = \begin{pmatrix} y^2z^3 \\ 2xyz^3 \\ 3xy^2z^2 \end{pmatrix}$$

Solution.

The function is defined on $\mathbb{R}^3$, a simply open set, then conservative field is potential field (f is continuously differentiable). We only need to check

$$\nabla \times (\nabla f) = 0$$


Thank You!