Limit Definition of the Derivative

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Contents

1 The basics

To understand the concept of slopes[1], limits[2], and derivatives[3], we have to know in clear terms what a function is. A function in mathematics is simply an expression to represent a relationship between two different variables: the independent variable, and the dependent variable, where for every input there is exactly one output. Functions are really useful, not just in pure mathematics, but in real-life situations too. Let's say that you started a savings account with an annually compounded interest rate of 0.5%, and you deposited $10,000, in that case, the best way to predict your future balance is by formulating a function. We want our function to be so accessible and easy to use that we would simply give it a number, and it would give us the balance after that many years. Let's call our function b, you can call it whatever you want, because it is your function, but I decided to call it b because it stands for the word "balance", and this function will be calculating our balance over time. Our function can look something like this

$$b(t) = 10000 * 1.005^t$$

This syntax means "balance is dependent on time, and the balance after a certain number of years (t) will be the initial amount (10000) multiplied by the interest rate (1.005) t times", which means that given this formula, we can calculate the balance based on a certain number of years by simply substituting that number into t and then simplifying it. Now let's use the function that we just made; let's say that we want to know the balance after the first year or in other words $b(1)$. We can simply do this by plugging 1 into t

$$b(1) = 10000 * 1.005^1$$

After simplifying this, we know that

$$b(1) = 10050$$

This basically means that after exactly a year, our balance will be 10050, which means that our profit after exactly a year is $50. Another useful feature of functions is that we can use them backward. We now know that $b(1) = 10050$, so we can say that

$$10050 = 10000 * 1.005^t$$

If we now solve for t in the above example, we will simply get 1, because that is the number of years that it would take for our account to have the balance of $10,050. Let's now say that we want to find the number of years that it would take to have a balance of $11,000. We can simply just set $11,000 equal to the function definition and then we can just solve for t

$$11000 = 10000 * 1.005^t$$

[1] The slope of a line is a number that describes both the direction and the steepness of the line.

[2] The limit is the value that a function approaches as the input approaches some value.

[3] The derivative of a function of a real variable measures the sensitivity to change of the function value with respect to a change in its argument.

If we solve for t we will get 19.10966, which means that after 19.10966 years our balance will be \$11,000. Now imagine having to do all that manually, it would be very tedious and unnecessary, and that is why we have functions. A good way to represent a function visually is a graph. The graph for our function above looks like this

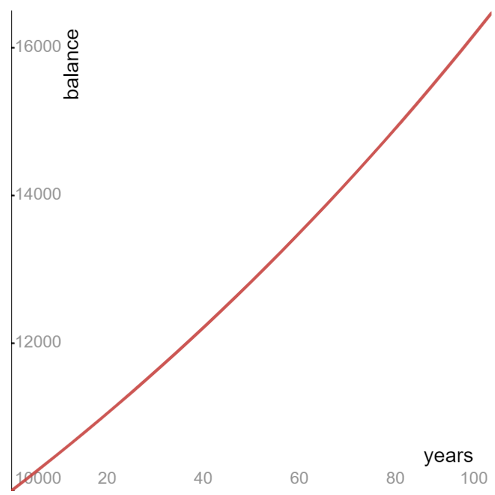

figure 1

Over time you will be able to formulate your own functions for any problem that you face, the best way to get this skill is with practice, the next time you face a problem, try to solve the problem with a function.

2 What is a slope?

A slope of a function is a very easy concept to understand. To put it simply, a slope of a function is the change of y over the change of x. For example, if you have a function of position over time, which is a function that tells you the position of a certain object at a certain time, the slope of this function will be the change of position over the change of time, which in other words is velocity. There are times when the slope is positive, like the graph below

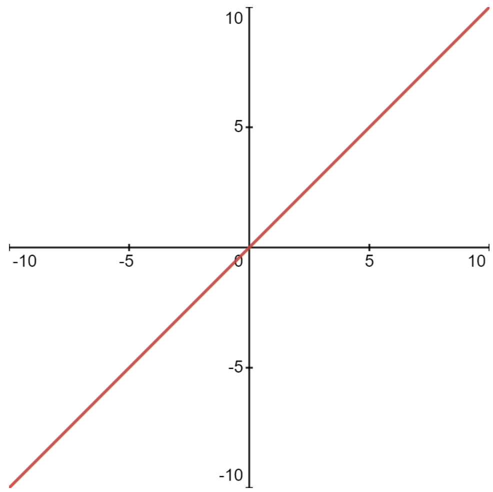

figure 2

If the graph above was a graph representing position over time, then that means that our velocity is positive, because our slope is positive, and while this is a good representation of position with respect to time, negative time must be ignored in this instance. There are other times where the slope can be negative, like the graph below

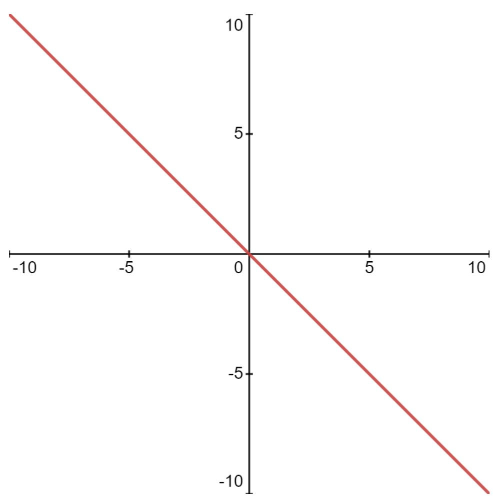

figure 3

If the graph above was a graph representing position over time, then that means that our velocity is negative, because our slope is negative. There are other times where the slope can just be zero, like the graph below

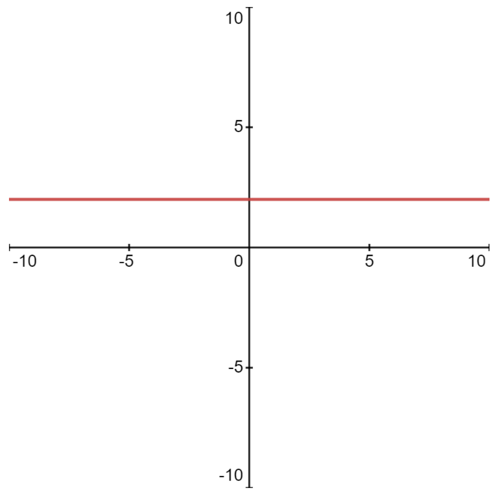

figure 4

If the graph above was a graph representing position over time, then that means that our velocity is zero, because our slope, or in other words our rate of change is zero because the line does not change in the y-direction.

3 Slope of linear functions

When it comes to linear functions there are two ways to find the slope. The first way is by finding the slope using the equation only, and the second way is by deriving information from the graph of the function and then using that information to mathematically find the slope. Let's look at this graph

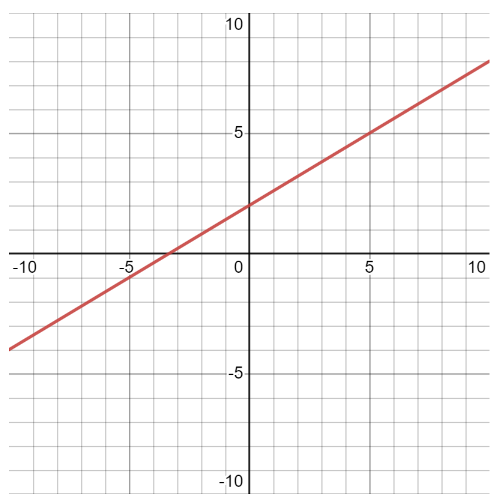

figure 5

We can easily find the slope of this function by finding the rate that y changes as x changes. We can do that by picking any two points in the graph and then dividing the difference of the y value of the second and first point by the difference of the x value of the second and first point. Let's pick point $(0, 2)$ and point $(5, 5)$ since they are both on the graph, and then let's plug the numbers into the slope of a line formula, which is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{5-0} = \frac{3}{5}$$

The top part of the final answer (3) is called rise, and the bottom part of the final answer (5) is called the run. The reason it is called rise is that when we look at the graph we can see that in that interval, this is how much the y value is going up by, and the reason it is called run is that when we look at the graph we can see that this is how much x is increasing to make that change happen in y. We can visualize this by looking at the graph below

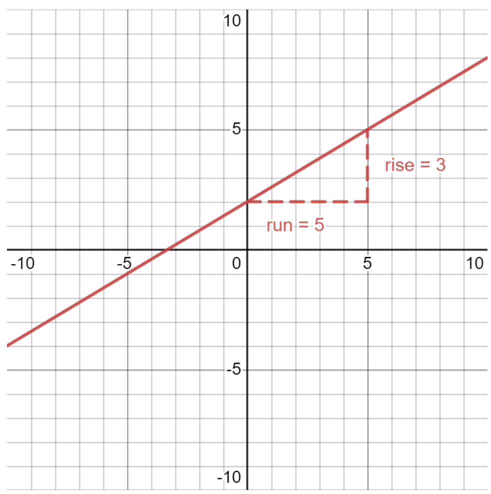

figure 6

After knowing this we can simplify the slope of a line formula into this

$$m = \frac{rise}{run}$$

We can now use the information that we know to find the slope of any linear function just using the graph, but there is another way to find the slope using the equation only. Let's say that you know the equation of a linear function to be

$$x + 0.5 - y = -0.5x$$

The best way and the fastest way to find the slope without looking at the graph is by utilizing the $y = mx + b$ form. If you change the form of a linear function to the $y = mx + b$ form, then your slope will be m. The way we can do this is by solving for y

$$y = x + 0.5 + 0.5x$$

And then simplifying the expression on the right

$$y = 1.5x + 0.5$$

In this case our slope is 1.5, which means that the rise is 1.5 and the run is 1 because the default denominator is always 1, and that means that for this function, as x increases by 1, y increases by 1.5. And we can confirm this by looking at the graph

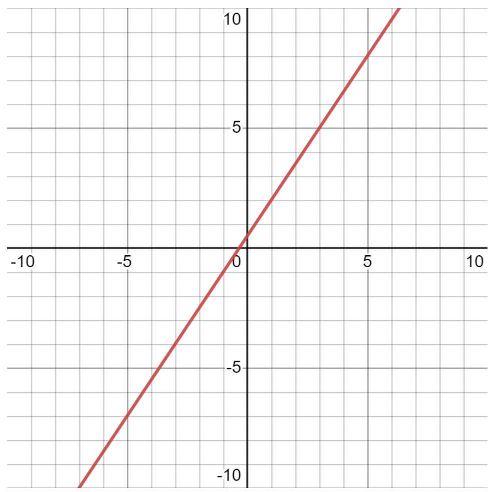

figure 7

4 Slope of non-linear functions

Now we know how to find the slope of a linear function; when it comes to linear functions, you can pick any two points in the graph and find the slope, and it will be the same as everywhere else in the graph; the reason for that is because the slope in a linear function is constant, which means it never changes, but let's take a look at this graph

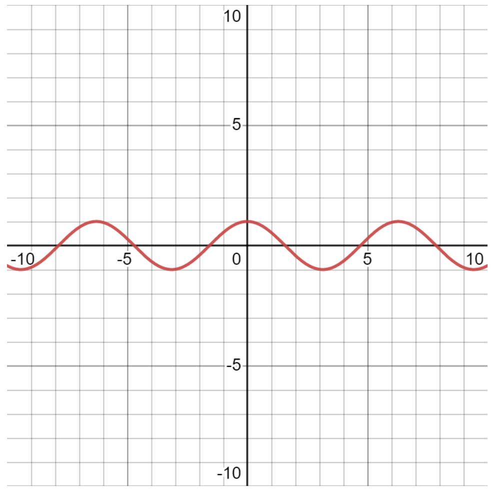

figure 8

If I were to ask you what the slope of this graph is, what would you say? Whatever your answer is, I can find a slope that contradicts your answer, because the slope of a non-linear function is not constant, which means that it is different at each specific point on the graph; we can try and guess the slope at $x = 0$ to be 0, because, at that point, the rate of change appears to be nonexistent, which is right, the rate of change at $x = 0$ is 0, but we will face a problem if we try to find the slope of a not very obvious point at the graph, for example at $x = 1$; if I were to ask you what the slope of this graph is at point $x = 1$ you wouldn't be able to give me an exact answer with the strategies we covered so far, but you might be able to give me an estimate by finding the slope of the secant line[4] of $x = 1$ and a point close to it; let's try to find an estimate by finding the slope of the secant line of $x = 1$ and $x = 4$. The function of this graph if you didn't notice by now is

$$f(x) = cos(x)$$

And we are trying to find the slope of this secant line

[4]A secant is a line that intersects a curve at a minimum of two distinct points.

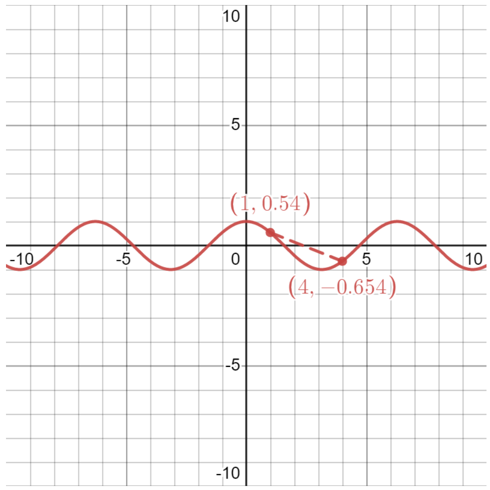

figure 9

We can now easily estimate the slope of this function at point $x = 1$ by using the slope of a line formula

$$m = \frac{-0.654 - 0.54}{4 - 1}$$

Which can be further simplified to

$$m = -0.398$$

Remember that this is merely the slope of the secant line and not the slope of the function at point $x = 1$, it is just an estimation, and something that we can do to improve this estimation is pick a closer value to 1 than 4, so let's try using 3

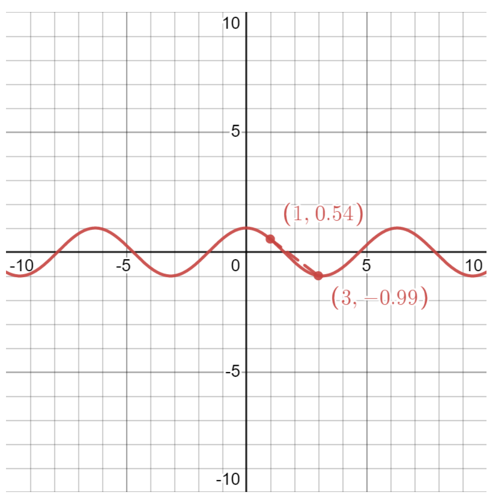

figure 10

We can now easily estimate the slope of this function at point $x = 1$ by using the slope of a line formula

$$m = \frac{-0.99 - 0.54}{3 - 1}$$

Which can be further simplified to

$$m = -0.765$$

Now we know that we can have a more accurate estimation just by calculating the slope of the secant line of $x = 1$ and a very close x value, but how close can we get? Can we somehow get the exact and accurate slope at a certain point in a graph? Yes.

5 Limit definition of the derivative

We now understand how the slope of a line formula works, it is the change of y over the change of x

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Now, hear me out for a second, let's write the same formula in this form

$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$

The x in this formula is the first point, and h is the offset of the second point to the first point, so if we try the example from the last section where we were trying to find the slope of the secant line of $x = 1$ and $x = 3$, this would look like

$$m = \frac{f(1+2) - f(1)}{(1+2) - 1}$$

This is just another way to represent the same formula from above but we can even simplify it further by removing the x's from the denominator since they will always divide each other out

$$m = \frac{f(x+h) - f(x)}{h}$$

We have already established that the smaller the secant line, the more accurate the slope is, so the smaller the offset, which is h, the more accurate the slope is, so we can use our understanding of limits to represent this idea, but before we do that, I want to introduce you to the idea of the derivative, the derivative is a function that is derived from a different function, that is simply why it is called a derivative, if you give x to the derivative function, the output (y) will be the slope of point x of the original function. A derivative is usually represented as "f prime of x" instead of "f of x" which is $f'(x)$ instead of $f(x)$. We can use all the information that we have so far to formulate the limit definition of the derivative, which is a definition that you can use to find the derivative of any function, and now we precisely understand how it works and it will help us find the derivative of any function so we won't have to find the slope of each point every time, we will have a function where we can simply substitute a point in, and it will give us the slope at that point

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let's now use this definition to find the derivative of a function

$$f(x) = x^2$$

The first step to do this is by finding what $f(x+h)$ is, and we can do that by simply switching every x in the function to $x+h$, and make sure to always put parentheses around the $x+h$ because it is a quantity. Now we can say that

$$f(x+h) = (x+h)^2$$

The next step is to plug our results in

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

From now on our only goal is to make it so we can plug in 0 as h without breaking the function, remember that our purpose of substituting in 0 is making the secant line as small as possible, and we concluded earlier that the smaller the offset (h) is, the more accurate the slope is, and we can be the most accurate by setting h equal to 0. At the moment we can't do that, because division by 0 is illegal in math, so we have to somehow get rid of the h at the denominator. We can do that by factoring the numerator to have an h on the outside

$$f'(x) = \lim_{h \to 0} \frac{(2x+h)h}{h}$$

We can now cancel out the h's and we will have

$$f'(x) = \lim_{h \to 0} 2x + h$$

And now the final step is finding the limit as h approaches 0 because only now we are allowed to do that, and that will leave us with

$$f'(x) = 2x$$

Now you may ask, okay what now? Now we can say that the exact and accurate slope of the function $f(x)$ at point a, is $f'(a)$. We now have a function that will give us the exact and accurate slope at any point instantly, without us having to do any unnecessary work. Now to double-check if all of the work we just did is accurate, let's graph the function $f(x) = x^2$ and use $f'(x) = 2x$ to find the slope at a certain point in the graph

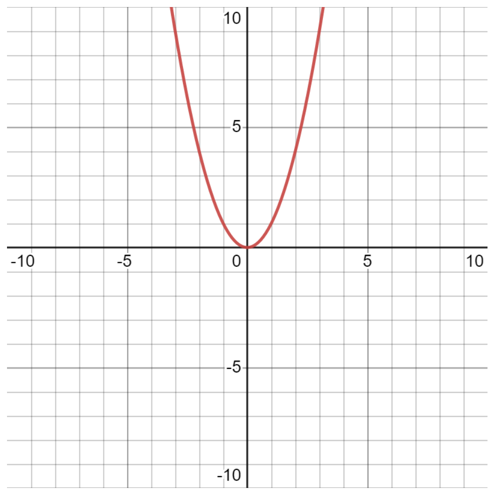

figure 11

Let's now assume that we want to find the exact slope at $x = 1$; in theory, we can plug in 1 into $f'(x)$ and it should give us the slope

$$f'(1) = 2$$

I am certain that this is the correct slope at this point, but just to make you more certain, let's try to graph a line that is going through point $f(1)$ which is $(1, 1)$ with the slope of 2 and let's see if it is correct; the equation for this line in $y = mx + b$ format is

$$y = 2x - 1$$

And the graph of this line with the graph of our function looks like this

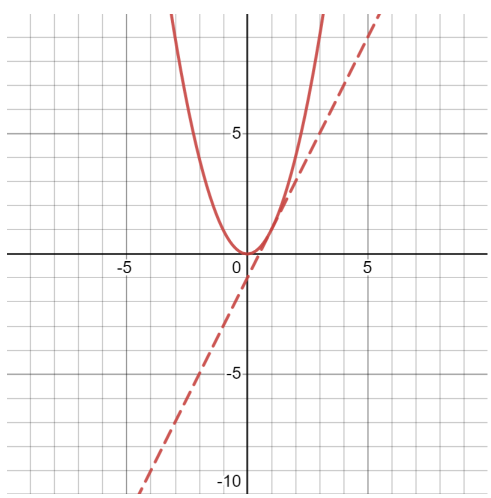

figure 12

Looking at the graph, we can see that the tangent line[5] is accurate, which means that we did everything right. Now you may ask, why are we doing all of this? What is the point? Did we do all of this just to draw a line? Those are very good questions that I asked myself when I first started learning about derivatives, but very soon I learned that there are a lot of real-life situations where finding the derivative is really useful, one of those many situations is finding the velocity. When you have a position over time function, the derivative of that function would be a velocity over time function, and if you think about it that makes a lot of sense, since velocity is the change of position over time, or in other words slope, and the derivative of a velocity over time function is an acceleration over time function, and that also makes sense because acceleration is the change of velocity over time. You don't have to do this process every time you want to find the derivative, you can always just use the derivative rules, but all of those rules are based on this, and it is essential to understand this concept, which will make using the rules even easier.

References

[1] Calculus note intro derivative. Berkeley City College. (n.d.). Retrieved May 6, 2022, from https://www.berkeleycitycollege.edu/wjeh/files/2012/08/calculus_note_intro_derivative.pdf

[2] Limit definition of the derivative. Harvey Mudd College Mathematics. (n.d.). Retrieved May 6, 2022, from https://math.hmc.edu/calculus/hmc-mathematics-calculus-online-tutorials/single-variable-calculus/limit-definition-of-the-derivative

[5] The tangent line to a plane curve at a given point is the straight line that "just touches" the curve at that point.