

Seifert – Van Kampen Theorem, Applications

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March 7, 2022

Groups; generators and relations

- ▶ A group is a set of elements G , with a $\cdot : G \times G \rightarrow G$ such that:

$$(a.b).c = a.(b.c)$$

There's an identity element 1 such that for all a ,

$$1.a = a.1 = a$$

For all a , there's an inverse element a^{-1} such that:

$$a^{-1}.a = a.a^{-1} = 1$$

- ▶ Notably, it doesn't have to be commutative ($a.b$ isn't necessarily $b.a$).
- ▶ Common examples include invertible functions under composition, integers under addition (but not multiplication), positive rationals under multiplication.
- ▶ A function between groups that preserves products, identity, and inverses – $\sigma(a.b) = \sigma(a).\sigma(b)$, $\sigma(1) = 1$, $\sigma(a^{-1}) = \sigma(a)^{-1}$ is called a homomorphism. These have other special properties that can be derived.
- ▶ A set a, b, \dots generates a group G if every element can be written in as a finite combination of elements (or inverses of elements) in the group. Ex. 1 generates integers under addition, prime numbers generate positive rationals under multiplication.
- ▶ Every group has a generator – the set of all elements in the group.

- ▶ A presentation of a group by its generators and relations is essentially the free group, together with a few equivalences:
 - ▶ $\langle a, b : a^5, b^3 \rangle$ means the free group $\langle a, b \rangle$, but a^5 and b^3 also have to be equal to the identity.
 - ▶ This can be expressed as $\langle a \rangle$ divided by the smallest normal subgroup that contains a^5 and b^3 .
 - ▶ If there's any group G generated by a and b , with $a^5 = b^3 = 1$, there's a homomorphism $\langle a, b : a^5, b^3 \rangle \rightarrow G$:
$$\sigma((a)) = a, \sigma((b))$$

And, this is well-defined, even though some sequences might be equivalent in $\langle a, b : a^5, b^3 \rangle$ without being equal, because whenever terms cancel out in $\langle a, b : a^5, b^3 \rangle$, they cancel out in G :

$$(a, a, a, b, b, b, a, a) = (a, a, a, a, a) = 1$$

$$\sigma((a, a, a, b, b, b, a, a)) = aaabbbaa \in G = aaaaa = 1$$

- ▶ So, for any group generated by the generators and satisfying the relations, there's a surjective homomorphism from the presentation onto the group.
- ▶ Note: Some relations might be simplified by adding an equal sign:

$$\langle a, b : aba^{-1}, b^2 \rangle = \langle a, b : ab = a, b^2 \rangle$$

Fundemental Groups

- ▶ Let X be a topological space, and x_0 be any point in it.
- ▶ A function $f : [0, 1] \rightarrow X$ represents a curve in X . If $f(0) = f(1) = x_0$, we'll call it a loop.
- ▶ Define equivalence classes where f is equivalent to g if the loops can be continuously deformed into each other. We'll refer to these equivalence classes as paths. If f is a loop, and a is the path f is in, I'll denote $a = [f]$.
- ▶ In \mathbb{R}^2 , $x_0 = (0, 0)$, any two loops are equivalent, and there's only one path. In $\mathbb{R}^2 - (0, 0)$, $x_0 = (1, 0)$, the two loops in the right aren't equivalent.
- ▶ More rigorously, $f \sim g$ iff there's a continuous $F : [0, 1] \times [0, 1] \rightarrow X$ where $F(0, t) = f(t)$, $F(1, t) = g(t)$, and $F(s, 0) = F(s, 1) = x_0$
- ▶ Given two loops f and g , we can compose them as follows:

$$f.g(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then, $(f.g).h \neq f.(g.h)$, but $(f.g).h \sim f.(g.h)$. Also, if $f \sim g$, $f.h \sim g.h$ for any h .

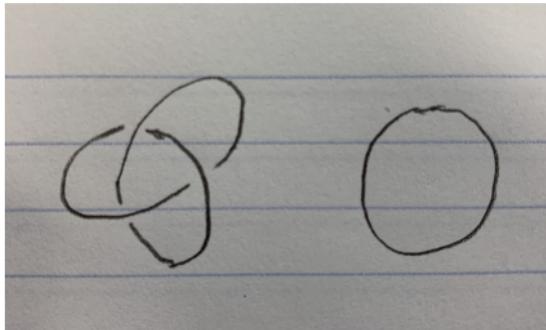
- ▶ We can define an identity element as $i(t) = x_0$ and inverse element as $f^{-1}(t) = f(1-t)$. (You can check $f.f^{-1} = f^{-1}.f = i$).
- ▶ So, paths in X with base point x_0 under this operation is a group, called the $\pi(X)$. The base point usually doesn't matter, since the groups are usually the same regardless of the base point.

Fundemental Groups

- ▶ Let σ be a continuous function between X and Y .
- ▶ Then, for loops $x, y : [0, 1] \rightarrow X$, If $x \sim y$, $\sigma(x) \sim \sigma(y)$ (If there's an appropriate continuous function F , $\sigma.F$ is a continuous function $[0, 1]^2 \rightarrow Y$ with $\sigma(F(0, t)) = \sigma(x)(t)$, $\sigma(F(1, t)) = \sigma(y)(t)$, $\sigma(F(s, 0)) = \sigma(F(s, 1)) = x_0$.
- ▶ So, σ defines a function $\sigma_* : \pi(X) \rightarrow \pi(Y)$.
- ▶ Also, $\sigma(1) = 1$. $\sigma(f.g) = \sigma(f).\sigma(g)$ for loops, so $\sigma(a.b) = \sigma(a).\sigma(b)$ for paths.
- ▶ $\sigma(f^{-1}) = \sigma(f)^{-1}$ for loops, so $\sigma(a^{-1}) = \sigma(a)^{-1}$ for paths.
- ▶ So, σ_* is a group homomorphism between $\pi(X)$ and $\pi(Y)$, called the induced homomorphism of σ .
- ▶ If $\sigma : X \rightarrow Y$ and $\mu : Y \rightarrow Z$, $\nu : X \rightarrow Z$ with $\nu = \mu \cdot \sigma$, $a = [f]$,
$$\mu_*(\sigma_*(a)) = \mu_*(\sigma_*([f])) = \mu_*([\sigma(f)]) = [\mu(\sigma(f))] = [\nu(f)] = \nu_*(a)$$
- ▶ Note: This means the fundamental group is a functor between the category of (pointed) topological spaces with (pointed) continuous functions and the category of groups and group homomorphisms.

Knots

- ▶ A knot is a subset of \mathbb{R}^3 homeomorphic to \mathbb{S}^1 (or more generally, any subset of a topological space homeomorphic to \mathbb{S}^n).



- ▶ A knot K is equivalent to K' if (K, \mathbb{R}^3) is homeomorphic to (K', \mathbb{R}^3)

In other words, there's a homeomorphism

$$h : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ so that } h(K) = K'$$

Then,

$$h|_{\mathbb{R}^3 - K} : \mathbb{R}^3 - K \rightarrow \mathbb{R}^3 - K'$$

is also a homeomorphism,

- ▶ ... and its induced homeomorphism from the fundamental groups

$$\pi_1(\mathbb{R}^3 - K) \rightarrow \pi_1(\mathbb{R}^3 - K')$$

is an isomorphism, so those groups are isomorphic.

- ▶ The group $\pi_1(\mathbb{R}^3 - K)$ is also called the fundamental group of a knot.
- ▶ The main argument we'll be making is, if the fundamental groups of two knots aren't isomorphic, then the knots aren't equivalent.
- ▶ Seifert – Van Kampen's Theorem helps determine the fundamental group of $A \cup B$ given the fundamental groups of A , B , and $A \cap B$.
- ▶ In this presentation, we'll be looking at the statement and proof of this theorem, and applying it to find the fundamental groups of a few knots.

- ▶ The 'unknot' is the following knot:
- ▶ To say that a given knot can't be unknotted, we want to say that that knot is not equal to the unknot. So, before developing Seifert-Van Kampen's theorem, let's look at the fundamental group of the unknot:
- ▶ One nontrivial path is wrapping around the circle as shown. Similar to $\mathbb{R}^2 - (0,0)$, it can be proved that any loops that wrap around the same number of times are equivalent (and loops that don't are not). So, the fundamental group is $< u >$.

Seifert – Van Kampen Theorem

- ▶ Let X be a path-connected topological space, x_0 be any point in X . Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X so that each U_λ contains x_0 and the intersection of any two elements in the cover is also in the cover.
- ▶ *here, $\{U_\lambda\}_{\lambda \in \Lambda}$ could be $\{A, B, A \cap B\}^*$
- ▶ Let ψ_λ be the homomorphism induced by the inclusion map $U_\lambda \rightarrow X$.
- ▶ For $U_\lambda \subseteq U_\mu$, let $\phi_{\lambda\mu}$ be the homomorphism induced by the inclusion map $U_\lambda \rightarrow U_\mu$. Clearly, the following commutes:
- ▶ Let H be any group and $\{p_\lambda\}_{\lambda \in \Lambda}$ be any family of homomorphisms so the following commutes:

$$\begin{array}{ccc} \pi_1(U_\lambda) & \xrightarrow{\phi_{\lambda\mu}} & \pi_1(U_\mu) \\ & \searrow \psi_\lambda & \downarrow \psi_\mu \\ & & \pi_1(X) \end{array}$$

- ▶ Then, there's a unique σ so that the following commutes:

$$\begin{array}{ccc} \pi_1(U_\lambda) & \xrightarrow{\psi_\lambda} & \pi_1(X) \\ & \searrow p_\lambda & \downarrow \sigma \\ & & H \end{array}$$

From this definition, we can tell:

- ▶ If $\alpha \in \pi_1(U_\lambda)$, $\sigma(\psi_\lambda(\alpha)) = p_\lambda(\alpha)$
 - ▶ If $\alpha \in \pi_1(U_\lambda)$, $\beta \in \pi_1(U_\mu)$,
- $$\sigma(\psi_\lambda(\alpha)\psi_\mu(\beta)) = \sigma(\psi_\lambda(\alpha))\sigma(\psi_\mu(\beta)) = p_\lambda(\alpha)p_\mu(\beta)$$
- ▶ For $\{\alpha_i\}_{i=1}^n$ so that $\alpha_i \in U_{\lambda_i}$,
- $$\sigma(\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\dots\psi_{\lambda_n}(\alpha_n)) = p_{\lambda_1}(\alpha_1)p_{\lambda_2}(\alpha_2)\dots p_{\lambda_n}(\alpha_n)$$
- ▶ We need to prove that σ is well defined. In other words, if
- $$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\dots\psi_{\lambda_n}(\alpha_n) \sim \psi_{\mu_1}(\beta_1)\psi_{\mu_2}(\beta_2)\dots\psi_{\mu_m}(\beta_m)$$
- Then, $\sigma(\psi_{\lambda_1}(\alpha_1)\dots\psi_{\lambda_n}(\alpha_n)) \sim \sigma(\psi_{\mu_1}(\beta_1)\dots\psi_{\mu_m}(\beta_m))$
 So, $p_{\lambda_1}(\alpha_1)\dots p_{\lambda_n}(\alpha_n) \sim p_{\mu_1}(\beta_1)\dots p_{\mu_m}(\beta_m)$
- ▶ Since this is all the restrictions on σ , but σ is unique, $\pi_1(X)$ must not have any elements which aren't in the form
- $$\psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\dots\psi_{\lambda_n}(\alpha_n)$$

We also need to prove this.

- ▶ We'll also look at when two elements of $\pi_1(X)$ are equal and when they're different.
- ▶ But hopefully it makes sense how this theorem determines $\pi_1(X)$!

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of $\pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\dots\psi_{\lambda_n}(\alpha_n)$$

for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.

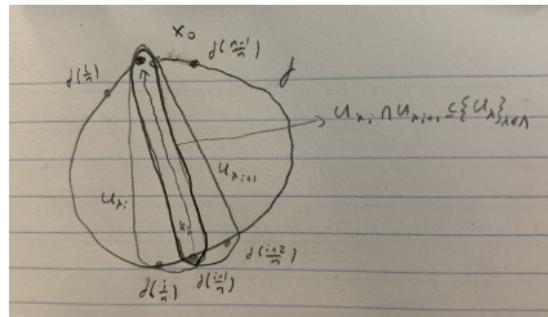
- We will use:

Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.

- For any $a \in \pi_1(X)$, find a path $f : [0, 1] \rightarrow X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is a cover of the compact metric space $[0, 1]$. It has a Lebesgue number δ .
- Find n so $\frac{1}{n} < \delta$, divide $[0, 1]$ into subintervals $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \dots, [\frac{n-1}{n}, 1]$. Each has diameter less than δ , so $[\frac{i}{n}, \frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i , and $f([\frac{i}{n}, \frac{i+1}{n}]) \in U_{\lambda_i}$.
- Let f_i be f from $f(\frac{i-1}{n})$ to $f(\frac{i}{n})$. So,

$$f \sim f_1 f_2 f_3 \dots f_n$$

- $f(\frac{i}{n}) \in U_{\lambda_i}, U_{\lambda_{i+1}}$. Since $U_{\lambda_i} \cap U_{\lambda_{i+1}} \in \{U_\lambda\}_{\lambda \in \Lambda}$, and all elements of $\{U_\lambda\}_{\lambda \in \Lambda}$ are path connected and include x_0 , there's a path k_i from $f(\frac{i}{n})$ to x_0 contained in $U_{\lambda_i} \cap U_{\lambda_{i+1}}$.



- We add the k_i to put each small piece starts and ends at x_0 , and so is in a fundamental group:

$$f \sim f_1 k_1 \cdot k_1^{-1} f_2 k_2 \cdot k_2^{-1} f_3 k_3 \cdot \dots \cdot k_{n-1}^{-1} f_n$$

$$a = [f]_{\pi_1(X)} = [f_1 k_1]_{\pi_1(X)} [k_1^{-1} f_2 k_2]_{\pi_1(X)} \dots [k_{n-1}^{-1} f_n]_{\pi_1(X)}$$

Now, $k_{i-1} f_i k_i \subseteq U_{\lambda_i}$, since $k_i \subseteq U_{\lambda_i}, U_{\lambda_{i+1}}$. Since ψ_{λ_i} is the homomorphism induced by an inclusion map,

$$a = \psi_{\lambda_1}([f_1 k_1]_{\pi_1(U_{\lambda_1})}) \dots \psi_{\lambda_n}([k_{n-1} f_n]_{\pi_1(U_{\lambda_n})})$$

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1) \dots \psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1) \dots \psi_{\mu_m}(\beta_m)$$

Implies

$$p_{\lambda_1}(\alpha_1) \dots p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1) \dots p_{\mu_1}(\beta_1)$$

Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1) \dots \psi_{\lambda_n}(\alpha_n) \psi_{\mu_m}(\beta_m^{-1}) \dots \psi_{\mu_1}(\beta_1^{-1}) = 1$$

Implies

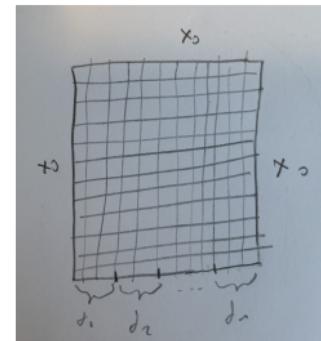
$$p_{\lambda_1}(\alpha_1) \dots p_{\lambda_n}(\alpha_n) p_{\mu_m}(\beta_m^{-1}) \dots p_{\mu_1}(\beta_1^{-1}) = 1$$

So, it'll suffice to show, $\psi_{\lambda_1}(\alpha_1) \dots \psi_{\lambda_n}(\alpha_n) = 1$ implies $p_{\lambda_1}(\alpha_1) \dots p_{\lambda_n}(\alpha_n) = 1$

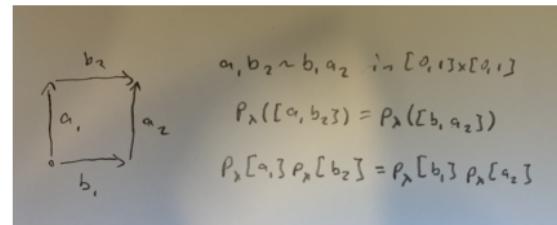
Let f_i represent $\psi_{\lambda_i}(\alpha_i)$ so $\psi_{\lambda_i}(\alpha_i) = [f_i]_{\pi_1(X)}$. If $\psi_{\lambda_1}(\alpha_1) \dots \psi_{\lambda_n}(\alpha_n) = 1$, there's a continuous function $F : [0, 1] \times [0, 1] \rightarrow X$ with $F(1, t) = F(s, 0) = F(s, 1) = x_0$,

$$F(0, t) = \begin{cases} f_1 & [0, \frac{1}{n}] \\ f_2 & [\frac{1}{n}, \frac{2}{n}] \\ \dots & \end{cases}$$

- Using the Lebesgue number, split up $[0, 1] \times [0, 1]$ into rectangles so each fits in a single U_λ , making sure that each $\frac{i}{n}$ is at a boundary:



- For each intersection, add a line k_{ij} going from the intersection to x_0 , contained in the U_λ 's of all four surrounding rectangles.
- In each line below, add k_{ij} 's as necessary to put them in the fundamental group:
For each rectangle, since it's contained in U_λ ,



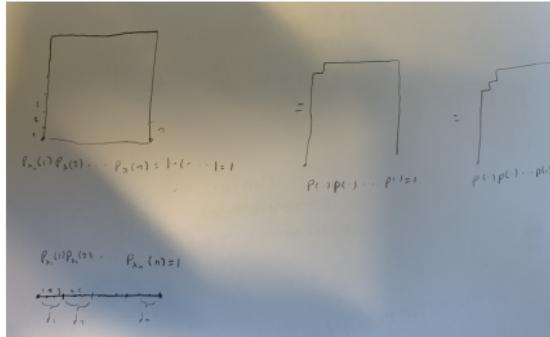
- ▶ Now, remember that

$$\begin{array}{ccc} \pi_1(U_\lambda) & \xrightarrow{\phi_{\lambda\mu}} & \pi_1(U_\mu) \\ & \searrow p_\lambda & \downarrow p_\mu \\ & H & \end{array}$$

$$a \in U_\lambda, U_\mu$$

$$\begin{aligned} p_{U_\lambda \cap U_\mu}(a) &= p_{U_\lambda}(\varphi_{U_\lambda \cap U_\mu, U_\lambda}(a)) \\ &= p_{U_\lambda}(\varphi_{U_\lambda \cap U_\mu, U_\mu}(a)) \\ p_{U_\lambda}([a]_{U_\lambda}) &= p_{U_\mu}([a]_{U_\mu}) \end{aligned}$$

- ▶ Notice that along the left, top, and right edges, p of the trivial loops is 1, so the whole composition is 1.
- ▶ Now, we can apply the previous two parts a finite number of times to move that composition to the bottom without changing its value:



- ▶ Finally, since each f_i is contained in a U_{μ_i} ,

$$\underbrace{p_{\lambda_1}([1]_{U_{\lambda_1}})p_{\lambda_2}([2]_{U_{\lambda_2}})p_{\lambda_3}([3]_{U_{\lambda_3}})}_{f_1} \underbrace{p_{\lambda_4}([4]_{U_{\lambda_4}})p_{\lambda_5}([5]_{U_{\lambda_5}})}_{f_2} \dots = 1$$

$$\underbrace{p_{\mu_1}([1]_{U_{\mu_1}})p_{\mu_1}([2]_{U_{\mu_1}})p_{\mu_1}([3]_{U_{\mu_1}})}_{f_1} \underbrace{p_{\mu_2}([4]_{U_{\mu_2}})p_{\mu_2}([5]_{U_{\mu_2}})}_{f_2} \dots = 1$$

$$\underbrace{p_{\mu_1}([123]_{U_{\mu_1}})}_{f_1} \underbrace{p_{\mu_2}([45]_{U_{\mu_2}})}_{f_2} \dots = 1$$

- ▶ Since we know F maps those sections to f_i ,

$$p_{\mu_1}([f_1])p_{\mu_2}([f_2]) \dots p_{\mu_n}([f_n]) = 1$$

Seifert – Van Kampen Theorem, corollary

- ▶ We will prove: If $X = U \cup V$, $\pi(X) = \frac{\pi(U)*\pi(V)}{N}$ where N is the smallest subgroup containing $\phi_{U \cap V, U}(x)\phi_{U \cap V, V}(x)^{-1}$ for all x in $U \cap V$.
- ▶ Elements of the free product $\pi(U) * \pi(V)$ are strings like $u_1 v_1 u_2 .. v_n$, where each u_i is in $\pi(U)$ and each v_i is in $\pi(V)$. If two adjacent elements are in the same group, you can apply that group's operation:

$$u_1 u_2 v_1 = (u_1 \cdot_{\pi_U} u_2) v_3$$

In terms of generators and relations, if

$$\pi(U) = < u_1, u_2, \dots : r_1, r_2, \dots >,$$

$$\pi(V) = < v_1, v_2, \dots : r'_1, r'_2, \dots >, \text{ and}$$

$$\pi(U \cap V) = < w_1, w_2, \dots : r''_1, r''_2, \dots >, \text{ then}$$

$$\pi(U) * \pi(V) = < u_1, u_2, \dots, v_1, v_2, \dots : r_1, r_2, \dots, rr'_1, r'_2, \dots >$$

$$\frac{\pi(U) * \pi(V)}{N} = < u_1, u_2, \dots, v_1, v_2, \dots : r_1, r_2, \dots, rr'_1, r'_2, \dots, i_U(w_1)i_V(w_1)^{-1}, i_U(w_2)i_V(w_2)^{-1}, \dots >$$

- ▶ Proof: We can define a function $F : \pi(U) * \pi(V) \rightarrow \pi(X)$, by

$$F(u_1 v_1 u_2 \dots v_n) = \psi_U(u_1) \psi_V(v_1) \psi_U(u_2) \dots \psi_V(v_n)$$

If $i_U(x)i_V(x)^{-1} \in N$,

$$F(i_U(x)i_V(x)^{-1}) = \psi_U(i_U(x)) \psi_V(i_V(x))^{-1} = 1$$

So, N is in the kernel of F and the function F is well

Calculating the group of the trefoil knot

- ▶ Let's consider the following knot:
- ▶ We can draw a torus that includes this knot:
- ▶ Then, split the space $R^3 - K$ into three sections:
 - U : inside the torus, including the boundary (but not K)
 - V : outside the torus, including the boundary (but not K)
 - $U \cap V$: the boundary of the torus, not including K .
- ▶ The fundamental group of U is essentially the same as the fundamental group of S^1 , which is $\langle u \rangle$, where u is a loop going around the hole
- ▶ The fundamental group of V is essentially the same as the fundamental group of $R^3 - S^1$, which we saw is $\langle v \rangle$ where v is the loop in the picture:
- ▶ $U \cap V$ is essentially just a strip in R^3 , so its fundamental group is going around once in the strip, $\langle w \rangle$.
- ▶ w goes around the torus two times, so $i_U(w) = u^2$. It rotates three times, around the outside, so $i_V(w) = v^3$.
- ▶ By Seifert – Van Kampen's theorem,
$$\pi(R^3 - K) = \langle u, v, u^2 = v^3 \rangle$$

- ▶ Take any 'drawing' of a knot as curves and put it in \mathbb{R}^3 as shown. Draw lines a_i going across every section of the knot.
- ▶ At every intersection, there are one of two intersections r_i .
- ▶ Divide up \mathbb{R}^3 as follows:
- ▶ $A = \{(x, y, z) : z \geq -1\}$. In A , each line segments forms a handle, and the fundamental group of A is going through any number of them in any order, so $\langle a_1, \dots, a_n \rangle$.
- ▶ B_i : rectangular box under every intersection, with its top at $z = -1$. Any path in the rectangular box is trivial, so its fundamental group is 1.
- ▶ C : Anything below A and all the B 's. Its fundamental group is 1 as well.

We'll take A and add the B_i 's one by one. $A \cap B_1$ is a rectangle with a line segment in the middle cut out. So, its fundamental group is generated by going around that line segment once, which, included in A is exactly r_1 . So, the fundamental group is $\langle a_1, \dots, a_n : r_1 \rangle$.

Adding the other B_i 's one by one, the fundamental group becomes

$$\langle a_1, a_2, \dots, a_n : r_1, r_2, \dots, r_n \rangle$$

The intersection of $A \cup B_1 \dots \cup B_n$ and C also has a trivial fundamental group so the fundamental group of the union of all the parts, $\mathbb{R}^3 - K$ is

$$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, \dots, a_n : r_1, r_2, \dots, r_n \rangle$$

Do we get the same group with the Wirtinger Presentation?

- ▶ For the trefoil, we get three line segments and three relations as shown.

$$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, a_3 : a_1 a_3 = a_2 a_1, a_2 a_1 = a_3 a_2, a_3 a_2 = a_2 a_1 \rangle$$

Clearly, the first two relations imply the third, so this group is equivalent to

$$\langle a_1, a_2, a_3 : a_1 a_3 = a_2 a_1, a_2 a_1 = a_3 a_2 \rangle$$

We can solve the first relation for a_3 to get $a_3 = a_1^{-1} a_2 a_1$. So, any string of a_1, a_2, a_3 is also generated by a_2 and a_3 with a_1 as this expression. So, we get:

$$\langle a_2, a_3 : a_2 a_1 = a_1^{-1} a_2 a_1 a_2 \rangle$$

$$\langle u, v : uvu = vuv \rangle$$

$$uvu * uvu = vu * vu * vu$$

Let $w = uvu$, $z = vu$. Then, $v = wz^{-1}$, $u = zu^{-1} = z^2 w^{-1}$. So, we can express any string of u 's and v 's as a string of w 's and z 's, so the group is also generated by w and z . Then, the relation we have becomes $w^2 = z^3$.

$$\langle w, z : w^2 = z^3 \rangle$$

And, this is the same group we got before.

Is the group we got actually different than the group of the unknot?

- The group of the unknot, $\langle a \rangle$, is commutative – for any $n, m \in \mathbb{Z}$,
$$a^n a^m = a^{n+m} = a^{m+n} = a^m a^n$$
- We'll argue that the group of the unknot, $B_3 = \langle a, b : a^2 = b^3 \rangle$ is not.
- Consider the set $\{1, 2, 3\}$, and the group of invertible functions $\{1, 2, 3\} \rightarrow \{1, 2, 3\}$, with the group operation being function composition, S_3 .
- Define f and g as follows:

$$f(1) = 2, f(2) = 1, f(3) = 3$$

$$g(1) = 2, g(2) = 3, g(3) = 1$$

Then,

$$f^2(1) = 1, f^2(2) = 2, f^2(3) = 3$$

$$g^2(1) = 3, g^2(2) = 1, g^2(3) = 2$$

$$g^3(1) = 1, g^3(2) = 2, g^3(3) = 3$$

So, both are invertible, and so in S_3 , and in the group, $f^2 = g^3 = 1$. Then, since any string of f 's and g 's is still in S_3 , and $f^2 = g^3$, we can define a homomorphism $\sigma : B_3 \rightarrow S_3$:

$$\sigma(a) = f, \sigma(b) = g$$

And this is a well-defined function – if two different strings are the same element in B_3 , the σ of both strings are equivalent in S_3 . But,

Sources

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- ▶ Massey, W. S. (1991). A basic course in algebraic topology. Springer.
- ▶ Rolfsen, D. (2003). Knots and links. AMS Chelsea Publ.
- ▶ You can find the source code for this presentation at:
<https://github.com/yahya-tamur/seifert-van-kampen-presentation>