1

Let B be a ring where $x = x^2$ for all x. Some immediate consequences are:

B is commutative:

Every element is its own additive inverse:

$$x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x$$

$$0 = x + x$$

Multiplication is commutative:

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$$

$$0 = xy + yx$$

$$xy = -yx = yx$$

Where the last identity follows from our previous result.

Now, define the following binary relation: $x \leq y$ iff $xy = x$. We get:

$$xx = x, \text{ so } x \leq x$$

$$xy = x, yz = y \to xz = x(yz) = (xy)z = yz = x, \text{ so } x \leq y, y \leq z \to x \leq z$$

$$xy = x, yx = y \to x = xy = yx = y, \text{ so } x \leq y, y \leq x \to x = y$$

So, this is a reflexive partial order. Furthermore,

$$0x = 0, x1 = x \to 0 \leq x, x \leq 1$$

So we have greatest and least elements. We also have least upper bounds and greatest upper bounds:

If $c \leq x, c \leq y$, then $c \leq xy \leq x, y$: By definition, $cx = c, cy = c$. Then,

$$cxy = cy = c \to c \leq xy$$

$$x(xy) = xy \to xy \leq x$$

$$y(xy) = (xy)y = xy \to xy \leq y$$

If $x \leq c, y \leq c$, $x, y \leq x + xy + y \leq c$: By definition, $xc = x, yc = y$. Then,

$$(x + y + xy)c = xc + yc + xyc = x + y + xy \to x + y + xy \leq c$$

$$(x+y+xy)x = x + yx + yx = x \to x \leq x + y + xy$$

$$(x+y+xy)y = xy + y + xy = y \to y \leq x + y + xy$$

Denoting the operations $x \frown y = xy, x \smile y = x + y + xy$, these operations a