1

Let B be a ring where $x = x^2$ for all x. Some immediate consequences are:

B is commutative:

Every element is its own additive inverse:

$$x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x$$

$$0 = x + x$$

Multiplication is commutative:

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$$

$$0 = xy + yx$$

$$xy = -yx = yx$$

Where the last identity follows from our previous result.

Now, define the following binary relation: $x \leq y$ iff $xy = x$. We get:

$$xx = x, \text{ so } x \leq x$$

$$xy = x, yz = y \to xz = x(yz) = (xy)z = yz = x, \text{ so } x \leq y, y \leq z \to x \leq z$$

$$xy = x, yx = y \to x = xy = yx = y, \text{ so } x \leq y, y \leq x \to x = y$$

So, this is a reflexive partial order. Furthermore,

$$0x = 0, x1 = x \to 0 \leq x, x \leq 1$$

So we have greatest and least elements. We also have least upper bounds and greatest upper bounds:

If $c \leq x, c \leq y$, then $c \leq xy \leq x, y$: By definition, $cx = c, cy = c$. Then,

$$cxy = cy = c \to c \leq xy$$

$$x(xy) = xy \to xy \leq x$$

$$y(xy) = (xy)y = xy \to xy \leq y$$

If $x \leq c, y \leq c$, $x, y \leq x + xy + y \leq c$: By definition, $xc = x, yc = y$. Then,

$$(x + y + xy)c = xc + yc + xyc = x + y + xy \to x + y + xy \leq c$$

$$(x+y+xy)x = x+yx+yx = x \to x \leq x+y+xy$$

$$(x+y+xy)y = xy+y+xy = y \to y \leq x+y+xy$$

Denoting the operations $x \frown y = xy, x \smile y = x+y+xy$, these operations a

This is object is called a boolean algebra. We will now proceed to define them by starting from a lattice and defining a ring structure. One advantage of that approach is that in order to verify a map between boolean algebras is a homomorphism, it will suffice to check that it respects the order relation ($x \leq y$ iff $f(x) \leq f(y)$). We will also need the definition based on rings to discuss ideals later.

2

A partial order with least upper bounds ('joins') and greatest lower bounds ('meets') is called a lattice; a lattice with a greatest and least element is called a bounded latice. A lattice where $\smile$ and $\frown$ distribute over each other is called a a distributive lattice, and a lattive where, for every x there's a x' with $x \smile x' = 1$, $x \frown x' = 0$ is called a complemented lattice. Such a complement is unique – let x', x'' be complements of x. Then,

$$\begin{aligned} x' \smile x'' &= (x' \smile x'') \frown 1 \\ &= (x' \smile x'') \frown (x \smile x'') \\ &= (x' \frown x) \smile x'' \\ &= 0 \smile x'' = x'' \end{aligned}$$

So, $x' \leq x''$, by applying the definition in reverse we get $x'' \leq x'$ so $x'\ x''$. In particular, this shows that the complement of the complement of x is x.

We can also use this to prove de Morgan's laws:

$$\begin{aligned} (x' \smile y') \frown (x \frown y) &= (x' \frown (x \frown y)) \smile (y' \frown (x \frown y)) \\ &= (0 \frown y) \smile (0 \frown x) = 0 \end{aligned}$$

$$\begin{aligned} (x' \smile y') \smile (x \frown y) &= ((x' \smile y') \smile x) \frown (x' \smile y') \smile y) \\ &= (1 \smile y') \frown (1 \smile x') = 1 \end{aligned}$$

So, $x' \smile y'$ is the complement of $x \frown y$:

$$(x' \smile y') = (x \frown y)'$$

Replacing x and y' by their respective complements, and taking the complement of both sides,

$$(x'' \smile y'')' = (x' \frown y')''$$

$$(x \smile y)' = x' \frown y'$$

To make this a ring, the multiplication is defined as $\frown$, and the addition is defined as:

$$x + y = (x \frown y') \smile (x' \frown y)$$

I'll omit the details, but this does form a ring, with $x^2 = x \frown x = x$, so a boolean algebra.

Our previous definition of $\smile$ is compatible:

$$A = \{X : X \subseteq \mathbb{N} : X \text{ or } \mathbb{N} - X \text{ is finite}\}$$

3 homomorphisms and isomorphisms

TODO: incomplete, and necessary for the final result (at least the fact that $\leq$ determines an isomorphism is)

A homomorphism of boolean algebras is a homomorphism of rings that happen to be boolean albegras – maps h so that

$$h(x + y) = h(x) + h(y)$$
$$h(xy) = h(x)h(y)$$
$$h(1) = 1$$

The condition $h(0) = 0$ can be derived from:

$$h(0) = h(0 + 0) = h(0) + h(0) = 0$$

From this definition, we can easily prove

$$h(x \frown y) = h(xy) = h(x)h(y) = h(x) \frown h(y)$$
$$h(x') = h(x + 1) = h(x) + h(1) = h(x) + 1 = h(x)'$$
$$h(x \smile y) = h(x + y + xy) = h(x) + h(y) + h(x)h(y) = h(x) \smile h(y)$$
$$x \leq y \to xy = x \to h(xy) = h(x) \to h(x) \leq h(y)$$

It also suffices to specify

$$h(x \frown y) = h(x) \frown h(y)$$
$$h(x') = h(x)'$$

Since

$$x + y = (x \frown y') \smile (x' \frown y) = (x \frown y')' \frown (x' \frown y)'$$

An isomorphism of Boolean algebras is a bijective homomorphism. An inverse of an isomorphism is also a homomorhpism:

4 Ideals

In the following, 'ideal' means proper ideal, or an ideal which is not the whole ring. For a subset I of a Boolean algebra A to be an ideal, the following are necessary and sufficient:

$$0 \in I, 1 \notin I$$

$$\text{for all } x, y \in I, x \smile y \in I$$

$$\text{for all } x \in I, y \in A, y \leq x \text{ implies } y \in I$$

Proof:

Let I be an ideal. 0 is in I, since it's an additive subgroup. 1 isn't, since then, for every $a \in A$, $a * 1 = a$ is in I, then I would have to be the entire ring.

If x and y are in I, by definition, xy is too. Then, since I is an additive subgroup, $x \smile y = x + y + xy$ is as well.

Finally, if x is in I, $xy \in I$. $y \leq x$, by definition means $xy = y \in I$.

Conversely, let these conditions be satisfied.

If $x, y \in I$, $x \smile y \in I$.

$$(x+y)(x \smile y) = (x+y)((x+y)+xy) = x+y+(x+y)xy = x+y+xy+xy = x+y$$

So, $x + y \leq x \smile y$, and $x + y \in I$ by the third condition. Since $-x = x$ in a boolean algebra, we conclude that I is an additive subgroup.

For $x \in I$, $xy = x \frown y \leq x$, so $xy \in I$. So, I is an ideal. It's proper, since it doesn't include the element 1.

The equivalent conditions define what's called an ideal in the context of partially ordered sets. We've proven that they're equivalent to ideals for rings in Boolean algebras.

Then, the following are equivalent:

(1)I is a maximal ideal

(2)A/I is isomorphic to $\{0, 1\}$

(3)I is the kernel of a homomorphism $A \to \{0, 1\}$

(4)For all $x, x \in I$ or $x + 1 \in I$

(5)For all $x, y,$ if $xy \in I,$ then $x \in I$ or $y \in I$

(6)For all $x_1 x_2 ... x_k \in I, x_1 \in I$ or $x_2 \in I$ or $... x_k \in I$

(1) $\to$ (2)

In fact, in any commutative ring, we can prove that A/I is a field if and only if I is maximal:

If I isn't maximal, let IJ, and $a \in J - I$. a isn't in I, so $[a]$ isn't the zero element in A/I. If it was invertible, there would be a b with

$$[a][b] = 1$$

$$[ab-1]=0$$

$$ab-1 \in I$$

$a \in I$, so $ab \in I$. Then,

$$ab-(ab-1)=1 \in I$$

This is a contradiction.

Conversely, assume I is maximal. Let $[a] \neq 0 \in A/I$. Form

$$K=\{ay+z : y \in A, z \in I\}$$

Clearly, $0=a*0+0 \in K$, if ay_1+z_1, $ay_2+z_2 \in K$, since the ring is commutative,

$$ay_1+z_1-(ay_2+z_2)=a(y_1-y_2)+(z_1-z_2) \in K$$

And, if $ay+z$ ∈K, $(ay+z)t=a(yt)+(zt) \in K$So, K is an ideal. It includes any element $a*0+y$ of I, and it includes $a*1+0=a$, which isn't in I. Since I is maximal, K must be the whole ring, and then, there must be y, z with

$$ay+z=1$$

Then,

$$ay+z=[a][y]+[z]=[a][y]=1$$

Note that z ∈I.$Then, any nonzero element of$ A/I$does have an inverse, and this proof is complete.$

Finally, note that $\{0,1\}$ is the only Boolean algebra which is a field: For every x,

$$(x+1)x=x+x=0$$

Which, if this is a field (or even an integral domain), implies that either $x=0$ or $x=1$.

(2) $\to$ (3) I is the kernel of the quotient map $A \to A/I=\{0,1\}$.

(3) $\to$ (4) Let h be the homomorphism $A \to \{0,1\}$ with kernel I. Then, for all x, $h(x)=0$ or $h(x)=1$. In the latter case, $h(1+x)=h(1)+h(x)=1+1=0$. So, x or $1+x$ is in the kernel, which is I.

(4) $\to$ (5) Let $x, y \notin I$. Then, $1+x, 1+y \in I$. Then,

$$(1+x) \smile (1+y)=1+(x \frown y)=1+xy \in I$$

So, $xy \in I$ implies $x \in I$ or $y \in I$.

(5) $\to$ (1) Suppose I isn't maximal. Then, let IJ be an ideal, $a \in J-I$. $a \in J$, so $1+a \notin J$, because then

$$a \smile (1+a)=1 \in J$$

But, $(1+a) \in I \subseteq J$, which is a contradiction.

(5) $\leftrightarrow$ (6) This is an obvious use of induction.

5 Filters

Filters in order theory are the dual notion of ideals:

A filter is a subset F of a Boolean algebra A such that:

$$0 \notin F, 1 \in F$$

$$x, y \in F \to x \frown y \in F$$

$$x \in F, y \geq x \to y \in F$$

In boolean algebras, we can equivalently define them as subsets F such that $\{x \in A : 1 + x \in F\}$ is a filter.

Proof:

Let

$$I = \{x \in A : 1 + x \in F\}$$

Assume the first set of conditions hold.

The first condition implies $0 \in I, 1 \notin I$.

The second condition implies for $x, y \in I$, $(1+x), (1+y) \in F$, then

$$(1+x) \frown (1+y) \in F$$

$$1 + (x \smile y) \in F$$

$$x \smile y \in I$$

Finally, notice that $x \leq y$ imples $1 + y \leq 1 + x$:

$$xy = x$$

$$(1+x)(1+y) = 1 + x + y + xy = 1 + x + y + x = 1 + y$$

Then, if $x \in I$, $y \leq x$, $1 + x \in F$, $1 + y \geq 1 + x$, so by the third condition, $1 + y \in F$, and $y \in I$. So, I is an ideal as we have defined.

Conversely, if I is an ideal, the proof is entirely the same:

The first condition is given by $0 \in I, 1 \notin I$.

If $x, y \in F$, $1 + x, 1 + y \in I$, then

$$(1+x) \smile (1+y) \in I$$

$$1 + (x \frown y) \in F$$

$$x \frown y\ inF$$

And finally, if $x \in F$, $y \geq x$,

$$1 + x \in I, 1 + y \leq 1 + x \to 1 + y \in I$$

$$y \in F$$

We also have a similar characterization for maximal filters, which are called ultrafilters. The following are equivalent:

$$(1')F \text{ is an ultrafilter}$$

$$(3') \text{ there's a homomorphism } g : A \to \{0,1\} \text{ with } F = h^{-1}(\{1\})$$

$$(4') \text{ for all x}, x \in F \text{ or } 1 + x \in F$$

$$(5') \text{ if } x \smile \in F, \text{ then } x \in F \text{ or } y \in F$$

$$(6') \text{ if } x_1 \smile ... \smile x_n \in F, \text{ then } x_1 \in F \text{ or } ...x_n \in F$$

We'll prove this be proving that if and only if F has one of these properties, the dual filter I as defined above has the corresponding property.

(1′) Saying there's no filter $F'F$ is equivalent to saying there's no ideal $I'I$, since a set is a filter if and only if the set of its complements is an ideal and vice versa.

(3′) For a boolean algebra homomorphism, $h(x + 1) = h(x) + 1$, so a homomorphism sending F to 1 is equivalent to a homomorphism sending I to 0.

(4′) This is obvious by the definition of $I - x \in I$ if and only if $x + 1 \in F$, etc.

(5′) If this condition is true, for $x \frown y \in I$, $1 + x \frown y = (1 + x) \smile (1 + y) =$ is in F, so $1 + x$ or $1 + y$ is in F, so x or y is in I.

Similarly, if the corresponding statement is true for I, let $x \smile y \in F$.

$$I \ni 1 + (x \smile y) = (1 + x) \frown (1 + y)$$

$$1 + x \in I \text{ or } 1 + y \; inI$$

And x or y is in F.

Finally, (6′) may be shown to be equivalent to (5′) by induction.

One fact that will be especially impactful later is that we've proven in properties (3) and (3′) that every homomorphism $A \to \{0, 1\}$ corresponds bijectively to a maximal ideal and an ultrafilter.

5.a Filterbases

skipped for now – I'd be copying the book too closely. we'll see what we need later.

6 Topological Spaces

6.a An equivalent condition for compactness

It will be convenient later to rephrase the definition of compactness as follows.

The usual definition states that every open cover of a topological space has a finite subcover. We'll also assume the topological space has to be Hausdorff. Consider the following condition:

Every open cover of a topological space by sets in some basis has a finite subcover.

Clearly the usual definition implies this; conversely, if this is true, for any open cover $\{U_i\}_{i \in I}$, we can form an open cover by basis elements by finding a cover $\{V_{ij}\}_{j \in J_i}$ of every U_i by basis elements, find a finite subcover $\{V_{i_k j_k}\}_{k=1}^{n}$ and then pick the open set these basic sets were in in the original cover:

$$\{U_{i_k}\}_{k=1}^{n}$$

This covers the topological space, since $\{V_{i_k j_k}\}_{k=1}^{n}$ did as well.

Finally, we can state this condition by equivalently talking about the complements:

For every set of (closed) complements of basic open sets whose total intersection is empty, we can find a finite subset whose intersection is still empty.

6.b Zero dimensional topological spaces

A topological space is said to be zero dimensional if it has a basis consisting of closed sets. We won't get into a definition of dimension in general for topological spaces here.

We can equivalently say that the family of sets which are both open and closed ('clopen') form a basis for the topology. This clearly implies the previous condition, and if the previous statement is true, the basis consisting of closed sets is a subset of the set of clopen sets, so the latter forms a basis as well.

A compact zero dimensional topological space is called a Boolean topological space.

Given an indexed family of topological spaces $(X_i)_{i \in I}$, we can define a topology on their product $\Pi_{i \in I} X_i$ – elements $\Pi_{i \in I} O_i$, where all the O_i are open in X_i and all but finitely many ones are equal to X_i. Tychonoff's Theorem, which is equivalent to the axoim of choice, states that if all the X_i are compact, this product space is too.

Take the discrete topology on two points $\{0, 1\}$, and form the product space $\{0, 1\}^I$ for some indexing set I. A basic open set Ω is either empty (in case when one of the open sets is the empty set), or has a finite number of indices which are either $\{0\}$ or $\{1\}$. Where the element of the product space are written as functions $I \to \{0, 1\}$, it's the set of functions whose values at a finite number of points is determined:

$$\{f : I \to \{0, 1\} : f(i_1) = \epsilon_1, ...f(i_n) = \epsilon_n\}$$

Where n is some natural number and each ϵ_i is either 0 or 1. By De Morgan's Law, the complement of this set is

$$\bigcup_{1 \leq j \leq n} \{f : f(i_j) = 1 - \epsilon_j\}$$

This is a finite union of open sets in the basis, so it's open. Then, Ω is closed as well. Since every set in the basis is clopen, this is a zero dimensional topological set. $\{0,1\}$ is clearly compact. By Tychonoff's Theorem, this space is compact as well, and therefore a Boolean topological space.

6.c The Stone space of a Boolean algebra

For a Boolean algebra A, the subset of $\{0,1\}^A$ which consists of homomorphisms $A \to \{0,1\}$ is denoted $S(A)$ and called the Stone space of A.

We've seen that $\{0,1\}^A$ is zero dimensional, so we can find a basis by clopen sets. With $S(A)$ under the subspace topology, the same basis forms a basis of clopen sets, so $S(A)$ is zero dimensional as well.

Now, we'll characterize the basis of $S(A)$. Note that we've characterized the basis of $\{0,1\}^A$ as functions which take a finite set of points to either 0 or 1.

The sets in the basis of $S(A)$ are exactly the sets defined as

$$\{h \in S(A) : h(a) = 1\}$$

for some $a \in A$. This element a is unique.

Clearly, every set defined this way is a set in the basis – $S(A) \cap \{f : A \to \{0,1\} : f(a) = 1\}$.

Every set in the basis can be defined this way:

Let Δ be an arbitrary element in the basis, given by

$$\Delta = \{h \in S(A) : h(a_1) = \epsilon_1, ...h(a_n) = \epsilon_n\}$$

Where $a_i \in A$, and ϵ_i are either zero or one. Define:

$$b_k = a_k \text{ if } \epsilon_k = 1, 1 + a_k \text{ otherwise}$$

Since the elements of $S(A)$ are homomorphisms, if $\epsilon_k = 1$,

$$h(b_k) = h(a_k)$$

and, if $\epsilon_k = 0$,

$$h(b_k) = h(1 + a_k) = 1 + h(a_k)$$

In other words, $h(b_k) = 1$ if and only if $h(a_k) = \epsilon_k$. Then, $h(a_1) = \epsilon_1, h(a_2) = \epsilon_2, ...h(a_n) = \epsilon_n$ if and only if

$$h(b_1) \frown h(b_2) \frown ... \frown h(b_n) = 1$$

Since h also preserves intersections, this happens if and only if

$$h(b_1 \frown b_2 \frown ... \frown b_n) = 1$$

As maybe a special case, notice that the empty set and $S(A)$ are given by $\{h \in S(A) : h(0) = 1\}$ and $\{h \in S(A) : h(1) = 1\}$ respectively.

For uniqueness, !!! requires principal filters + ultrafilter theorem

As a corollary, the complements of the sets in the basis are exactly the sets in the basis:

$$\{h \in S(A) : h(a) = 1\}^c = \{h \in S(A) : h(1 + a) = 1\}$$

Finally, here's a proof that the $S(A)$ is closed. Remember that a function between Boolean algebras is a homomorphism if and only if it preserves intersections and complements.

Let

$$\Omega(a, b) = \{f : A \to \{0, 1\} : f(ab) = f(a)f(b), f(1 + a) = 1 + f(a)\}$$

Then, by definition,

$$S(A) = \bigcap_{a \in A, b \in A} \Omega(a, b)$$

Finally, we can write $\Omega(a, b)$ as the following union of closed sets:

$$\Omega(a, b) = \{f : f(a) = 0, f(b) = 0, f(ab) = 0), f(1+a) = 1\} \cup \{f : f(a) = 0, f(b) = 1, f(ab) = 0), f(1+a) = 1\} \cup \{f$$

So, $S(A)$ is some arbitrary intersection of closed sets, and is therefore closed itself. As a closed subset of a compact space, it's compact. Then, since it's also zero-dimensional, it's a Boolean topological space.

We can also show that the clopen sets are exactly the sets in this basis.

Every basis set we've found is already both open and closed. Given an open and closed set Γ,

Since it's open, it's covered by sets in the basis $\{\Gamma_i\}_{i \in I}$ Since Γ is a closed subset of a compact space, it's compact. Then, find a finite subcover $\Gamma_1, ...\Gamma_n$. Each has form:

$$\Gamma_i = \{h : h(x_i) = 1\}$$

Then,

$$\Gamma = \{h : h(x_1 \smile x_2 \smile ... \smile x_n)\}$$

7

The Boolean algebra of clopen subsets of a topological space S will be denoted by $B(S)$. This is a Boolean algebra with regular set operations union, intersection, complement by S, and is clearly closed under all of these.

We'll now prove that every Boolean algebra is isomorphic to the Boolean algebra of clopen sets of its Stone space.

Let H be the map $A \to \mathcal{P}(S(A))$

$$H(a) = \{h \in S(A) : h(a) = 1\}$$

We've done a lot of the work in the previous section: We know that $H(a)$ is a clopen subset of $S(A)$ for any a, and that any clopen subset is in the image, so the map is surjective onto $B(S(A))$. We also know that this map is injective, since we've seen that a is unique.

To show that it's an isomorphism, we'll show that $H(a) \leq H(b)$ if and only if $a \leq b$.

If $x \leq y$, and h is any homomorphism with $h(x) = 1$, $h(x) \leq h(y)$ implies that $h(y) = 1$. Then, $H(x) \subseteq H(y)$.

Now, assume x is not less than or equal to y. Then, $xy \neq x$, $x(1+y) \neq 0$. Consider the ultrafilter which includes this element, and the homomorphism associated with it.

$$h(x(1+y)) = 1$$

So

$$h(x) = 1, h(y) = 0$$

And then, $h \in H(x)$, $h \notin H(y)$, and $H(x)H(y)$.

We can also now prove that every Boolean topological space X is homeomorphic to the Stone space of the Boolean algebra of clopen subsets of X.

We've seen that the Boolean algebra of clopen sets of X is a basis for the topology on any Boolean topological space. For all $x \in X$, define

$$f_x : B(X) \to \{0,1\} = \Omega \mapsto \{1 \text{if} x \in \Omega, 0 \text{otherwise}\}$$

First, we'll show that f_x is really a homomorphism of Boolean algebras, so in $S(B(X))$.

For clopen subsets Ω, Δ,

$$f_x(\Omega \cap \Delta) = 1 \text{ iff } x \in \Omega \cap \Delta$$

$$= 1 \iff f_x(\Omega) = 1 \text{ and } f_x(\Delta) = 1$$
$$= f_x(\Omega) f_x(\Delta)$$

$$f_x(\Omega^c) = 1 \text{ iff } x \in \Omega^c$$
$$= 0 \text{ iff } x \in \Omega$$
$$= f_x(\Omega)^c$$

So, f_x is a Boolean algebra homomorphism from $B(X)$ onto $\{0,1\}$, so is in $S(B(X))$.

Now, we'll show that it's injective.

If $x \neq y$, because X is Hausdorrf, we can find an open set O with $x \in O$, $y \notin O$. O is a union of basic open sets, so there's a clopen set $\Omega \in B(X)$ with $x \in \Omega$, $y \notin \Omega$. Then,

$$f_x(\Omega) = 1 \neq f_y(\Omega) = 0$$

So, $f_x \neq f_y$, and f is injective.

Now, we'll show that it's surjective.

Let h be an arbitrary element of $S(B(X))$, so a homomorphism $B(X) \to \{0,1\}$.

To Do:

– section on homomorphisms + isomorphisms – principal filter / ultrafilter – organize better – x + 1 for complements isn't great, introduce + use x^c notation

– future content: – equivalent forms of aoc – atomless + countable boolean algebra