Seifert – Van Kampen Theorem, Applications

Yahya Tamur

March 7, 2022

Groups; generators and relations

- A group is a set of elements G, with a $.: GxG \to G$ such that:

$$(a.b).c = a.(b.c)$$

There's an identity element 1 such that for all a,

$$1.a = a.1 = a$$

For all a, there's an inverse element a^{-1} such that:

$$a^{-1}.a = a.a^{-1} = 1$$

- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).

Groups; generators and relations

- A group is a set of elements G, with a $. : GxG \to G$ such that:

$$(a.b).c = a.(b.c)$$

  There's an identity element 1 such that for all a,

$$1.a = a.1 = a$$

  For all a, there's an inverse element a^{-1} such that:

$$a^{-1}.a = a.a^{-1} = 1$$

- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.

Groups; generators and relations

- A group is a set of elements G, with a $. : GxG \to G$ such that:

$$(a.b).c = a.(b.c)$$

There's an identity element 1 such that for all a,

$$1.a = a.1 = a$$

For all a, there's an inverse element a^{-1} such that:

$$a^{-1}.a = a.a^{-1} = 1$$

- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- Another example is the one-element group $\{1\}$ with $1.1 = 1$.

Groups; generators and relations

- A group is a set of elements G, with a . : $GxG \to G$ such that:

$$(a.b).c = a.(b.c)$$

There's an identity element 1 such that for all a,

$$1.a = a.1 = a$$

For all a, there's an inverse element a^{-1} such that:

$$a^{-1}.a = a.a^{-1} = 1$$

- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- Another example is the one-element group $\{1\}$ with $1.1 = 1$.
- A function between groups that preserves products, identity, and inverses – $\sigma(a.b) = \sigma(a).\sigma(b)$, $\sigma(1) = 1$, $\sigma(a^{-1}) = \sigma(a)^{-1}$ is called a homomorphism. These have other special properties that can be derived.

Groups; generators and relations

- A group is a set of elements G, with a $. : GxG \to G$ such that:

  $$(a.b).c = a.(b.c)$$

  There's an identity element 1 such that for all a,

  $$1.a = a.1 = a$$

  For all a, there's an inverse element a^{-1} such that:

  $$a^{-1}.a = a.a^{-1} = 1$$

- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- Another example is the one-element group $\{1\}$ with $1.1 = 1$.
- A function between groups that preserves products, identity, and inverses – $\sigma(a.b) = \sigma(a).\sigma(b)$, $\sigma(1) = 1$, $\sigma(a^{-1}) = \sigma(a)^{-1}$ is called a homomorphism. These have other special properties that can be derived.
- A set $a, b, ...$ generates a group G if every element can be written in as a finite combination of elements (or inverses of elements) in the group. Ex. 1 generates integers under addition, prime numbers generate positive rationals under multiplication.

Groups; generators and relations

- A group is a set of elements G, with a $. : GxG \to G$ such that:
$$(a.b).c = a.(b.c)$$
There's an identity element 1 such that for all a,
$$1.a = a.1 = a$$
For all a, there's an inverse element a^{-1} such that:
$$a^{-1}.a = a.a^{-1} = 1$$
- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- Another example is the one-element group $\{1\}$ with $1.1 = 1$.
- A function between groups that preserves products, identity, and inverses – $\sigma(a.b) = \sigma(a).\sigma(b)$, $\sigma(1) = 1$, $\sigma(a^{-1}) = \sigma(a)^{-1}$ is called a homomorphism. These have other special properties that can be derived.
- A set $a, b, \ldots$ generates a group G if every element can be written in as a finite combination of elements (or inverses of elements) in the group. Ex. 1 generates integers under addition, prime numbers generate positive rationals under multiplication.
- Every group has a generator – the set of all elements in the group.

Groups; generators and relations

- A group is a set of elements G, with a $.: GxG \to G$ such that:

$$(a.b).c = a.(b.c)$$

There's an identity element 1 such that for all a,

$$1.a = a.1 = a$$

For all a, there's an inverse element a^{-1} such that:

$$a^{-1}.a = a.a^{-1} = 1$$

- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- Another example is the one-element group $\{1\}$ with $1.1 = 1$.
- A function between groups that preserves products, identity, and inverses – $\sigma(a.b) = \sigma(a).\sigma(b)$, $\sigma(1) = 1$, $\sigma(a^{-1}) = \sigma(a)^{-1}$ is called a homomorphism. These have other special properties that can be derived.
- A set $a, b, \ldots$ generates a group G if every element can be written in as a finite combination of elements (or inverses of elements) in the group. Ex. 1 generates integers under addition, prime numbers generate positive rationals under multiplication.
- Every group has a generator – the set of all elements in the group.
- Given a set of elements, you can define the group of finite sequences of elements (and inverses of elements) in the set under concatenation. For example:

$$(a, b, c^{-1}).(c, a^{-1}) = (a, b, a^{-1})$$

$$(a, b, c).(a, b, c)^{-1} = (a, b, c).(c^{-1}, b^{-1}, a^{-1}) = ()$$

This is called the free group of the set.

Groups; generators and relations

- A group is a set of elements G, with a $. : GxG \to G$ such that:
$$(a.b).c = a.(b.c)$$
There's an identity element 1 such that for all a,
$$1.a = a.1 = a$$
For all a, there's an inverse element a^{-1} such that:
$$a^{-1}.a = a.a^{-1} = 1$$
- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- Another example is the one-element group $\{1\}$ with $1.1 = 1$.
- A function between groups that preserves products, identity, and inverses – $\sigma(a.b) = \sigma(a).\sigma(b)$, $\sigma(1) = 1$, $\sigma(a^{-1}) = \sigma(a)^{-1}$ is called a homomorphism. These have other special properties that can be derived.
- A set $a, b, ...$ generates a group G if every element can be written in as a finite combination of elements (or inverses of elements) in the group. Ex. 1 generates integers under addition, prime numbers generate positive rationals under multiplication.
- Every group has a generator – the set of all elements in the group.
- Given a set of elements, you can define the group of finite sequences of elements (and inverses of elements) in the set under concatenation. For example:
$$(a, b, c^{-1}).(c, a^{-1}) = (a, b, a^{-1})$$
$$(a, b, c).(a, b, c)^{-1} = (a, b, c).(c^{-1}, b^{-1}, a^{-1}) = ()$$
This is called the free group of the set.
- Then, if you have any group that's generated by that set, there's a surjective homomorphism from the free group onto the group, defined as
$$\sigma((x)) = x$$
and extended to other elements using properties of homomorphisms:
$$\sigma((a, b, c, a^{-1})) = \sigma((a)).\sigma((b)).\sigma((c)).\sigma((a^{-1})) = a.b.c.a^{-1}$$

Groups; generators and relations

- A group is a set of elements G, with a $. : GxG \to G$ such that:
$$(a.b).c = a.(b.c)$$
There's an identity element 1 such that for all a,
$$1.a = a.1 = a$$
For all a, there's an inverse element a^{-1} such that:
$$a^{-1}.a = a.a^{-1} = 1$$
- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- Another example is the one-element group $\{1\}$ with $1.1 = 1$.
- A function between groups that preserves products, identity, and inverses – $\sigma(a.b) = \sigma(a).\sigma(b)$, $\sigma(1) = 1$, $\sigma(a^{-1}) = \sigma(a)^{-1}$ is called a homomorphism. These have other special properties that can be derived.
- A set $a, b, \ldots$ generates a group G if every element can be written in as a finite combination of elements (or inverses of elements) in the group. Ex. 1 generates integers under addition, prime numbers generate positive rationals under multiplication.
- Every group has a generator – the set of all elements in the group.
- Given a set of elements, you can define the group of finite sequences of elements (and inverses of elements) in the set under concatenation. For example:
$$(a, b, c^{-1}).(c, a^{-1}) = (a, b, a^{-1})$$
$$(a, b, c).(a, b, c)^{-1} = (a, b, c).(c^{-1}, b^{-1}, a^{-1}) = ()$$
This is called the free group of the set.
- Then, if you have any group that's generated by that set, there's a surjective homomorphism from the free group onto the group, defined as
$$\sigma((x)) = x$$
and extended to other elements using properties of homomorphisms:
$$\sigma((a, b, c, a^{-1})) = \sigma((a)).\sigma((b)).\sigma((c)).\sigma((a^{-1})) = a.b.c.a^{-1}$$
- By the First Isomorphism Theorem, any group is the free group of its generators quotiented by the kernel of this homomorphism.

Groups; generators and relations

- A group is a set of elements G, with a $. : GxG \to G$ such that:
$$(a.b).c = a.(b.c)$$
There's an identity element 1 such that for all a,
$$1.a = a.1 = a$$
For all a, there's an inverse element a^{-1} such that:
$$a^{-1}.a = a.a^{-1} = 1$$
- Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- Another example is the one-element group $\{1\}$ with $1.1 = 1$.
- A function between groups that preserves products, identity, and inverses – $\sigma(a.b) = \sigma(a).\sigma(b)$, $\sigma(1) = 1$, $\sigma(a^{-1}) = \sigma(a)^{-1}$ is called a homomorphism. These have other special properties that can be derived.
- A set $a, b, \ldots$ generates a group G if every element can be written in as a finite combination of elements (or inverses of elements) in the group. Ex. 1 generates integers under addition, prime numbers generate positive rationals under multiplication.
- Every group has a generator – the set of all elements in the group.
- Given a set of elements, you can define the group of finite sequences of elements (and inverses of elements) in the set under concatenation. For example:
$$(a, b, c^{-1}).(c, a^{-1}) = (a, b, a^{-1})$$
$$(a, b, c).(a, b, c)^{-1} = (a, b, c).(c^{-1}, b^{-1}, a^{-1}) = ()$$
This is called the free group of the set.
- Then, if you have any group that's generated by that set, there's a surjective homomorphism from the free group onto the group, defined as
$$\sigma((x)) = x$$
and extended to other elements using properties of homomorphisms:
$$\sigma((a, b, c, a^{-1})) = \sigma((a)).\sigma((b)).\sigma((c)).\sigma((a^{-1})) = a.b.c.a^{-1}$$
- By the First Isomorphism Theorem, any group is the free group of its generators quotiented by the kernel of this homomorphism.

Groups; generators and relations

- A presentation of a group by its generators and relations is essentially the free group, together with a few equivalences:

Groups; generators and relations

- A presentation of a group by its generators and relations is essentially the free group, together with a few equivalences:
- $\langle a, b : a^5, b^3 \rangle$ means the free group $\langle a, b \rangle$, but a^5 and b^3 also have to be equal to the identity.

Groups; generators and relations

- A presentation of a group by its generators and relations is essentially the free group, together with a few equivalences:
- $\langle a, b : a^5, b^3 \rangle$ means the free group $\langle a, b \rangle$, but a^5 and b^3 also have to be equal to the identity.
- This can be expressed as $\langle a, b \rangle$ divided by the smallest normal subgroup that contains a^5 and b^3.

Groups; generators and relations

- A presentation of a group by its generators and relations is essentially the free group, together with a few equivalences:
- $\langle a, b : a^5, b^3 \rangle$ means the free group $\langle a, b \rangle$, but a^5 and b^3 also have to be equal to the identity.
- This can be expressed as $\langle a, b \rangle$ divided by the smallest normal subgroup that contains a^5 and b^3.
- If G is any group generated by a and b, with $a^5 = b^3 = 1$, we can try to define a homomorphism

$$\sigma((a)) = a, \sigma((b)) = b$$

And, this is well-defined, even though some sequences might be equivalent in $\langle a, b : a^5, b^3 \rangle$ without being equal, because whenever terms cancel out in $\langle a, b : a^5, b^3 \rangle$, they cancel out in G:

$$(a, a, a, b, b, b, a, a) = (a, a, a, a, a) = 1$$

$$\sigma((a, a, a, b, b, b, a, a)) = aaabbbaa \in G = aaaaa = 1$$

Groups; generators and relations

- A presentation of a group by its generators and relations is essentially the free group, together with a few equivalences:
- $\langle a, b : a^5, b^3 \rangle$ means the free group $\langle a, b \rangle$, but a^5 and b^3 also have to be equal to the identity.
- This can be expressed as $\langle a, b \rangle$ divided by the smallest normal subgroup that contains a^5 and b^3.
- If G is any group generated by a and b, with $a^5 = b^3 = 1$, we can try to define a homomorphism

  $$\sigma((a)) = a, \sigma((b)) = b$$

  And, this is well-defined, even though some sequences might be equivalent in $\langle a, b : a^5, b^3 \rangle$ without being equal, because whenever terms cancel out in $\langle a, b : a^5, b^3 \rangle$, they cancel out in G:

  $$(a, a, a, b, b, b, a, a) = (a, a, a, a, a) = 1$$

  $$\sigma((a, a, a, b, b, b, a, a)) = aaabbbaa \in G = aaaaa = 1$$

- So, for any group generated by the generators and satisfying the relations, there's a surjective homomorphism from the presentation onto the group.
- Since there's a surjective homomorphism from the free group of a set of generators to any group, we can express any group as a set of generators and relations (choose the whole kernel of the homomorphism for example).

Groups; generators and relations

- A presentation of a group by its generators and relations is essentially the free group, together with a few equivalences:
- $\langle a, b : a^5, b^3 \rangle$ means the free group $\langle a, b \rangle$, but a^5 and b^3 also have to be equal to the identity.
- This can be expressed as $\langle a, b \rangle$ divided by the smallest normal subgroup that contains a^5 and b^3.
- If G is any group generated by a and b, with $a^5 = b^3 = 1$, we can try to define a homomorphism

$$\sigma((a)) = a, \sigma((b)) = b$$

  And, this is well-defined, even though some sequences might be equivalent in $\langle a, b : a^5, b^3 \rangle$ without being equal, because whenever terms cancel out in $\langle a, b : a^5, b^3 \rangle$, they cancel out in G:

$$(a, a, a, b, b, b, a, a) = (a, a, a, a, a) = 1$$

$$\sigma((a, a, a, b, b, b, a, a)) = aaabbbaa \in G = aaaaa = 1$$

- So, for any group generated by the generators and satisfying the relations, there's a surjective homomorphism from the presentation onto the group.
- Since there's a surjective homomorphism from the free group of a set of generators to any group, we can express any group as a set of generators and relations (choose the whole kernel of the homomorphism for example).
- Note: Some relations might be simplified by adding an equal sign:

$$\langle a, b : aba^{-1}, b^2 \rangle = \langle a, b : ab = a, b^2 \rangle$$

Fundemental Groups

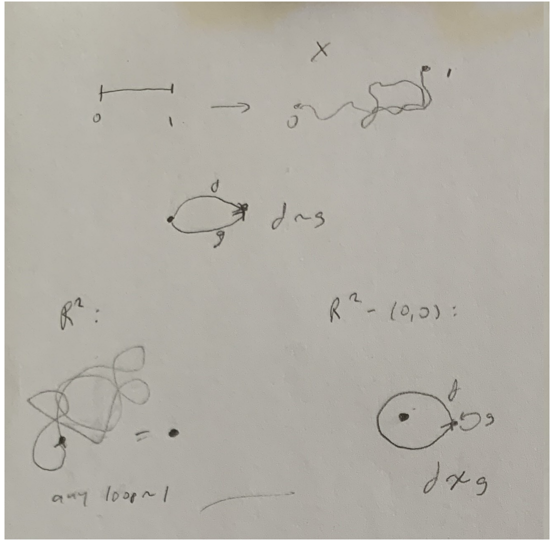
- Let X be a topological space, and x_0 be any point in it.

Fundemental Groups

- Let X be a topological space, and x_0 be any point in it.
- A continuous function $f : [0, 1] \to X$ represents a curve in X. If $f(0) = f(1) = x_0$, we'll call it a loop.

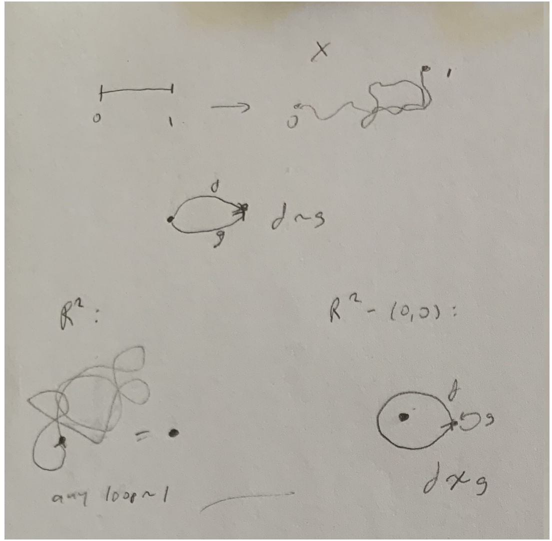
Fundemental Groups

- Let X be a topological space, and x_0 be any point in it.
- A continuous function $f : [0, 1] \to X$ represents a curve in X. If $f(0) = f(1) = x_0$, we'll call it a loop.



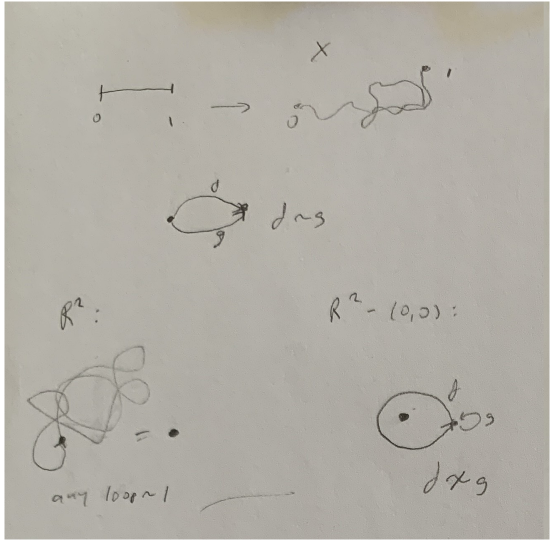
Fundemental Groups

- Let X be a topological space, and x_0 be any point in it.
- A continuous function $f : [0, 1] \to X$ represents a curve in X. If $f(0) = f(1) = x_0$, we'll call it a loop.
- Define equivalence classes where f is equivalent to g if the loops can be continuously deformed into each other. We'll refer to these equivalence classes as paths. If f is a loop, and a is the path f is in, I'll denote $a = [f]$.



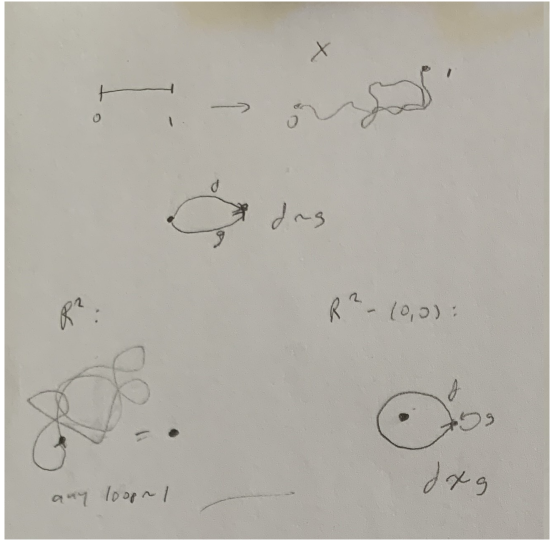
Fundemental Groups

- Let X be a topological space, and x_0 be any point in it.
- A continuous function $f : [0,1] \to X$ represents a curve in X. If $f(0) = f(1) = x_0$, we'll call it a loop.
- Define equivalence classes where f is equivalent to g if the loops can be continuously deformed into each other. We'll refer to these equivalence classes as paths. If f is a loop, and a is the path f is in, I'll denote $a = [f]$.
- In $\mathbb{R}^2$, $x_0 = (0,0)$, any two loops are equivalent, and there's only one path. In $\mathbb{R}^2 - (0,0)$, $x_0 = (1,0)$, the two loops on the right aren't equivalent.



Fundemental Groups

- Let X be a topological space, and x_0 be any point in it.
- A continuous function $f : [0, 1] \to X$ represents a curve in X. If $f(0) = f(1) = x_0$, we'll call it a loop.
- Define equivalence classes where f is equivalent to g if the loops can be continuously deformed into each other. We'll refer to these equivalence classes as paths. If f is a loop, and a is the path f is in, I'll denote $a = [f]$.
- In $\mathbb{R}^2$, $x_0 = (0, 0)$, any two loops are equivalent, and there's only one path. In $\mathbb{R}^2 - (0, 0)$, $x_0 = (1, 0)$, the two loops on the right aren't equivalent.
- More rigorously, $f \sim g$ iff there's a continuous $F : [0, 1]x[0, 1] \to X$ where $F(0, t) = f(t)$, $F(1, t) = g(t)$, and $F(s, 0) = F(s, 1) = x_0$

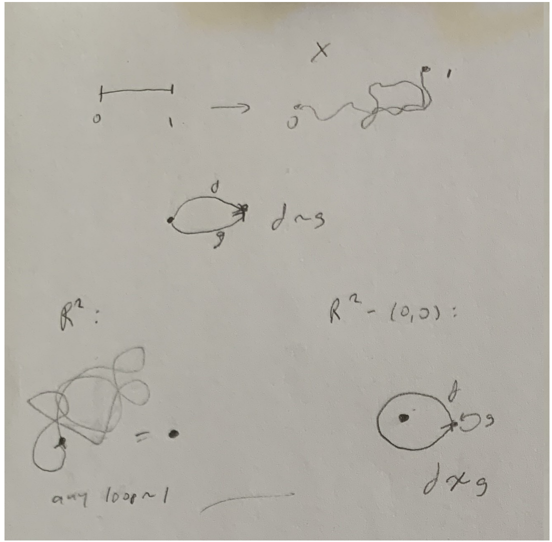


Fundemental Groups

- Let X be a topological space, and x_0 be any point in it.
- A continuous function $f : [0,1] \to X$ represents a curve in X. If $f(0) = f(1) = x_0$, we'll call it a loop.
- Define equivalence classes where f is equivalent to g if the loops can be continuously deformed into each other. We'll refer to these equivalence classes as paths. If f is a loop, and a is the path f is in, I'll denote $a = [f]$.
- In $\mathbb{R}^2$, $x_0 = (0,0)$, any two loops are equivalent, and there's only one path. In $\mathbb{R}^2 - (0,0)$, $x_0 = (1,0)$, the two loops on the right aren't equivalent.
- More rigorously, $f \sim g$ iff there's a continuous $F : [0,1]x[0,1] \to X$ where $F(0,t) = f(t)$, $F(1,t) = g(t)$, and $F(s,0) = F(s,1) = x_0$
- Given two loops f and g, we can compose them as follows:

$$f.g(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Then, $(f.g).h \neq f.(g.h)$, but $(f.g).h \sim f.(g.h)$. Also, if $f \sim g$, $f.h \sim g.h$ for any h.

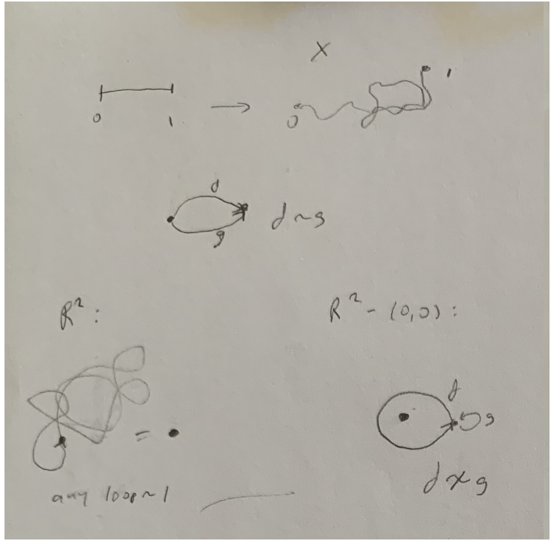


Fundemental Groups

- Let X be a topological space, and x_0 be any point in it.
- A continuous function $f : [0,1] \to X$ represents a curve in X. If $f(0) = f(1) = x_0$, we'll call it a loop.
- Define equivalence classes where f is equivalent to g if the loops can be continuously deformed into each other. We'll refer to these equivalence classes as paths. If f is a loop, and a is the path f is in, I'll denote $a = [f]$.
- In $\mathbb{R}^2$, $x_0 = (0,0)$, any two loops are equivalent, and there's only one path. In $\mathbb{R}^2 - (0,0)$, $x_0 = (1,0)$, the two loops on the right aren't equivalent.
- More rigorously, $f \sim g$ iff there's a continuous $F : [0,1]x[0,1] \to X$ where $F(0,t) = f(t)$, $F(1,t) = g(t)$, and $F(s,0) = F(s,1) = x_0$
- Given two loops f and g, we can compose them as follows:

$$f.g(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

  Then, $(f.g).h \neq f.(g.h)$, but $(f.g).h \sim f.(g.h)$. Also, if $f \sim g$, $f.h \sim g.h$ for any h.
- We can define an identity element as $i(t) = x_0$ and inverse element as $f^{-1}(t) = f(1-t)$. (Again, $f.f^{-1} \neq 1$ as loops but $[f].[f^{-1}] = [f^{-1}].[f] = 1$).



Fundemental Groups

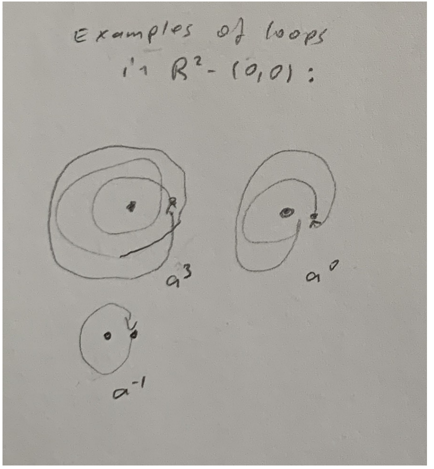
- So, paths in X with base point x_0 under this operation is a group, called the $\pi(X)$. The base point usually doesn't matter, since the groups are usually the same regardless of the base point.

Fundemental Groups

- So, paths in X with base point x_0 under this operation is a group, called the $\pi(X)$. The base point usually doesn't matter, since the groups are usually the same regardless of the base point.
- It can be proven that in $\mathbb{R}^2 - (0,0)$, any two loops that go around the center the same number of times are equivalent. So, if a is the path that represents going aroung once clockwise, the group $\pi(\mathbb{R}^2 - (0,0))$ is $\langle a \rangle$, so a^n for any $n \in \mathbb{Z}$.

Fundemental Groups

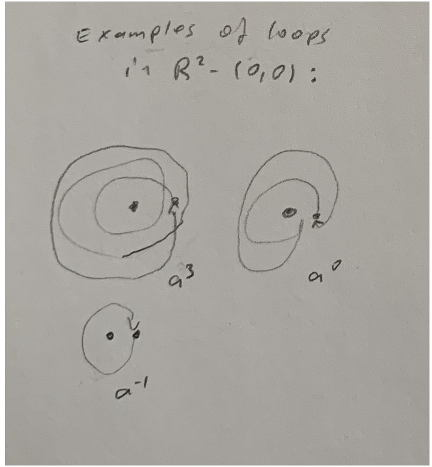
- So, paths in X with base point x_0 under this operation is a group, called the $\pi(X)$. The base point usually doesn't matter, since the groups are usually the same regardless of the base point.
- It can be proven that in $\mathbb{R}^2 - (0,0)$, any two loops that go around the center the same number of times are equivalent. So, if a is the path that represents going aroung once clockwise, the group $\pi(\mathbb{R}^2 - (0,0))$ is $\langle a \rangle$, so a^n for any $n \in \mathbb{Z}$.



Fundemental Groups

- So, paths in X with base point x_0 under this operation is a group, called the $\pi(X)$. The base point usually doesn't matter, since the groups are usually the same regardless of the base point.
- It can be proven that in $\mathbb{R}^2 - (0,0)$, any two loops that go around the center the same number of times are equivalent. So, if a is the path that represents going aroung once clockwise, the group $\pi(\mathbb{R}^2 - (0,0))$ is $\langle a \rangle$, so a^n for any $n \in \mathbb{Z}$.
- Here, a^{-n} signifies going around counterclockwise n times and a^0 signifies not going around the point. Then,

$$a^w . a^z = a^{w+z}$$



Fundemental Groups

- Let σ be a continuous function between X and Y.

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0, 1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0,1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(s,0) = F(s,1) = x_0$,

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0,1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(s,0) = F(s,1) = x_0$,
- $\sigma.F$ is a continuous function $[0,1]^2 \to Y$ with

$$\sigma(F(0,t)) = \sigma(f)(t)$$

$$\sigma(F(1,t)) = \sigma(g)(t)$$

$$\sigma(F(s,0)) = \sigma(F(s,1)) = x_0$$

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0,1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(s,0) = F(s,1) = x_0$,
- $\sigma.F$ is a continuous function $[0,1]^2 \to Y$ with

$$\sigma(F(0,t)) = \sigma(f)(t)$$

$$\sigma(F(1,t)) = \sigma(g)(t)$$

$$\sigma(F(s,0)) = \sigma(F(s,1)) = x_0$$

- So, σ defines a function $\sigma_* : \pi(X) \to \pi(Y)$.

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0,1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(s,0) = F(s,1) = x_0$,
- $\sigma.F$ is a continuous function $[0,1]^2 \to Y$ with

$$\sigma(F(0,t)) = \sigma(f)(t)$$

$$\sigma(F(1,t)) = \sigma(g)(t)$$

$$\sigma(F(s,0)) = \sigma(F(s,1)) = x_0$$

- So, σ defines a function $\sigma_* : \pi(X) \to \pi(Y)$.
- Also, $\sigma(1) = 1$.

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0,1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(s,0) = F(s,1) = x_0$,
- $\sigma.F$ is a continuous function $[0,1]^2 \to Y$ with

$$\sigma(F(0,t)) = \sigma(f)(t)$$

$$\sigma(F(1,t)) = \sigma(g)(t)$$

$$\sigma(F(s,0)) = \sigma(F(s,1)) = x_0$$

- So, σ defines a function $\sigma_* : \pi(X) \to \pi(Y)$.
- Also, $\sigma(1) = 1$.
- $\sigma(f.g) = \sigma(f).\sigma(g)$ for loops, so $\sigma(a.b) = \sigma(a).\sigma(b)$ for paths.
- $\sigma(f^{-1}) = \sigma(f)^{-1}$ for loops, so $\sigma(a^{-1}) = \sigma(a)^{-1}$ for paths.

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0, 1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0, t) = f(t)$, $F(1, t) = g(t)$, $F(s, 0) = F(s, 1) = x_0$,
- $\sigma.F$ is a continuous function $[0, 1]^2 \to Y$ with

$$\sigma(F(0, t)) = \sigma(f)(t)$$

$$\sigma(F(1, t)) = \sigma(g)(t)$$

$$\sigma(F(s, 0)) = \sigma(F(s, 1)) = x_0$$

- So, σ defines a function $\sigma_* : \pi(X) \to \pi(Y)$.
- Also, $\sigma(1) = 1$.
- $\sigma(f.g) = \sigma(f).\sigma(g)$ for loops, so $\sigma(a.b) = \sigma(a).\sigma(b)$ for paths.
- $\sigma(f^{-1}) = \sigma(f)^{-1}$ for loops, so $\sigma(a^{-1}) = \sigma(a)^{-1}$ for paths.
- So, σ_* is a group homomorphism between $\pi(X)$ and $\pi(Y)$, called the induced homomorphism of σ.

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0,1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(s,0) = F(s,1) = x_0$,
- $\sigma.F$ is a continuous function $[0,1]^2 \to Y$ with

$$\sigma(F(0,t)) = \sigma(f)(t)$$

$$\sigma(F(1,t)) = \sigma(g)(t)$$

$$\sigma(F(s,0)) = \sigma(F(s,1)) = x_0$$

- So, σ defines a function $\sigma_* : \pi(X) \to \pi(Y)$.
- Also, $\sigma(1) = 1$.
- $\sigma(f.g) = \sigma(f).\sigma(g)$ for loops, so $\sigma(a.b) = \sigma(a).\sigma(b)$ for paths.
- $\sigma(f^{-1}) = \sigma(f)^{-1}$ for loops, so $\sigma(a^{-1}) = \sigma(a)^{-1}$ for paths.
- So, σ_* is a group homomorphism between $\pi(X)$ and $\pi(Y)$, called the induced homomorphism of σ.
- If $\sigma : X \to Y$ and $\mu : Y \to Z$, $\nu : X \to Z$ with $\nu = \mu \cdot \sigma$, $a = [f]$,

$$\mu_*(\sigma_*(a)) = \mu_*(\sigma_*([f])) = \mu_*([\sigma(f)]) = [\mu(\sigma(f))]$$

$$= [\nu(f)] = \nu_*([f]) = \nu_*(a)$$

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0,1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(s,0) = F(s,1) = x_0$,
- $\sigma.F$ is a continuous function $[0,1]^2 \to Y$ with

$$\sigma(F(0,t)) = \sigma(f)(t)$$

$$\sigma(F(1,t)) = \sigma(g)(t)$$

$$\sigma(F(s,0)) = \sigma(F(s,1)) = x_0$$

- So, σ defines a function $\sigma_* : \pi(X) \to \pi(Y)$.
- Also, $\sigma(1) = 1$.
- $\sigma(f.g) = \sigma(f).\sigma(g)$ for loops, so $\sigma(a.b) = \sigma(a).\sigma(b)$ for paths.
- $\sigma(f^{-1}) = \sigma(f)^{-1}$ for loops, so $\sigma(a^{-1}) = \sigma(a)^{-1}$ for paths.
- So, σ_* is a group homomorphism between $\pi(X)$ and $\pi(Y)$, called the induced homomorphism of σ.

- If $\sigma : X \to Y$ and $\mu : Y \to Z$, $\nu : X \to Z$ with $\nu = \mu \cdot \sigma$, $a = [f]$,

$$\mu_*(\sigma_*(a)) = \mu_*(\sigma_*([f])) = \mu_*([\sigma(f)]) = [\mu(\sigma(f))]$$

$$= [\nu(f)] = \nu_*([f]) = \nu_*(a)$$

- In terms of a commutative diagram, if

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\nu"] & Y \arrow[d, "\mu"] \\ & Z \end{tikzcd}$$

Then,

$$\begin{tikzcd} \pi(X) \arrow[r, "\sigma_*"] \arrow[dr, "\nu_*"] & \pi(Y) \arrow[d, "\mu_*"] \\ & \pi(Z) \end{tikzcd}$$

Fundemental Groups

- Let σ be a continuous function between X and Y.
- Then, for loops $f, g : [0,1] \to X$, If $f \sim g$, $\sigma(f) \sim \sigma(g)$:
- If $f \sim g$,there's an F with $F(0,t) = f(t)$, $F(1,t) = g(t)$, $F(s,0) = F(s,1) = x_0$,
- $\sigma.F$ is a continuous function $[0,1]^2 \to Y$ with

$$\sigma(F(0,t)) = \sigma(f)(t)$$

$$\sigma(F(1,t)) = \sigma(g)(t)$$

$$\sigma(F(s,0)) = \sigma(F(s,1)) = x_0$$

- So, σ defines a function $\sigma_* : \pi(X) \to \pi(Y)$.
- Also, $\sigma(1) = 1$.
- $\sigma(f.g) = \sigma(f).\sigma(g)$ for loops, so $\sigma(a.b) = \sigma(a).\sigma(b)$ for paths.
- $\sigma(f^{-1}) = \sigma(f)^{-1}$ for loops, so $\sigma(a^{-1}) = \sigma(a)^{-1}$ for paths.
- So, σ_* is a group homomorphism between $\pi(X)$ and $\pi(Y)$, called the induced homomorphism of σ.

- If $\sigma : X \to Y$ and $\mu : Y \to Z$, $\nu : X \to Z$ with $\nu = \mu \cdot \sigma$, $a = [f]$,

$$\mu_*(\sigma_*(a)) = \mu_*(\sigma_*([f])) = \mu_*([\sigma(f)]) = [\mu(\sigma(f))]$$

$$= [\nu(f)] = \nu_*([f]) = \nu_*(a)$$

- In terms of a commutative diagram, if

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\nu"] & Y \arrow[d, "\mu"] \\ & Z \end{tikzcd}$$

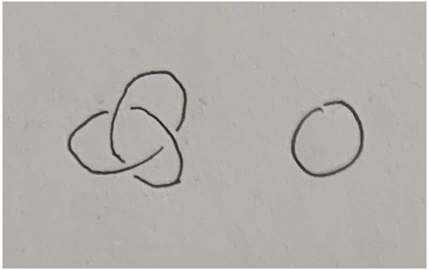
Then,

$$\begin{tikzcd} \pi(X) \arrow[r, "\sigma_*"] \arrow[dr, "\nu_*"] & \pi(Y) \arrow[d, "\mu_*"] \\ & \pi(Z) \end{tikzcd}$$

- Note: This means the fundemental group is a functor between the category of (pointed) topological spaces with (pointed) continuous functions and the category of groups and group homomorphisms.

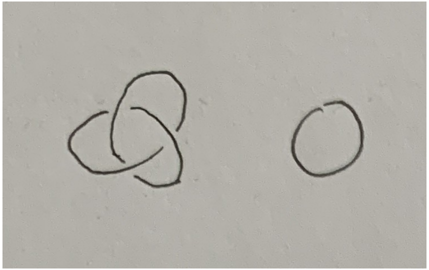
Knots

- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).

¿ 

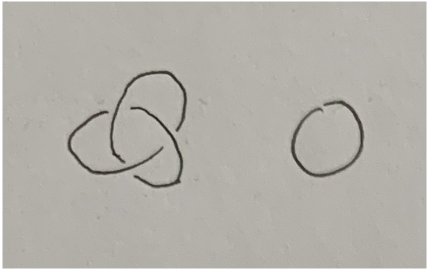
Knots

- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).

¿ 

Knots

- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).

¿ 

- A knot K is equivalent to K' if $(K, \mathbb{R}^3)$ is homeomorphic to $(K', \mathbb{R}^3)$
  In other words, there's a homeomorphism

$$h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ so that } h(K) = K'$$

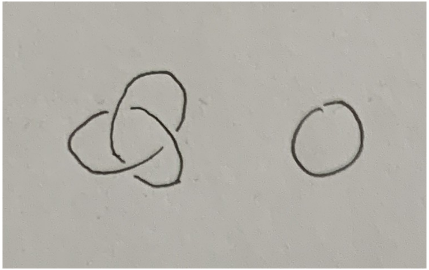
  Then,

$$h|_{\mathbb{R}^3 - K} : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$$

  is also a homeomorphism,

Knots

- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).

¿ 

- A knot K is equivalent to K' if $(K, \mathbb{R}^3)$ is homeomorphic to $(K', \mathbb{R}^3)$
  In other words, there's a homeomorphism

  $$h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ so that } h(K) = K'$$

  Then,

  $$h|_{\mathbb{R}^3 - K} : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$$

  is also a homeomorphism,

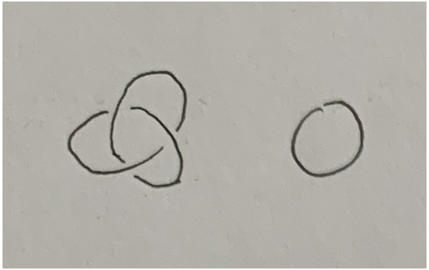
- ... and its induced homeomorphism from the fundemental groups

  $$\pi_1(\mathbb{R}^3 - K) \to \pi_1(\mathbb{R}^3 - K')$$

  is an isomorphism, so those groups are isomorphic.

Knots

- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).

¿ 

- A knot K is equivalent to K' if $(K, \mathbb{R}^3)$ is homeomorphic to $(K', \mathbb{R}^3)$
  In other words, there's a homeomorphism

$$h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ so that } h(K) = K'$$

  Then,

$$h|_{\mathbb{R}^3 - K} : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$$

  is also a homeomorphism,

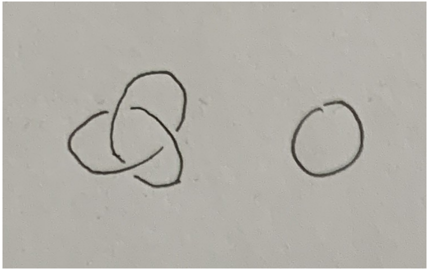
- ... and its induced homeomorphism from the fundemental groups

$$\pi_1(\mathbb{R}^3 - K) \to \pi_1(\mathbb{R}^3 - K')$$

  is an isomorphism, so those groups are isomorphic.
- The group $\pi_1(\mathbb{R}^3 - K)$ is also called the fundemental group of a knot.

Knots

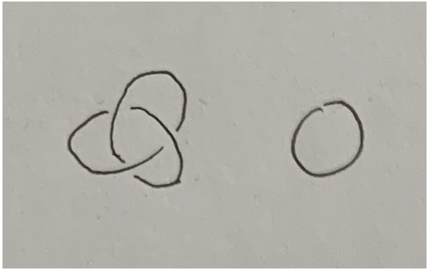
- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).

  ¿ 

- A knot K is equivalent to K' if $(K, \mathbb{R}^3)$ is homeomorphic to $(K', \mathbb{R}^3)$
  In other words, there's a homeomorphism
  $$h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ so that } h(K) = K'$$
  Then,
  $$h|_{\mathbb{R}^3 - K} : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$$
  is also a homeomorphism,
- ... and its induced homeomorphism from the fundemental groups
  $$\pi_1(\mathbb{R}^3 - K) \to \pi_1(\mathbb{R}^3 - K')$$
  is an isomorphism, so those groups are isomorphic.
- The group $\pi_1(\mathbb{R}^3 - K)$ is also called the fundemental group of a knot.
- The main argument we'll be making is, if the fundemental groups of two knots aren't isomorphic, then the knots aren't equivalent.

Knots

- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).

¿ 

- A knot K is equivalent to K' if $(K, \mathbb{R}^3)$ is homeomorphic to $(K', \mathbb{R}^3)$
  In other words, there's a homeomorphism

$$h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ so that } h(K) = K'$$

  Then,

$$h|_{\mathbb{R}^3 - K} : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$$

  is also a homeomorphism,

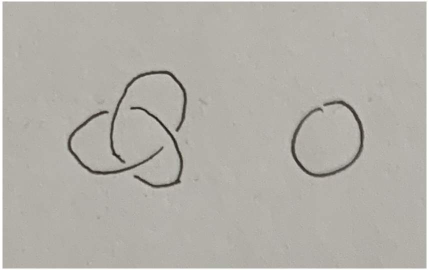
- ... and its induced homeomorphism from the fundemental groups

$$\pi_1(\mathbb{R}^3 - K) \to \pi_1(\mathbb{R}^3 - K')$$

  is an isomorphism, so those groups are isomorphic.
- The group $\pi_1(\mathbb{R}^3 - K)$ is also called the fundemental group of a knot.
- The main argument we'll be making is, if the fundemental groups of two knots aren't isomorphic, then the knots aren't equivalent.
- Seifert – Van Kampen's Theorem helps determine the fundemental group of $A \cup B$ given the fundemental groups of A, B, and $A \cap B$.

Knots

- A knot is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}^1$ (or more generally, any subset of a topological space homeomorphic to $\mathbb{S}^n$).

¿ 

- A knot K is equivalent to K' if $(K, \mathbb{R}^3)$ is homeomorphic to $(K', \mathbb{R}^3)$
  In other words, there's a homeomorphism

$$h : \mathbb{R}^3 \to \mathbb{R}^3 \text{ so that } h(K) = K'$$

  Then,

$$h|_{\mathbb{R}^3 - K} : \mathbb{R}^3 - K \to \mathbb{R}^3 - K'$$

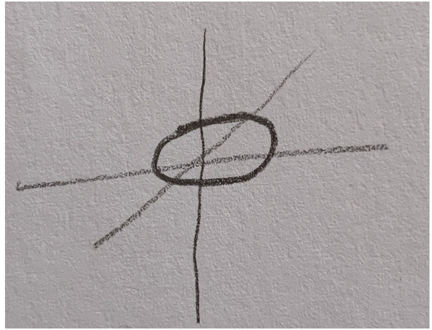
  is also a homeomorphism,

- ... and its induced homeomorphism from the fundemental groups

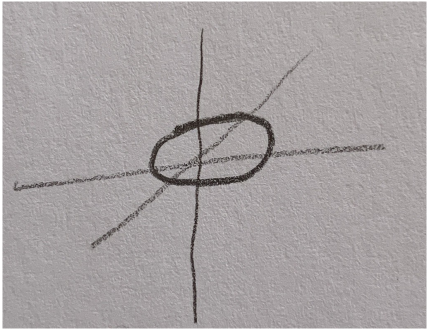
$$\pi_1(\mathbb{R}^3 - K) \to \pi_1(\mathbb{R}^3 - K')$$

  is an isomorphism, so those groups are isomorphic.
- The group $\pi_1(\mathbb{R}^3 - K)$ is also called the fundemental group of a knot.
- The main argument we'll be making is, if the fundemental groups of two knots aren't isomorphic, then the knots aren't equivalent.
- Seifert – Van Kampen's Theorem helps determine the fundemental group of $A \cup B$ given the fundemental groups of A, B, and $A \cap B$.
- In this presentation, we'll be looking at the statement and proof of this theorem, and applying it to find the fundemental groups of a few knots.

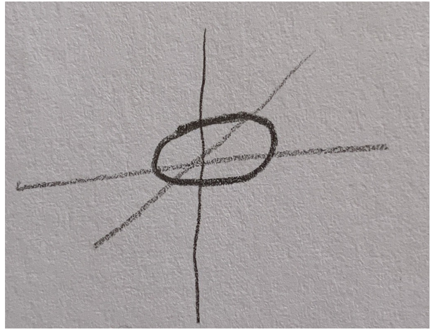
- The 'unknot' is the following knot:

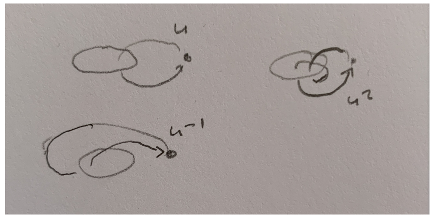


- The 'unknot' is the following knot:
- To say that a given knot can't be unknotted, we want to say that that knot is not equal to the unknot. So, before developing Seifert-Van Kampen's theorem, let's look at the fundemental group of the unknot:



- The 'unknot' is the following knot:
- To say that a given knot can't be unknotted, we want to say that that knot is not equal to the unknot. So, before developing Seifert-Van Kampen's theorem, let's look at the fundemental group of the unknot:
- One nontrivial path is wrapping around the circle as shown. Similar to $\mathbb{R}^2 - (0,0)$, it can be proved that any loops that wrap around the same number of times are equivalent (and loops that don't are not). So, the fundemental group is $\langle u \rangle$.





Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda \in \Lambda}$ could be $\{A, B, A \cap B\}$*

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda \in \Lambda}$ could be $\{A, B, A \cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda \in \Lambda}$ could be $\{A, B, A \cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"'] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- From this definition, we can tell:

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"'] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"'] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- From this definition, we can tell:
- If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- From this definition, we can tell:
- If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$
- If $\alpha \in \pi_1(U_\lambda)$, $\beta \in \pi_1(U_\mu)$,

$$\sigma(\psi_\lambda(\alpha)\psi_\mu(\beta)) = \sigma(\psi_\lambda(\alpha))\sigma(\psi_\mu(\beta)) = p_\lambda(\alpha)p_\mu(\beta)$$

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"'] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"'] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- From this definition, we can tell:
- If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$
- If $\alpha \in \pi_1(U_\lambda)$, $\beta \in \pi_1(U_\mu)$,

$$\sigma(\psi_\lambda(\alpha)\psi_\mu(\beta)) = \sigma(\psi_\lambda(\alpha))\sigma(\psi_\mu(\beta)) = p_\lambda(\alpha)p_\mu(\beta)$$

- For $\{\alpha_i\}_{i=1}^n$ so that $\alpha_i \in U_{\lambda_i}$,

$$\sigma(\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) = p_{\lambda_1}(\alpha_1)p_{\lambda_2}(\alpha_2)...p_{\lambda_n}(\alpha_n)$$

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- From this definition, we can tell:
- If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$
- If $\alpha \in \pi_1(U_\lambda)$, $\beta \in \pi_1(U_\mu)$,

$$\sigma(\psi_\lambda(\alpha)\psi_\mu(\beta)) = \sigma(\psi_\lambda(\alpha))\sigma(\psi_\mu(\beta)) = p_\lambda(\alpha)p_\mu(\beta)$$

- For $\{\alpha_i\}_{i=1}^n$ so that $\alpha_i \in U_{\lambda_i}$,

$$\sigma(\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) = p_{\lambda_1}(\alpha_1)p_{\lambda_2}(\alpha_2)...p_{\lambda_n}(\alpha_n)$$

- We need to prove that σ is well defined, In other words, if

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n) \sim \psi_{\mu_1}(\beta_1)\psi_{\mu_2}(\beta_2)...\psi_{\mu_m}(\beta_m)$$

Then, $\sigma(\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) \sim \sigma(\psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m))$
So, $\quad p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) \sim p_{\mu_1}(\beta_1)...p_{\mu_m}(\beta_m)$

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda \in \Lambda}$ could be $\{A, B, A \cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"'] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda \in \Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"'] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- From this definition, we can tell:
- If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$
- If $\alpha \in \pi_1(U_\lambda)$, $\beta \in \pi_1(U_\mu)$,

$$\sigma(\psi_\lambda(\alpha)\psi_\mu(\beta)) = \sigma(\psi_\lambda(\alpha))\sigma(\psi_\mu(\beta)) = p_\lambda(\alpha)p_\mu(\beta)$$

- For $\{\alpha_i\}_{i=1}^n$ so that $\alpha_i \in U_{\lambda_i}$,

$$\sigma(\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) = p_{\lambda_1}(\alpha_1)p_{\lambda_2}(\alpha_2)...p_{\lambda_n}(\alpha_n)$$

- We need to prove that σ is well defined, In other words, if

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n) \sim \psi_{\mu_1}(\beta_1)\psi_{\mu_2}(\beta_2)...\psi_{\mu_m}(\beta_m)$$

  Then, $\sigma(\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) \sim \sigma(\psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m))$
  So, $\quad p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) \sim p_{\mu_1}(\beta_1)...p_{\mu_m}(\beta_m)$
- Since this is all the restrictions on σ, but σ is unique, $\pi_1(X)$ must not have any elements which aren't in the form

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

  We also need to prove this.

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"'] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"'] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- From this definition, we can tell:
- If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$
- If $\alpha \in \pi_1(U_\lambda)$, $\beta \in \pi_1(U_\mu)$,

$$\sigma(\psi_\lambda(\alpha)\psi_\mu(\beta)) = \sigma(\psi_\lambda(\alpha))\sigma(\psi_\mu(\beta)) = p_\lambda(\alpha)p_\mu(\beta)$$

- For $\{\alpha_i\}_{i=1}^n$ so that $\alpha_i \in U_{\lambda_i}$,

$$\sigma(\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) = p_{\lambda_1}(\alpha_1)p_{\lambda_2}(\alpha_2)...p_{\lambda_n}(\alpha_n)$$

- We need to prove that σ is well defined, In other words, if

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n) \sim \psi_{\mu_1}(\beta_1)\psi_{\mu_2}(\beta_2)...\psi_{\mu_m}(\beta_m)$$

  Then, $\sigma(\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) \sim \sigma(\psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m))$
  So, $\quad p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) \sim p_{\mu_1}(\beta_1)...p_{\mu_m}(\beta_m)$
- Since this is all the restrictions on σ, but σ is unique, $\pi_1(X)$ must not have any elements which aren't in the form

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

  We also need to prove this.
- We'll also look at when two elements of $\pi_1(X)$ are equal and when they're different.

Seifert – Van Kampen Theorem

- Let X be a path-connected topological space, x_0 be any point in X. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- *here, $\{U_\lambda\}_{\lambda\in\Lambda}$ could be $\{A, B, A\cap B\}$*
- Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \to X$.
- For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \to U_\mu$. The following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "\psi_\lambda"] & \pi_1(U_\mu) \arrow[d, "\psi_\mu"] \\ & \pi_1(X) \end{tikzcd}$$

- Let H be any group and $\{p_\lambda\}_{\lambda\in\Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\ & H \end{tikzcd}$$

- Then, there's a unique σ so that the following commutes:

$$\begin{tikzcd} \pi_1(U_\lambda) \arrow[r, "\psi_\lambda"] \arrow[dr, "p_\lambda"] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- From this definition, we can tell:
- If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$
- If $\alpha \in \pi_1(U_\lambda)$, $\beta \in \pi_1(U_\mu)$,

$$\sigma(\psi_\lambda(\alpha)\psi_\mu(\beta)) = \sigma(\psi_\lambda(\alpha))\sigma(\psi_\mu(\beta)) = p_\lambda(\alpha)p_\mu(\beta)$$

- For $\{\alpha_i\}_{i=1}^n$ so that $\alpha_i \in U_{\lambda_i}$,

$$\sigma(\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) = p_{\lambda_1}(\alpha_1)p_{\lambda_2}(\alpha_2)...p_{\lambda_n}(\alpha_n)$$

- We need to prove that σ is well defined, In other words, if

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n) \sim \psi_{\mu_1}(\beta_1)\psi_{\mu_2}(\beta_2)...\psi_{\mu_m}(\beta_m)$$

  Then, $\sigma(\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)) \sim \sigma(\psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m))$
  So, $\quad p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) \sim p_{\mu_1}(\beta_1)...p_{\mu_m}(\beta_m)$
- Since this is all the restrictions on σ, but σ is unique, $\pi_1(X)$ must not have any elements which aren't in the form

$$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

  We also need to prove this.
- We'll also look at when two elements of $\pi_1(X)$ are equal and when they're different.
- But hopefully it makes sense how this theorem determines $\pi_1(X)$!

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of a $\pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of a $\pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

  for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
  Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of a $\pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

  for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
  Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0, 1] \to X$ so that $a = [f]_{\pi_1(X)}$.

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of a $\pi_1(X)$ can be expressed as
$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$
for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0,1] \to X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is a cover of the compact metric space $[0,1]$. It has a Lebesgue number δ.

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of a $\pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

  for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
  Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0,1] \to X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is a cover of the compact metric space $[0,1]$. It has a Lebesgue number δ.
- Find n so $\frac{1}{n} < \delta$, divide $[0,1]$ into subintervals $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, ..., $[\frac{n-1}{n}, 1]$. Each has diameter less than δ, so $[\frac{i}{n}, \frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i, and $f([\frac{i}{n}, \frac{i+1}{n}]) \in U_{\lambda_i}$.

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of a $\pi_1(X)$ can be expressed as
$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$
for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0,1] \to X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is a cover of the compact metric space $[0,1]$. It has a Lebesgue number δ.
- Find n so $\frac{1}{n} < \delta$, divide $[0,1]$ into subintervals $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, ..., $[\frac{n-1}{n}, 1]$. Each has diameter less than δ, so $[\frac{i}{n}, \frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i, and $f([\frac{i}{n}, \frac{i+1}{n}]) \in U_{\lambda_i}$.
- Let f_i be f from $f(\frac{i-1}{n})$ to $f(\frac{i}{n})$. So,
$$f \sim f_1 f_2 f_3 ... f_n$$

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of a $\pi_1(X)$ can be expressed as

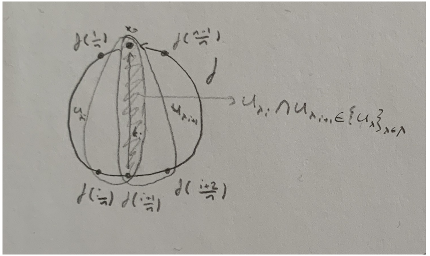
$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

  for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
  Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0,1] \to X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is a cover of the compact metric space $[0,1]$. It has a Lebesgue number δ.
- Find n so $\frac{1}{n} < \delta$, divide $[0,1]$ into subintervals $[0,\frac{1}{n}]$, $[\frac{1}{n},\frac{2}{n}]$, ..., $[\frac{n-1}{n},1]$. Each has diameter less than δ, so $[\frac{i}{n},\frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i, and $f([\frac{i}{n},\frac{i+1}{n}]) \in U_{\lambda_i}$.
- Let f_i be f from $f(\frac{i-1}{n})$ to $f(\frac{i}{n})$. So,

$$f \sim f_1 f_2 f_3 ... f_n$$

- $f(\frac{i}{n}) \in U_{\lambda_i}, U_{\lambda i+1}$. Since $U_{\lambda_i} \cap U_{\lambda_{i+1}} \in \{U_\lambda\}_{\lambda\in\Lambda}$, and all elements of $\{U_\lambda\}_{\lambda\in\Lambda}$ are path connected and include x_0, there's a path k_i from $f(\frac{i}{n})$ to x_0 contained in $U_{\lambda_i} \cap U_{\lambda_{i+1}}$.

Seifert – Van Kampen Theorem Proof – Part 1



- To Prove: Every element of $a\ \pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

  for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
  Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0,1] \to X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is a cover of the compact metric space $[0,1]$. It has a Lebesgue number δ.
- Find n so $\frac{1}{n} < \delta$, divide $[0,1]$ into subintervals $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, ..., $[\frac{n-1}{n}, 1]$. Each has diameter less than δ, so $[\frac{i}{n}, \frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i, and $f([\frac{i}{n}, \frac{i+1}{n}]) \in U_{\lambda_i}$.
- Let f_i be f from $f(\frac{i-1}{n})$ to $f(\frac{i}{n})$. So,

$$f \sim f_1 f_2 f_3 ... f_n$$

- $f(\frac{i}{n}) \in U_{\lambda_i}, U_{\lambda i+1}$. Since $U_{\lambda_i} \cap U_{\lambda_{i+1}} \in \{U_\lambda\}_{\lambda\in\Lambda}$, and all elements of $\{U_\lambda\}_{\lambda\in\Lambda}$ are path connected and include x_0, there's a path k_i from $f(\frac{i}{n})$ to x_0 contained in $U_{\lambda_i} \cap U_{\lambda_{i+1}}$.

Seifert – Van Kampen Theorem Proof – Part 1

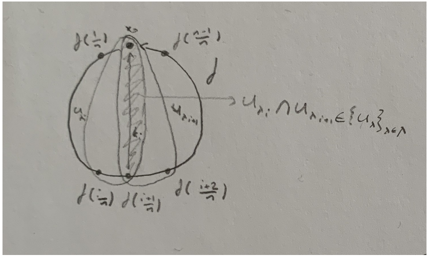
- To Prove: Every element of a $\pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\ldots\psi_{\lambda_n}(\alpha_n)$$

  for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
  Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0,1] \to X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is a cover of the compact metric space $[0,1]$. It has a Lebesgue number δ.
- Find n so $\frac{1}{n} < \delta$, divide $[0,1]$ into subintervals $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, ..., $[\frac{n-1}{n}, 1]$. Each has diameter less than δ, so $[\frac{i}{n}, \frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i, and $f([\frac{i}{n}, \frac{i+1}{n}]) \in U_{\lambda_i}$.
- Let f_i be f from $f(\frac{i-1}{n})$ to $f(\frac{i}{n})$. So,

$$f \sim f_1 f_2 f_3 \ldots f_n$$

- $f(\frac{i}{n}) \in U_{\lambda_i}, U_{\lambda i+1}$. Since $U_{\lambda_i} \cap U_{\lambda_{i+1}} \in \{U_\lambda\}_{\lambda\in\Lambda}$, and all elements of $\{U_\lambda\}_{\lambda\in\Lambda}$ are path connected and include x_0, there's a path k_i from $f(\frac{i}{n})$ to x_0 contained in $U_{\lambda_i} \cap U_{\lambda_{i+1}}$.



- We add the k_i to put each small piece starts and ends at x_0, and so is in a fundemental group:

$$f \sim f_1 k_1 \cdot k_1^{-1} f_2 k_2 \cdot k_2^{-1} f_3 k_3 \cdot \ldots \cdot k_{n-1}^{-1} f_n$$

$$a = [f]_{\pi_1(X)} = [f_1 k_1]_{\pi_1(X)}[k_1^{-1} f_2 k_2]_{\pi_1(X)} \ldots [k_{n-1}^{-1} f_n]_{\pi_1(X)}$$

  Now, $k_{i-1} f_i k_i \subseteq U_{\lambda_i}$, since $k_i \subseteq U_{\lambda_i}, U_{\lambda_{i+1}}$. Since ψ_{λ_i} is the homomorphism induced by an inclusion map,

$$a = \psi_{\lambda_1}([f_1 k_1]_{\pi_1(U_{\lambda_1})}) \ldots \psi_{\lambda_n}([k_{n-1} f_n]_{\pi_1(U_{\lambda_n})})$$

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

  Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

- So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

- So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$
- Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$.

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

  Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

- So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$
- Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$.
- If $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$, there's a continuous function $F : [0,1]x[0,1] \to X$ with $F(1,t) = F(s,0) = F(s,1) = x_0$,

$$F(0,t) = \begin{cases} f_1 & [0, \frac{1}{n}) \\ f_2 & [\frac{1}{n}, \frac{2}{n}) \\ ... \end{cases}$$

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

  Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

- So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$
- Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$.
- If $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$, there's a continuous function $F : [0,1]x[0,1] \to X$ with $F(1,t) = F(s,0) = F(s,1) = x_0$,

$$F(0,t) = \begin{cases} f_1 & [0, \frac{1}{n}) \\ f_2 & [\frac{1}{n}, \frac{2}{n}) \\ ... \end{cases}$$

- Using the Lebesgue number, split up $[0,1]x[0,1]$ into rectangles so each fits in a single U_λ, making sure that each $\frac{i}{n}$ is at a boundary:

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

  Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

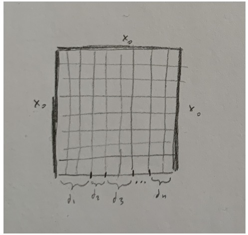
  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

- So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$
- Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$.
- If $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$, there's a continuous function $F : [0,1]x[0,1] \to X$ with $F(1,t) = F(s,0) = F(s,1) = x_0$,

$$F(0,t) = \begin{cases} f_1 & [0, \frac{1}{n}) \\ f_2 & [\frac{1}{n}, \frac{2}{n}) \\ ... & \end{cases}$$

- Using the Lebesgue number, split up $[0,1]x[0,1]$ into rectangles so each fits in a single U_λ, making sure that each $\frac{i}{n}$ is at a boundary:



Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

  Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

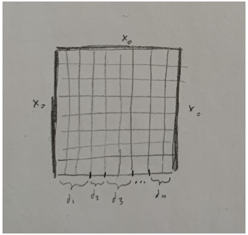
  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

- So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$
- Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$.
- If $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$, there's a continuous function $F : [0,1]x[0,1] \to X$ with $F(1,t) = F(s,0) = F(s,1) = x_0$,

$$F(0,t) = \begin{cases} f_1 & [0, \frac{1}{n}) \\ f_2 & [\frac{1}{n}, \frac{2}{n}) \\ ... & \end{cases}$$

- Using the Lebesgue number, split up $[0,1]x[0,1]$ into rectangles so each fits in a single U_λ, making sure that each $\frac{i}{n}$ is at a boundary:



- For each intersection, add a line k_{ij} going from the intersection to x_0, contained in the U_λ's of all four surrounding rectangles.

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

  Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

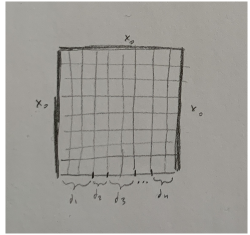
  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

- So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$
- Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$.
- If $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$, there's a continuous function $F : [0,1]x[0,1] \to X$ with $F(1,t) = F(s,0) = F(s,1) = x_0$,

$$F(0,t) = \begin{cases} f_1 & [0, \frac{1}{n}) \\ f_2 & [\frac{1}{n}, \frac{2}{n}) \\ ... & \end{cases}$$

- Using the Lebesgue number, split up $[0,1]x[0,1]$ into rectangles so each fits in a single U_λ, making sure that each $\frac{i}{n}$ is at a boundary:



- For each intersection, add a line k_{ij} going from the intersection to x_0, contained in the U_λ's of all four surrounding rectangles.
- In each line below, add k_{ij}'s as necessary to put them in the fundemental group:
  For each rectangle, since it's contained in U_λ,

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

  Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

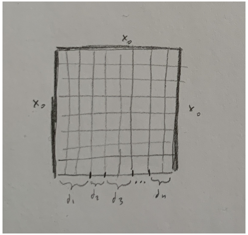
  Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n)p_{\mu_m}(\beta_m^{-1})...p_{\mu_1}(\beta_1^{-1}) = 1$$

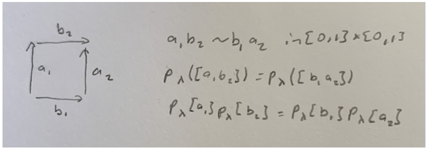
- So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = 1$
- Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$.
- If $\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = 1$, there's a continuous function $F : [0,1]x[0,1] \to X$ with $F(1,t) = F(s,0) = F(s,1) = x_0$,

$$F(0,t) = \begin{cases} f_1 & [0, \frac{1}{n}) \\ f_2 & [\frac{1}{n}, \frac{2}{n}) \\ ... \end{cases}$$

- Using the Lebesgue number, split up $[0,1]x[0,1]$ into rectangles so each fits in a single U_λ, making sure that each $\frac{i}{n}$ is at a boundary:



- For each intersection, add a line k_{ij} going from the intersection to x_0, contained in the U_λ's of all four surrounding rectangles.
- In each line below, add k_{ij}'s as necessary to put them in the fundemental group:
  For each rectangle, since it's contained in U_λ,

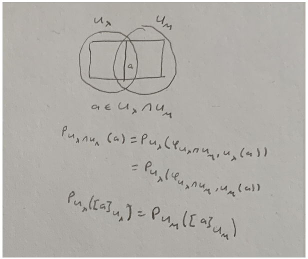


- Now, remember that

$$
\begin{tikzcd}
\pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\
& H
\end{tikzcd}
$$

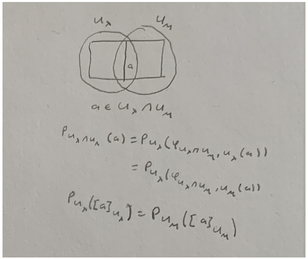
- Now, remember that

$$
\begin{tikzcd}
\pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\
 & H
\end{tikzcd}
$$



- Now, remember that

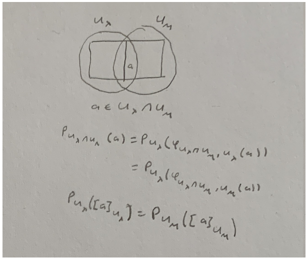
$$
\begin{tikzcd}
\pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\
 & H
\end{tikzcd}
$$



- Notice that along the left, top, and right edges, p of the trivial loops is 1, so the whole composition is 1.

- Now, remember that

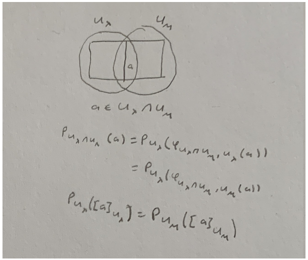
$$\begin{tikzcd}
\pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\
 & H
\end{tikzcd}$$



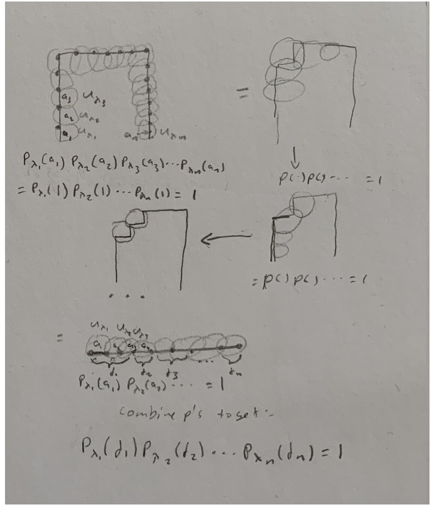
- Notice that along the left, top, and right edges, p of the trivial loops is 1, so the whole composition is 1.
- Now, we can apply the previous two parts a finite number of times to move that composition to the bottom without changing its value:

- Now, remember that

$$\begin{tikzcd}
\pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"'] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\
 & H
\end{tikzcd}$$



- Notice that along the left, top, and right edges, p of the trivial loops is 1, so the whole composition is 1.
- Now, we can apply the previous two parts a finite number of times to move that composition to the bottom without changing its value:



- And that concludes the proof of Seifert-Van Kampen theorem: We proved that σ is well-defined, so it exists, and we know that it's unique.

Seifert – Van Kampen Theorem, corollary

- We will prove: If $X = U \cup V$, $\pi(X) = \frac{\pi(U) * \pi(V)}{N}$ where N is the smallest subgroup containing $\phi_{U\cap V,U}(x)\phi_{U\cap V,V}(x)^{-1}$ for all x in $U \cap V$.

Seifert – Van Kampen Theorem, corollary

- We will prove: If $X = U \cup V$, $\pi(X) = \frac{\pi(U)*\pi(V)}{N}$ where N is the smallest subgroup containing $\phi_{U\cap V,U}(x)\phi_{U\cap V,V}(x)^{-1}$ for all x in $U \cap V$.
- Elements of the free product $\pi(U) * \pi(V)$ are strings like $u_1 v_1 u_2 .. v_n$, where each u_i is in $\pi(U)$ and each v_i is in $\pi(V)$. If two adjacent elements are in the same group, you can apply that group's operation:

$$u_1 u_2 v_1 = (u_1 \cdot_{\pi_U} u_2) v_3$$

In terms of generators and relations, if $\pi(U) = \langle u_1, u_2, ... : r_1, r_2, ... \rangle$, $\pi(V) = \langle v_1, v_2, ... : r_1', r_2', ... \rangle$, and $\pi(U \cap V) = \langle w_1, w_2, ... : r_1'', r_2'', ... \rangle$, then

$$\pi(U) * \pi(V) = \langle u_1, u_2, ..., v_1, v_2, ... : r_1, r_2, ..., r_1', r_2', ... \rangle$$

$$\frac{\pi(U) * \pi(V)}{N} = \langle u_1, ..., v_1, ... : r_1, ..., r_1', ..., i_U(w_1) i_V(w_1)^{-1}, ... \rangle$$

Seifert – Van Kampen Theorem, corollary

- We will prove: If $X = U \cup V$, $\pi(X) = \frac{\pi(U) * \pi(V)}{N}$ where N is the smallest subgroup containing $\phi_{U\cap V,U}(x)\phi_{U\cap V,V}(x)^{-1}$ for all x in $U \cap V$.
- Elements of the free product $\pi(U) * \pi(V)$ are strings like $u_1 v_1 u_2 .. v_n$, where each u_i is in $\pi(U)$ and each v_i is in $\pi(V)$. If two adjacent elements are in the same group, you can apply that group's operation:

$$u_1 u_2 v_1 = (u_1 \cdot_{\pi_U} u_2) v_3$$

In terms of generators and relations, if $\pi(U) = \langle u_1, u_2, ... : r_1, r_2, ... \rangle$, $\pi(V) = \langle v_1, v_2, ... : r_1', r_2', ... \rangle$, and $\pi(U \cap V) = \langle w_1, w_2, ... : r_1'', r_2'', ... \rangle$, then

$$\pi(U) * \pi(V) = \langle u_1, u_2, ..., v_1, v_2, ... : r_1, r_2, ..., r_1', r_2', ... \rangle$$

$$\frac{\pi(U) * \pi(V)}{N} = \langle u_1, ..., v_1, ... : r_1, ..., r_1', ..., i_U(w_1) i_V(w_1)^{-1}, ... \rangle$$

- Proof: We can define a function $F : \pi(U) * \pi(V) \to \pi(X)$, by

$$F(u_1 v_1 u_2 ... v_n) = \psi_U(u_1)\psi_V(v_1)\psi_U(u_2)...\psi_V(v_n)$$

If $i_U(x) i_V(x)^{-1} \in N$,

$$F(i_U(x) i_V(x)^{-1}) = \psi_U(i_U(x))\psi_V(i_V(x))^{-1} = 1$$

Seifert – Van Kampen Theorem, corollary

- We will prove: If $X = U \cup V$, $\pi(X) = \frac{\pi(U)*\pi(V)}{N}$ where N is the smallest subgroup containing $\phi_{U\cap V,U}(x)\phi_{U\cap V,V}(x)^{-1}$ for all x in $U \cap V$.
- Elements of the free product $\pi(U) * \pi(V)$ are strings like $u_1 v_1 u_2..v_n$, where each u_i is in $\pi(U)$ and each v_i is in $\pi(V)$. If two adjacent elements are in the same group, you can apply that group's operation:

$$u_1 u_2 v_1 = (u_1 \cdot_{\pi_U} u_2) v_3$$

In terms of generators and relations, if $\pi(U) = \langle u_1, u_2, ... : r_1, r_2, ...\rangle$, $\pi(V) = \langle v_1, v_2, ... : r_1', r_2', ...\rangle$, and $\pi(U \cap V) = \langle w_1, w_2, ... : r_1'', r_2'', ...\rangle$, then

$$\pi(U) * \pi(V) = \langle u_1, u_2, ..., v_1, v_2, ... : r_1, r_2, ..., r_1', r_2', ...\rangle$$

$$\frac{\pi(U) * \pi(V)}{N} = \langle u_1, ..., v_1, ... : r_1, ..., r_1', ..., i_U(w_1) i_V(w_1)^{-1}, ...\rangle$$

- Proof: We can define a function $F : \pi(U) * \pi(V) \to \pi(X)$, by

$$F(u_1 v_1 u_2 ... v_n) = \psi_U(u_1)\psi_V(v_1)\psi_U(u_2)...\psi_V(v_n)$$

If $i_U(x) i_V(x)^{-1} \in N$,

$$F(i_U(x) i_V(x)^{-1}) = \psi_U(i_U(x))\psi_V(i_V(x))^{-1} = 1$$

- So, N is in the kernel of F and the function F is well defined $\frac{\pi(U)*\pi(V)}{N} \to \pi(X)$. By the proof of Seifert - Van Kampen's theorem, part 1, we know that elements $\psi_U(u_1)\psi_V(v_1)\psi_U(u_2)...\psi_V(v_n)$ generate $\pi(X)$, so F is surjective onto $\pi(X)$. We also have:

$$i_1 : \pi(U) \to \frac{\pi(U) * \pi(V)}{N}, i_1(u) = u$$

$$i_2 : \pi(V) \to \frac{\pi(U) * \pi(V)}{N}, i_1(v) = v$$

$$i_3 : \pi(U \cap V) \to \frac{\pi(U) * \pi(V)}{N}, i_1(x) = i_U(x) = i_V(x)$$

where the last two elements are equal in the quotient group since $i_U(x) i_V(x)^{-1} \in N$.

Seifert – Van Kampen Theorem, corollary

- We will prove: If $X = U \cup V$, $\pi(X) = \frac{\pi(U)*\pi(V)}{N}$ where N is the smallest subgroup containing $\phi_{U\cap V,U}(x)\phi_{U\cap V,V}(x)^{-1}$ for all x in $U \cap V$.
- Elements of the free product $\pi(U) * \pi(V)$ are strings like $u_1 v_1 u_2 .. v_n$, where each u_i is in $\pi(U)$ and each v_i is in $\pi(V)$. If two adjacent elements are in the same group, you can apply that group's operation:

$$u_1 u_2 v_1 = (u_1 \cdot_{\pi_U} u_2) v_3$$

In terms of generators and relations, if $\pi(U) = \langle u_1, u_2, \ldots : r_1, r_2, \ldots \rangle$, $\pi(V) = \langle v_1, v_2, \ldots : r'_1, r'_2, \ldots \rangle$, and $\pi(U \cap V) = \langle w_1, w_2, \ldots : r''_1, r''_2, \ldots \rangle$, then

$$\pi(U) * \pi(V) = \langle u_1, u_2, \ldots, v_1, v_2, \ldots : r_1, r_2, \ldots, r'_1, r'_2, \ldots \rangle$$

$$\frac{\pi(U) * \pi(V)}{N} = \langle u_1, \ldots, v_1, \ldots : r_1, \ldots, r'_1, \ldots, i_U(w_1) i_V(w_1)^{-1}, \ldots \rangle$$

- Proof: We can define a function $F : \pi(U) * \pi(V) \to \pi(X)$, by

$$F(u_1 v_1 u_2 \ldots v_n) = \psi_U(u_1)\psi_V(v_1)\psi_U(u_2)\ldots\psi_V(v_n)$$

If $i_U(x) i_V(x)^{-1} \in N$,

$$F(i_U(x) i_V(x)^{-1}) = \psi_U(i_U(x))\psi_V(i_V(x))^{-1} = 1$$

- So, N is in the kernel of F and the function F is well defined $\frac{\pi(U)*\pi(V)}{N} \to \pi(X)$. By the proof of Seifert - Van Kampen's theorem, part 1, we know that elements $\psi_U(u_1)\psi_V(v_1)\psi_U(u_2)\ldots\psi_V(v_n)$ generate $\pi(X)$, so F is surjective onto $\pi(X)$. We also have:

$$i_1 : \pi(U) \to \frac{\pi(U) * \pi(V)}{N}, i_1(u) = u$$

$$i_2 : \pi(V) \to \frac{\pi(U) * \pi(V)}{N}, i_1(v) = v$$

$$i_3 : \pi(U \cap V) \to \frac{\pi(U) * \pi(V)}{N}, i_1(x) = i_U(x) = i_V(x)$$

where the last two elements are equal in the quotient group since $i_U(x) i_V(x)^{-1} \in N$.
- Then, we can apply Seifert - Van Kampen's theorem and get $\sigma : \pi(X) \to \frac{\pi(U)*\pi(V)}{N}$. The following commutes:

$$\begin{tikzcd}
\pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"'] & \pi_1(X) \arrow[d, "\sigma"] \\
 & H
\end{tikzcd}$$

- The following commutes:

$$\begin{tikzcd} \pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"'] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- The following commutes:

$$\begin{tikzcd}
\pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"] & \pi_1(X) \arrow[d, "\sigma"] \\
& H
\end{tikzcd}$$

- Elements of $\pi(U)$ and $\pi(V)$ generate $\pi(U) * \pi(V)$, which generates $\frac{\pi(U)*\pi(V)}{N}$.

- The following commutes:

$$\begin{tikzcd} \pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- Elements of $\pi(U)$ and $\pi(V)$ generate $\pi(U) * \pi(V)$, which generates $\frac{\pi(U)*\pi(V)}{N}$.
- If $u \in \pi(U)$,

$$(\sigma \cdot F)(u) = \sigma(i_U(u)) = i_1(u) = u$$

Similarly, if $v \in \pi(U)$,

$$(\sigma \cdot F)(v) = \sigma(i_V(u)) = i_2(v) = v$$

- The following commutes:

$$\begin{tikzcd} \pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"'] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- Elements of $\pi(U)$ and $\pi(V)$ generate $\pi(U) * \pi(V)$, which generates $\frac{\pi(U)*\pi(V)}{N}$.
- If $u \in \pi(U)$,

$$(\sigma \cdot F)(u) = \sigma(i_U(u)) = i_1(u) = u$$

  Similarly, if $v \in \pi(U)$,

$$(\sigma \cdot F)(v) = \sigma(i_V(u)) = i_2(v) = v$$

- Then, this means $\sigma \cdot F$ is the identity function.

- The following commutes:

$$\begin{tikzcd} \pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- Elements of $\pi(U)$ and $\pi(V)$ generate $\pi(U) * \pi(V)$, which generates $\frac{\pi(U)*\pi(V)}{N}$.
- If $u \in \pi(U)$,

$$(\sigma \cdot F)(u) = \sigma(i_U(u)) = i_1(u) = u$$

Similarly, if $v \in \pi(U)$,

$$(\sigma \cdot F)(v) = \sigma(i_V(u)) = i_2(v) = v$$

- Then, this means $\sigma \cdot F$ is the identity function.
- Then, F is injective.

- The following commutes:

$$\begin{tikzcd} \pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"'] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- Elements of $\pi(U)$ and $\pi(V)$ generate $\pi(U) * \pi(V)$, which generates $\frac{\pi(U)*\pi(V)}{N}$.
- If $u \in \pi(U)$,

$$(\sigma \cdot F)(u) = \sigma(i_U(u)) = i_1(u) = u$$

  Similarly, if $v \in \pi(U)$,

$$(\sigma \cdot F)(v) = \sigma(i_V(u)) = i_2(v) = v$$

- Then, this means $\sigma \cdot F$ is the identity function.
- Then, F is injective.
- We already knew F was surjective, so F is an isomorphism and the groups $\pi(X)$ and $\frac{\pi(U)*\pi(V)}{N}$ are isomorphic.

- The following commutes:

$$\begin{tikzcd} \pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"'] & \pi_1(X) \arrow[d, "\sigma"] \\ & H \end{tikzcd}$$

- Elements of $\pi(U)$ and $\pi(V)$ generate $\pi(U) * \pi(V)$, which generates $\frac{\pi(U)*\pi(V)}{N}$.
- If $u \in \pi(U)$,

$$(\sigma \cdot F)(u) = \sigma(i_U(u)) = i_1(u) = u$$

  Similarly, if $v \in \pi(U)$,

$$(\sigma \cdot F)(v) = \sigma(i_V(u)) = i_2(v) = v$$

- Then, this means $\sigma \cdot F$ is the identity function.
- Then, F is injective.
- We already knew F was surjective, so F is an isomorphism and the groups $\pi(X)$ and $\frac{\pi(U)*\pi(V)}{N}$ are isomorphic.

- To conclude,

$$\pi(X) = \langle u_1, ..., v_1, ... : r_1, ..., r_1', ..., i_U(w_1) i_V(w_1)^{-1}, ... \rangle$$
$$= \langle u_1, ..., v_1, ... : r_1, ..., r_1', ..., i_U(w_1) = i_V(w_1), ... \rangle$$

where

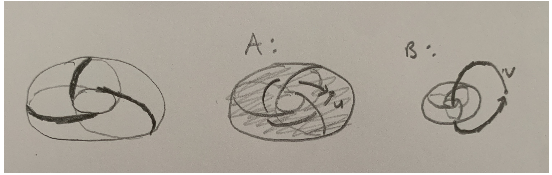
$$\pi(U) = \langle u_1, u_2, ... : r_1, r_2, ... \rangle$$
$$\pi(V) = \langle v_1, v_2, ... : r_1', r_2', ... \rangle$$
$$\pi(U \cap V) = \langle w_1, w_2, ... : r_1'', r_2'', ... \rangle$$

Calculating the group of the trefoil knot

- We'll consider the trefoil knot we saw in the introduction.

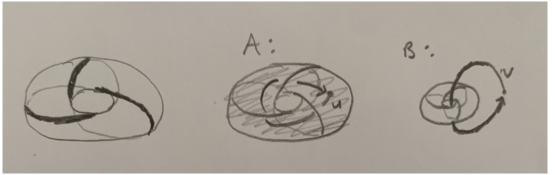
Calculating the group of the trefoil knot

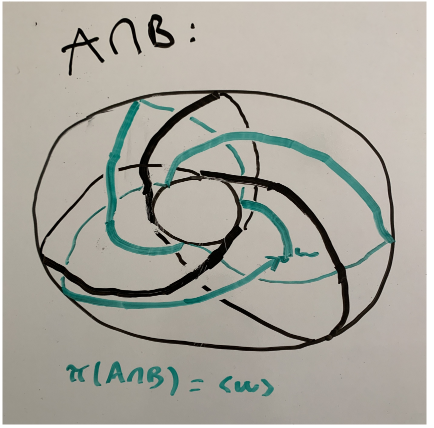
- We'll consider the trefoil knot we saw in the introduction.
- We can draw it on a torus:



Calculating the group of the trefoil knot

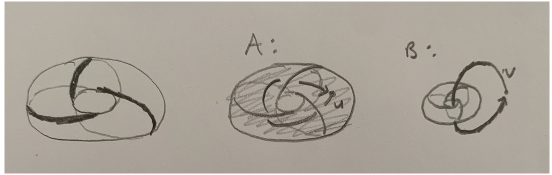
- We'll consider the trefoil knot we saw in the introduction.
- We can draw it on a torus:
- Then, split the space $R^3 - K$ into three sections:
  A: inside the torus, including the boundary (but not K)
  B: outside the torus, including the boundary (but not K)
  $A \cap B$: the boundary of the torus, not including K.

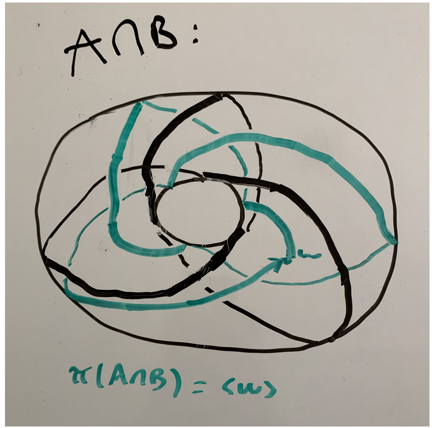




Calculating the group of the trefoil knot

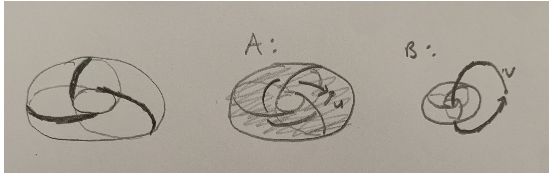
- We'll consider the trefoil knot we saw in the introduction.
- We can draw it on a torus:
- Then, split the space $R^3 - K$ into three sections:
  A: inside the torus, including the boundary (but not K)
  B: outside the torus, including the boundary (but not K)
  $A \cap B$: the boundary of the torus, not including K.
- The fundemental group of A is essentially the same as the fundemental group of S^1, which is $\langle u \rangle$, where u is a loop going aroung the hole

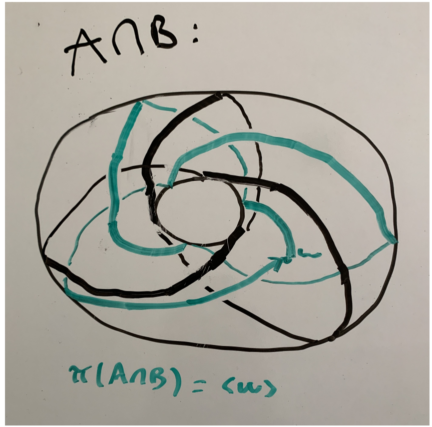




Calculating the group of the trefoil knot

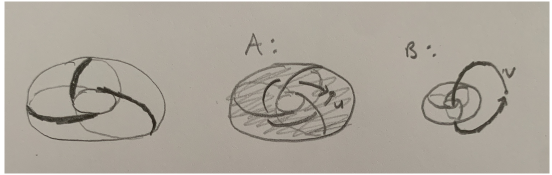
- We'll consider the trefoil knot we saw in the introduction.
- We can draw it on a torus:
- Then, split the space $R^3 - K$ into three sections:
  A: inside the torus, including the boundary (but not K)
  B: outside the torus, including the boundary (but not K)
  $A \cap B$: the boundary of the torus, not including K.
- The fundemental group of A is essentially the same as the fundemental group of S^1, which is $\langle u \rangle$, where u is a loop going aroung the hole
- The fundemental group of B is essentially the same as the fundemental group of $R^3 - S^1$, which we saw is $\langle v \rangle$ where v is the loop in the picture:

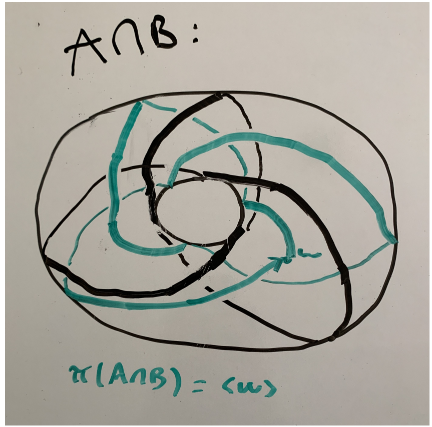




Calculating the group of the trefoil knot

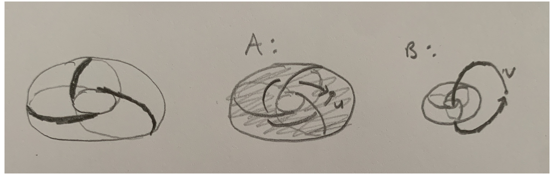
- We'll consider the trefoil knot we saw in the introduction.
- We can draw it on a torus:
- Then, split the space $R^3 - K$ into three sections:
  A: inside the torus, including the boundary (but not K)
  B: outside the torus, including the boundary (but not K)
  $A \cap B$: the boundary of the torus, not including K.
- The fundemental group of A is essentially the same as the fundemental group of S^1, which is $\langle u \rangle$, where u is a loop going aroung the hole
- The fundemental group of B is essentially the same as the fundemental group of $R^3 - S^1$, which we saw is $\langle v \rangle$ where v is the loop in the picture:
- $A \cap B$ is essentially just a strip in R^3, so its fundemental group is going around once in the strip, $\langle w \rangle$.

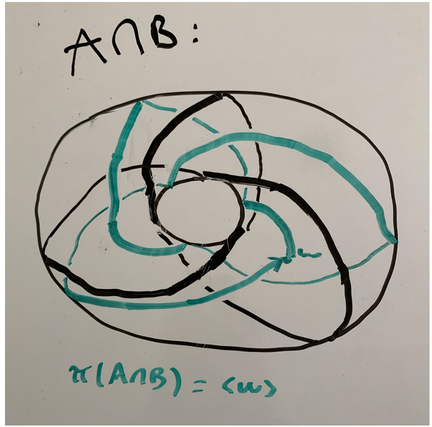




Calculating the group of the trefoil knot

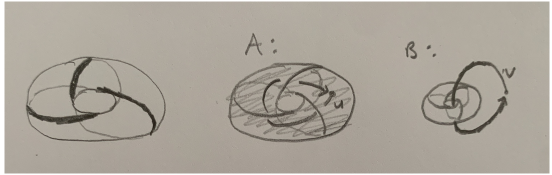
- We'll consider the trefoil knot we saw in the introduction.
- We can draw it on a torus:
- Then, split the space $R^3 - K$ into three sections:
  A: inside the torus, including the boundary (but not K)
  B: outside the torus, including the boundary (but not K)
  $A \cap B$: the boundary of the torus, not including K.
- The fundemental group of A is essentially the same as the fundemental group of S^1, which is $\langle u \rangle$, where u is a loop going aroung the hole
- The fundemental group of B is essentially the same as the fundemental group of $R^3 - S^1$, which we saw is $\langle v \rangle$ where v is the loop in the picture:
- $A \cap B$ is essentially just a strip in R^3, so its fundemental group is going around once in the strip, $\langle w \rangle$.
- w goes around the torus two times, so $i_U(w) = u^2$. It rotates three times, around the outside, so $i_V(w) = v^3$.

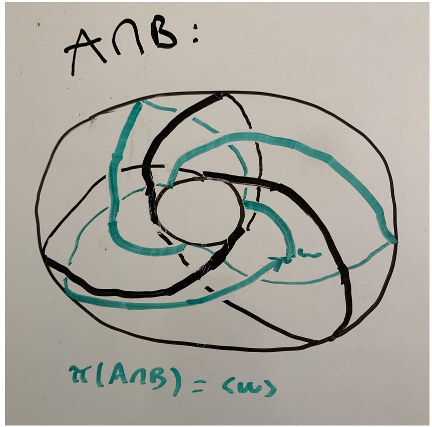




Calculating the group of the trefoil knot

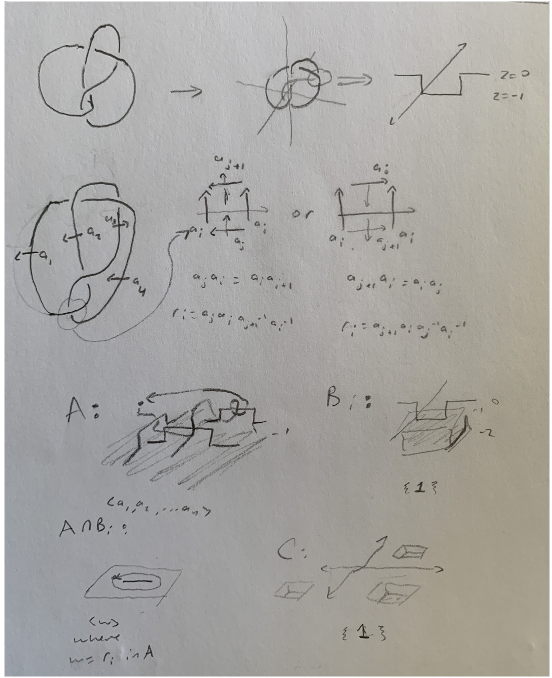
- We'll consider the trefoil knot we saw in the introduction.
- We can draw it on a torus:
- Then, split the space $R^3 - K$ into three sections:
  A: inside the torus, including the boundary (but not K)
  B: outside the torus, including the boundary (but not K)
  $A \cap B$: the boundary of the torus, not including K.
- The fundemental group of A is essentially the same as the fundemental group of S^1, which is $\langle u \rangle$, where u is a loop going aroung the hole
- The fundemental group of B is essentially the same as the fundemental group of $R^3 - S^1$, which we saw is $\langle v \rangle$ where v is the loop in the picture:
- $A \cap B$ is essentially just a strip in R^3, so its fundemental group is going around once in the strip, $\langle w \rangle$.
- w goes around the torus two times, so $i_U(w) = u^2$. It rotates three times, around the outside, so $i_V(w) = v^3$.
- By Seifert – Van Kampen's theorem, $\pi(R^3 - K) = \langle u, v, u^2 = v^3 \rangle$





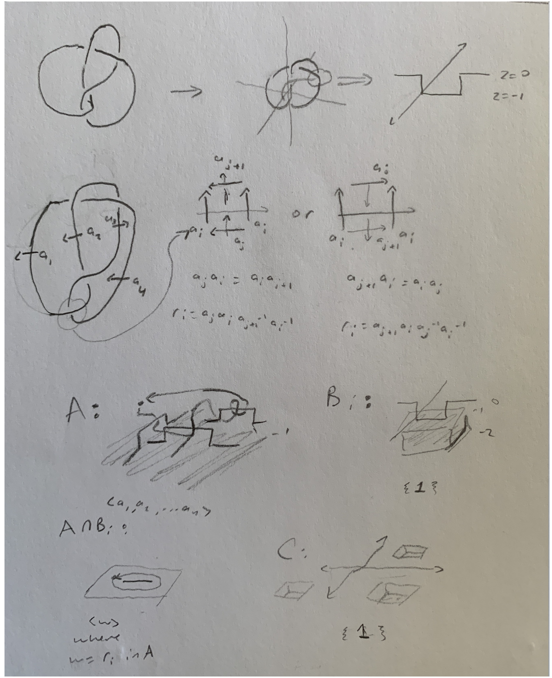
Wirtinger Presentation

- Take any 'drawing' of a knot as curves and put it in $\mathbb{R}^3$ as shown. Draw lines a_i going across every section of the knot.



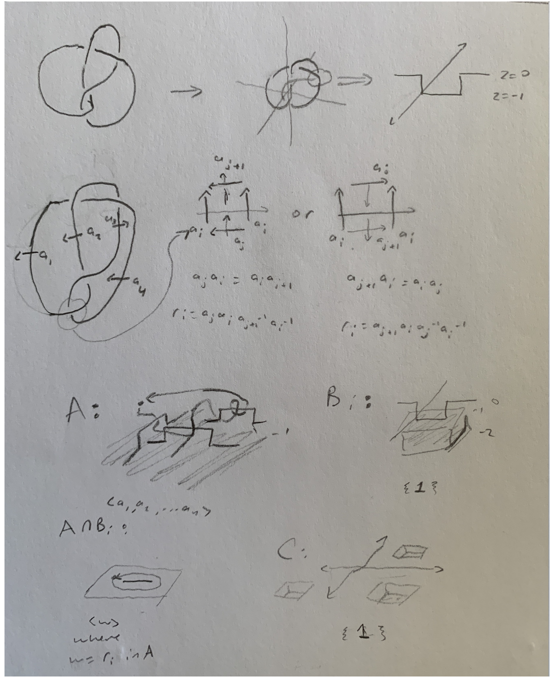
Wirtinger Presentation

- Take any 'drawing' of a knot as curves and put it in $\mathbb{R}^3$ as shown. Draw lines a_i going across every section of the knot.
- At every intersection, there are one of two intersections r_i.



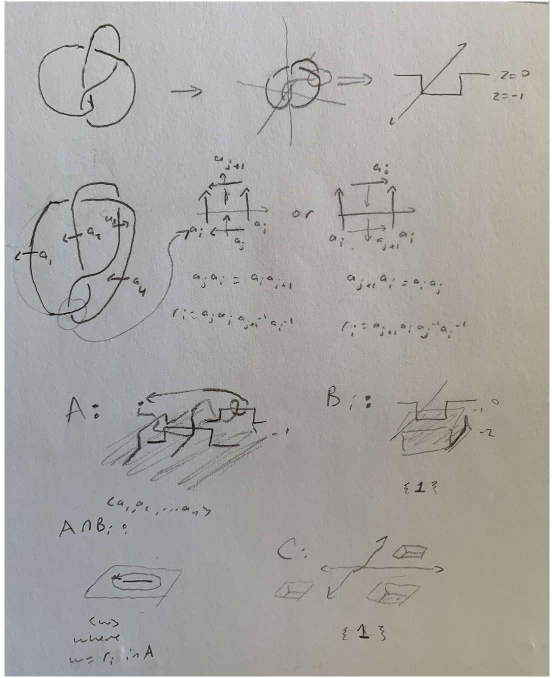
Wirtinger Presentation

- Take any 'drawing' of a knot as curves and put it in $\mathbb{R}^3$ as shown. Draw lines a_i going across every section of the knot.
- At every intersection, there are one of two intersections r_i.
- Divide up $\mathbb{R}^3$ as follows:



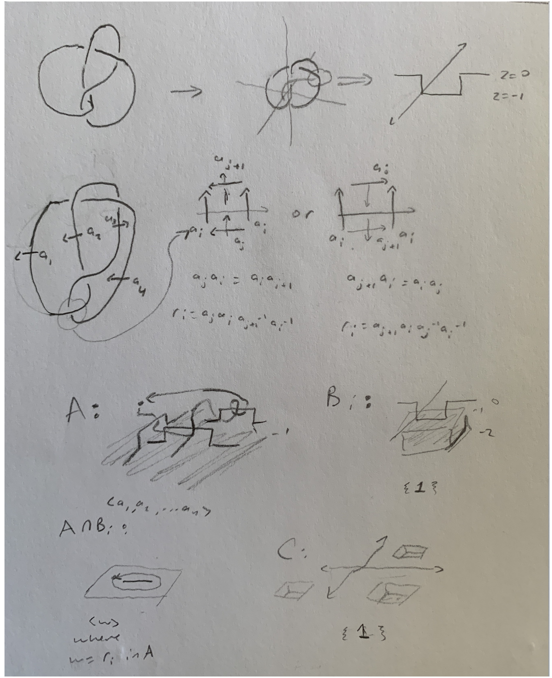
Wirtinger Presentation

- Take any 'drawing' of a knot as curves and put it in $\mathbb{R}^3$ as shown. Draw lines a_i going across every section of the knot.
- At every intersection, there are one of two intersections r_i.
- Divide up $\mathbb{R}^3$ as follows:
- $A = \{(x, y, z) : z >= -1\}$. In A, each line segments forms a handle, and the fundemental group of A is going through any number of them in any order, so $\langle a_1, ...a_n \rangle$.



Wirtinger Presentation

- Take any 'drawing' of a knot as curves and put it in $\mathbb{R}^3$ as shown. Draw lines a_i going across every section of the knot.
- At every intersection, there are one of two intersections r_i.
- Divide up $\mathbb{R}^3$ as follows:
- $A = \{(x, y, z) : z >= -1\}$. In A, each line segments forms a handle, and the fundemental group of A is going through any number of them in any order, so $\langle a_1, ...a_n \rangle$.
- B_i: rectangular box under every intersection, with its top at $z = -1$. Any path in the rectangular box is trivial, so its fundemental group is 1.



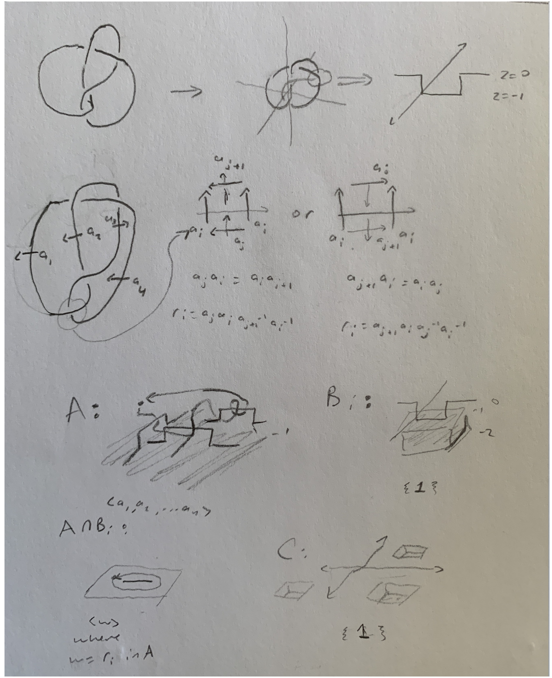
Wirtinger Presentation

- Take any 'drawing' of a knot as curves and put it in $\mathbb{R}^3$ as shown. Draw lines a_i going across every section of the knot.
- At every intersection, there are one of two intersections r_i.
- Divide up $\mathbb{R}^3$ as follows:
- $A = \{(x, y, z) : z >= -1\}$. In A, each line segments forms a handle, and the fundemental group of A is going through any number of them in any order, so $\langle a_1, ...a_n \rangle$.
- B_i: rectangular box under every intersection, with its top at $z = -1$. Any path in the rectangular box is trivial, so its fundemental group is 1.
- C: Anything below A and all the B's. Its fundemental group is 1 as well.
  We'll take A and add the B_i's one by one. $A \cap B_1$ is a rectangle with a line segment in the middle cut out. So, its fundemental group is generated by going around that line segment once, which, included in A is exactly r_i. So, the fundemental group is $\langle a_1, ...a_n : r_1 \rangle$.
  Adding the other B_i's one by one, the fundemental group becomes

  $$\langle a_1, a_2, ...a_n : r_1, r_2, ...r_n \rangle$$

  The intersection of $A \cup B_1 ... \cup B_n$ and C also has a trivial fundemental group. So the fundemental group of the union of all the parts, $\mathbb{R}^3 - K$ is:

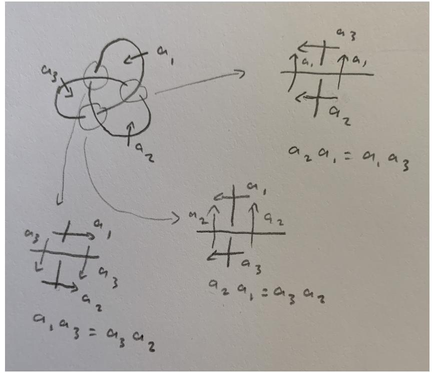
  $$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, ...a_n : r_1, r_2, ...r_n \rangle$$



Do we get the same group with the Wirtinger Presentation?

- For the trefoil, we get three line segments and three relations as shown.

  $$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, a_3 : a_1 a_3 = a_3 a_2, a_2 a_1 = a_3 a_2, a_2 a_1 = a_1 a_3 \rangle$$



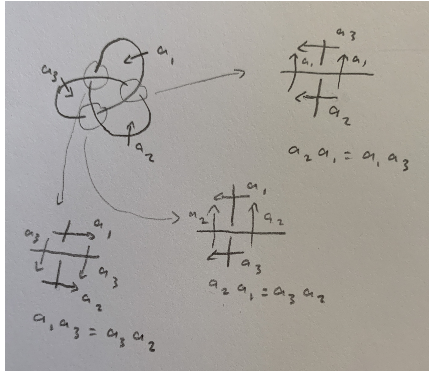
Do we get the same group with the Wirtinger Presentation?

- For the trefoil, we get three line segments and three relations as shown.

$$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, a_3 : a_1 a_3 = a_3 a_2, a_2 a_1 = a_3 a_2, a_2 a_1 = a_1 a_3 \rangle$$

- Clearly, the first two relations imply the third, so this group is equivalent to

$$\langle a_1, a_2, a_3 : a_1 a_3 = a_3 a_2, a_2 a_1 = a_1 a_3 \rangle$$



Do we get the same group with the Wirtinger Presentation?

- For the trefoil, we get three line segments and three relations as shown.

$$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, a_3 : a_1 a_3 = a_3 a_2, a_2 a_1 = a_3 a_2, a_2 a_1 = a_1 a_3 \rangle$$

- Clearly, the first two relations imply the third, so this group is equivalent to

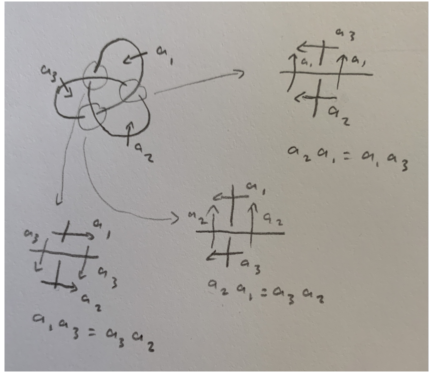
$$\langle a_1, a_2, a_3 : a_1 a_3 = a_3 a_2, a_2 a_1 = a_1 a_3 \rangle$$

- We can solve the first relation for a_1 to get $a_1 = a_3 a_2 a_3^{-1}$. So, any string of a_1,a_2, a_3 is also generated by a_2 and a_3 with a_1 as this expression. So, we get:

$$\langle a_2, a_3 : a_2(a_3 a_2 a_3^{-1}) = (a_3 a_2 a_3^{-1}) a_3 = a_3 a_2 \rangle$$

$$\langle u, v : uvu = vuv \rangle$$

$$uvu * uvu = vu * vu * vu$$



Do we get the same group with the Wirtinger Presentation?

- For the trefoil, we get three line segments and three relations as shown.

$$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, a_3 : a_1 a_3 = a_3 a_2, a_2 a_1 = a_3 a_2, a_2 a_1 = a_1 a_3 \rangle$$

- Clearly, the first two relations imply the third, so this group is equivalent to

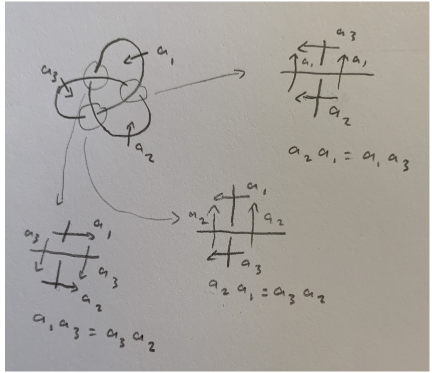
$$\langle a_1, a_2, a_3 : a_1 a_3 = a_3 a_2, a_2 a_1 = a_1 a_3 \rangle$$

- We can solve the first relation for a_1 to get $a_1 = a_3 a_2 a_3^{-1}$. So, any string of a_1,a_2, a_3 is also generated by a_2 and a_3 with a_1 as this expression. So, we get:

$$\langle a_2, a_3 : a_2(a_3 a_2 a_3^{-1}) = (a_3 a_2 a_3^{-1}) a_3 = a_3 a_2 \rangle$$

$$\langle u, v : uvu = vuv \rangle$$

$$uvu * uvu = vu * vu * vu$$



Do we get the same group with the Wirtinger Presentation?

- For the trefoil, we get three line segments and three relations as shown.

$$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, a_3 : a_1a_3 = a_3a_2, a_2a_1 = a_3a_2, a_2a_1 = a_1a_3 \rangle$$

- Clearly, the first two relations imply the third, so this group is equivalent to

$$\langle a_1, a_2, a_3 : a_1a_3 = a_3a_2, a_2a_1 = a_1a_3 \rangle$$

- We can solve the first relation for a_1 to get $a_1 = a_3a_2a_3^{-1}$. So, any string of a_1,a_2, a_3 is also generated by a_2 and a_3 with a_1 as this expression. So, we get:

$$\langle a_2, a_3 : a_2(a_3a_2a_3^{-1}) = (a_3a_2a_3^{-1})a_3 = a_3a_2 \rangle$$

$$\langle u, v : uvu = vuv \rangle$$

$$uvu * uvu = vu * vu * vu$$

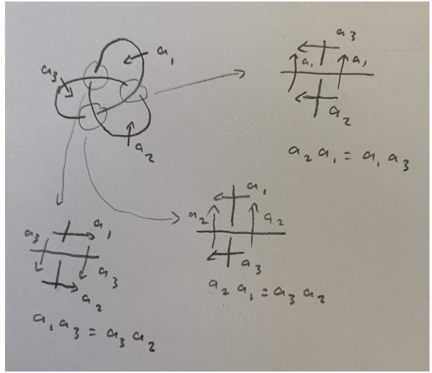
- Let $w = uvu$, $z = vu$. Then,

$$u = wz^{-1}, v = zu^{-1} = z(wz^{-1})^{-1} = z^2w^{-1}$$

So, we can express any string of u's and v's as a string of w's and z's, so the group is also generated by w and z. Then, the relation we have becomes $w^2 = z^3$.

$$\langle w, z : w^2 = z^3 \rangle$$

And this is the same group we got before.



Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.
- Consider the set $\{1, 2, 3\}$, and the group of invertible functions $\{1, 2, 3\} \to \{1, 2, 3\}$, with the group operation being function composition, S_3.

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.
- Consider the set $\{1, 2, 3\}$, and the group of invertible functions $\{1, 2, 3\} \to \{1, 2, 3\}$, with the group operation being function composition, S_3.
- Define f and g as follows:
$$f(1) = 2, f(2) = 1, f(3) = 3$$
$$g(1) = 2, g(2) = 3, g(3) = 1$$

Then,
$$f^2(1) = 1, f^2(2) = 2, f^2(3) = 3$$
$$g^2(1) = 3, g^2(2) = 1, g^2(3) = 2$$
$$g^3(1) = 1, g^3(2) = 2, g^3(3) = 3$$

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.
- Consider the set $\{1, 2, 3\}$, and the group of invertible functions $\{1, 2, 3\} \to \{1, 2, 3\}$, with the group operation being function composition, S_3.
- Define f and g as follows:
$$f(1) = 2, f(2) = 1, f(3) = 3$$
$$g(1) = 2, g(2) = 3, g(3) = 1$$
Then,
$$f^2(1) = 1, f^2(2) = 2, f^2(3) = 3$$
$$g^2(1) = 3, g^2(2) = 1, g^2(3) = 2$$
$$g^3(1) = 1, g^3(2) = 2, g^3(3) = 3$$
- So, both are invertible, and so in S_3, and in the group, $f^2 = g^3 = 1$. Then, since any string of f's and g's is still in S_3, and $f^2 = g^3$, we can define a homomorphism $\sigma : B_3 \to S_3$:
$$\sigma(a) = f, \sigma(b) = g$$
And this is a well-defined function – if two different strings are the same element in B_3, the σ of both strings are equivalent in S_3.

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.
- Consider the set $\{1, 2, 3\}$, and the group of invertible functions $\{1, 2, 3\} \to \{1, 2, 3\}$, with the group operation being function composition, S_3.
- Define f and g as follows:
$$f(1) = 2, f(2) = 1, f(3) = 3$$
$$g(1) = 2, g(2) = 3, g(3) = 1$$
Then,
$$f^2(1) = 1, f^2(2) = 2, f^2(3) = 3$$
$$g^2(1) = 3, g^2(2) = 1, g^2(3) = 2$$
$$g^3(1) = 1, g^3(2) = 2, g^3(3) = 3$$
- So, both are invertible, and so in S_3, and in the group, $f^2 = g^3 = 1$. Then, since any string of f's and g's is still in S_3, and $f^2 = g^3$, we can define a homomorphism $\sigma : B_3 \to S_3$:
$$\sigma(a) = f, \sigma(b) = g$$
And this is a well-defined function – if two different strings are the same element in B_3, the σ of both strings are equivalent in S_3.
- But,
$$(fg)(1) = 1, (fg)(2) = 3, (fg)(3) = 2$$
$$(gf)(1) = 3, (gf)(2) = 3, (gf)(3) = 1$$

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.
- Consider the set $\{1, 2, 3\}$, and the group of invertible functions $\{1, 2, 3\} \to \{1, 2, 3\}$, with the group operation being function composition, S_3.
- Define f and g as follows:
$$f(1) = 2, f(2) = 1, f(3) = 3$$
$$g(1) = 2, g(2) = 3, g(3) = 1$$
Then,
$$f^2(1) = 1, f^2(2) = 2, f^2(3) = 3$$
$$g^2(1) = 3, g^2(2) = 1, g^2(3) = 2$$
$$g^3(1) = 1, g^3(2) = 2, g^3(3) = 3$$
- So, both are invertible, and so in S_3, and in the group, $f^2 = g^3 = 1$. Then, since any string of f's and g's is still in S_3, and $f^2 = g^3$, we can define a homomorphism $\sigma : B_3 \to S_3$:
$$\sigma(a) = f, \sigma(b) = g$$
And this is a well-defined function – if two different strings are the same element in B_3, the σ of both strings are equivalent in S_3.
- But,
$$(fg)(1) = 1, (fg)(2) = 3, (fg)(3) = 2$$
$$(gf)(1) = 3, (gf)(2) = 3, (gf)(3) = 1$$
- So,
$$fg \neq gf$$
$$\sigma(ab) \neq \sigma(ba)$$
$$ab \neq ba$$

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.
- Consider the set $\{1, 2, 3\}$, and the group of invertible functions $\{1, 2, 3\} \to \{1, 2, 3\}$, with the group operation being function composition, S_3.
- Define f and g as follows:
$$f(1) = 2, f(2) = 1, f(3) = 3$$
$$g(1) = 2, g(2) = 3, g(3) = 1$$
Then,
$$f^2(1) = 1, f^2(2) = 2, f^2(3) = 3$$
$$g^2(1) = 3, g^2(2) = 1, g^2(3) = 2$$
$$g^3(1) = 1, g^3(2) = 2, g^3(3) = 3$$
- So, both are invertible, and so in S_3, and in the group, $f^2 = g^3 = 1$. Then, since any string of f's and g's is still in S_3, and $f^2 = g^3$, we can define a homomorphism $\sigma : B_3 \to S_3$:
$$\sigma(a) = f, \sigma(b) = g$$
And this is a well-defined function – if two different strings are the same element in B_3, the σ of both strings are equivalent in S_3.
- But,
$$(fg)(1) = 1, (fg)(2) = 3, (fg)(3) = 2$$
$$(gf)(1) = 3, (gf)(2) = 3, (gf)(3) = 1$$
- So,
$$fg \neq gf$$
$$\sigma(ab) \neq \sigma(ba)$$
$$ab \neq ba$$
- Since there are elements a, b of B_3 such that $ab \neq ba$, B_3 is not commutative, and can't equal $\langle a \rangle$.

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.
- Consider the set $\{1, 2, 3\}$, and the group of invertible functions $\{1, 2, 3\} \to \{1, 2, 3\}$, with the group operation being function composition, S_3.
- Define f and g as follows:
$$f(1) = 2, f(2) = 1, f(3) = 3$$
$$g(1) = 2, g(2) = 3, g(3) = 1$$
Then,
$$f^2(1) = 1, f^2(2) = 2, f^2(3) = 3$$
$$g^2(1) = 3, g^2(2) = 1, g^2(3) = 2$$
$$g^3(1) = 1, g^3(2) = 2, g^3(3) = 3$$
- So, both are invertible, and so in S_3, and in the group, $f^2 = g^3 = 1$. Then, since any string of f's and g's is still in S_3, and $f^2 = g^3$, we can define a homomorphism $\sigma : B_3 \to S_3$:
$$\sigma(a) = f, \sigma(b) = g$$
And this is a well-defined function – if two different strings are the same element in B_3, the σ of both strings are equivalent in S_3.
- But,
$$(fg)(1) = 1, (fg)(2) = 3, (fg)(3) = 2$$
$$(gf)(1) = 3, (gf)(2) = 3, (gf)(3) = 1$$
- So,
$$fg \neq gf$$
$$\sigma(ab) \neq \sigma(ba)$$
$$ab \neq ba$$
- Since there are elements a, b of B_3 such that $ab \neq ba$, B_3 is not commutative, and can't equal $\langle a \rangle$.
- Therefore, the trefoil is a different knot than the unknot.

Sources

- Gallagher, K. (n.d.). The fundamental group and Seifert-van Kampen's theorem. Retrieved March 6, 2022, from https://www.math.uchicago.edu/~may/REU2016/REUPapers/Gallagher.pdf
- Massey, W. S. (1991). A basic course in algebraic topology. Springer.
- Rolfsen, D. (2003). Knots and links. AMS Chelsea Publ.

You can find the source code for this presentation at:
https://github.com/yahya-tamur/seifert-van-kampen-presentation