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Let B be a ring where $x = x^2$ for all x. Some immediate consequences are:

B is commutative:

Every element is its own additive inverse:

$$x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x$$
$$0 = x + x$$

Multiplication is commutative:

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$$
$$0 = xy + yx$$
$$xy = -yx = yx$$

Where the last identity follows from our previous result.

Now, define the following binary relation: $x \leq y$ iff $xy = x$. We get:

$$xx = x, \text{ so } x \leq x$$
$$xy = x, yz = y \to xz = x(yz) = (xy)z = yz = x, \text{ so } x \leq y, y \leq z \to x \leq z$$

$$xy = x, yx = y \to x = xy = yx = y, \text{ so } x \leq y, y \leq x \to x = y$$

So, this is a reflexive partial order. Furthermore,

$$0x = 0, x1 = x \to 0 \leq x, x \leq 1$$

So we have greatest and least elements. We also have least upper bounds and greatest upper bounds:

If $c \leq x, c \leq y$, then $c \leq xy \leq x, y$: By definition, $cx = c, cy = c$. Then,

$$cxy = cy = c \to c \leq xy$$
$$x(xy) = xy \to xy \leq x$$
$$y(xy) = (xy)y = xy \to xy \leq y$$

If $x \leq c, y \leq c$, $x, y \leq x + xy + y \leq c$: By definition, $xc = x, yc = y$. Then,

$$(x + y + xy)c = xc + yc + xyc = x + y + xy \to x + y + xy \leq c$$

$$(x+y+xy)x = x + yx + yx = x \to x \leq x+y+xy$$

$$(x+y+xy)y = xy + y + xy = y \to y \leq x+y+xy$$

Denoting the operations $x \frown y = xy, x \smile y = x + y + xy$, these operations a

This is object is called a boolean algebra. We will now proceed to define them by starting from a lattice and defining a ring structure. One advantage of that approach is that in order to verify a map between boolean algebras is a homomorphism, it will suffice to check that it respects the order relation ($x \leq y$ iff $f(x) \leq f(y)$). We will also need the definition based on rings to discuss ideals later.

A partial order with least upper bounds ('joins') and greatest lower bounds ('meets') is called a lattice; a lattice with a greatest and least element is called a bounded latice. A lattice where $\smile$ and $\frown$ distribute over each other is called a a distributive lattice, and a lattive where, for every x there's a x' with $x \smile x' = 1$, $x \frown x' = 0$ is called a complemented lattice. Such a complement is unique – let x', x'' be complements of x. Then,

$$\begin{aligned} x' \smile x'' &= (x' \smile x'') \frown 1 \\ &= (x' \smile x'') \frown (x \smile x'') \\ &= (x' \frown x) \smile x'' \\ &= 0 \smile x'' = x'' \end{aligned}$$

So, $x' \leq x''$, by applying the definition in reverse we get $x'' \leq x'$ so $x'\ x''$. In particular, this shows that the complement of the complement of x is x.

We can also use this to prove de Morgan's laws:

$$\begin{aligned} (x' \smile y') \frown (x \frown y) &= (x' \frown (x \frown y)) \smile (y' \frown (x \frown y)) \\ &= (0 \frown y) \smile (0 \frown x) = 0 \end{aligned}$$

$$\begin{aligned} (x' \smile y') \smile (x \frown y) &= ((x' \smile y') \smile x) \frown (x' \smile y') \smile y) \\ &= (1 \smile y') \frown (1 \smile x') = 1 \end{aligned}$$

So, $x' \smile y'$ is the complement of $x \frown y$:

$$(x' \smile y') = (x \frown y)'$$

Replacing x and y' by their respective complements, and taking the complement of both sides,

$$(x'' \smile y'')' = (x' \frown y')''$$

$$(x \smile y)' = x' \frown y'$$

To make this a ring, the multiplication is defined as $\frown$, and the addition is defined as:

$$x + y = (x \frown y') \smile (x' \frown y)$$

I'll omit the details, but this does form a ring, with $x^2 = x \frown x = x$, so a boolean algebra.

Our previous definition of $\smile$ is compatible:

$$A = \{X : X \subseteq \mathbb{N} : X \text{ or } \mathbb{N} - X \text{ is finite}\}$$