Seifert – Van Kampen Theorem, Applications

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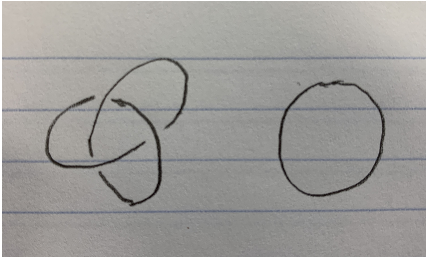
Knots

- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).



- A knot K is equivalent to K' if $(K, \mathbb{R}^3)$ is homeomorphic to $(K', \mathbb{R}^3)$
  In other words, there's a homeomorphism

$$h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ so that } h(K) = K'$$

  Then,

$$h|_{\mathbb{R}^3 - K} : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$$

  is also a homeomorphism,

- ... and its induced homeomorphism from the fundemental groups

$$\pi_1(\mathbb{R}^3 - K) \to \pi_1(\mathbb{R}^3 - K')$$

  is an isomorphism, so those groups are isomorphic.
- The group $\pi_1(\mathbb{R}^3) - K$ is also called the fundemental group of a knot.
- The main argument we'll be making is, if the fundemental groups of two knots aren't isomorphic, then the knots aren't equivalent.
- Seifert – Van Kampen's Theorem helps determine the fundemental group of $A \cup B$ given the fundemental groups of A, B, and $A \cap B$.
- In this presentation, we'll be looking at the statement and proof of this theorem, and applying it to find the fundemental groups of a few knots.

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. Clearly, the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

From this definition, we can tell:

- If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$
- If $\alpha \in \pi_1(U_\lambda)$, $\beta \in \pi_1(U_\mu)$,

$$\sigma(\psi_\lambda(\alpha)\psi_\mu(\beta)) = \sigma(\psi_\lambda(\alpha))\sigma(\psi_\mu(\beta)) = p_\lambda(\alpha)p_\mu(\beta)$$

- For $\{\alpha_i\}_{i=1}^n$ so that $\alpha_i \in U_{\lambda_i}$,

$$\sigma(\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_1)\ldots\psi_{\lambda_n}(\alpha_n)) = p_{\lambda_1}(\alpha_1)p_{\lambda_2}(\alpha_2)\ldots p_{\lambda_n}(\alpha_n)$$

- We need to prove that σ is well defined, In other words, if

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\ldots\psi_{\lambda_n}(\alpha_n) \sim \psi_{\mu_1}(\beta_1)\psi_{\mu_2}(\beta_2)\ldots\psi_{\mu_m}(\beta_m)$$

  Then, $\sigma(\psi_{\lambda_1}(\alpha_1)\ldots\psi_{\lambda_n}(\alpha_n)) \sim \sigma(\psi_{\mu_1}(\beta_1)\ldots\psi_{\mu_m}(\beta_m))$
  So, $p_{\lambda_1}(\alpha_1)\ldots p_{\lambda_n}(\alpha_n) \sim p_{\mu_1}(\beta_1)\ldots p_{\mu_m}(\beta_m)$
- Since this is all the restrictions on σ, but σ is unique, $\pi_1(X)$ must not have any elements which aren't in the form

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\ldots\psi_{\lambda_n}(\alpha_n)$$

  We also need to prove this.
- We'll also look at when two elements of $\pi_1(X)$ are equal and when they're different.
- But hopefully it makes sense how this theorem determines $\pi_1(X)$!

Seifert – Van Kampen Theorem Proof – Part 1

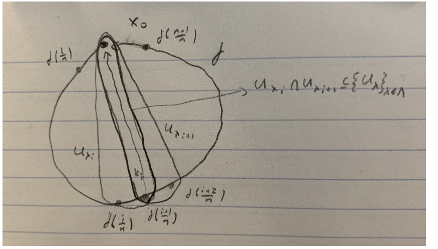
- To Prove: Every element of $a\ \pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\dots\psi_{\lambda_n}(\alpha_n)$$

  for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
  Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0,1] \to X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is a cover of the compact metric space $[0,1]$. It has a Lebesgue number δ.
- Find n so $\frac{1}{n} < \delta$, divide $[0,1]$ into subintervals $[0,\frac{1}{n}]$, $[\frac{1}{n},\frac{2}{n}]$, ..., $[\frac{n-1}{n},1]$. Each has diameter less than δ, so $[\frac{i}{n},\frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i, and $f([\frac{i}{n},\frac{i+1}{n}]) \in U_{\lambda_i}$.
- Let f_i be f from $f(\frac{i-1}{n})$ to $f(\frac{i}{n})$. So,

$$f \sim f_1 f_2 f_3 \dots f_n$$

- $f(\frac{i}{n}) \in U_{\lambda_i}, U_{\lambda i+1}$. Since $U_{\lambda_i} \cap U_{\lambda_{i+1}} \in \{U_\lambda\}_{\lambda\in\Lambda}$, and all elements of $\{U_\lambda\}_{\lambda\in\Lambda}$ are path connected and include x_0, there's a path k_i from $f(\frac{i}{n})$ to x_0 contained in $U_{\lambda_i} \cap U_{\lambda_{i+1}}$.



- We add the k_i to put each small piece starts and ends at x_0, and so is in a fundemental group:

$$f \sim f_1 k_1 \cdot k_1^{-1} f_2 k_2 \cdot k_2^{-1} f_3 k_3 \cdot \dots \cdot k_{n-1}^{-1} f_n$$

$$a = [f]_{\pi_1(X)} = [f_1 k_1]_{\pi_1(X)}[k_1^{-1} f_2 k_2]_{\pi_1(X)} \dots [k_{n-1}^{-1} f_n]_{\pi_1(X)}$$

  Now, $k_{i-1} f_i k_i \subseteq U_{\lambda_i}$, since $k_i \subseteq U_{\lambda_i}, U_{\lambda_{i+1}}$. Since ψ_{λ_i} is the homomorphism induced by an inclusion map,

$$a = \psi_{\lambda_1}([f_1 k_1]_{\pi_1(U_{\lambda_1})}) \dots \psi_{\lambda_n}([k_{n-1} f_n]_{\pi_1(U_{\lambda_n})})$$

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

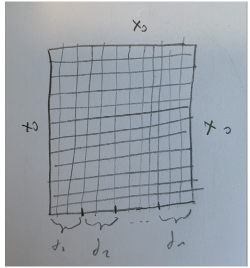
Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

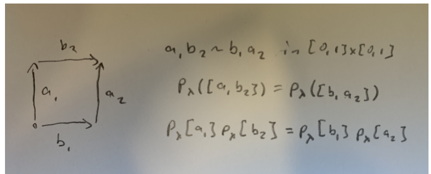
So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$
Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$. If $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$, there's a continuous function $F : [0,1]x[0,1] \to X$ with $F(1,t) = F(s,0) = F(s,1) = x_0$,

$$F(0,t) = \begin{cases} f_1 & [0, \frac{1}{n}) \\ f_2 & [\frac{1}{n}, \frac{2}{n}) \\ \dots & \end{cases}$$

- Using the Lebesgue number, split up $[0,1]x[0,1]$ into rectangles so each fits in a single U_λ, making sure that each $\frac{i}{n}$ is at a boundary:

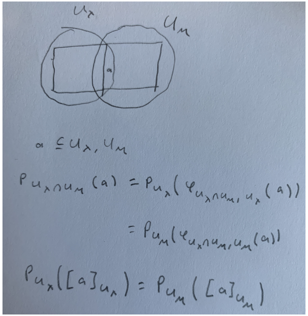


- For each intersection, add a line k_{ij} going from the intersection to x_0, contained in the U_λ's of all four surrounding rectangles.
- In each line below, add k_{ij}'s as necessary to put them in the fundemental group:
For each rectangle, since it's contained in U_λ,

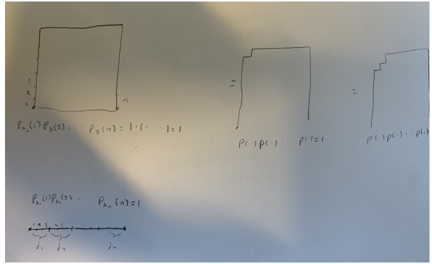


- Now, remember that

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$



- Notice that along the left, top, and right edges, p of the trivial loops is 1, so the whole composition is 1.
- Now, we can apply the previous two parts a finite number of times to move that composition to the bottom without changing its value:



- Finally, since each f_i is contained in a U_{μ_i},

$$\underbrace{p_{\lambda_1}([1]_{U_{\lambda_1}})p_{\lambda_2}([2]_{U_{\lambda_2}})p_{\lambda_3}([3]_{U_{\lambda_3}})}_{f_1}\underbrace{p_{\lambda_4}([4]_{U_{\lambda_4}})p_{\lambda_5}([5]_{U_{\lambda_5}})}_{f_2}\ldots = 1$$

$$\underbrace{p_{\mu_1}([1]_{U_{\mu_1}})p_{\mu_1}([2]_{U_{\mu_1}})p_{\mu_1}([3]_{U_{\mu_1}})}_{f_1}\underbrace{p_{\mu_2}([4]_{U_{\mu_2}})p_{\mu_2}([5]_{U_{\mu_2}})}_{f_2}\ldots = 1$$

$$\underbrace{p_{\mu_1}([123]_{U_{\mu_1}})}_{f_1}\underbrace{p_{\mu_2}([45]_{U_{\mu_2}})}_{f_2}\ldots = 1$$

- Since we know F maps those sections to f_i,

$$p_{\mu_1}([f_1])p_{\mu_2}([f_2])\ldots p_{\mu_n}([f_n]) = 1$$