Analytical Solution of Two-Layer Beam Taking into account Interlayer Slip and Shear Deformation

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Abstract: A mathematical model is proposed and its analytical solution derived for the analysis of the geometrically and materially linear two-layer beams with different material and geometric characteristics of an individual layer. The model takes into account the effect of the transverse shear deformation on displacements in each layer. The analytical study is carried out to evaluate the influence of the transverse shear deformation on the static and kinematic quantities. We study a simply supported two-layer planar beam subjected to the uniformly distributed load. Parametric studies have been performed to investigate the influence of shear by varying material and geometric parameters, such as interlayer slip modulus (K), flexural-to-shear moduli ratios (E/G) and span-to-depth ratios (L/h). The comparison of the results for vertical deflections shows that shear deformations are more important for high slip modulus, for "short" beams with small L/h ratios, and beams with high E/G ratios. In these cases, the effect of the shear deformations becomes significant and has to be addressed in design. It also becomes apparent that models, which consider the partial interaction between the layers, should be employed if beams have very flexible connections.

Introduction

Due to their economy of construction and high bearing capacity, layered systems are widely used to optimize the performance of components in structural engineering. Classic cases are steel–concrete composite beams in buildings and bridges, wood–concrete floor systems, coupled shear walls, concrete beams externally reinforced with laminates, sandwich beams, and many more. The behavior of these structures largely depends on the type of the connection between the layers. Mechanical shear connectors are usually employed to provide a desired composite action. With the use of the rigid shear connectors, a full shear connection and, consequently, a full composite action between the individual components can be achieved. The result is that conventional principles of the solid beam analysis can be employed. Unfortunately, the rigid shear connectors can hardly be realized in

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practice. Therefore, most of shear connections result in only a partial composite action. As a result, an interlayer slip often develops; if it has a sufficient magnitude, it significantly effects the deformation and the stress distribution of the composite system.

The first appreciation of a composite construction probably originated from observations of highway bridges in service. After the experimental studies had indicated the absence of full composite interaction between the layers in composite beams, new theories were presented accounting for the slip between the layers. These early theories of incomplete or partial interaction between the layers of a composite beam were developed independently during the 1940s in Switzerland, Sweden, and the United States (Leon and Viest 1998). These theories were based on the assumptions of linear elastic material models and the Euler–Bernoulli hypothesis of plane sections. Perhaps the first but certainly the most quoted partial action theory was developed by Newmark et al. (1951). The subsequent theories differ in one or more aspects regarding the additional assumptions and resulted in similar second-order differential equations. Until now, a number of elastic theories with fewer simplifying assumptions and of greater sophistication have been developed. Several exact analytical solutions of simply supported, layered planar beams for different combinations of simple loading cases and simple boundary conditions have been presented in professional literature, e.g., (Girhammar and Gopu 1993; Goodman and Popov 1968; Goodman and Popov 1969; Jasim 1997; Jasim and Ali 1997; Ranzi et al. 2003). With the development of computational tools and computers over the last few decades, these elastic theories have been refined to incorporate numerous aspects of nonlinear geometric and material behavior, as well as time-dependent effects, fatigue, and load reversals, e.g., (Ayoub 2001; Čas et al. 2004a,b; Čas 2004; Fabbrocino et al. 2002; Faella et al. 2002; Gattesco 1999; Smith and Teng 2001).

One of the basic assumptions of all the above-mentioned exact analytical models with partial interaction theory of composite

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beams was the Euler–Bernoulli hypothesis of plane sections of each individual layer. It is well known that the classical Euler–Bernoulli theory of beam bending, also known as the elementary theory of bending, disregards the effects of the shear deformation. The theory is based on the assumption that the cross section remains perpendicular to the deformed centroidal axis of the beam during bending. This assumption implies a zero shear strain and an infinite shear stiffness. In reality, no material exists that possesses such a property. Since the Euler–Bernoulli theory neglects the transverse shear deformation, its suitability for composite beams can be questioned. This is particularly true in circumstances where shear effects can be significant, as in thick and short composite beams, where flexural-to-shear rigidity ratio parameters are large and the span-to-depth ratio is small. Timoshenko (1921) was the pioneering investigator to include refined effects, such as the shear deformation, in the beam theory. In the literature, this theory is now widely referred to as the Timoshenko beam theory. The effect of shear deformation in Timoshenko's theory is accounted for by an additional rotation angle of transverse cross sections. Consequently, the distribution of the transverse shear deformation is assumed to be constant through the beam thickness. In the beginning of the 1970s, Reissner (1972) has introduced a similar shear distortion in his one-dimensional finite-strain beam model.

To improve the accuracy of the transverse stress prediction, nonclassical higher-order shear-deformable iterative models have been proposed (Gorik 2003; Matsunaga 2002; Piskunov and Grinevitskii 2004; Soldatos and Watson 1997). According to these propositions, a zero-iteration model corresponds to the classical Euler–Bernoulli theory, while the above-mentioned nonclassical Timoshenko theory corresponds to the first-iteration model. Higher-order iteration models introduce further deformation modes, such as cross-sectional bulging and warping, which are important in the modeling of thin-walled composites structures, employed in the aerospace industry. It is not the goal of this paper to model the higher-order deformations. Only the first-iteration model will be considered and the Reissner one-dimensional finite-strain beam model used in the present analytical model. To the best of the authors' knowledge, no report seems to exist on the exact analytical solution of the Timoshenko composite beams with the partial interaction between the layers. In this paper, we aim to fill this gap, and present an exact analytical model of the composite beam, which takes into account the effect of the shear deformation. Then we make the parametric studies on the influence of shear deformation effects on the mechanical behavior of composite beams with the partial interaction between the layers. This way we show when the effect of the shear deformation in the individual layer can be neglected.

Analytical Model

Assumptions

Our formulation of the planar Timoshenko two-layer composite beam model uses the following assumptions: (1) material is linear elastic; (2) displacements, strains, and rotations are small; (3) shear deformations are taken into account (the "Timoshenko beam"); (4) strains vary linearly over each layer (the "Bernoulli hypothesis"); (5) the layers are continuously connected and the slip modulus of the connection is constant; (6) friction between the layers is not considered; and (7) the shapes of the cross sections are symmetrical with respect to the plane of deformation

and remain unchanged in the form and size during deformation. Our further assumption (8) is that an interlayer tangential slip can occur at the interface between the layers, but no delamination or the transverse separation between them is possible.

Governing System of Equations

We consider an initially straight, planar, two-layer composite beam element of undeformed length L. Layers are marked by letters a and b (see Fig. 1). The beam element is placed in the (x,z) plane of a spatial Cartesian coordinate system with coordinates (x,y,z) and unit base vectors $\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z$. The undeformed reference axis of the layered beam element is common to both layers and lies in their contact plane. The layered beam element is subjected to the action of the conservative distributed load $\boldsymbol{p} = p_x \mathbf{E}_x + p_z \mathbf{E}_z$ and the distributed moment $\boldsymbol{m} = m_y \mathbf{E}_y$ along the span, and to external point forces and moments S_i^a and S_i^b ($i = 1, 2, \ldots, 6$) at its ends, respectively; see, e.g., (Čas et al. 2004).

The deformed configurations of both layers are defined by vector-valued functions

$$\mathbf{R}^a(x,z) = (x + u^a(x) + z\varphi^a(x))\mathbf{E}_x + (z + w^a(x))\mathbf{E}_z$$

$$\mathbf{R}^b(x^*,z) = (x^* + u^b(x^*) + z\varphi^b(x^*))\mathbf{E}_x + (z + w^b(x^*))\mathbf{E}_z \tag{1}$$

where x^* represents a material, undeformed coordinate of that point of layer b which, in the deformed state, gets in contact with the point of layer a with coordinate x (see Fig. 1). In Eq. (1) and in all further expressions, the notations $(\cdot)^a$ and $(\cdot)^b$ refer to layers a and b. Functions u^a, w^a, φ^a denote the components of the displacement vector and the rotation angle of layer a at the reference axis with respect to the base vectors $\mathbf{E}_x$, $\mathbf{E}_z$, and $\mathbf{E}_y$, respectively. Variables u^b, w^b, φ^b are related to layer b.

The system of governing equations of the two-layer composite beam consists of kinematic, equilibrium, and constitutive equations with accompanying boundary conditions for each of the two layers, and the constraining equations that assemble each layer into a two-layer composite beam. Since deformations, displacements, and rotations are assumed to be small quantities, the generalized equilibrium equations can be simplified using the following two assumptions [see, e.g., (Čas et al. 2004b)]: (i) $dx \approx dx^*$; (ii) vertical deflections of the reference axis of individual layers are equal $w^a(x) = w^b(x^*) = w(x)$ and $\mathcal{I}^a \approx \mathcal{I}^b = [0, L]$. Thus, $(\cdot)^b(x^*) = (\cdot)^b(x)$ holds true for any quantity of layer b, e.g., $u^b(x^*) = u^b(x)$. Kinematic, equilibrium, and constraining equations can now be considerably simplified. After considering the assumptions mentioned above, we can decompose the basic equations of the two-layer beam with an interlayer slip into two separate systems of differential and algebraic equations

$$u^{a\prime} - \varepsilon^a = 0, \quad u^{b\prime} - \varepsilon^b = 0$$

$$w^{a\prime} + \varphi^a - \gamma^a = 0, \quad w^{b\prime} + \varphi^b - \gamma^b = 0 \tag{2}$$

$$\varphi^{a\prime} - \kappa^a = 0, \quad \varphi^{b\prime} - \kappa^b = 0$$

$$w^a - w^b = 0 \tag{3}$$

$$\mathcal{N}^{a\prime} - p_t = 0, \quad \mathcal{N}^{b\prime} + p_t + p_x = 0$$

$$\mathcal{Q}^{a\prime} + p_n = 0, \quad \mathcal{Q}^{b\prime} - p_n + p_z = 0 \tag{4}$$

$$\mathcal{M}^{a\prime} - \mathcal{Q}^a = 0, \quad \mathcal{M}^{b\prime} - \mathcal{Q}^b + m_y = 0$$

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$$\mathcal{N}^a - \mathcal{N}^a_C = 0, \quad \mathcal{N}^b - \mathcal{N}^b_C = 0$$

$$\mathcal{Q}^a - \mathcal{Q}^a_C = 0, \quad \mathcal{Q}^b - \mathcal{Q}^b_C = 0 \tag{5}$$

$$\mathcal{M}^a - \mathcal{M}^a_C = 0, \quad \mathcal{M}^b - \mathcal{M}^b_C = 0$$

$$\Delta = u^a - u^b \tag{6}$$

$$p_t = \mathcal{F}(\Delta) = K\Delta \tag{7}$$

and

$$x + u^a = x^* + u^b \rightarrow x^* = x + \Delta$$
$$\tag{8}$$
$$\mathcal{N} = \mathcal{N}^a + \mathcal{N}^b$$

$$\mathcal{Q} = \mathcal{Q}^a + \mathcal{Q}^b \tag{9}$$

$$\mathcal{M} = \mathcal{M}^a + \mathcal{M}^b \tag{10}$$

In Eqs. (2)–(7), ε^a and ε^b=extensional strains of the reference axes of layers a and b; κ^a, and κ^b=pseudocurvatures; while γ^a and γ^b=transverse shear strains of the corresponding cross sections of layers a and b, respectively. $\mathcal{N}^a, \mathcal{N}^b, \mathcal{Q}^a, \mathcal{Q}^b$, and $\mathcal{M}^a, \mathcal{M}^b$ represent equilibrium axial forces, equilibrium shear forces, and equilibrium bending moments of both layers. p_t and p_n denote the tangential and the normal interlayer contact tractions in the contact plane between the layers. $\mathcal{N}^a_C, \mathcal{N}^b_C, \mathcal{Q}^a_C, \mathcal{Q}^b_C$, and $\mathcal{M}^a_C, \mathcal{M}^b_C$=constitutive axial forces, constitutive shear forces, and constitutive bending moments of layers a and b, respectively. In the case of linear elastic material, the constitutive forces are assumed to be given by the linear relations with respect to ε^a, ε^b, κ^a, κ^b, γ^a, and γ^b and, therefore, take the following notation:

$$\mathcal{N}^a_C = E^a A^a \varepsilon^a + E^a S^a \kappa^a = C^a_{11}\varepsilon^a + C^a_{12}\kappa^a$$

$$\mathcal{Q}^a_C = k_y G^a A^a \gamma^a = C^a_{33}\gamma^a$$

$$\mathcal{M}^a_C = E^a S^a \varepsilon^a + E^a J^a \kappa^a = C^a_{21}\varepsilon^a + C^a_{22}\kappa^a$$
$$\tag{11}$$
$$\mathcal{N}^b_C = E^b A^b \varepsilon^b + E^b S^b \kappa^b = C^b_{11}\varepsilon^b + C^b_{12}\kappa^b$$

$$\mathcal{Q}^b_C = k_y G^b A^b \gamma^b = C^b_{33}\gamma^b$$

$$\mathcal{M}^b_C = E^b S^b \varepsilon^b + E^b J^b \kappa^b = C^b_{21}\varepsilon^b + C^b_{22}\kappa^b$$

where E^a, E^b=elastic and G^a, G^b=shear moduli of layers a and b; A^a, A^b denote the areas of the cross-sections of layers a and b; S^a, S^b=static moments; and J^a, J^b=moments of inertia of layers a and b with respect to the interlayer contact line. k_y=shear coefficient of the cross section of the layer. In the case of a rectangular cross section and isotropic material, the shear coefficient is 5/6 (Cowper 1966).

The system of Eqs. (2)–(7) consists of 21 equations for 21 unknown functions u^a, u^b, w^a, w^b, φ^a, φ^b, ε^a, ε^b, κ^a, κ^b, γ^a, γ^b, $\mathcal{N}^a$, $\mathcal{N}^b$, $\mathcal{Q}^a$, $\mathcal{Q}^b$, $\mathcal{M}^a$, $\mathcal{M}^b$, Δ, p_t, and p_n, whereas Eqs. (8)–(10) constitute a system of three equations for three unknown functions x^*, $\mathcal{Q}$, $\mathcal{M}$. In Eq. (7), K denotes the slip modulus at the interlayer surface.
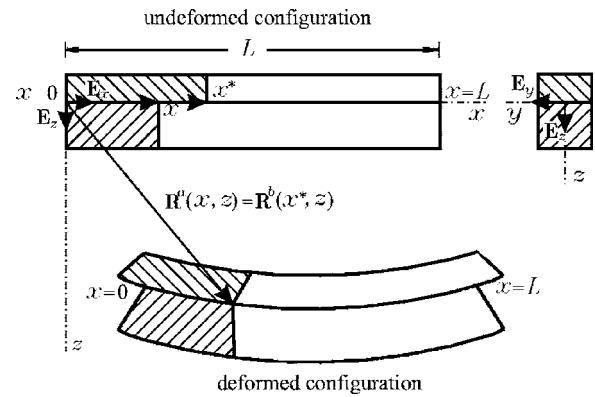
Fig. 1. Undeformed and deformed configuration of the two-layer beam

Solution Algorithm

If the slip Δ and normal traction p_n between the layers is a known function of x, the solution of the system of Eqs. (2)–(7) can easily be obtained with the following sequence of steps.

In the first step, we differentiate Eqs. (3) and (6) twice with respect to x and insert Eq. (2). The following differential equations for the interlayer slip and the pseudocurvatures are derived:

$$\Delta'' = \varepsilon^{a\prime} - \varepsilon^{b\prime} \tag{12}$$

$$\kappa^b = \kappa^a + \gamma^{b\prime} - \gamma^{a\prime} \tag{13}$$

The derivatives $\varepsilon^{a\prime}$, $\varepsilon^{b\prime}$, $\gamma^{a\prime}$, and $\gamma^{b\prime}$ are obtained from Eq. (5), if differentiated with respect to x. Solving the differentiated Eq. (5) for $\varepsilon^{a\prime}$, $\varepsilon^{b\prime}$, $\gamma^{a\prime}$, $\gamma^{b\prime}$, $\kappa^{a\prime}$, and $\kappa^{b\prime}$ gives

$$\begin{Bmatrix} \varepsilon^{a\prime} \\ \gamma^{a\prime} \\ \kappa^{a\prime} \\ \varepsilon^{b\prime} \\ \gamma^{b\prime} \\ \kappa^{b\prime} \end{Bmatrix} = \mathbf{C}^{-1} \begin{Bmatrix} \mathcal{N}^{a\prime} \\ \mathcal{Q}^{a\prime} \\ \mathcal{M}^{a\prime} \\ \mathcal{N}^{b\prime} \\ \mathcal{Q}^{b\prime} \\ \mathcal{M}^{b\prime} \end{Bmatrix} \tag{14}$$

$\mathbf{C}$=matrix of constitutive constants [see Eq. (11)], and $\mathbf{C}^{-1}$=its inverse
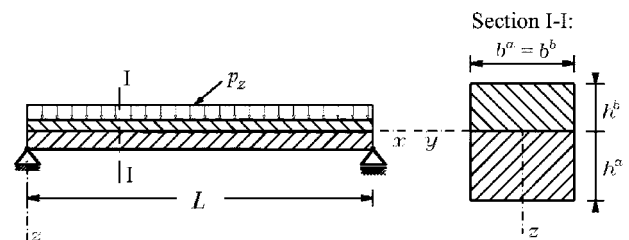

Fig. 2. Simply supported two-layer beam

$$\mathbf{C}^{-1} = \begin{bmatrix} C_{11}^a & 0 & C_{12}^a & 0 & 0 & 0 \\ 0 & C_{33}^a & 0 & 0 & 0 & 0 \\ C_{21}^a & 0 & C_{22}^a & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{11}^b & 0 & C_{12}^b \\ 0 & 0 & 0 & 0 & C_{33}^b & 0 \\ 0 & 0 & 0 & C_{21}^b & 0 & C_{22}^b \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} D_{11}^a & 0 & D_{12}^a & 0 & 0 & 0 \\ 0 & D_{33}^a & 0 & 0 & 0 & 0 \\ D_{21}^a & 0 & D_{22}^a & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11}^b & 0 & D_{12}^b \\ 0 & 0 & 0 & 0 & D_{33}^b & 0 \\ 0 & 0 & 0 & D_{21}^b & 0 & D_{22}^b \end{bmatrix} \qquad (15)$$

Furthermore, we differentiate Eq. (13) twice with respect to x. By insertion of Eq. (4) into Eq. (14), the second and third derivatives of strains are obtained by differentiation of Eq. (14) twice and three times with respect to x. Introducing $\varepsilon^{a\prime}$, $\varepsilon^{b\prime}$, $\kappa^{a\prime\prime}$, $\kappa^{b\prime\prime}$, $\gamma^{a\prime\prime\prime}$, and $\gamma^{b\prime\prime\prime}$ in Eq. (12) and differentiated Eq. (13), results in a coupled system of two higher-order linear differential equations with constant coefficients for the slip and the normal interlayer traction between layers a and b

$$\Delta''' + K_1\Delta' + K_2 p_n = D_{12}^b p_z$$
$$K_5 p_n'' + K_4 p_n + K_3 \Delta' = -D_{22}^b p_z \qquad (16)$$

with K_1, K_2, K_3, K_4, and K_5 being constants

$$K_1 = -K(D_{11}^a + D_{11}^b), \quad K_2 = D_{12}^a + D_{12}^b, \quad K_3 = K(D_{21}^a + D_{21}^b)$$
$$K_4 = -(D_{22}^a + D_{22}^b), \quad K_5 = D_{33}^a + D_{33}^b \qquad (17)$$

Boundary conditions associated with Eq. (16) are the values of the interlayer slip and its first two derivatives, and the values of the normal interlayer traction and its first derivative at the edge $x=0$ of the beam element. An exact solution of Eq. (16) was easily obtained by MATHEMATICA (Wolfram 2003). Due to its length and complexity, the closed-form expressions are not shown throughout the paper. When the slip and the normal interlayer traction have been obtained, the remaining equations of the system (2)–(7) can simply be solved. We first determine the boundary rotations and displacements from the system of linear equations

$$\mathbf{K}_T\mathbf{u} = \mathbf{g} \qquad (18)$$

for the composite structure. In Eq. (18), $\mathbf{K}_T$ denotes the tangent stiffness matrix, $\mathbf{u}$=vector of end-point displacements, and $\mathbf{g}$=load vector. Once $\mathbf{u}$ is known, the values of the end forces can easily be computed. By integrating Eqs. (2) and (4) and considering Eqs. (3), (6), and (7), the solution for unknown functions u^a, u^b, w^a, w^b, φ^a, φ^b, ε^a, ε^b, κ^a, κ^b, γ^a, γ^b, $\mathcal{N}^a$, $\mathcal{N}^b$, $\mathcal{Q}^a$, $\mathcal{Q}^b$, $\mathcal{M}^a$, $\mathcal{M}^b$, Δ, p_t, and p_n can easily be obtained. Finally, the unknown functions x^*, $\mathcal{Q}$, and $\mathcal{M}$ are obtained from Eqs. (8)–(10).

Parametric Studies

This section presents parametric studies performed on a simply supported two-layer planar beam subjected to a uniformly distributed load (see Fig. 2) with the aim to investigate the influence of
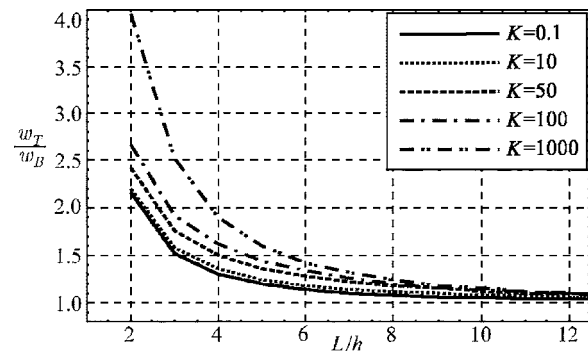

Fig. 3. Influence of slip modulus and L/h ratios on vertical deflections for $E/G=16$

the shear deformation in an individual layer and a variety of other material and geometric parameters, such as flexural-to-shear and span-to-depth ratios, interlayer slip modulus, etc., on the mechanical behavior of the Timoshenko composite beams.

The main interest was focused on the assessment of the contribution effect of the transverse shear deformation to the deformation and stresses in composite beams with partial interaction between the layers. To this end, the vertical deflections were calculated for different values of parameters $(K, E/G, L/h)$ and compared to those obtained by the analytical model of Euler–Bernoulli composite beams with partial interaction between the individual components. Results are given in Figs. 3 and 4, and Table 1.

In Fig. 3, the vertical deflections (w_T) of the Timoshenko composite beam with the partial interlayer interaction are compared to the vertical deflections (w_B) obtained by the Euler–Bernoulli composite beam model with the same partial interlayer interaction (here also called the classical composite beam model), for different L/h ratios and various interlayer slip moduli K. It can be observed in Fig. 3 that decreasing the L/h ratios and increasing interlayer slip modulus K, increases the influence of the transverse shear deformation on vertical deflections. This influence is considerable in the case of timber composite beams $(E/G=16)$ even for relatively slender beams $(L/h=10)$, as can be seen from Fig. 4 and Table 1. The effect is even more pronounced for timber composite beams with $L/h=5$, where the shear deformations increase the vertical deflections for the values in the range from 19.2 to 59.5%.

The effect of shear deformation on the vertical deflections at the midspan of a composite beam has been investigated for vari-
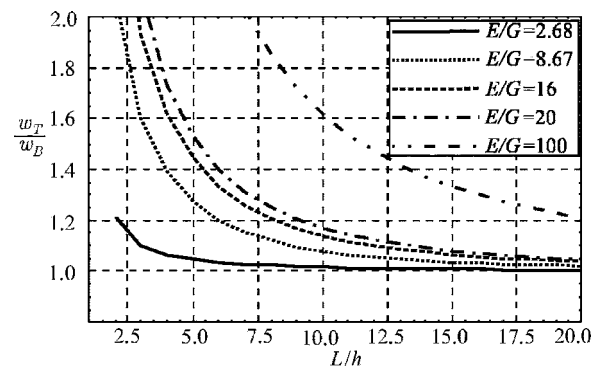

Fig. 4. Influence of E/G and L/h ratios on vertical deflections for $K=100$

Table 1. Influence of K, E/G, and L/h on Vertical Deflections (ω_T/ω_B)

K [kN/cm^2]	$E/G=2.68$			$E/G=8.67$			$E/G=16$		
	I	II	III	I	II	III	I	II	III
0.001	1.090	1.032	1.008	1.287	1.104	1.026	1.524	1.192	1.048
0.01	1.090	1.032	1.008	1.287	1.104	1.026	1.524	1.192	1.048
0.1	1.090	1.032	1.008	1.288	1.105	1.027	1.524	1.192	1.049
1	1.091	1.033	1.008	1.293	1.110	1.032	1.529	1.197	1.054
10	1.092	1.034	1.010	1.341	1.152	1.056	1.581	1.244	1.087
50	1.098	1.040	1.014	1.494	1.237	1.075	1.767	1.370	1.128
100	1.105	1.046	1.017	1.603	1.274	1.080	1.931	1.445	1.139
1000	1.182	1.083	1.024	1.875	1.329	1.084	2.534	1.595	1.154

Note: I: $L/h=3$; II: $L/h=5$; and III: $L/h=10$.

ous E/G and L/h ratios and different interlayer slip moduli K. Here, we present (see Table 1) only the results for beams with $E/G=2.68$ (the ratio, typical for isotropic materials such as steel, aluminum, and copper) and beams with $E/G=8.67$ (transversely isotropic glass-fiber-reinforced unidirectional composite beams) and for anisotropic wood beams with $E/G=16\div20$. The results for a beam with $E/G=100$, which is not a realistic value for material, have been added in Fig. 4 as well.

A comparative analysis of the analytical results for the vertical deflections at the midspan of a simply supported composite beam with the partial interlayer interaction shows that the influence of shear effects is significant for composite beams where $E/G\geq16$, particularly in the case of rather stiff connections with very high interlayer slip moduli K, where the influence of shear effects on the increase of deflections can be as high as 15.4% for $L/h=10$, and more than 250% for short beams with $L/h=3$.

The contribution of the shear effects to the vertical deflections ranges, in the case of steel, aluminum, and copper composite beams with $E/G=2.86$, from 0.3 to 8.3% for $5\leq L/h\leq15$. Therefore, for such composite beams, the shear effects are insignificant, except for short beams with $L/h\leq3$ and high Ks.

In the case of glass-fiber-reinforced composite beams with $E/G=8.67$, the influence for beams with $L/h\geq10$ is still beyond 8.4% and, thus less significant, while the influence on beams with $L/h\leq5$ becomes important (it ranges from 10.4 to 32.9%).

It is illustrative to study the influence of the shear deformation in an individual layer on the static and kinematic quantities rather than w, such as Δ, p_n, ε^a, $\mathcal{N}^a$, etc.
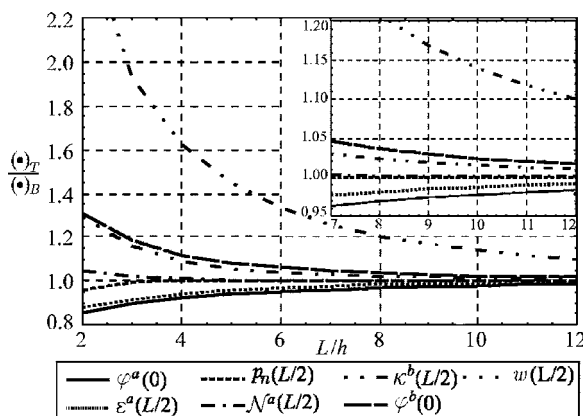
These quantities have been calculated for timber composite beams with $E/G=16$ and for various values of parameters L/h and K. The results are presented in Fig. 5 and in Table 2, where the notations $(\cdot)_T$ and $(\cdot)_B$ mark that the quantities have been calculated by different beam models. Thus, $(\cdot)_B$ represents quantities calculated by the Euler–Bernoulli composite beam model with the partial interlayer interaction, and $(\cdot)_T$ represents quantities obtained by the Timoshenko composite beam model.

The examination of analytical results in Table 2 and in Fig. 5 reveals that the shear deformation has an important influence not only on the vertical deflections, but also on the other mechanical quantities of composite beams with the partial interlayer interaction.

An interesting detail at this point is that the influence of the shear deformation has different effects on different quantities. Some quantities, like w, φ^b, κ^b, and $\mathcal{N}^a$ increase due to the shear deformation, while the others, like p_n, ε^a, and φ^a decrease when compared to the quantities calculated by the classical beam model. For example, an increasing influence of shear effects on $\varphi^b(0)$ for interlayer slip modulus $K=0.1$ kN/cm^2 and $L/h=3$ is found to be 25.0%, while for $L/h=15$ it is only 1.2%. On the other hand, a decreasing influence of shear effects on $\varepsilon^a(L/2)$ is found (-10.3%) for $K=100$ kN/cm^2 and $L/h=3$, and it is -2.2% for $L/h=10$. It is clear from the results depicted in Fig. 6 that, in addition to the increase in the L/h, the increase of K leads to a significant enlargement of tangential interlayer tractions with respect to normal interlayer tractions in the contact plane between the layers. For example, in the case of relatively slender two-layer composite beams ($L/h\geq10$) with $K\geq10$ kN/cm^2, the tangential

Fig. 5. Static and kinematic quantities as a function of L/h for $E/G=16$ and $K=100$ kN/cm^2

Table 2. Static and Kinematic Quantities as Functions of K and L/h for $E/G=16$

$(\cdot)_T/(\cdot)_B$	$K=0.1$ kN/cm^2			$K=100$ kN/cm^2		
	I	II	III	I	II	III
$w(L/2)$	1.524	1.192	1.049	1.931	1.445	1.139
$\Delta(0)$	1.046	1.020	1.006	1.026	1.012	1.005
$\varphi^a(0)$	0.939	0.973	0.992	0.919	0.960	0.988
$\varphi^b(0)$	1.250	1.092	1.026	1.185	1.083	1.030
$\kappa^a(L/2)$	0.944	0.977	0.994	0.927	0.969	0.992
$\kappa^b(L/2)$	1.188	1.077	1.020	1.156	1.059	1.015
$\mathcal{N}^a(L/2)$	1.044	1.019	1.005	1.022	1.007	1.001
$\varepsilon^a(L/2)$	0.944	0.977	0.994	0.897	0.942	0.978
$p_n(L/2)$	0.968	0.994	1.000	1.993	1.004	1.001

Note: I: $L/h=3$; II: $L/h=5$; and III: $L/h=10$.

Fig. 6. Tangential and normal tractions as a function of L/h for $E/G=16$ and different Ks
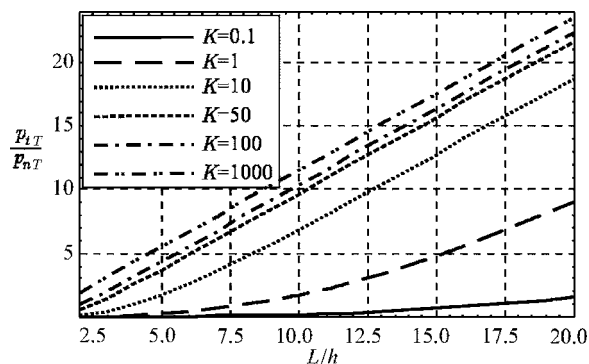
Fig. 8. Vertical deflections as a function of h^a/h^b for $L/h=10$ and $E/G=16$ and different Ks

interlayer tractions may be as high as about 20 times the normal interlayer tractions. Although it has been shown that the influence of shear effects on the interlayer tractions is negligible, it is apparent from the graphs in Fig. 7, that the shear deformation has an important impact on the ratio of the interlayer tractions, especially for the two-layer beams with $L/h \le 10$, where the influence of shear effects is more than 29.0%.

A parametric study has also been conducted to assess the effects of different parameters, such as h^a/h^b and K on the vertical deflections and shear forces. For this purpose, the vertical deflections at the midspan and the shear forces at the edge $x=0$ of the two-layer composite beam have been calculated for various h^a/h^b and K. In the case of relatively slender timber beams with $L/h=10$ and $E/G=16$, the parametric study reveals, that minimum shear effects occur when layers have approximately equal depths. In Fig. 8, it is shown that the corresponding discrepancies are higher for smaller values of K and can be, in the case of a rather flexible connection ($K \le 1$ kN/cm²), as much as about four times smaller than in the case of stiff interaction between the layers.

Figs. 9 and 10 show that the contribution to the shear forces due to shear effects can be considerable, especially when the depth of one layer is very small compared to the depth of the other one and for small values of K.

It is observed that for a very thin bottom layer a ($h^a/h^b \le 0.1$) and the nonstiff interlayer contact ($K=0.1$ kN/cm²), the shear force Q^a can be due to shear effects about 2.8 times bigger than the one obtained by the Euler–Bernoulli model. By contrast, for high values of K, the shear force Q^a may be about twice as small as in the case of the Euler–Bernoulli beam. It is also
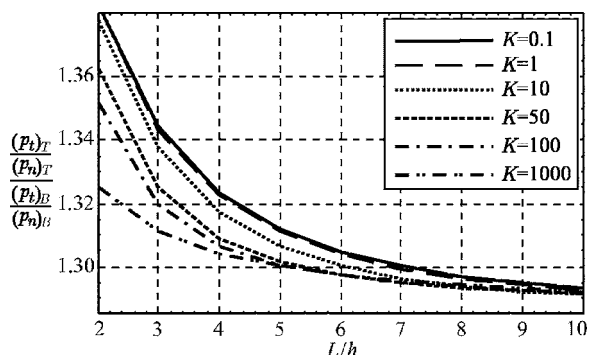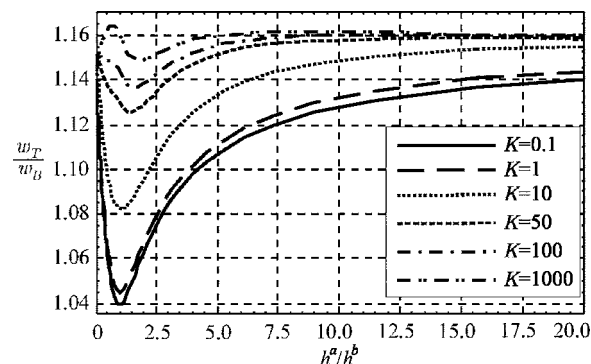
apparent from Fig. 10, that the value of the shear force Q^b in Timoshenko's theory is in the nonstiff connection about 2.5 times bigger if the top layer is very thin compared to the bottom one. Thus, it has been shown, that the shear deformations have considerable influence on static shear forces of individual layers of the two-layer composite beam and hence should not be neglected in the analysis of such structures.

In addition, the vertical deflections have been calculated for timber composite beams with $E/G=16$ and $L/h=10$ by different beam models: (1) using empirical formulas given in the European code for timber structures Eurocode 5 (Eurocode 5 1993); (2) with the classical Euler–Bernoulli beam model with and without considering the interlayer slip; (3) with the Timoshenko beam model for beams without the interlayer slip; and finally (4) with the present analytical Timoshenko composite beam model with the consideration of the partial interaction between the layers.

In Table 3, the results of different beam models and for a wide range of slip modulus from 0.001 to 1000 kN/cm² are presented and compared. Observe that the influence of the interlayer slip modulus on the vertical deflections is negligible for the range of the slip modulus from 0.001 to 0.1 kN/cm², and that the influence of solely slip modulus on the deflections due to the interlayer slip (w_B/w_B^*) decreases with the increase of the slip modulus. On the other hand, the combined influence of the shear deformation and slip (w_T/w_B) on the deflections increases with the increase of the slip modulus between the layers. Thus, for high values of K, deflections obtained by both Euler–Bernoulli and Timoshenko models with the consideration of the partial interaction differ from deflections obtained by the complete inter-

Fig. 7. Ratio of tangential and normal tractions as a function of L/h for $E/G=16$ and different Ks

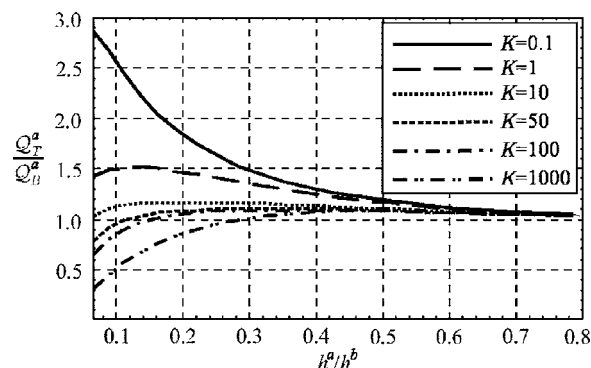**Fig. 9.** Shear force Q^a as a function of h^a/h^b for $L/h=10$, $E/G=16$, and different Ks
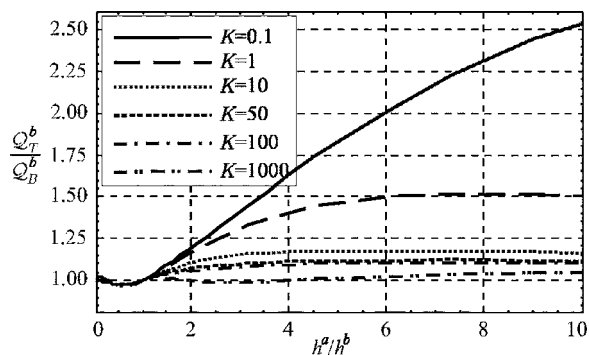
Fig. 10. Shear force Q^b as a function of h^a/h^b for $L/h=10$, $E/G=16$, and different Ks

Fig. 11. Comparisons of vertical deflections calculated by different beam models, for different Ks with $L/h=10$ and $E/G=16$

action models by less than 1.2%. Note that the deflections, obtained by the formula given in Eurocode (Eurocode 5 1993) for composite beams with interlayer slip between the layers, agree with the present results for the Euler–Bernoulli beam for $0.001<K<10$ kN/m^2 and $100<K<1000$ kN/m^2, but not for $10<K<100$ kN/m^2, where the maximum difference is 22.5%. The comparison with the Timoshenko beam is more promising.

Fig. 11 shows comparisons of vertical deflections obtained by different beam models. It can be seen from the results in Table 3 and Fig. 11 that the models with the partial interaction are essential for the accurate prediction of vertical deflections, especially for more flexible connections between the layers, i.e., for the range of slip moduli from 0.001 to 1 kN/cm^2, since the comparisons of w_B/w_B^* and w_T/w_T^* show that the deflections may be as high as about 3.5 times of the deflections of a rigidly connected layered beam. Furthermore, based on the comparisons between the analytical results for vertical deflections shown in Table 3 and Fig. 11, it is clear that the proposed model needs to be employed even for the range of the slip modulus $K>50$ kN/cm^2. The comparison of vertical deflections w_T/w_B shows that the effect of the shear deformation on the increase of vertical deflections can be about 15.4%.

Next, let us inspect the stress distributions over the depth of the two-layer composite beam. The longitudinal normal stresses σ_{xx} have been evaluated at the midspan section, and the tangential stresses σ_{xz} at the edge section of the beam shown in Fig. 2. From Fig. 12, it can be seen that the distributions and the values of the normal and the tangential stresses in the layers are very much affected by the stiffness of the contact. The effect is depicted for various stiffnesses and the "zig-zag" linear variation of normal stresses and the quadratic distribution of tangential stresses is
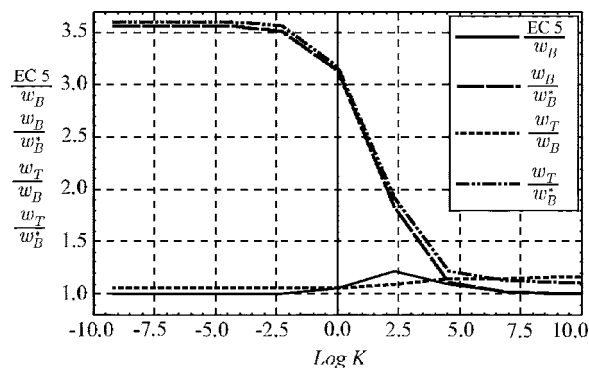
obtained. Note that for small Ks the maximum tangential stresses considerably exceed the stresses obtained from the classical solid beam model. It is apparent that the classical beam theory underestimates both the normal and the tangential stresses in layered beams. In the case of the nonstiff connection between the layers, the tangential stresses σ_{xz} may increase up to 25% compared to the stresses in the "solid beam."

Conclusions

A mathematical model is proposed and its analytical solution is found for the analysis of the geometrically and materially linear layered beams with different material and geometric characteristics of each layer. The proposed analytical model takes into account the transverse shear deformation of each layer of a multilayer beam. The analytical study is carried out for evaluating the influence of the transverse shear deformation on the static and kinematic quantities. Particular emphasis is given to the vertical deflections at the midspan of a simply supported two-layer planar beam subjected to the uniformly distributed load. For this purpose, several parametric studies have been performed to investigate the influence of shear effects and various material and geometric parameters, such as flexural-to-shear rigidity ratios and span-to-depth ratios, on the mechanical behavior of the layered Timoshenko beams.

Based on the results of this analytical study and the parametric evaluations undertaken, the following conclusions can be drawn:
1. The present mathematical model is general and relatively easy to comprehend.

Table 3. Vertical Deflections Calculated by Different Beam Models for Different Ks with $L/h=10$ and $E/G=16$

K [kN/cm^2]	EC 5 [cm]	w_B^* [cm]	w_B [cm]	w_T^* [cm]	w_T [cm]	$\dfrac{EC5}{w_B}$	$\dfrac{w_T}{w_B}$	$\dfrac{w_B}{w_B^*}$	$\dfrac{w_T}{w_T^*}$	$\dfrac{w_T}{w_B^*}$
0.001	3.875	1.085	3.875	1.252	4.062	1.000	1.048	3.571	3.243	3.743
0.01	3.872	1.085	3.869	1.252	4.057	1.000	1.048	3.566	3.239	3.738
0.1	3.845	1.085	3.818	1.252	4.005	1.007	1.049	3.518	3.197	3.691
1	3.602	1.085	3.391	1.252	3.573	1.062	1.054	3.125	2.853	3.293
10	2.427	1.085	1.982	1.252	2.154	1.225	1.087	1.826	1.720	1.985
50	1.526	1.085	1.325	1.252	1.494	1.153	1.128	1.221	1.193	1.377
100	1.326	1.085	1.230	1.252	1.379	1.096	1.139	1.115	1.101	1.270
1000	1.111	1.085	1.098	1.252	1.267	1.012	1.154	1.012	1.012	1.168

Note: *=without interlayer slip.

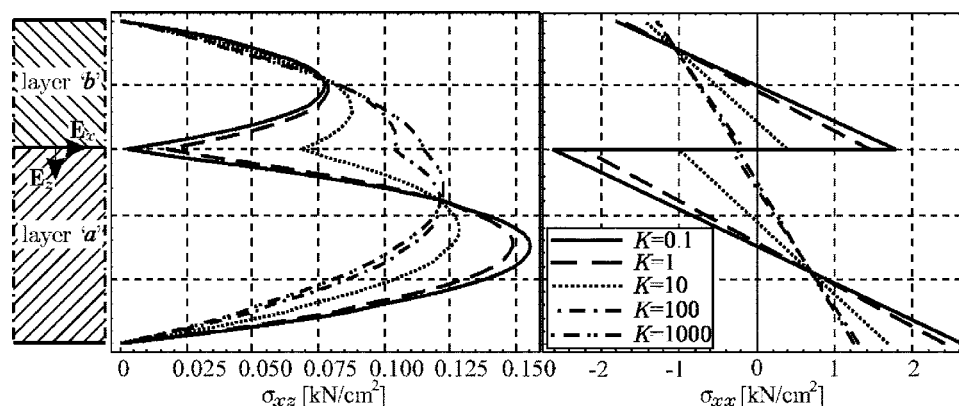**892** / JOURNAL OF STRUCTURAL ENGINEERING © ASCE / JUNE 2007

Fig. 12. The distribution of normal and tangential stresses over the cross section for different Ks

2. The influence of the shear deformation on vertical deflections is increasing with decreasing L/h ratios and increasing K. In the case of a timber composite beam ($E/G=16$), the contribution of shear deformations to vertical deflections can be about 15% for ratios $L/h=10$. The effect is even more pronounced for beams with $L/h=5$, where the effect of shear deformation on vertical deflections ranges between 19 and 60%.

3. The influence of shear effects is significant for composite beams with $E/G \geq 16$, particularly in the case of very high interlayer slip moduli, where the influence is about 15% for $L/h=10$, and about 250% for short beams with $L/h=3$.

4. In the case of steel, aluminum, and copper composite beams with $E/G=2.86$, the extra contribution to the vertical deflections due to shear effects ranges from 0.3 to 8% for $5 \leq L/h \leq 15$. Therefore, for such composite beams, shear effects are insignificant, except possibly for short beams with $L/h \leq 3$ and higher values of K.

5. In the case of glass-fiber reinforced unidirectional composite beams with $E/G=8.67$, the influence of shear effects on vertical deflections increases with an increase in K and a decrease in L/h. Thus, the influence for beams with $L/h \geq 10$ is still beyond 8% and, hence, insignificant, in contrast to beams with $L/h=5$, where the increase of the deflection ranges from 10 to 33%, and particularly for very short and rigidly connected composite beams, where the influence of shear effects can reach values up to 85%.

6. In the case of relatively slender two-layer composite beams ($L/h \geq 10$) with $K \geq 10$ kN/cm^2, the tangential interlayer tractions are about 20 times bigger than the related normal interlayer tractions. The shear deformation has an important impact on the actual ratio of the interlayer tractions. For the two-layer beams with $L/h \leq 10$, the influence of shear effects is more than 29%.

7. In the case of one very thin layer and a rather flexible connection ($K=0.1$ kN/cm^2), the corresponding shear force in the thin layer can be considerably bigger than in the classical theory. This is, $\mathcal{Q}_T$ is about 2.8 times larger for nonstiff and about twice smaller than $\mathcal{Q}_B$ obtained by the classical Euler–Bernoulli beam model. Similarly, the shear force $\mathcal{Q}_T$ of a very thin top layer is 2.5 times larger than that of the classical theory. Thus, we have shown, that the shear has a considerable impact on the values of the shear forces in the layers, and therefore should not be neglected.

8. The influence of shear deformation on vertical deflections is negligible, if $0.001 \leq K \leq 0.1$ kN/cm^2, $E/G=16$, and $L/h=10$.

9. The results of the deflection formula given in Eurocode 5 (Eurocode 5 1993) agree completely with the present results if $0.001 \leq K \leq 0.1$ kN/cm^2, while discrepancies may occur for other values of K.

10. The comparison of the results w_B/w_B^* and w_T/w_T^* shows that larger shear deformations develop for large slip moduli K, for short beams with small L/h ratios and for materials with high E/G ratios. In all these cases, the role of shear deformations is significant and they have to be addressed in design. It also becomes clear that the beam models should consider the partial interaction between the layers if K takes small values.

11. The zig-zag linear variation of the normal stresses and the piecewise quadratic distribution of the tangential stresses over the composite cross section has been obtained. For $K \leq 0.1$ kN/cm^2, the maximum tangential stresses σ_{xz} may exceed the values obtained from the classical solid beam model for about 25%. It is apparent then that the classical solid beam model provides nonconservative estimates for the tangential and normal stresses in layered beams.

Acknowledgments

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