Seifert – Van Kampen Theorem, Applications

Yahya Tamur

March 7, 2022

Groups; generators and relations

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- Given a set of elements, you can define the group of finite sequences of elements (and inverses of elements) in the set under concatenation. For example:
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This is called the free group of the set.
- Then, if you have any group that's generated by that set, there's a surjective homomorphism from the free group onto the group, defined as
$$\sigma((x)) = x$$
and extended to other elements using properties of homomorphisms:
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- If G is any group generated by a and b, with $a^5 = b^3 = 1$, we can try to define a homomorphism

$$\sigma((a)) = a, \sigma((b)) = b$$

  And, this is well-defined, even though some sequences might be equivalent in $\langle a, b : a^5, b^3 \rangle$ without being equal, because whenever terms cancel out in $\langle a, b : a^5, b^3 \rangle$, they cancel out in G:

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- Note: Some relations might be simplified by adding an equal sign:

$$\langle a, b : aba^{-1}, b^2 \rangle = \langle a, b : ab = a, b^2 \rangle$$

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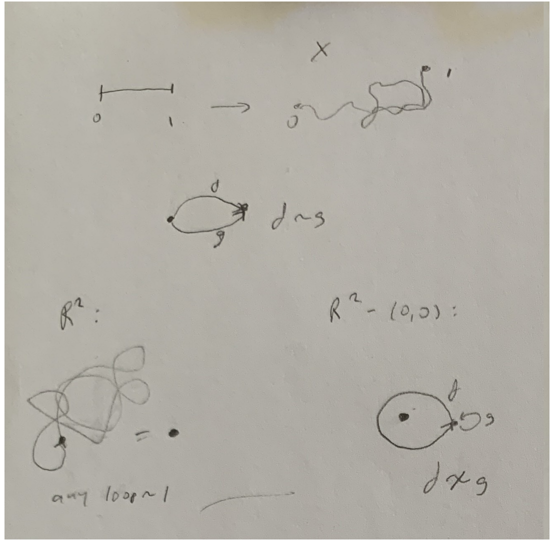

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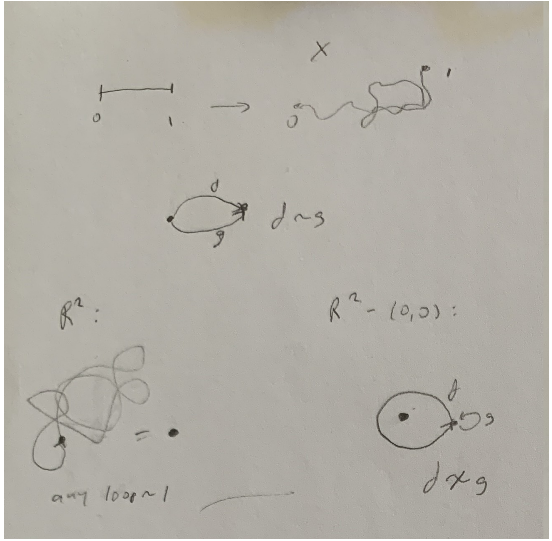

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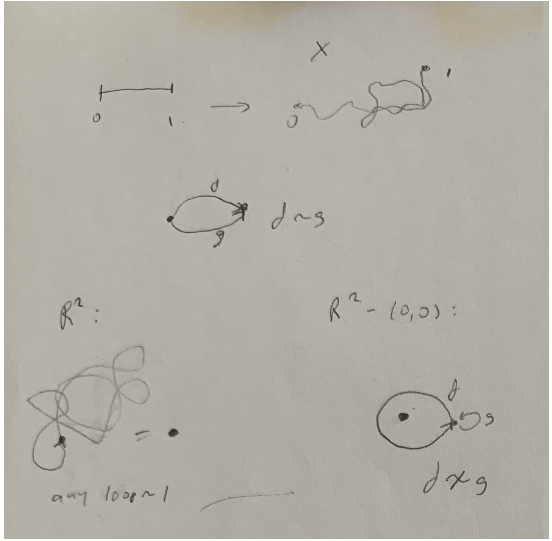

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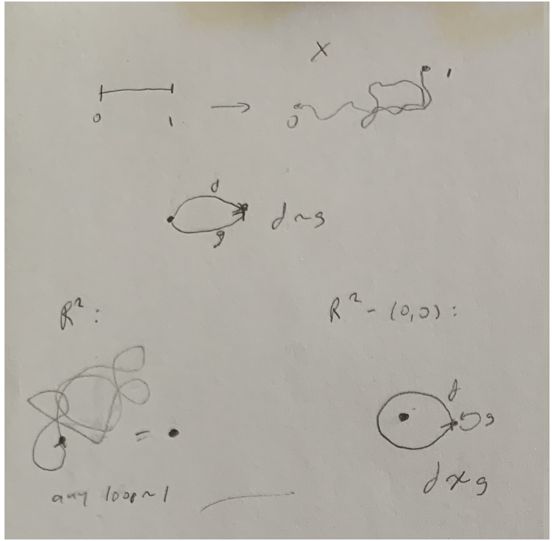

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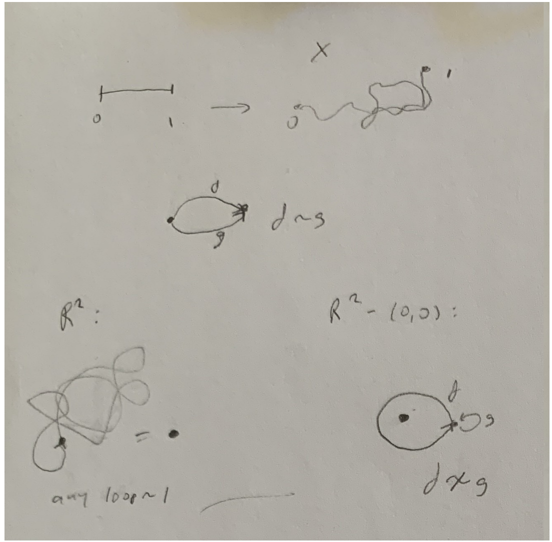
  Then, $(f.g).h \neq f.(g.h)$, but $(f.g).h \sim f.(g.h)$. Also, if $f \sim g$, $f.h \sim g.h$ for any h.



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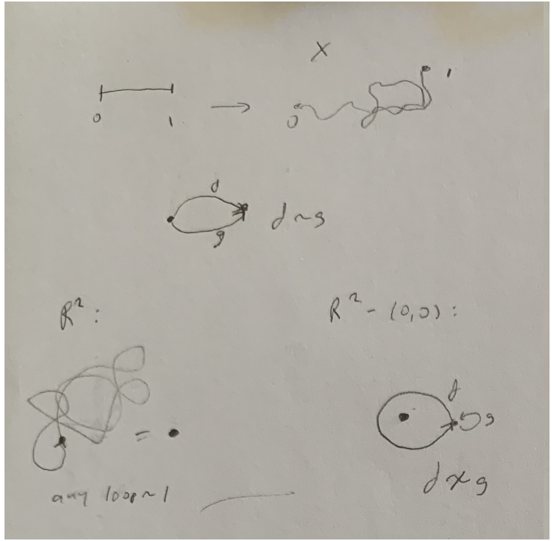
- We can define an identity element as $i(t) = x_0$ and inverse element as $f^{-1}(t) = f(1-t)$. (Again, $f.f^{-1} \neq 1$ as loops but $[f].[f^{-1}] = [f^{-1}].[f] = 1$).



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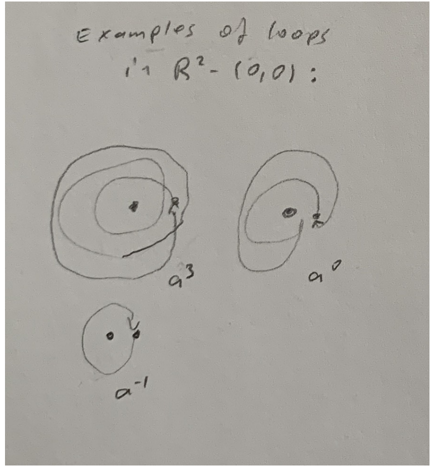

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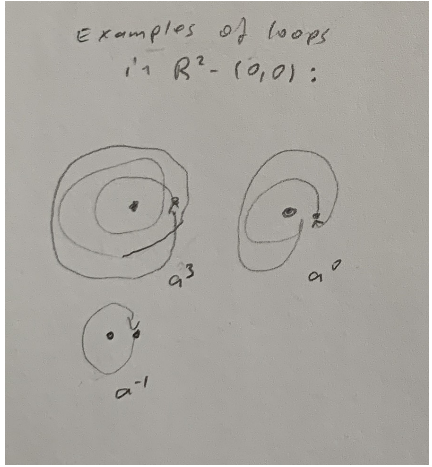
- Here, a^{-n} signifies going around counterclockwise n times and a^0 signifies not going around the point. Then,

$$a^w . a^z = a^{w+z}$$



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- Note: This means the fundemental group is a functor between the category of (pointed) topological spaces with (pointed) continuous functions and the category of groups and group homomorphisms.

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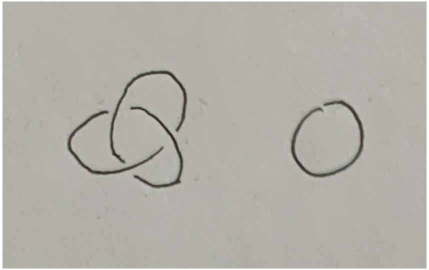
¿ 

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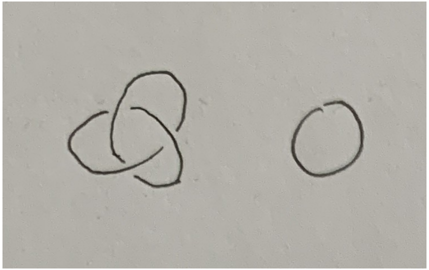
¿ 

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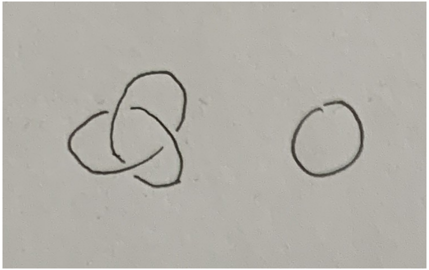
¿ 

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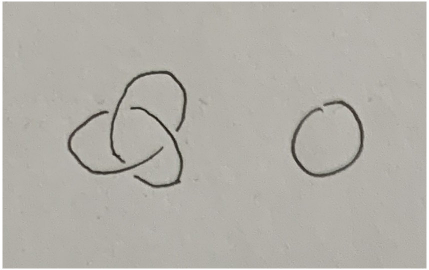
  ¿ 

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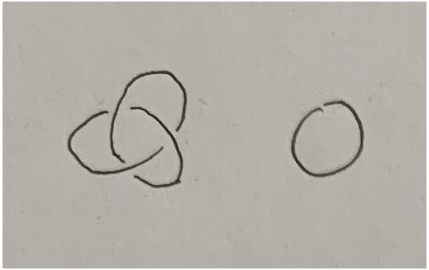
¿ 

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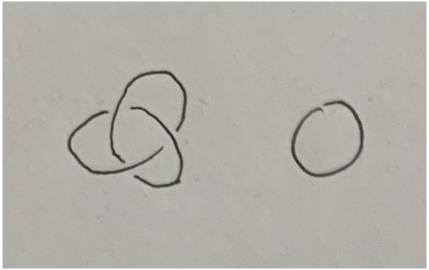
¿ 

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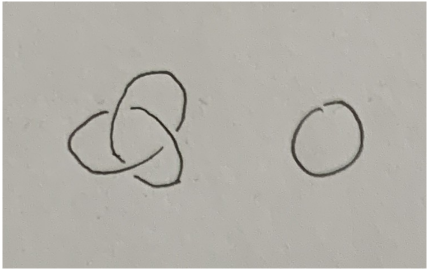
¿ 

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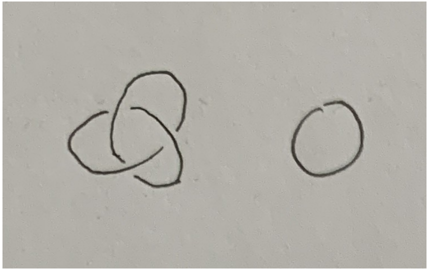
¿ 

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- In this presentation, we'll be looking at the statement and proof of this theorem, and applying it to find the fundemental groups of a few knots.

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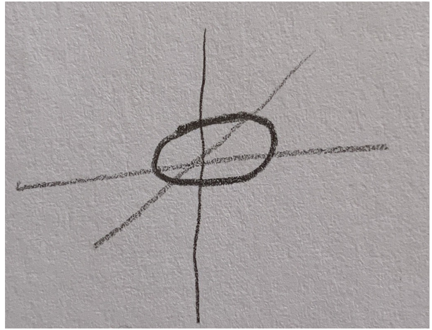

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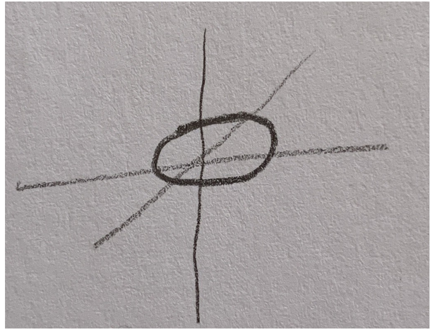

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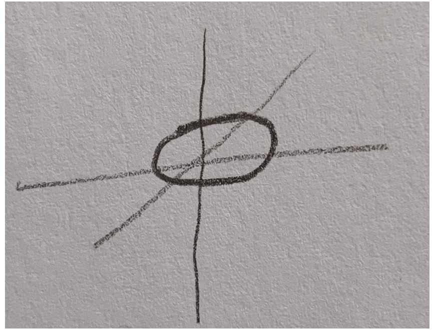
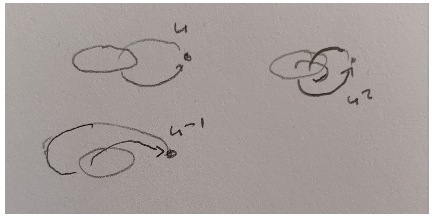
- One nontrivial path is wrapping around the circle as shown. Similar to $\mathbb{R}^2 - (0,0)$, it can be proved that any loops that wrap around the same number of times are equivalent (and loops that don't are not). So, the fundemental group is $\langle u \rangle$.





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- We'll also look at when two elements of $\pi_1(X)$ are equal and when they're different.
- But hopefully it makes sense how this theorem determines $\pi_1(X)$!

Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of a $\pi_1(X)$ can be expressed as

$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$

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- Find n so $\frac{1}{n} < \delta$, divide $[0, 1]$ into subintervals $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, ..., $[\frac{n-1}{n}, 1]$. Each has diameter less than δ, so $[\frac{i}{n}, \frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i, and $f([\frac{i}{n}, \frac{i+1}{n}]) \in U_{\lambda_i}$.

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- $f(\frac{i}{n}) \in U_{\lambda_i}, U_{\lambda i+1}$. Since $U_{\lambda_i} \cap U_{\lambda_{i+1}} \in \{U_\lambda\}_{\lambda\in\Lambda}$, and all elements of $\{U_\lambda\}_{\lambda\in\Lambda}$ are path connected and include x_0, there's a path k_i from $f(\frac{i}{n})$ to x_0 contained in $U_{\lambda_i} \cap U_{\lambda_{i+1}}$.

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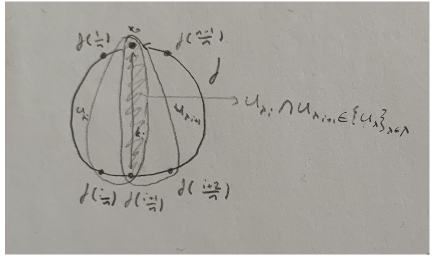
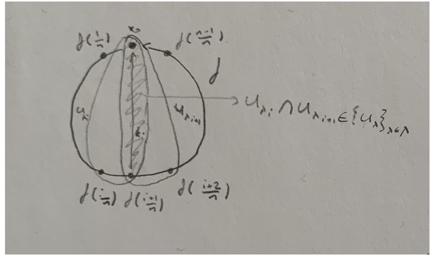
- $f(\frac{i}{n}) \in U_{\lambda_i}, U_{\lambda i+1}$. Since $U_{\lambda_i} \cap U_{\lambda_{i+1}} \in \{U_\lambda\}_{\lambda\in\Lambda}$, and all elements of $\{U_\lambda\}_{\lambda\in\Lambda}$ are path connected and include x_0, there's a path k_i from $f(\frac{i}{n})$ to x_0 contained in $U_{\lambda_i} \cap U_{\lambda_{i+1}}$.



Seifert – Van Kampen Theorem Proof – Part 1

- To Prove: Every element of $a\ \pi_1(X)$ can be expressed as
$$a = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)...\psi_{\lambda_n}(\alpha_n)$$
for $\lambda_i \in \Lambda, \alpha_i \in U_{\lambda_i}$.
- We will use:
Lebesgue's Number Lemma: Every open cover of a compact metric space has a δ so that any subset of the metric space with diameter less than δ is contained in a single element of the cover. δ is called the Lebesgue number of the cover.
- For any $a \in \pi_1(X)$, find a path $f : [0,1] \to X$ so that $a = [f]_{\pi_1(X)}$.
- $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is a cover of the compact metric space $[0,1]$. It has a Lebesgue number δ.
- Find n so $\frac{1}{n} < \delta$, divide $[0,1]$ into subintervals $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, ..., $[\frac{n-1}{n}, 1]$. Each has diameter less than δ, so $[\frac{i}{n}, \frac{i+1}{n}] \in f^{-1}(U_{\lambda_i})$ for some λ_i, and $f([\frac{i}{n}, \frac{i+1}{n}]) \in U_{\lambda_i}$.
- Let f_i be f from $f(\frac{i-1}{n})$ to $f(\frac{i}{n})$. So,
$$f \sim f_1 f_2 f_3 ... f_n$$
- $f(\frac{i}{n}) \in U_{\lambda_i}, U_{\lambda i+1}$. Since $U_{\lambda_i} \cap U_{\lambda_{i+1}} \in \{U_\lambda\}_{\lambda\in\Lambda}$, and all elements of $\{U_\lambda\}_{\lambda\in\Lambda}$ are path connected and include x_0, there's a path k_i from $f(\frac{i}{n})$ to x_0 contained in $U_{\lambda_i} \cap U_{\lambda_{i+1}}$.



- We add the k_i to put each small piece starts and ends at x_0, and so is in a fundemental group:
$$f \sim f_1 k_1 \cdot k_1^{-1} f_2 k_2 \cdot k_2^{-1} f_3 k_3 \cdot ... \cdot k_{n-1}^{-1} f_n$$
$$a = [f]_{\pi_1(X)} = [f_1 k_1]_{\pi_1(X)} [k_1^{-1} f_2 k_2]_{\pi_1(X)} ... [k_{n-1}^{-1} f_n]_{\pi_1(X)}$$
Now, $k_{i-1} f_i k_i \subseteq U_{\lambda_i}$, since $k_i \subseteq U_{\lambda_i}, U_{\lambda_{i+1}}$. Since ψ_{λ_i} is the homomorphism induced by an inclusion map,
$$a = \psi_{\lambda_1}([f_1 k_1]_{\pi_1(U_{\lambda_1})}) ... \psi_{\lambda_n}([k_{n-1} f_n]_{\pi_1(U_{\lambda_n})})$$

Seifert – Van Kampen Theorem Proof – Part 2

- We'd like to show that

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n) = \psi_{\mu_1}(\beta_1)...\psi_{\mu_m}(\beta_m)$$

Implies

$$p_{\lambda_1}(\alpha_1)...p_{\lambda_n}(\alpha_n) = p_{\mu_1}(\beta_1)...p_{\mu_1}(\beta_1)$$

Since paths are invertible, and everything involved is a homomorphism, we can put everything on one side:

$$\psi_{\lambda_1}(\alpha_1)...\psi_{\lambda_n}(\alpha_n)\psi_{\mu_m}(\beta_m^{-1})...\psi_{\mu_1}(\beta_1^{-1}) = 1$$

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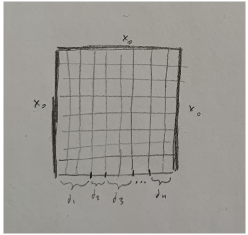

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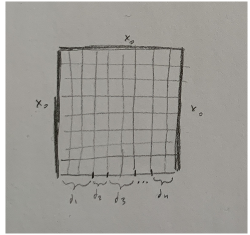

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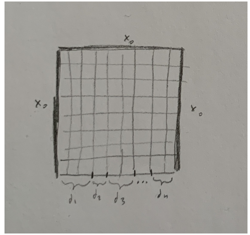

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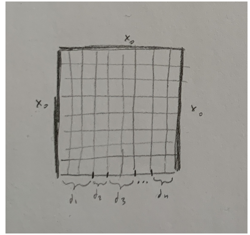

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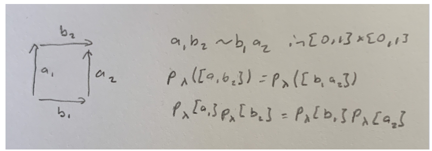

- Now, remember that

$$
\begin{tikzcd}
\pi_1(U_\lambda) \arrow[r, "\phi_{\lambda\mu}"] \arrow[dr, "p_\lambda"] & \pi_1(U_\mu) \arrow[d, "p_\mu"] \\
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\end{tikzcd}
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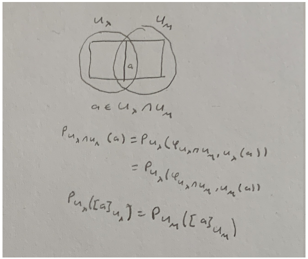

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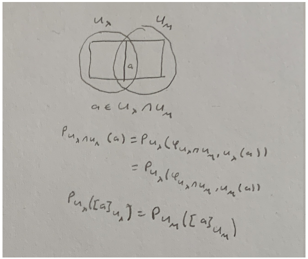

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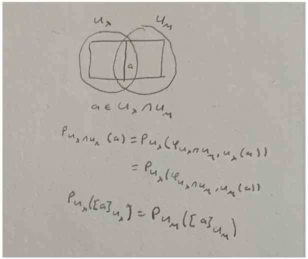

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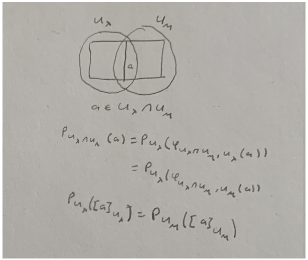

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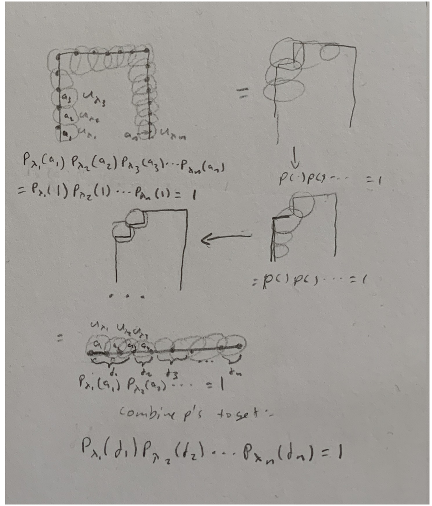

- And that concludes the proof of Seifert-Van Kampen theorem: We proved that σ is well-defined, so it exists, and we know that it's unique.

Seifert – Van Kampen Theorem, corollary

- We will prove: If $X = U \cup V$, $\pi(X) = \frac{\pi(U) * \pi(V)}{N}$ where N is the smallest subgroup containing $\phi_{U \cap V, U}(x) \phi_{U \cap V, V}(x)^{-1}$ for all x in $U \cap V$.

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- Elements of the free product $\pi(U) * \pi(V)$ are strings like $u_1 v_1 u_2..v_n$, where each u_i is in $\pi(U)$ and each v_i is in $\pi(V)$. If two adjacent elements are in the same group, you can apply that group's operation:

$$u_1 u_2 v_1 = (u_1 \cdot_{\pi_U} u_2) v_3$$

In terms of generators and relations, if $\pi(U) = \langle u_1, u_2, ... : r_1, r_2, ...\rangle$, $\pi(V) = \langle v_1, v_2, ... : r_1', r_2', ...\rangle$, and $\pi(U \cap V) = \langle w_1, w_2, ... : r_1'', r_2'', ...\rangle$, then

$$\pi(U) * \pi(V) = \langle u_1, u_2, ..., v_1, v_2, ... : r_1, r_2, ..., r_1', r_2', ...\rangle$$

$$\frac{\pi(U) * \pi(V)}{N} = \langle u_1, ..., v_1, ... : r_1, ..., r_1', ..., i_U(w_1) i_V(w_1)^{-1}, ...\rangle$$

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- Proof: We can define a function $F : \pi(U) * \pi(V) \to \pi(X)$, by

$$F(u_1 v_1 u_2 ... v_n) = \psi_U(u_1)\psi_V(v_1)\psi_U(u_2)...\psi_V(v_n)$$

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$$F(i_U(x) i_V(x)^{-1}) = \psi_U(i_U(x))\psi_V(i_V(x))^{-1} = 1$$

Seifert – Van Kampen Theorem, corollary

- We will prove: If $X = U \cup V$, $\pi(X) = \frac{\pi(U) * \pi(V)}{N}$ where N is the smallest subgroup containing $\phi_{U\cap V,U}(x)\phi_{U\cap V,V}(x)^{-1}$ for all x in $U \cap V$.
- Elements of the free product $\pi(U) * \pi(V)$ are strings like $u_1 v_1 u_2..v_n$, where each u_i is in $\pi(U)$ and each v_i is in $\pi(V)$. If two adjacent elements are in the same group, you can apply that group's operation:

$$u_1 u_2 v_1 = (u_1 \cdot_{\pi_U} u_2) v_3$$

  In terms of generators and relations, if $\pi(U) = \langle u_1, u_2, ... : r_1, r_2, ...\rangle$, $\pi(V) = \langle v_1, v_2, ... : r_1', r_2', ...\rangle$, and $\pi(U \cap V) = \langle w_1, w_2, ... : r_1'', r_2'', ...\rangle$, then

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- So, N is in the kernel of F and the function F is well defined $\frac{\pi(U) * \pi(V)}{N} \to \pi(X)$. By the proof of Seifert - Van Kampen's theorem, part 1, we know that elements $\psi_U(u_1)\psi_V(v_1)\psi_U(u_2)...\psi_V(v_n)$ generate $\pi(X)$, so F is surjective onto $\pi(X)$. We also have:

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- Then, we can apply Seifert - Van Kampen's theorem and get $\sigma : \pi(X) \to \frac{\pi(U)*\pi(V)}{N}$. The following commutes:

$$\begin{tikzcd}
\pi_1(U) \arrow[r, "i_U"] \arrow[dr, "i_1"'] & \pi_1(X) \arrow[d, "\sigma"] \\
& H
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- To conclude,

$$\pi(X) = \langle u_1, ..., v_1, ... : r_1, ..., r_1', ..., i_U(w_1)i_V(w_1)^{-1}, ...\rangle$$
$$= \langle u_1, ..., v_1, ... : r_1, ..., r_1', ..., i_U(w_1) = i_V(w_1), ...\rangle$$

where

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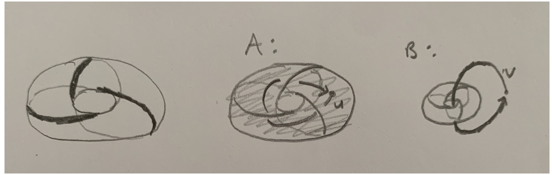

Calculating the group of the trefoil knot

- We'll consider the trefoil knot we saw in the introduction.
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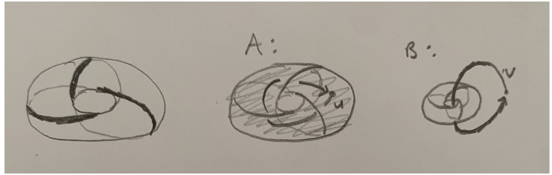

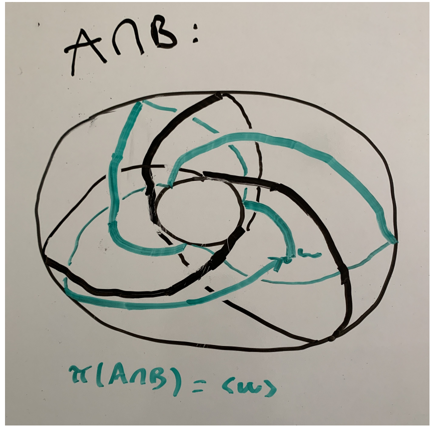

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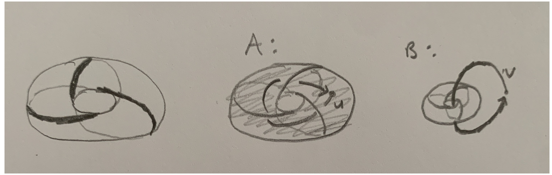

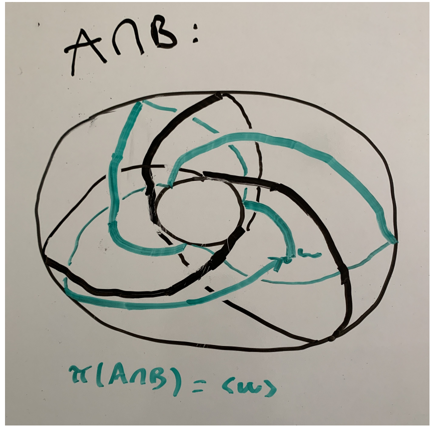

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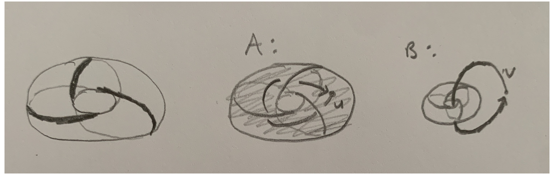

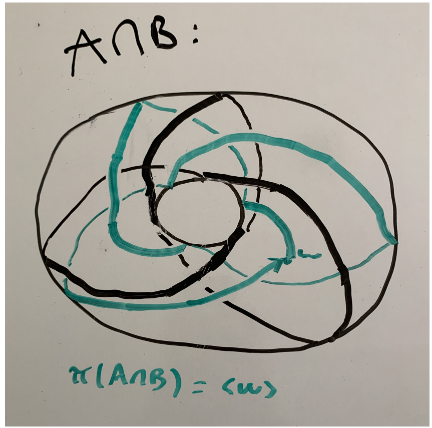

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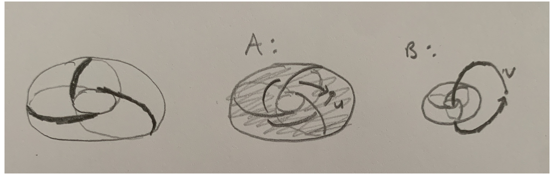

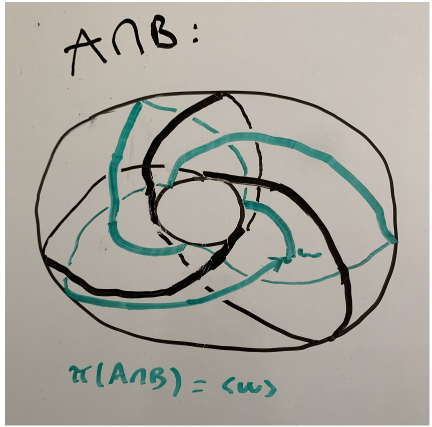

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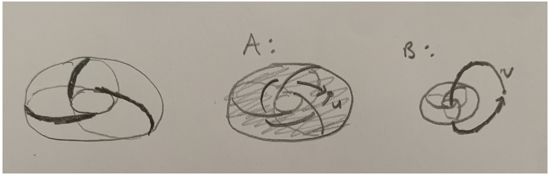
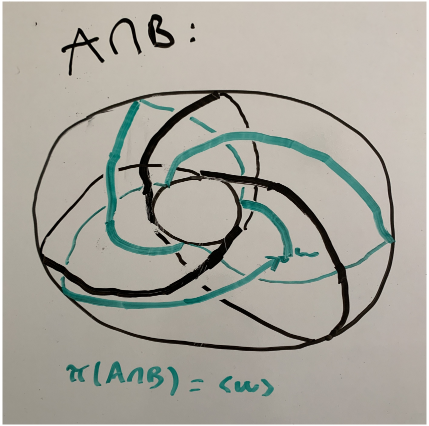
- w goes around the torus two times, so $i_U(w) = u^2$. It rotates three times, around the outside, so $i_V(w) = v^3$.





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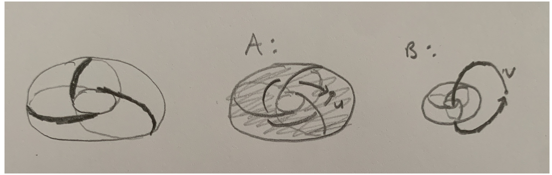
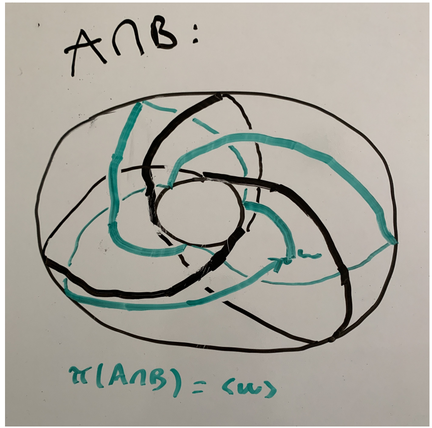
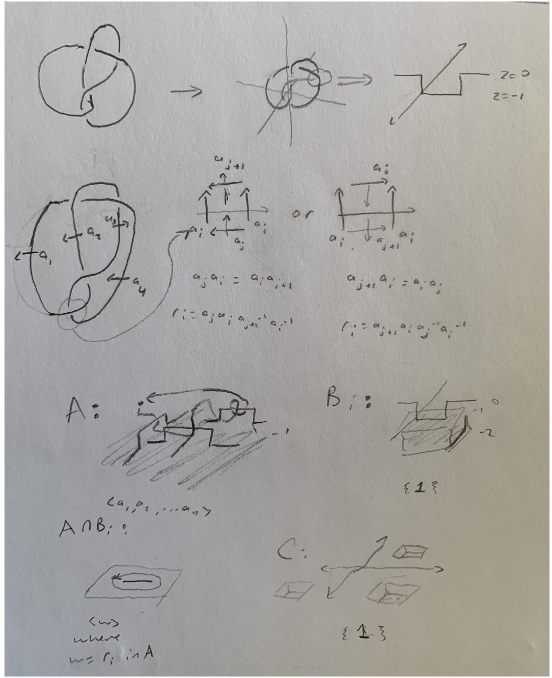
- By Seifert – Van Kampen's theorem,
  $\pi(R^3 - K) = \langle u, v, u^2 = v^3 \rangle$





Wirtinger Presentation

- Take any 'drawing' of a knot as curves and put it in $\mathbb{R}^3$ as shown. Draw lines a_i going across every section of the knot.



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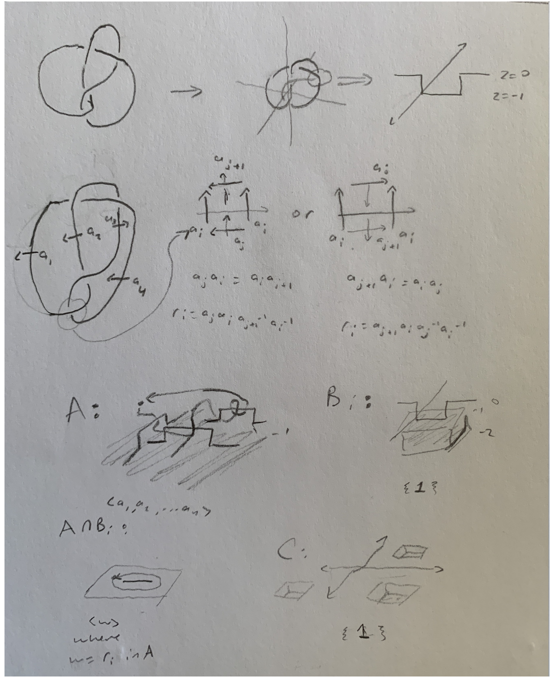

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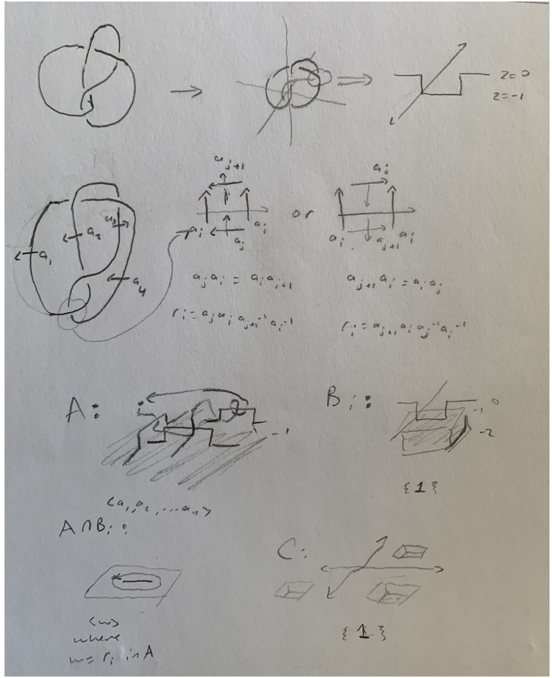

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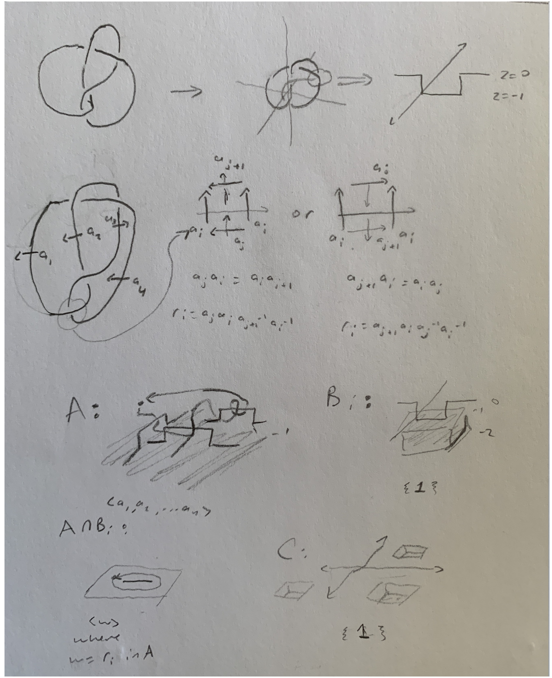

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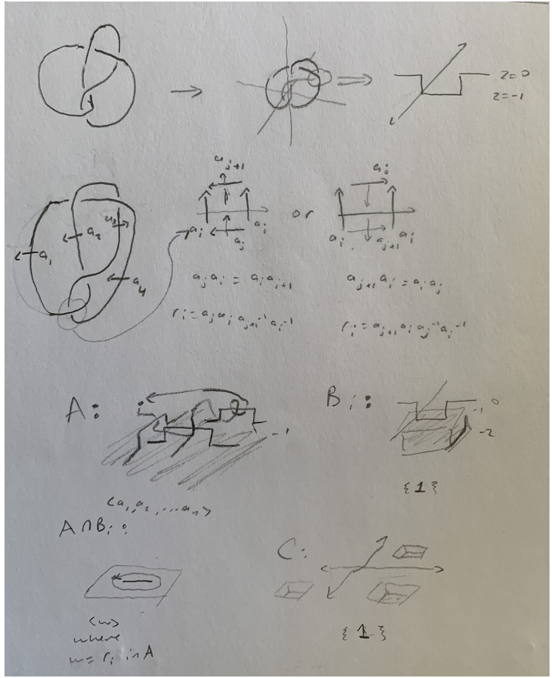
- B_i: rectangular box under every intersection, with its top at $z = -1$. Any path in the rectangular box is trivial, so its fundemental group is 1.



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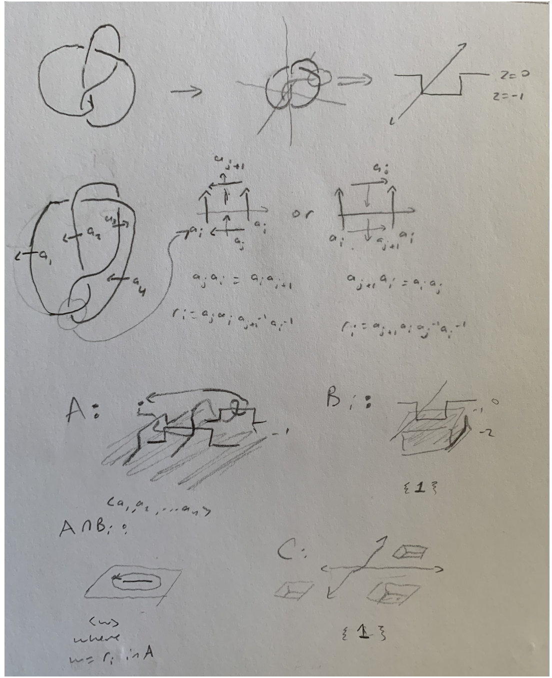
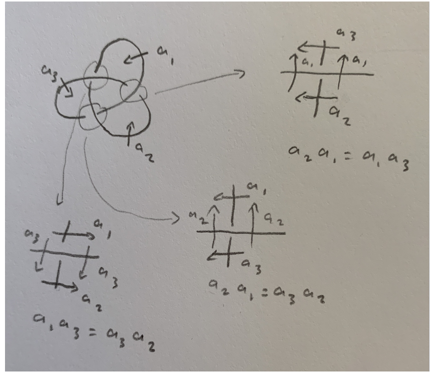
- C: Anything below A and all the B's. Its fundemental group is 1 as well.
  We'll take A and add the B_i's one by one. $A \cap B_1$ is a rectangle with a line segment in the middle cut out. So, its fundemental group is generated by going around that line segment once, which, included in A is exactly r_i. So, the fundemental group is $\langle a_1, ... a_n : r_1 \rangle$.
  Adding the other B_i's one by one, the fundemental group becomes
  $$\langle a_1, a_2, ... a_n : r_1, r_2, ... r_n \rangle$$
  The intersection of $A \cup B_1 ... \cup B_n$ and C also has a trivial fundemental group. So the fundemental group of the union of all the parts, $\mathbb{R}^3 - K$ is:
  $$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, ... a_n : r_1, r_2, ... r_n \rangle$$



Do we get the same group with the Wirtinger Presentation?

- For the trefoil, we get three line segments and three relations as shown.

  $$\pi(\mathbb{R}^3 - K) = \langle a_1, a_2, a_3 : a_1 a_3 = a_3 a_2, a_2 a_1 = a_3 a_2, a_2 a_1 = a_1 a_3 \rangle$$



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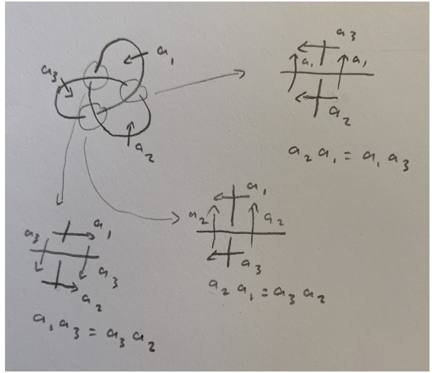

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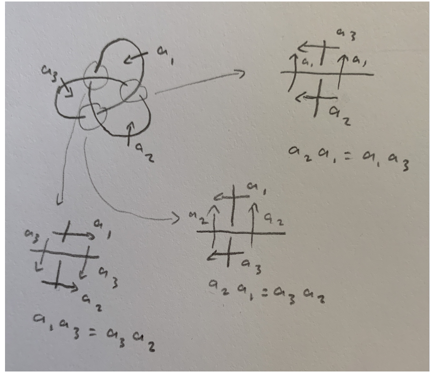
$$uvu * uvu = vu * vu * vu$$



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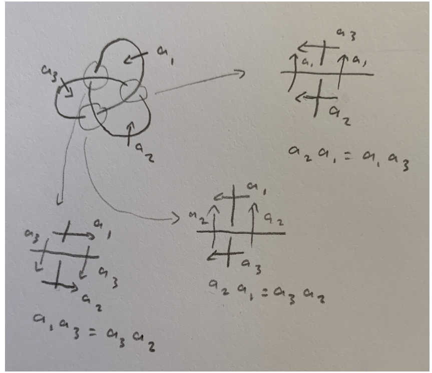

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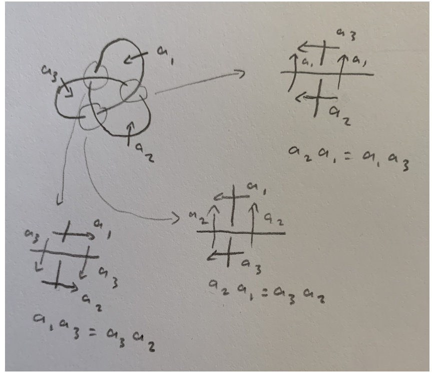
- Let $w = uvu$, $z = vu$. Then,

$$u = wz^{-1}, v = zu^{-1} = z(wz^{-1})^{-1} = z^2 w^{-1}$$

So, we can express any string of u's and v's as a string of w's and z's, so the group is also generated by w and z. Then, the relation we have becomes $w^2 = z^3$.

$$\langle w, z : w^2 = z^3 \rangle$$

And this is the same group we got before.



Is the group we got actually different than the group of the unknot?

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- Therefore, the trefoil is a different knot than the unknot.

Sources

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You can find the source code for this presentation at:
`https://github.com/yahya-tamur/seifert-van-kampen-presentation`