Week 3 Recitation

Inner Product Spaces and Linear Map

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Definition

Let V be a real or complex vector space. Then a map $<\cdot,\cdot>$: $V \times V \to F$ is called a scalar product or inner product if for all u,v,w $\in V$ and all $\lambda \in F$

(i) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$,
(ii) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,
(iii) $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$,
(iv) $\langle u, v \rangle = \overline{\langle v, u \rangle}$.

The pair (V, $<\cdot,\cdot>$) is called an inner product space.

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\end{aligned}$$

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Comment.

Properties (iii) and (iv) imply that

$$\langle \lambda u, v\rangle = \overline{\langle v, \lambda u\rangle} = \overline{\lambda\langle v, u\rangle} = \overline{\lambda}\langle u, v\rangle.$$

Example

- In $\mathbb{C}^n$ we can define the inner product

$$\langle x, y \rangle := \sum_{i=1}^{n} \overline{x_i} y_i, \qquad x, y \in \mathbb{C}^n.$$

- In C([a, b]), the space of complex-valued, continuous functions on the interval [a, b], we can define an inner product by

$$\langle f, g \rangle := \int_a^b \overline{f(x)} g(x)\, dx, \qquad f, g \in C([a, b]).$$

The Induced Norm

Definition

Let (V, $< \cdot, \cdot >$) be an inner product space. The map

$$\| \cdot \| : V \to \mathbb{R}, \qquad \|v\| = \sqrt{\langle v, v \rangle}$$

is called the induced norm on V.

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Example

The induced norm on C([a, b]) is

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b |f(x)|^2\, dx} = \|f\|_2$$

Cauchy-Schwarz Inequality

Theorem

Let $(V, < \cdot, \cdot >)$ be an inner product vector space. Then

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\| \qquad \text{for all } u, v \in V$$

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Proof.

Let e := v/$||$v$||$. Then <e, e> = <v, v>/$||v||^2$=1 and

$$\begin{aligned} 0 &\leq \|u - \langle e, u\rangle e\|^2 = \langle u - \langle e, u\rangle e, u - \langle e, u\rangle e\rangle \\ &= \|u\|^2 - |\langle e, u\rangle|^2 \end{aligned}$$

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It follows that

$$|\langle u, v\rangle|^2 = \|v\|^2 \cdot |\langle u, e\rangle|^2 \leq \|u\|^2 \cdot \|v\|^2.$$

□

The Induced Norm

Theorem

The induced norm is actually a norm

i. $||v|| = 0, ||v|| = 0 \Leftrightarrow v = 0$

ii. $||\lambda v|| = |\lambda| ||v||$

iii. $||u + v|| \leq ||u|| + ||v||$

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Proof.

All properties except for the triangle inequality are easily checked. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v\rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v\rangle| \\ &\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\ &= (\|u\| + \|v\|)^2. \end{aligned}$$

Angle Between Vectors

Definition

Let V be a real inner product space and u,v$\in V$. We define the angle α(u,v)$\in [0, \pi]$ between u and v by

$$\cos \alpha(u, v) = \frac{\langle u, v \rangle}{\|u\| \|v\|}.$$

Orthogonality

Definition

Let (V, $< \cdot, \cdot >$) be an inner product vector space.

(i) Two vectors u, v $\in$ V are called orthogonal or perpendicular if $< u, v >= 0$, we then write $u \perp v$.

(ii) We call

$$M^{\perp} := \left\{ v \in V : \underset{m \in M}{\forall} \langle m, v \rangle = 0 \right\}$$

the orthogonal complement of a set M$\subset$V.

For short, we sometimes write v$\perp$M instead of v$\in M^{\perp}$.

Pythagoras's Theorem.

Theorem

Let $(V, < \cdot, \cdot >)$ be an inner product space and M some subset of V. Let $z = x+y$, where $x \in M$ and $y \in M^{\perp}$. Then

$$\|z\|^2 = \|x\|^2 + \|y\|^2.$$

Orthonormal Systems

Theorem

Let $(V, <\cdot,\cdot>)$ be an inner product vector space. A tuple of vectors $(v_1, ..., v_r)$ $\subset V$ is called a (finite) orthonormal system if

$$\langle v_j, v_k \rangle = \delta_{jk} := \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases}, \qquad j, k = 1, \dots, r,$$

Orthonormal Systems

Theorem

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product vector space and $F = (v_1, \ldots, v_r) \subset V$ an orthonormal *system. Then the elements of $\mathcal{F}$ are linearly independent.*

Proof.

$$\sum_{i=0}^{r} \lambda_i v_i = 0$$

$$0 = \langle v_i, 0\rangle = \langle v_i, \sum_{i=0}^{r} \lambda_i v_i\rangle = \lambda_i \langle v_i, v_i\rangle = \lambda_i$$

□

Orthonormal Bases

Definition

Let (V, $\langle \cdot, \cdot \rangle$) be a finite-dimensional inner product vector space and $\mathcal{B} = (e_1, ..., e_n)$ a basis of V. If $\mathcal{B}$ is also an orthonormal system, we say that $\mathcal{B}$ is an orthonormal basis (ONB).

Theorem

Basis representation

$$v = \sum_{j=1}^{n} \langle e_j, v \rangle e_j.$$

Projection of v onto e_i

$$\pi_{e_i} v := \langle e_i, v \rangle e_i$$

Orthonormal Bases

Proof.

$$v = \sum_{i=0}^{n} \lambda_i e_i$$

$$\langle v, e_i \rangle = \langle \sum_{i=0}^{n} \lambda_i e_i, e_i \rangle = \lambda_i \langle e_i, e_i \rangle = \lambda_i$$

□

Orthonormal Bases

Proof.

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□

Theorem

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product vector space and $\mathcal{B} = (e_1, ..., e_n)$ a basis of V.

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2$$

The Projection Theorem

Definition

Let (V, $\langle\cdot,\cdot\rangle$) be a (possibly infinite-dimensional) inner product vector space and $(e_1, ..., e_r)$, $r \in N$, be an orthonormal system in V. Denote $U :=$ span$\{e_1, ..., e_r\}$.Then for every $v \in V$ there exists a unique representation

$$v = u + w \qquad \text{where } u \in U \text{ and } w \in U^\perp$$

and

$$u = \sum_{i=1}^{r} \langle e_i, v\rangle e_i, \; w := v - u.$$

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Proof.

Slides 94-95. Repeat the proof by yourself as a practice. □

Bessel's Inequality

Theorem

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and $(e_1, \ldots, e_r)$ an orthonormal system in V. Then, for every $v \in V$ and any $r \leq n$,

$$\sum_{k=1}^{r} |\langle e_k, v\rangle|^2 \leq \|v\|^2$$

Theorem

Best approximation

$$v \approx \sum_{i=1}^{r} \lambda_i e_i,$$

Gram-Schmidt Orthonormalization

Theorem

Assume that we have a system of vectors (perhaps a basis) $(v_1, \ldots, v_r)$ in an inner product vector space V. We wish to construct a new system $(w_1, \ldots, w_r)$ that is orthonormal.

$$w_k := \frac{v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j}{\| v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j \|}$$

Linear Map

Definition

Let $(U,\oplus,\odot)$ and $(U,\boxplus,\boxdot)$ be vector spaces that are either both real or both complex. Then a map $L: U \to V$ is said to be linear if it is both homogeneous

$$L(\lambda \odot u) = \lambda \boxdot L(u)$$

and additive

$$L(u \oplus u') = L(u) \boxplus L(u'),$$

The set of all map $L: U \to V$ is $\mathcal{L}(U, V)$.

Comment.

- Usually, $\oplus$ and $\boxplus$ are just $+$, $\odot$ and $\boxdot$ are just $\cdot$.
- A linear map $L: U \to V$: $L(0) = 0$.

Linear Map

Example

(i) All linear maps $\mathbb{R} \to \mathbb{R}$ are of the form $x \mapsto \alpha x$ for some $\alpha \in \mathbb{R}$.

(ii) For $I \subset \mathbb{R}$, the map $\frac{d}{dx} : f \mapsto f'$ is a linear map $C^1(I) \to C(I)$.

(iii) The map $(a_n) \mapsto a_0$ is a linear map from the space of all sequences to $\mathbb{C}$.

(iv) The map $(a_n) \mapsto \lim_{n\to\infty} a_n$ is linear map from the space of all convergent sequences to $\mathbb{C}$.

(v) If $\mathbb{C}$ is regarded as a real vector space, the map $z \mapsto \overline{z}$ is linear $\mathbb{C} \to \mathbb{C}$. It is not linear if $\mathbb{C}$ is regarded as a complex vector space.

(vi) For any real or complex vector space V, the map $V \ni x \mapsto c \in \mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is linear if and only if $c = 0$.

Linear Map

Theorem

Let U, V be real or complex vector spaces and $(b_1, ..., b_n)$ a basis of U (in particular, it is assumed that $dim U = n$). Then for every n-tuple $(v_1, ..., v_n \in V^n)$ there exists a unique linear map $L : U \to V$ such that $Lb_k = v_k$, $k = 1, ..., n$.

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Proof.

Slides 111-112. □

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Proof.

Slides 111-112. □

Comment.

- The identity map $id : V \to V, id(v) = v$, is linear.
- The set $L(U, V)$ is again a vector space when endowed with pointwise addition and scalar multiplication.
- The composition of linear maps is linear.

Dual Basis

Let V be a real or complex vector space. Then $L(V, \mathcal{F})$ is known as the dual space of V and denoted by V. The dual space of V is of course itself a vector space. Let $\dim V = n$ and $\mathcal{B} = (b_1, ..., b_n)$ be a basis of V. Then for every $k = 1, ..., n$ there exists a unique map

$$b_k^* : V \to \mathbb{F}, \qquad b_k^*(b_j) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Comment.

It turns out (see exercises) that the tuple of maps $\mathcal{B}^* = (b_1^*, ..., b_n^*)$ is a basis of $V^* = L(V, \mathbb{F})$ (called the dual basis of $\mathcal{B}$) and thus $\dim\ V = \dim\ V = n$.

Range and Kernel

$$\operatorname{ran} L := \left\{ v \in V : \underset{u \in U}{\exists}\ v = Lu \right\}$$

$$\ker L := \{ u \in U : \ Lu = 0 \}.$$

Comment.

$L \in \mathcal{L}(U, V)$ is injective if and only if $\ker L = \{0\}$.

Nomenclature

- an *isomorphism* if L is bijective;
- an *endomorphism* if $U = V$;
- an *automorphism* if $U = V$ and L is bijective;
- *epimorph* if L is surjective;
- *monomorph* if L is injective.

Comment.

If L is an isomorphism, the its inverse, is also linear and hence also an isomorphism.

Isomorphisms

Theorem

L is an isomorphism if and only if for every basis $(b_1, ..., b_n)$ of U the tuple $(Lb_1, ..., Lb_n)$ is a basis of V.

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Proof.

Slides 117-118. $\square$

Isomorphisms

Theorem

L is an isomorphism if and only if for every basis $(b_1, ..., b_n)$ of U the tuple $(Lb_1, ..., Lb_n)$ is a basis of V.

Proof.

Slides 117-118. $\square$

Theorem

Two finite-dimensional vector spaces U and V are isomorphic if and only if they have the same dimension:

The Dimension Formula

$$\dim \operatorname{ran} L + \dim \ker L = \dim U.$$

Proof.

Slides 120-121. $\square$

The Dimensional Theorem

Theorem

dim $U = \dim V$. *Then a linear map* $L \in \mathcal{L}(U, V)$ *is injective if and only if it is surjective.*

The Dimensional Theorem

Theorem

$\dim U = \dim V$. Then a linear map $L \in \mathcal{L}(U, V)$ is injective if and only if it is surjective.

Proof.

$$\begin{aligned} L \text{ injective} \quad &\Leftrightarrow \quad \ker L = \{0\} \\ &\Leftrightarrow \quad \dim \ker L = 0 \\ &\Leftrightarrow \quad \dim \operatorname{ran} L = \dim U = \dim V \\ &\Leftrightarrow \quad \operatorname{ran} L = V \\ &\Leftrightarrow \quad L \text{ surjective} \end{aligned}$$

□

Bounded Linear Maps and The Operator Norm

Definition

Bounded Linear Maps

$$\|Lu\|_V \le c \cdot \|u\|_U \qquad\qquad \text{for all } u \in U.$$

Definition

The Operator Norm

$$\|L\| := \sup_{\substack{u\in U\\ u\neq 0}} \frac{\|Lu\|_V}{\|u\|_U} = \sup_{\substack{u\in U\\ \|u\|_U=1}} \|Lu\|_V$$

Thank you for your attention!