The Fundamental Theorem of Calculus

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1 The Theorem

We always hear the the phrase "The Fundamental Theorem[1] of Calculus" but we don't always understand what it truly means, and why is it important. The Fundamental Theorem of Calculus has two main parts and I will go over them in detail in the coming sections. On a basic level, the theorem simply connects and relates the two main operations of calculus: differentiating a function, and integrating a function. It states that those two operations are inverse to each other, which is why sometimes we say "finding the anti-derivative" instead of "finding the integral". We know that finding the integral is finding the area under a curve, and that finding the derivative is finding its slope. This theorem is very important because now that we know the relationship between the two operations, it is much easier for us to find the integral of anything. Without the theorem we would have to sum a series every time we want to find an integral, but now that we know that finding the integral is the same as finding the anti-derivative, we can use the our knowledge about derivatives to try to apply that on integration and reverse them. Without this theorem we wouldn't have calculus, because integration and differentiation would be considered two different mathematical fields. There are of course real life applications (will go more in detail in coming sections), like going from acceleration to velocity, and from velocity to position easily, those are the most basic examples that we may encounter, but in reality this theorem is involved in many more things in more advanced levels. Let's take a look at this graph for example

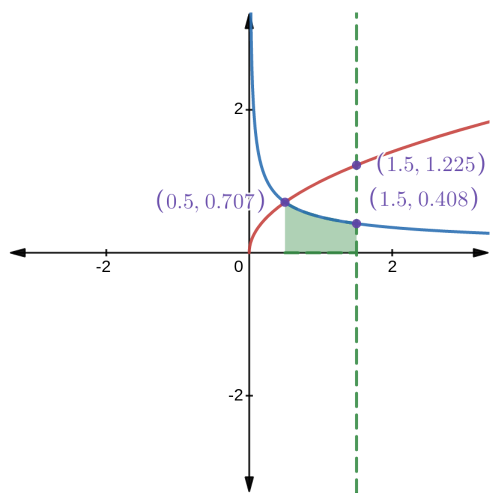

figure 1

Let's call the red curve $f(x)$, and let's call the blue curve $f'(x)$, which means that the blue curve is the derivative of the red curve. According to the theorem, the area under the blue curve between $x = 0.5$ and $x = 1.5$, should be exactly the same as $f(1.5) - f(0.5)$. Before we do the mathematics to prove this, let me

[1]a formula or statement in mathematics that was proved to be true by rigorous proof.

explain to you the two different parts of the theorem. And remember that this is meant to explain the idea of the theorem and its uses, so you won't see any mathematical proofs here, that is beyond the scope of this paper.

1.1 The First Part

The first part of the theorem states that

$$\int_a^b F'(x)dx = F(b) - F(a)$$

This may look scary and confusing at the beginning, but if you think about it that is exactly what we did had earlier

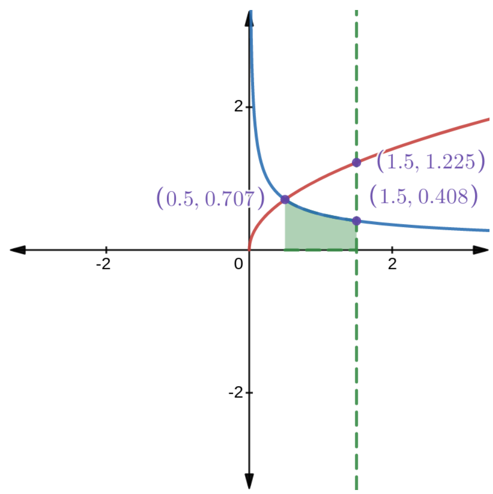

figure 2

It is simply saying that the area between two points of a derivative of a function (blue curve) is equal to the difference between that function's (red curve) output at those two points, that is why in physics when we have a velocity over time graph, and we want to find the displacement, we find the area under the velocity over time graph in the certain interval that we are examining, and that gives us $F(b) - F(a)$, which is the difference of the two outputs in the original function, or in other words displacement. The proof for this theorem is not that long and complicated, but since this paper is made for calculus students rather than analysis students, there is no point of getting into it, but that doesn't mean that we can't prove the specific case that we have in the graph above. Let's say that the red curve is defined as

$$f(x) = \sqrt{x}$$

That the blue curve as defined as

$$f'(x) = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

Let's take a second look at the theorem

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Let's substitute in our givens

$$\int_{0.5}^{1.5} \frac{1}{2\sqrt{x}}dx = \sqrt{1.5} - \sqrt{0.5}$$

In theory, the statement above should be absolutely correct, so let's check

$$\sqrt{x} \mid_{0.5}^{1.5} = \sqrt{1.5} - \sqrt{0.5}$$

$$\sqrt{1.5} - \sqrt{0.5} = \sqrt{1.5} - \sqrt{0.5}$$

$$0.518 = 0.518$$

If $f'(x)$ was a velocity overtime function, that means between 0.5 and 1.5 seconds, out object's displacement was about 0.518.

1.2 The Second Part

The second part of the theorem states that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Let's take a look at this graph

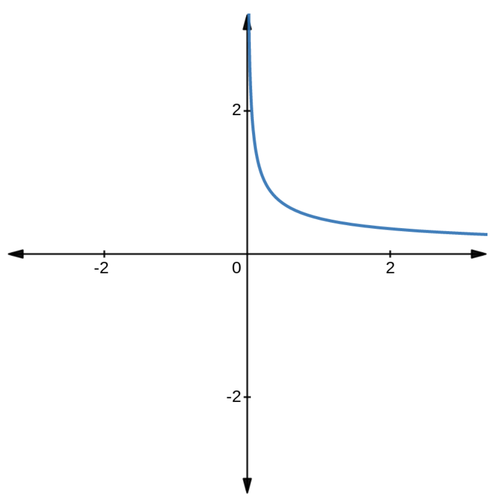

figure 3

Let's say that the blue curve is $f(x)$. Although the first part of the theorem showed us the relationship between the area under a curve and the function's antiderivative, this second part is more concrete, because it is literally telling us, using direct mathematical language, that the two operations: differentiation and integration are each others inverse operations. If you pay close attention to the second part of the theorem, it is simply saying "the derivative of the integral of a function, is equal to the same original function" which explains very simply how these two operations are each others inverse operations. Let's find the integral of the graph above, which is defined as

$$f(x) = \frac{1}{2\sqrt{x}}$$

We find that

$$\int f(x)dx = \sqrt{x}$$

Now, according to the second part of the theorem if we differentiate $\sqrt{x}$ we should get $\frac{1}{2\sqrt{x}}$, and we know for a fact that is the case. You may now ask yourself why are we doing all of this? it seems not useful to integrate a function, and then to differentiate the answer, just to get the original function back, shouldn't that be obvious? Well that is where the fundamental theorem of calculus comes into play, without it we wouldn't be able to prove that, which means the way we do integration today would be much harder than it should be, but now that we know how these two operations are each others inverse, when we try to integrate a function, we can just think about reversing the steps that another function took to get here, and that is the main purpose of the Fundamental Theorem of Calculus.

2 Examples

Below are some useful calculus questions related to this topic, feel free to try them (answers are provided in the next section).

2.1 Pure Math

1. $$\int_1^5 3x^2\,dx$$

2. $$\frac{d}{dx}\left[\int_2^x e^{t^2}\,dt\right]$$

3. $$\frac{d}{dx}\left[\int_x^{-3} \sqrt{1+\sin t}\,dt\right]$$

2.2 Physics

1. A car's acceleration over time is given by the function $a(t) = x^2 + x + 2$, assuming that the car's position and velocity when $t = 0$ are equal to 0, find the car's position when $t = 5$.

2. A ball's velocity over time is given by the function $v(t) = \frac{1}{2\sqrt{t}}$, find the ball's displacement from $t = 1$ to $t = 2$.

3 Answers

3.1 Pure Math

1. $$124$$

   Remember the two operations are each others inverses

2. $$e^{t^2}$$

3. $$-\sqrt{1 + \sin x}$$

   This was a trick question, because since the x is at the bottom and not at the top, it doesn't go along with the theorem that states that

   $$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

   And to make the question conform to the theorem we can switch the x and the -3 by multiplying the whole thing by -1 We can say that

   $$\frac{d}{dx} \left[\int_x^{-3} \sqrt{1 + \sin t}\, dt \right] = -\frac{d}{dx} \left[\int_{-3}^x \sqrt{1 + \sin t}\, dt \right]$$

   Which gives us

   $$-\sqrt{1 + \sin x}$$

3.2 Physics

1. $$\frac{1175}{12}$$

2. $$\sqrt{2} - 1$$

4 References

[1] 5.3 the fundamental theorem of calculus - calculus volume 1. OpenStax. (n.d.). Retrieved October 13, 2022, from `https://openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus`

[2] Merriam-Webster. (n.d.). Theorem definition & meaning. Merriam-Webster. Retrieved October 13, 2022, from `https://www.merriam-webster.com/dictionary/theorem`

[3] Sam Nelson - theorems and theories. (n.d.). Retrieved October 13, 2022, from `https://www1.cmc.edu/pages/faculty/VNelson/tth.html`

[4] 37.Fundamental theorem of calculus - auburn university. (n.d.). Retrieved October 13, 2022, from `https://web.auburn.edu/holmerr/1617/Textbook/fundthmcalc-screen.pdf`